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# Probability Density Methods for the Analysis of Power Grids Under Uncertainty

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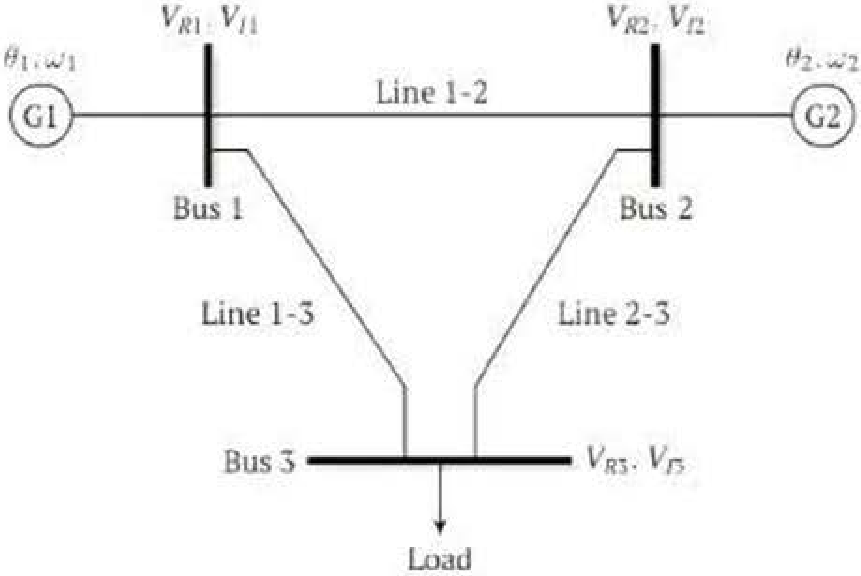
ENERGY

# Equations for generators of wind turbines




Index-1 DAE system for $N$ machines, $i \in [1, N]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = \omega_{\mathrm{B}} \Delta\omega_i,$$

$$2H_i \frac{\mathrm{d}}{\mathrm{d}t}\omega_i(t) = -D_i \Delta\omega_i + P_{\mathrm{e},i} + P_{\mathrm{m},i}(t),$$

$$0 = \mathbf{g}(\boldsymbol{\theta}, \mathbf{V})$$

| | |
|---|---|
| $\theta_i$ | Phase angle |
| $\omega_i$ | Angular speed |
| $V_{R,i}, V_{I,i}$ | Bus voltages |
| $P_{\mathrm{e},i}(\boldsymbol{\theta}, \mathbf{V})$ | Electric power |
| $D_i$ | Damping factor |
| $H_i$ | Inertia |
| $\omega_{\mathrm{B}}$ | Base angular speed |

Mechanical power (uncertain):

$$P_{\mathrm{m},i}(t) = \langle P_{\mathrm{m},i}(t) \rangle + P'_{\mathrm{m},i}(t)$$

$$\langle P'_{\mathrm{m},i}(t) P'_{\mathrm{m},j}(s) \rangle = \sigma_{ij}^2 \rho_{ij}(t, s)$$

# PDF method

**Objective:** Derive a PDE governing the joint PDF of phase angles and angular speeds, $p(\mathbf{x}, t)$, $\mathbf{x} = (\theta_1, \omega_1, \ldots, \theta_N, \omega_N)^\top \in \mathbb{R}^{2N}$

**Mathematical challenges:**

1. $\rho_{ij}(t, s) \neq \delta(t - s)$ (colored noise)
2. Various timescales need to be considered
   - Relaxation timescale $\gamma_i^{-1} = 2H_i / D_i$
   - Correlation timescale $\lambda_{ij}$

For $\rho_{ij}(t, s) = q_{ij}\delta(t - s)$ (white noise),

$$\frac{\partial p}{\partial t} + \nabla \cdot (\mathbf{v}(\mathbf{x})p) = \sum_{i=1}^{N} D_{\omega_i\omega_i} \sum_{j=1}^{N} \frac{\partial^2 p}{\partial \omega_i \omega_j} = \nabla \cdot (\mathbf{D}\nabla p)$$

$\mathbf{v}(\mathbf{x})$: Drift velocity (deterministic part of DAE system)
$D_{\omega_i\omega_j} = \sigma_{ij}^2 / 2$: Components of diffusion tensor $\mathbf{D}$

# Classical LED closure

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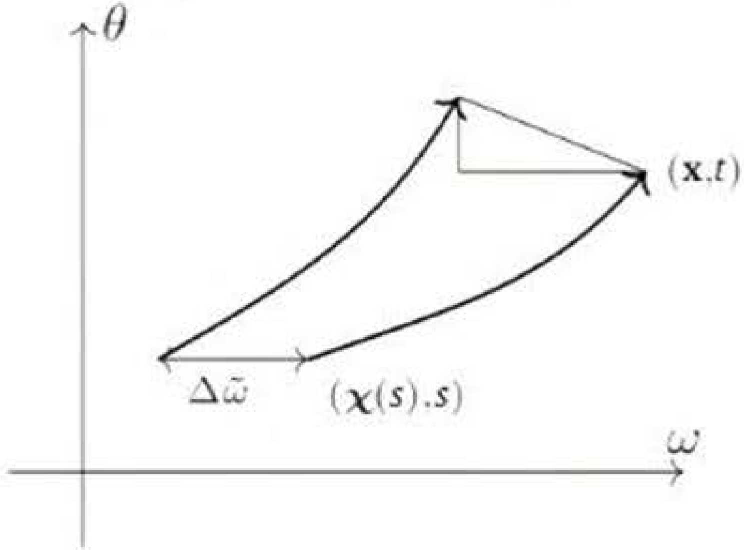
We employ a large-eddy diffusivity (LED) closure, $O(\sigma^2)$ accurate:

$$\frac{\partial p}{\partial t} + \nabla \cdot (\mathbf{v}p) = \sum_{i=1}^{N} \mathcal{D}_i[p]$$

$$\mathcal{D}_i[p(\mathbf{x},t)] = \int_0^t \left| \frac{\partial \chi(s)}{\partial \mathbf{x}} \right| \langle P'_{\mathrm{m},i}(t) P'_{\mathrm{m},j}(s) \rangle \frac{\partial}{\partial \tilde{\omega}_j} p(\chi(s), s)\, \mathrm{d}s,$$

where $\chi = (\tilde{\theta}_1, \tilde{\omega}_1, \ldots, \tilde{\theta}_N, \tilde{\omega}_N)^\top$, $\dot{\chi}(s) = \mathbf{v}(\chi)$, $\chi(t) = \mathbf{x}$




*Classical localization (Kraichnan)*:
Assume $\nabla_\chi p(\chi(s), s) \approx \nabla p(\mathbf{x}, t)$ for $|t - s| < \lambda_{ij}$, so that

$$\mathcal{D}_i[p] \approx \left( \int_0^t \langle P'_{\mathrm{m},i}(t) P'_{\mathrm{m},j}(s) \rangle \right) \frac{\partial}{\partial \omega_j} p$$

# Classical LED closure



Classical localization results in the PDF equation

$$\frac{\partial p}{\partial t} + \nabla \cdot (\mathbf{v}(\mathbf{x})p) = \sum_{i=1}^{N} D_{\omega_i \omega_j}(t) \sum_{j=1}^{N} \frac{\partial^2 p}{\partial \omega_i \omega_j} = \nabla \cdot (\mathbf{D}(t) \nabla p)$$

For $\lambda_{ij}\gamma_i \gg 1$, the SDAEs exhibit time-scale separation

- $\omega_i$: Fast variables, $\to \omega_S$ over the time scale $\gamma_i^{-1} \ll \lambda_{ij}$
- $\theta_i$: Slow variables

The classical approximation does not recover such behavior

# Modified LED closure



Deconvolve by considering only the advective contribution to the evolution of $p$ from $s$ to $t$:

$$\left|\frac{\partial \chi(s)}{\partial \mathbf{x}}\right| p(\chi(s), s) = p(\mathbf{x}, t) \to \mathcal{D}_i[p] = \left(\int_0^t \mathrm{d}s \langle P'_{\mathrm{m},i}(t) P'_{\mathrm{m},j}(s)\rangle \frac{\partial}{\partial \tilde{\omega}_j}\right) p(\mathbf{x}, t)$$

$$\mathcal{D}_i[p] = \left(\int_0^t \mathrm{d}s \langle P'_{\mathrm{m},i}(t) P'_{\mathrm{m},j}(s)\rangle \sum_{k=1}^{N} \left\{\frac{\partial \omega_k}{\partial \tilde{\omega}_j}\frac{\partial}{\partial \omega_k} + \frac{\partial \theta_k}{\partial \tilde{\omega}_j}\frac{\partial}{\partial \theta_k}\right\}\right) p(\mathbf{x}, t)$$

$$\frac{\partial p}{\partial t} + \nabla \cdot (\boldsymbol{v}(\mathbf{x})p) = \sum_{i=1}^{N} \frac{\partial}{\partial \omega_i}\left\{\sum_{j=1}^{N} D_{\omega_i\omega_j}(\mathbf{x}, t)\frac{\partial p}{\partial \omega_j} + D_{\omega_i\theta_j}(\mathbf{x}, t)\frac{\partial p}{\partial \theta_j}\right\}.$$

The challenge is to estimate the DAE sensitivities $\partial\omega_k/\partial\tilde{\omega}_j$, $\partial\theta_k/\partial\tilde{\omega}_j$, i.e., components of the sensitivity matrix

$$\Psi(t|\chi(s), s) = \mathcal{T}\exp\left(\int_s^t \mathbf{J}(\chi(s'), s')\,\mathrm{d}s'\right), \quad J_{ij}(\mathbf{x}, t) = \partial v_i(\mathbf{x}, t)/\partial x_j$$

# Diffusion coefficients

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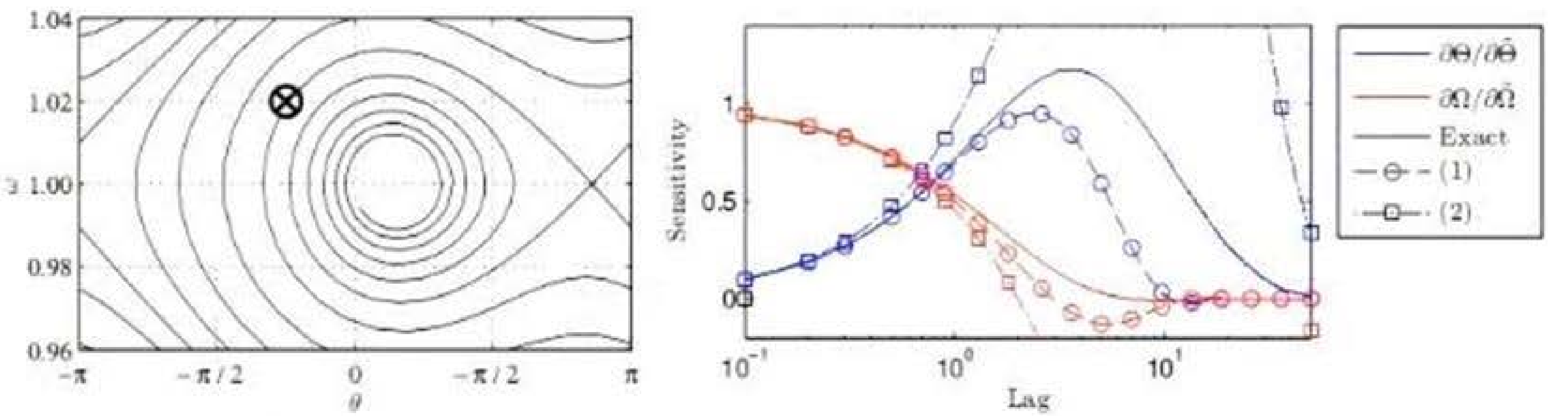
Various closed-form approximations to diffusion coefficients $\mathcal{D}_{\omega_i\omega_j}$, $\mathcal{D}_{\omega_i\theta_j}$.

1. $\lambda_{ij}\gamma_i > 1$: Linearize time-order exp. around $(\mathbf{x}, t)$:

$$\Psi(t|\chi(s), s) = \mathcal{T} \exp\left(\int_s^t \mathbf{J}(\chi(s'), s')\, \mathrm{d}s'\right) \approx \exp\left\{(t-s)\mathbf{J}(\mathbf{x}, t)\right\}$$

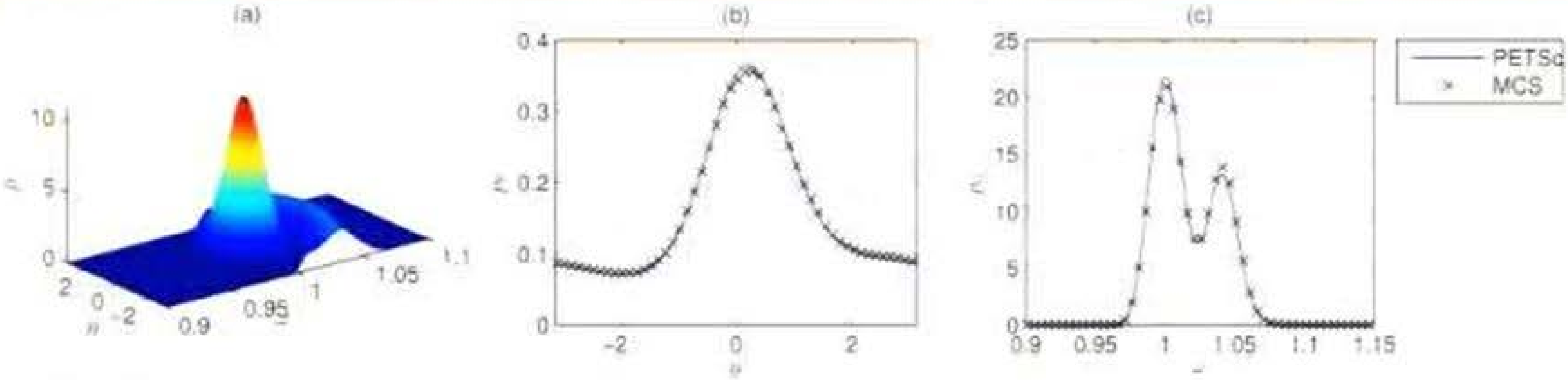
2. $\lambda_{ij}\gamma_i \ll 1$: Power series in $\lambda_{ij}$ of time-ordered exponential



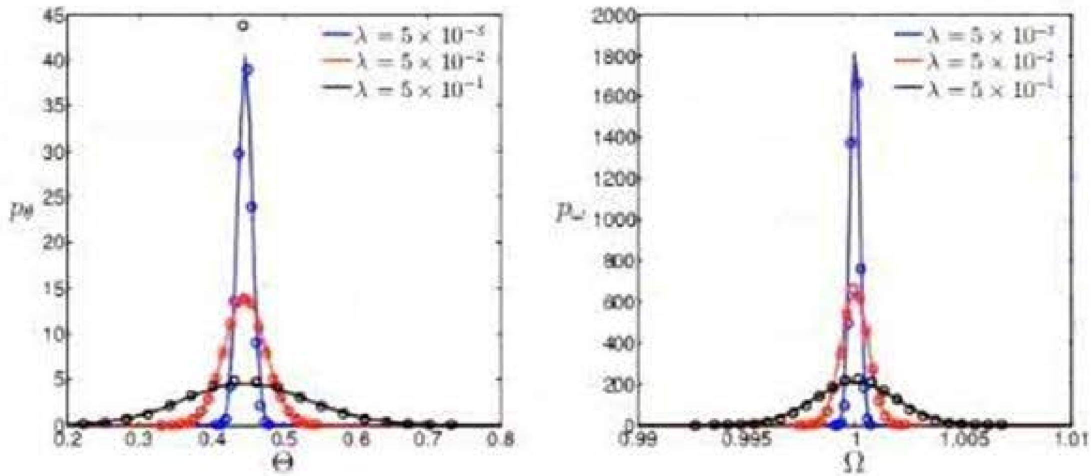

Comparison of various approximations to diffusion coefficients, $\lambda\gamma = 5$

# Numerical experiments






(a) Joint PDF of SMIB system. (b)–(c) Marginal PDFs of $\theta$ and $\omega$




Marginal PDF snapshots

# Numerical experiments

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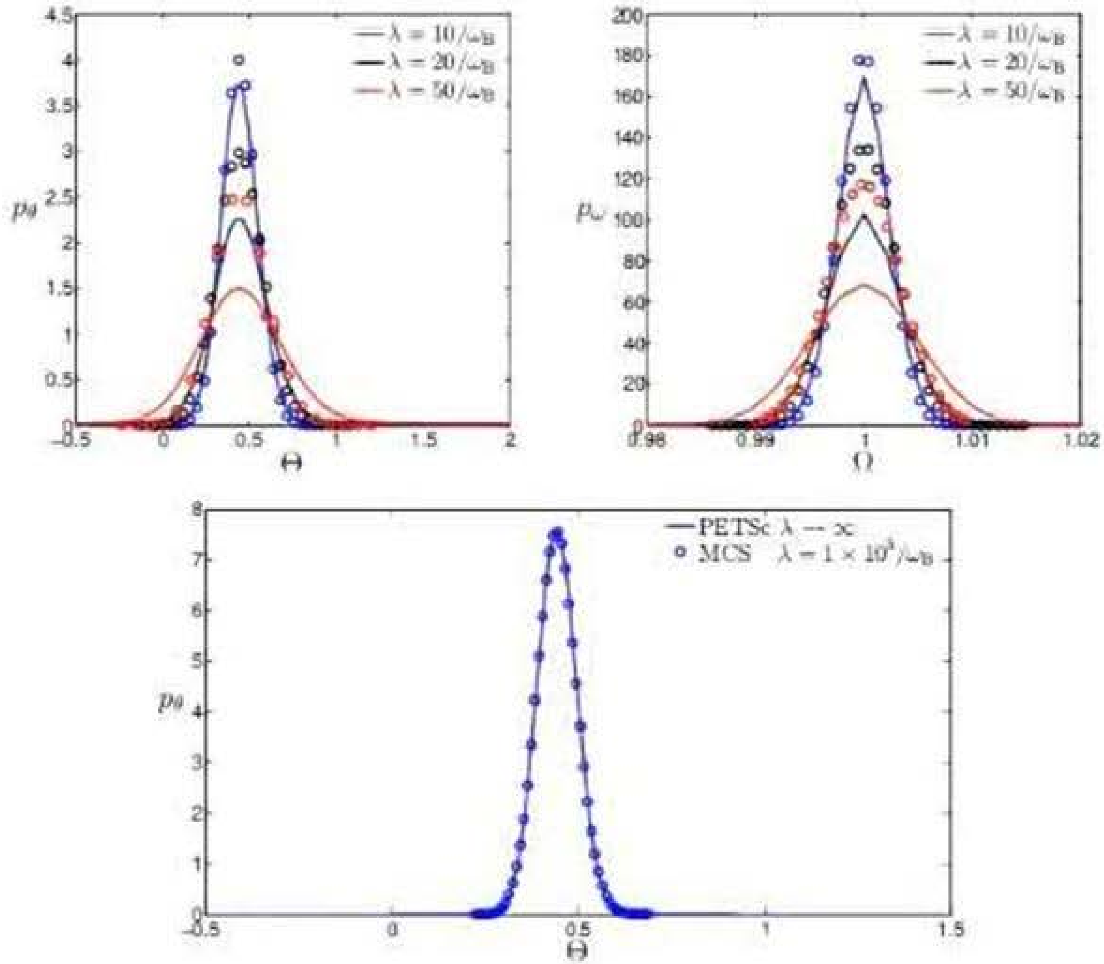


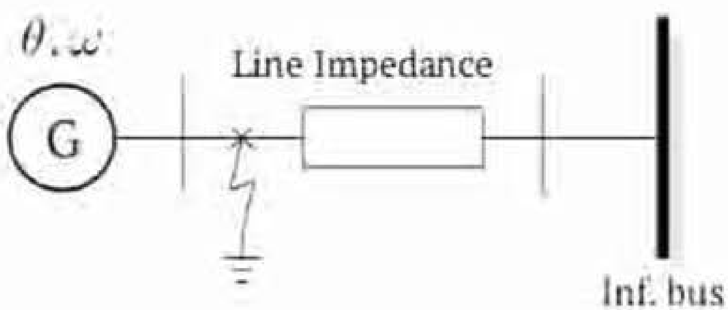

Stationary marginal PDFs

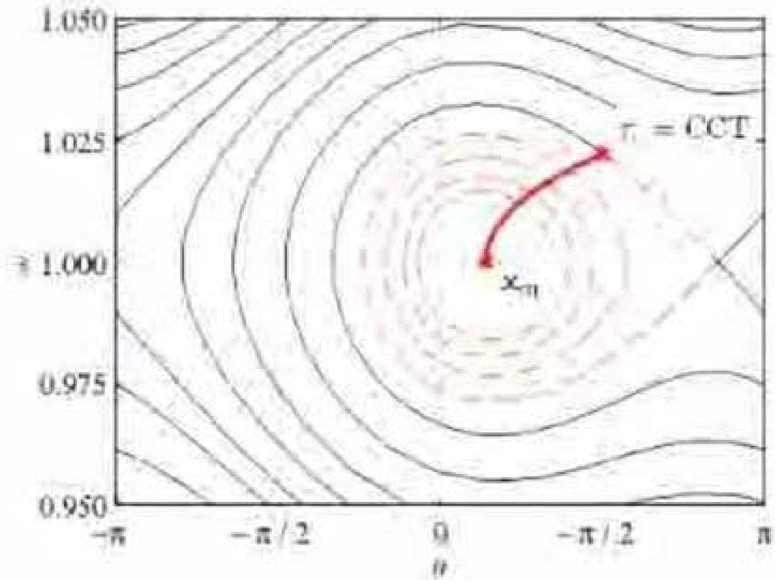
# Transient stability under uncertainty

**Failure**: Catastrophic, unrecoverable event (i.e., relay triggering after fault clearing)

Due to uncertainty in $P_m$, a power system has non-trivial probability of failure after clearing a fault in sub-critical time $\tau_c$

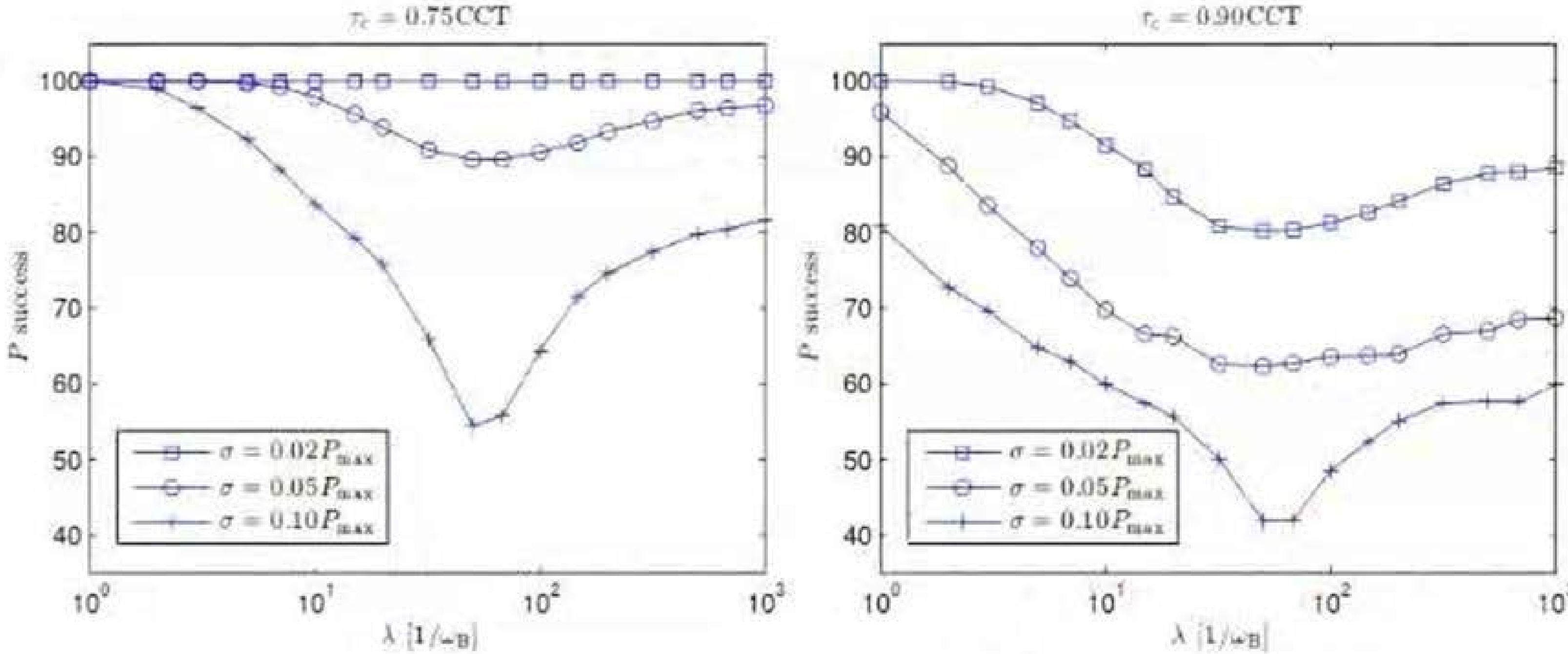
**Example:** For SMIB, define failure as $|\omega - \omega_s| \geq 0.05\omega_s$



# Transient stability under uncertainty






Probability of maintaining stability after clearing of fault, $\tau_c <$ CCT

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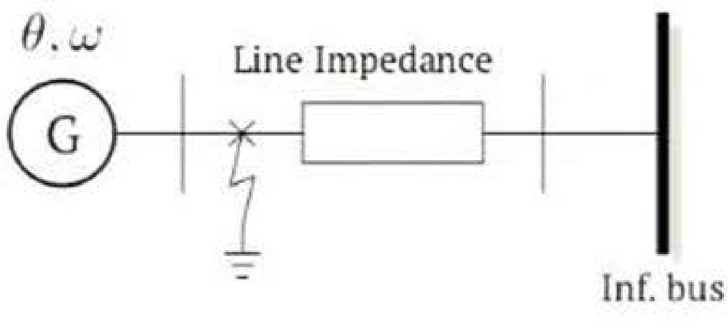


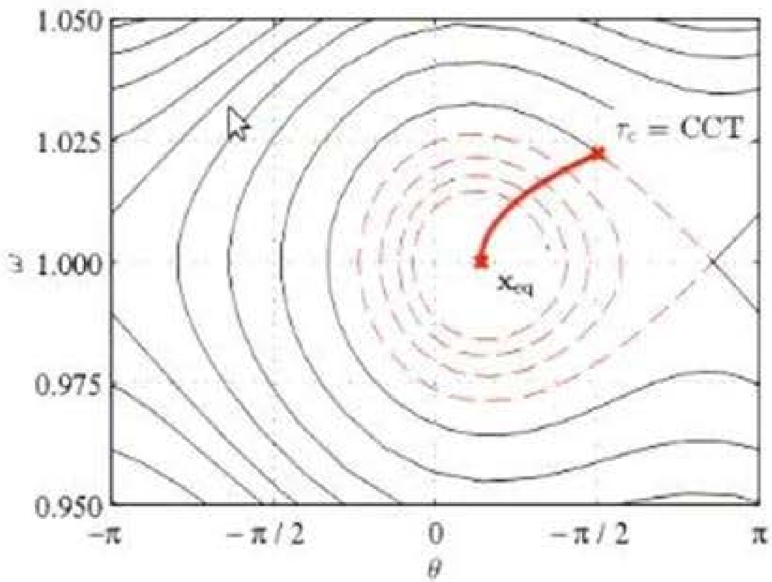

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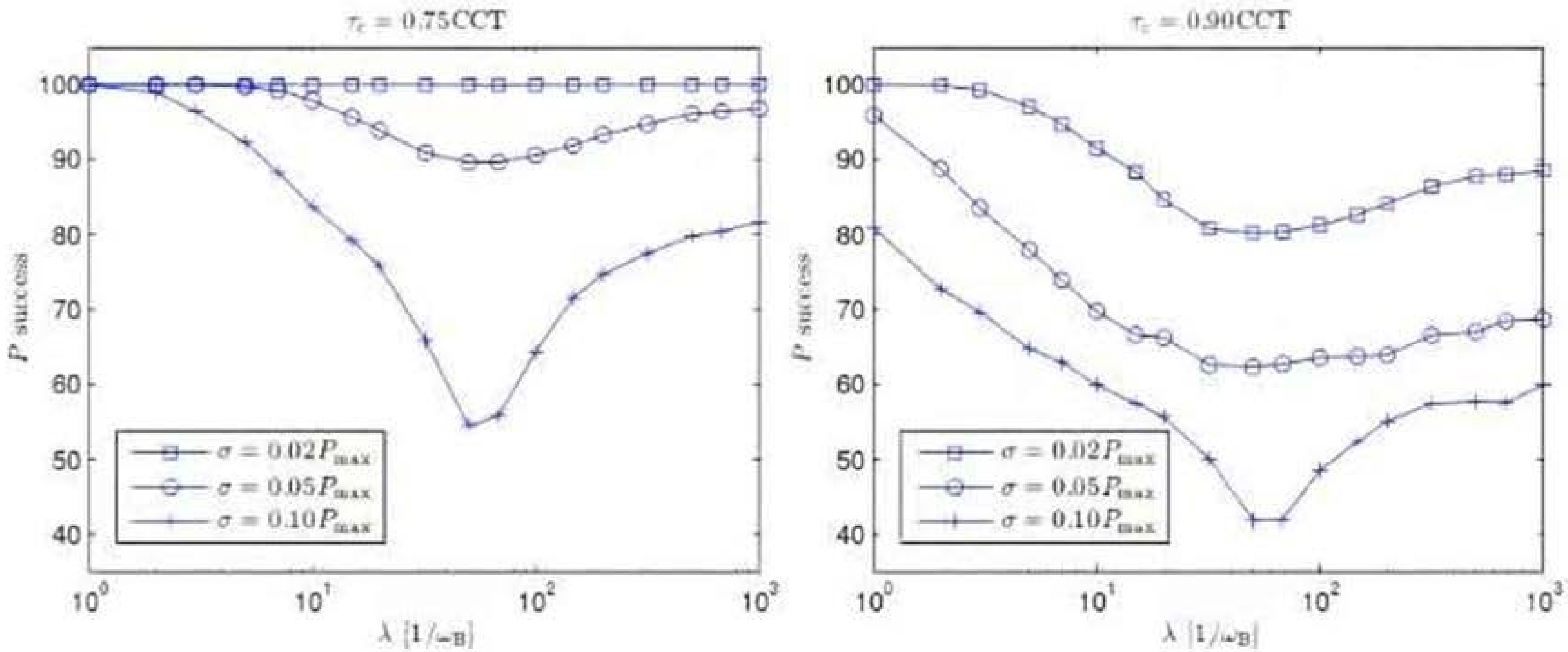


Probability of maintaining stability after clearing of fault, $\tau_c <$ CCT