Dark-Bright, Dark-Dark, Vortex-Bright: Multi-Component NLS Beasts

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References

- 1d GPE:
  - New J. Phys. **14**, 055006 (2012); PRA **84**, 053626 (2011); Physica D **240**, 767 (2011).
  - PRA **77**, 033612 (2008); PRA **91**, 043637 (2015); DCDS-S **4**, 1199 (2011).
  - PRE **91**, 032905 (2015); PRA **91**, 023619 (2015); PRE **91**, 012924 (2015).
- Experimental Results:
  - PLA **375**, 642 (2011); PRA **84**, 053630 (2011);
  - J. Phys. B **45**, 115301 (2012); J. Phys. B **46**, 065302 (2013).
- Generalizations to Vortices & Higher Dimensions:
  - J. Phys. B **44**, 191003 (2011); PRA **86**, 053601 (2012);
  - PRL **105**, 160405 (2010).

Brief Introduction to BECs

- 1924: S. Bose and A. Einstein realize that Bose statistics predicts a Maximum Atom Number in the Excited States: a Quantum Phase Transition.

- 1995: E. Cornell, C. Wieman and W. Ketterle realize BEC in a dilute gas of ^{87}Rb and ^{23}Na: 2001 Nobel Prize.

- Today:
  - ~ 35 Experimental Groups have achieved BEC (in 100-10^8 atoms of Rb, Li, Na, H).
  - O(10^3) Theoretical and O(10^2) Experimental papers ! Check out: http://amo.phy.gasou.edu/bec.html/bibliography.html

Mean-Field Models of BEC: why do we care ?

BEC

- Many Body Hamiltonian

$$\hat{H} = \int d\mathrm{r}\hat{\Psi}^{\dagger}\left[-\frac{\hbar^2}{2m}\Delta + V_{\mathrm{ext}}(\mathrm{r})\right]\hat{\Psi} + \frac{1}{2}\int d\mathrm{r}d\mathrm{r}'\hat{\Psi}^{\dagger}(\mathrm{r})\hat{\Psi}^{\dagger}(\mathrm{r}')V(\mathrm{r}-\mathrm{r}')\hat{\Psi}(\mathrm{r}')\hat{\Psi}(\mathrm{r}) \tag{1}$$

- Bogoliubov Decomposition:

$$\hat{\Psi} = \Phi(\mathrm{r},t) + \hat{\Psi}'(\mathrm{r},t) \tag{2}$$

- Φ is now a regular wavefunction (the expectation value of the field operator). Its equation:

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Phi + V_{\mathrm{ext}}(\mathrm{r})\Phi + g|\Phi|^2\Phi \tag{3}$$

- for dilute, cold, binary collision gas.
- But: This is 3D NLS with a Potential: GP !

Low Dimensional Reductions

- 1d Magnetic Trap and/or Optical Lattice

$$V(x) = \frac{1}{2}\Omega^2 x^2 + V_0 \sin^2(kx+\theta) \tag{4}$$

- 2d Magnetic Trap and/or Optical Lattice

$$V(x,y) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2\right) + V_0\left(\sin^2(kx+\theta) + \sin^2(ky+\theta)\right) \tag{5}$$

- Typical 1d Scenario: $g > 0 \Rightarrow$ Exact Prototypical Solutions: Dark Solitons

$$\Phi(x,t) = e^{-it}\tanh(x-x_0) \Rightarrow n = |\Phi|^2 = \tanh^2(x-x_0) \tag{6}$$

- It is also possible to have Multiple Spin States of a Bose Gas (such as ^{87}Rb or ^{23}Na or mixtures thereof) $\Rightarrow$ This will be our Focus here.

$$i\frac{\partial \psi_n}{\partial t} = -\frac{1}{2}\nabla^2\psi_n + V_n(\mathrm{r})\psi_n + \sum_{k=1}^{\mathcal{N}}\left[g_{nk}|\psi_k|^2\psi_n - \kappa_{nk}\psi_k + \Delta_{nk}\psi_n\right]. \tag{7}$$

Mean-Field Models of BEC: why do we care ?

BEC

- Many Body Hamiltonian

$$\hat{H} = \int d\mathrm{r} \hat{\Psi}^\dagger \left[-\frac{\hbar^2}{2m}\Delta + V_{\mathrm{ext}}(\mathrm{r}) \right] \hat{\Psi} + \frac{1}{2} \int d\mathrm{r} d\mathrm{r}' \hat{\Psi}^\dagger(\mathrm{r}) \hat{\Psi}^\dagger(\mathrm{r}') V(\mathrm{r}-\mathrm{r}') \hat{\Psi}(\mathrm{r}') \hat{\Psi}(\mathrm{r}) \tag{1}$$

- Bogoliubov Decomposition:

$$\hat{\Psi} = \Phi(\mathrm{r}, t) + \hat{\Psi}'(\mathrm{r}, t) \tag{2}$$

- Φ is now a regular wavefunction (the expectation value of the field operator). Its equation:

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- for dilute, cold, binary collision gas.
- But: This is 3D NLS with a Potential: GP !

Early Motivation: Dark-Bright Solitons in Nonlinear Optics

- Dark-Bright Solitons were shown to Robustly Persist in Photorefractive Crystals

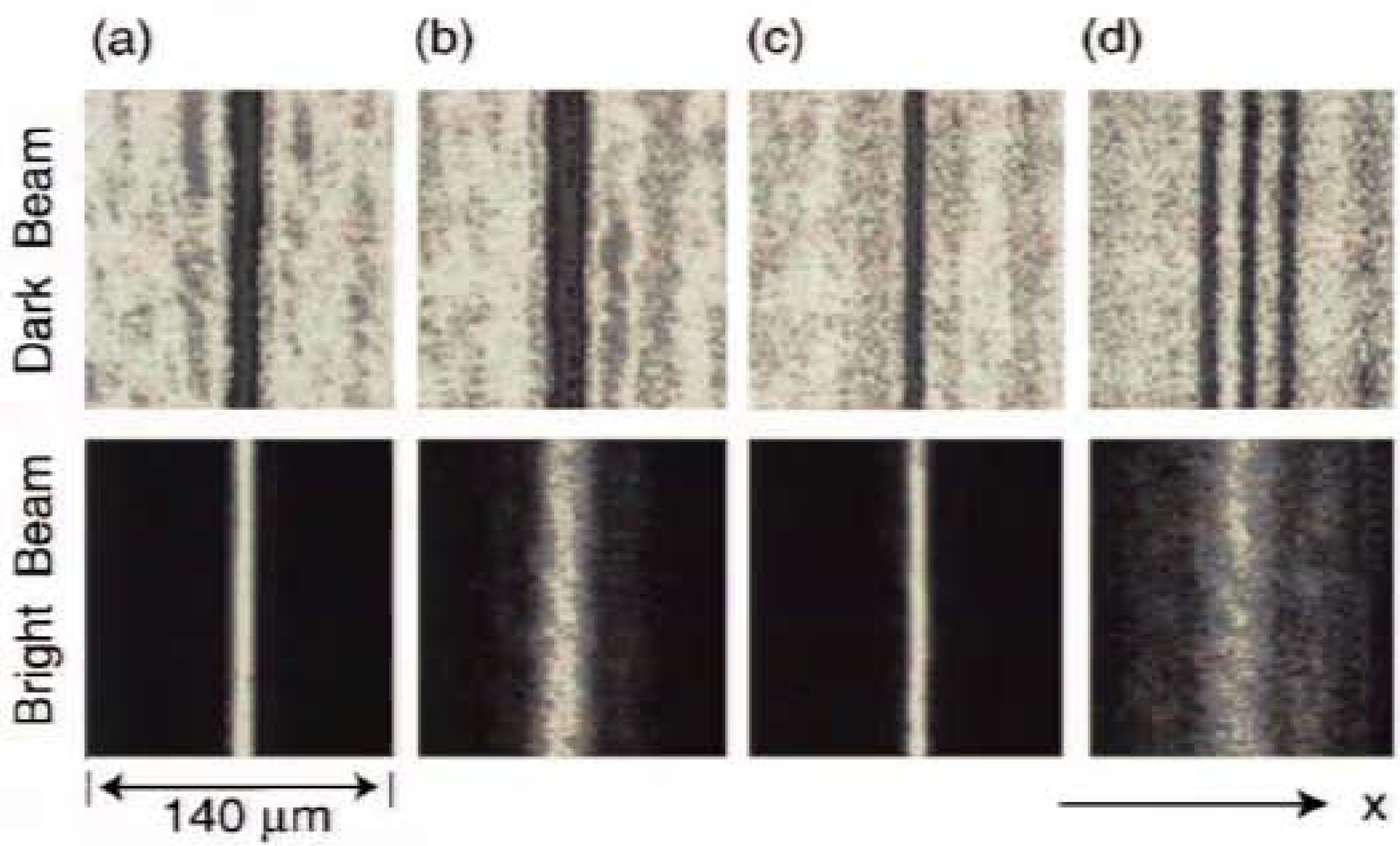


Fig. 7

Citation
Zhigang Chen, Mordechai Segev, Tamer H. Coskun, Demetrios N. Christodoulides, Yuri S. Kivshar, "Coupled photorefractive spatial-soliton pairs," J. Opt. Soc. Am. B 14, 3066-3077 (1997);
http://www.opticsinfobase.org/josab/abstract.cfm?URI=josab-14-11-3066

Further Early Motivation: Dark-Bright Soliton Pairs in Photorefractives

- Optical (Dark) Solitons were found to be Glued Together by Attraction between the Non-Soliton Beams they Guide
- This gave rise to the notion of Solitonic Gluons

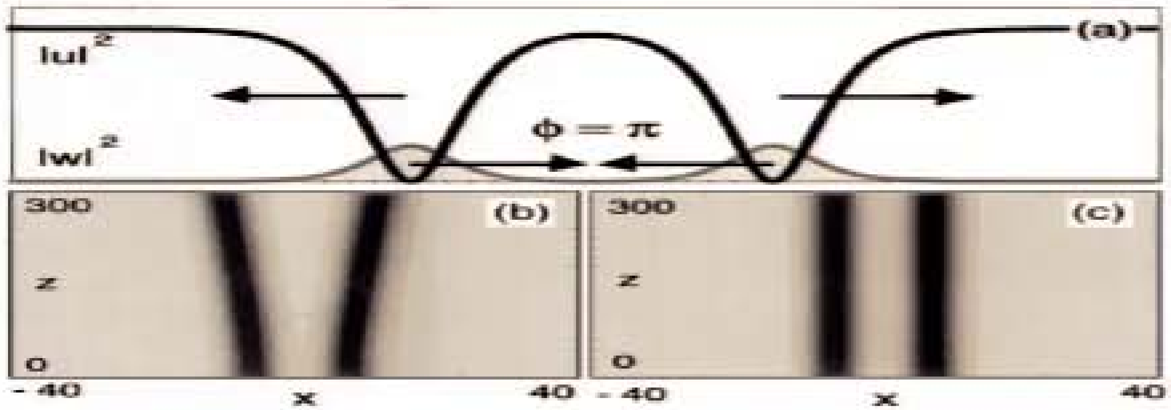


Fig. 1

Citation
Elena A. Ostrovskaya, Yuri S. Kivshar, Zhigang Chen, Mordechai Segev, "Interaction between vector solitons and solitonic gluons," Opt. Lett. 24, 327-329 (1999);

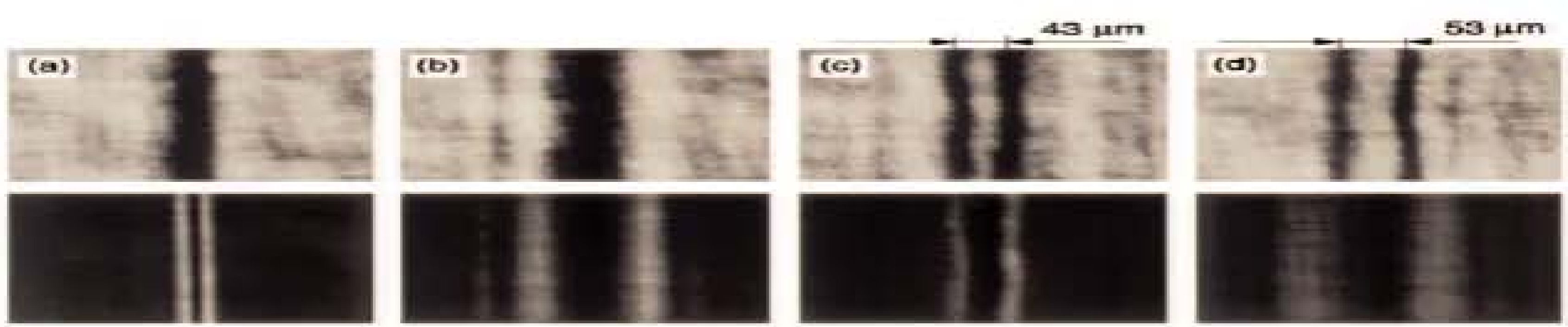
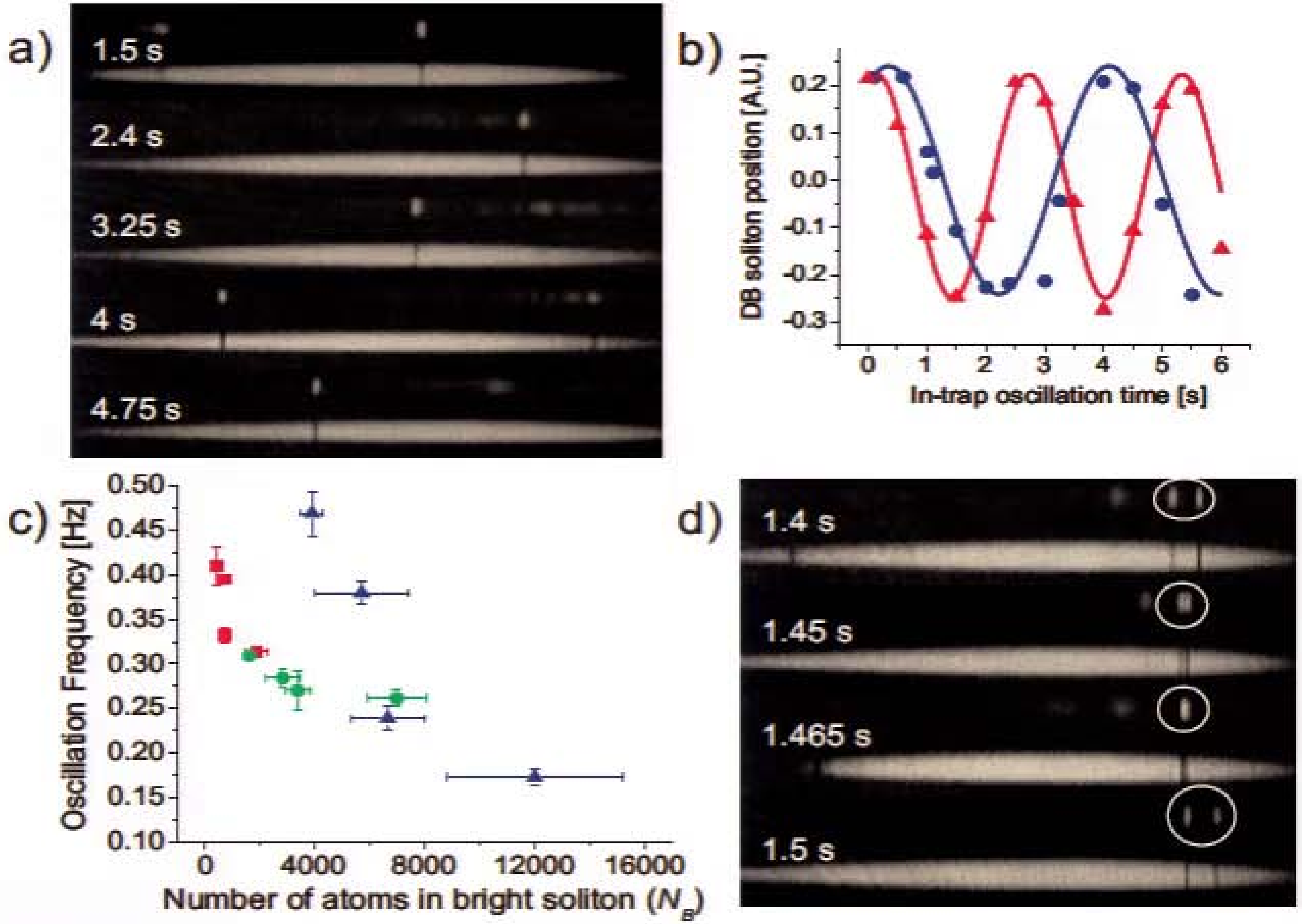


Fig. 2

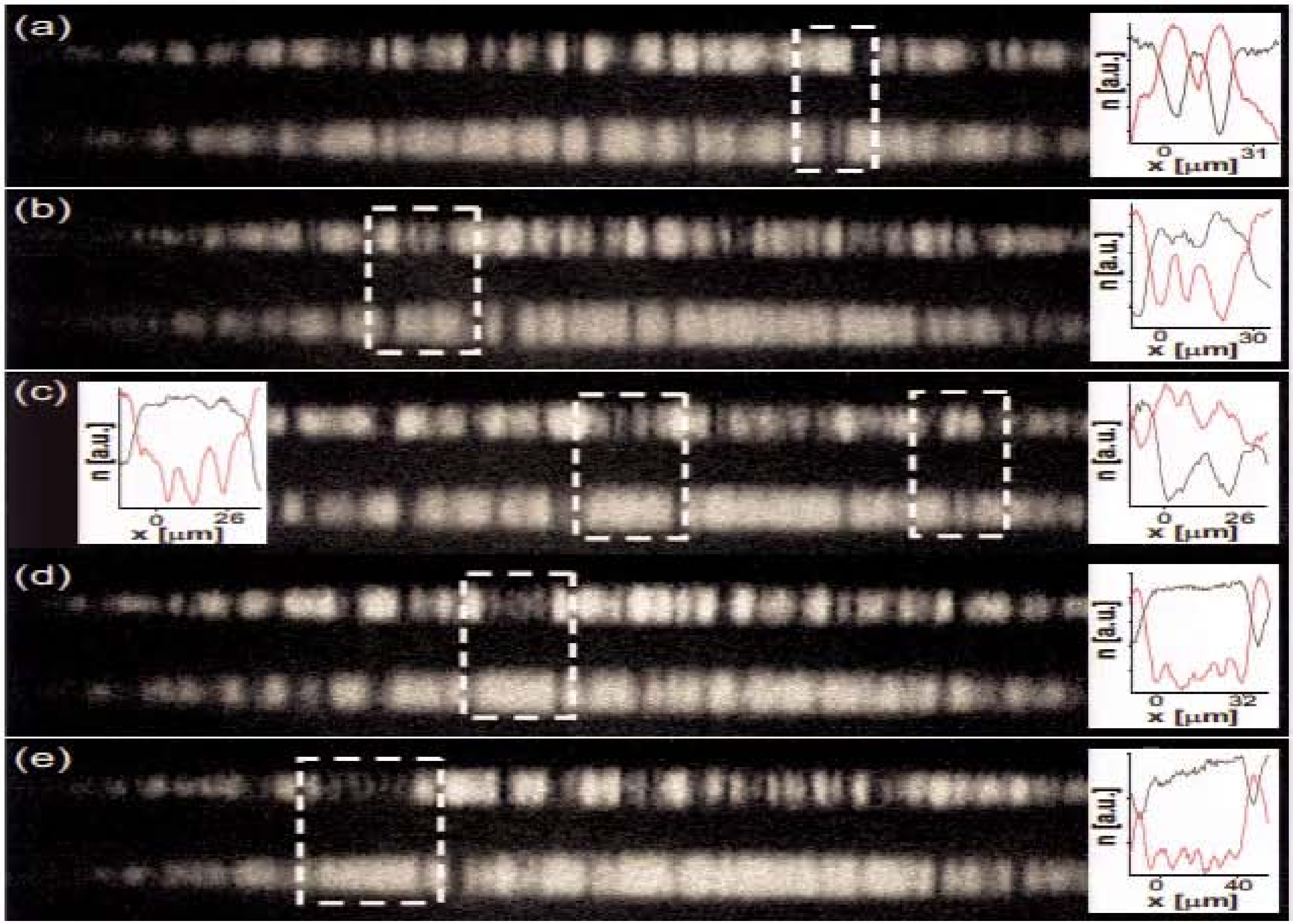
Citation
Elena A. Ostrovskaya, Yuri S. Kivshar, Zhigang Chen, Mordechai Segev, "Interaction between vector solitons and solitonic gluons," Opt. Lett. 24, 327-329 (1999);

More Recent Motivation: (Pseudo)-Spinor Experiments in BECs

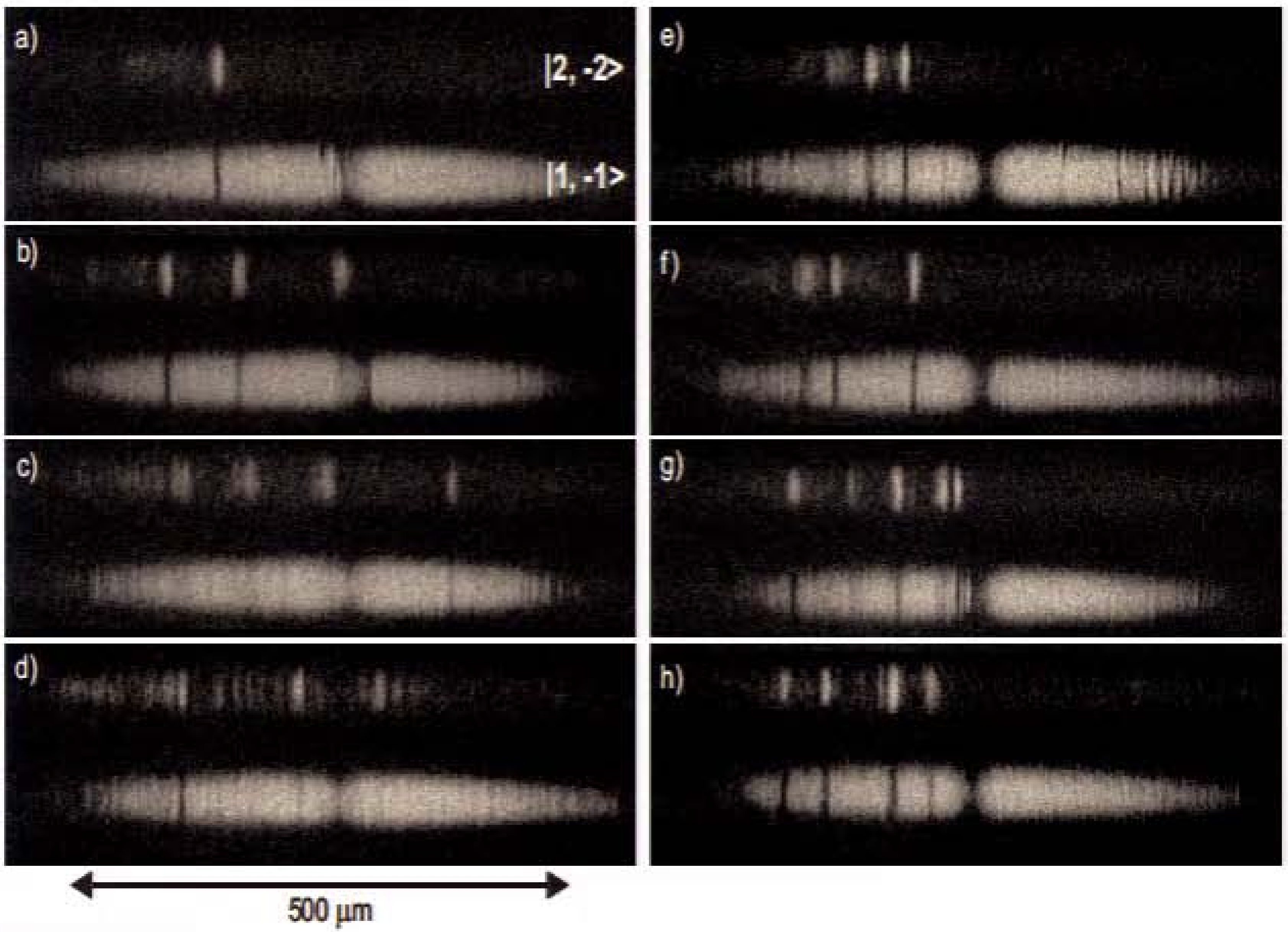
2-Components, 1-dimension: Dark-Bright Solitons in Pullman



More Complex Configurations: Multi-Dark-Bright Solitons in BECs (2, 3, 4, 5,...)

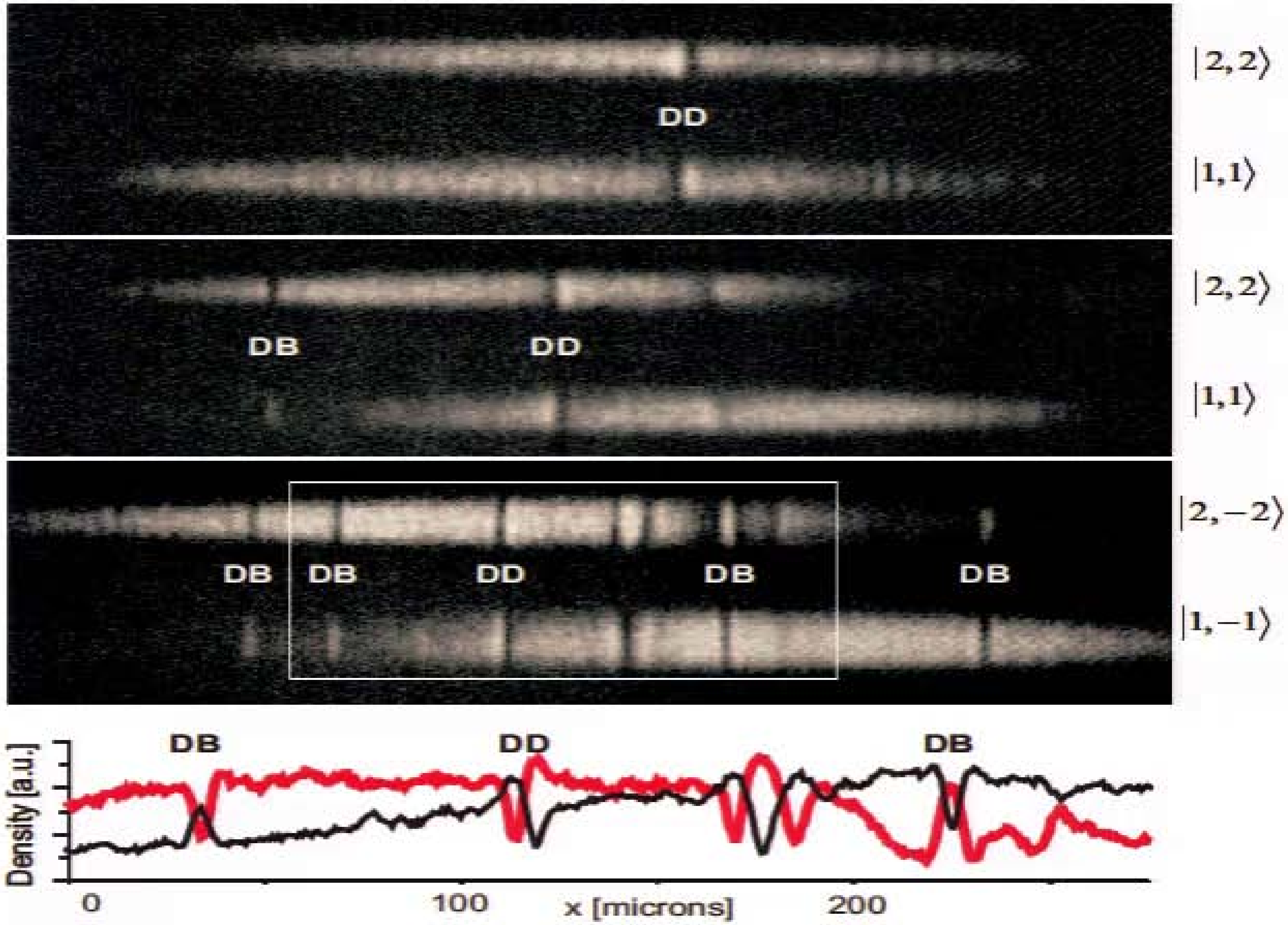


More Complex Dynamics: Interaction of Dark-Bright Solitons with Barriers



(Even) More Complex Dynamics: Counter-Flow Experiments

Spontaneous Production of Dark-Bright and Dark-Dark Solitons



Theoretical Analysis of the States: The Single Dark-Bright Soliton

- The General Model (for Two Hyperfine State of ^{87}Rb, or ^{23}Na reads:

$$i\frac{\partial\psi_n}{\partial t} = -\frac{1}{2}\nabla^2\psi_n + V_n(\mathrm{r})\psi_n + \sum_{k=1}^{\mathcal{N}}\left[g_{nk}|\psi_k|^2\psi_n - \kappa_{nk}\psi_k + \Delta_{nk}\psi_n\right]. \quad (8)$$

- Constraining to the Simpler Case of Purely Nonlinear Coupling, we (Approximately, since e.g. $g_{11} : g_{12} : g_{22} \approx 1.03 : 1 : 0.97$) have:

$$i\partial_t\psi_1 = -\frac{1}{2}\partial_x^2\psi_1 + V(x)\psi_1 + (|\psi_1|^2 + |\psi_2|^2 - \mu)\psi_1, \quad (9)$$

$$i\partial_t\psi_2 = -\frac{1}{2}\partial_x^2\psi_2 + V(x)\psi_2 + (|\psi_1|^2 + |\psi_2|^2 - \mu)\psi_2, \quad (10)$$

- Further, assuming Absence of Potential $V(x) = 0$, we obtain the Integrable Manakov Model. Special Solutions in the form of Dark-Bright Solitons read:

$$\psi_1(x,t) = \sqrt{\mu}(\cos\phi\,\tanh\xi + i\sin\phi), \quad (11)$$

$$\psi_2(x,t) = \eta\,\mathrm{sech}\,\xi\,\exp[ikx + i\theta(t)], \quad (12)$$

where $\xi = D(x - x_0(t))$, ϕ is the phase, $\cos\phi$ and η are the amplitude of the dark and bright solitons, and D and $x_0(t)$ describe the inverse width and the center position of the DB; $D^2 = \mu\cos^2\phi - \eta^2$, $\dot{x}_0 = k = D\tan\phi$ and $\theta(t) = (1/2)(D^2 - k^2)t + \theta_0$.

A Cousin of the DB States: The Single Dark-Dark Soliton

- The Manakov Model is Invariant under SU(2) Rotations with Matrices:

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

- Applying this to a DB, we obtain the Rotated Solutions:

$$\psi_1(x,t) = \alpha\sqrt{\mu}\{\cos\phi \tanh\xi + i\sin\phi\} - \beta^*\eta \operatorname{sech}\xi \exp\{ikx + i\theta(t)\}, \quad (13)$$

$$\psi_2(x,t) = \beta\sqrt{\mu}\{\cos\phi \tanh\xi + i\sin\phi\} + \alpha^*\eta \operatorname{sech}\xi \exp\{ikx + i\theta(t)\}. \quad (14)$$

- In the Special Case of SO(2) Rotation, $\alpha = \cos(\delta)$ and $\beta = \sin(\delta)$. Then, the Respective (Time-Dependent !) Densities become:

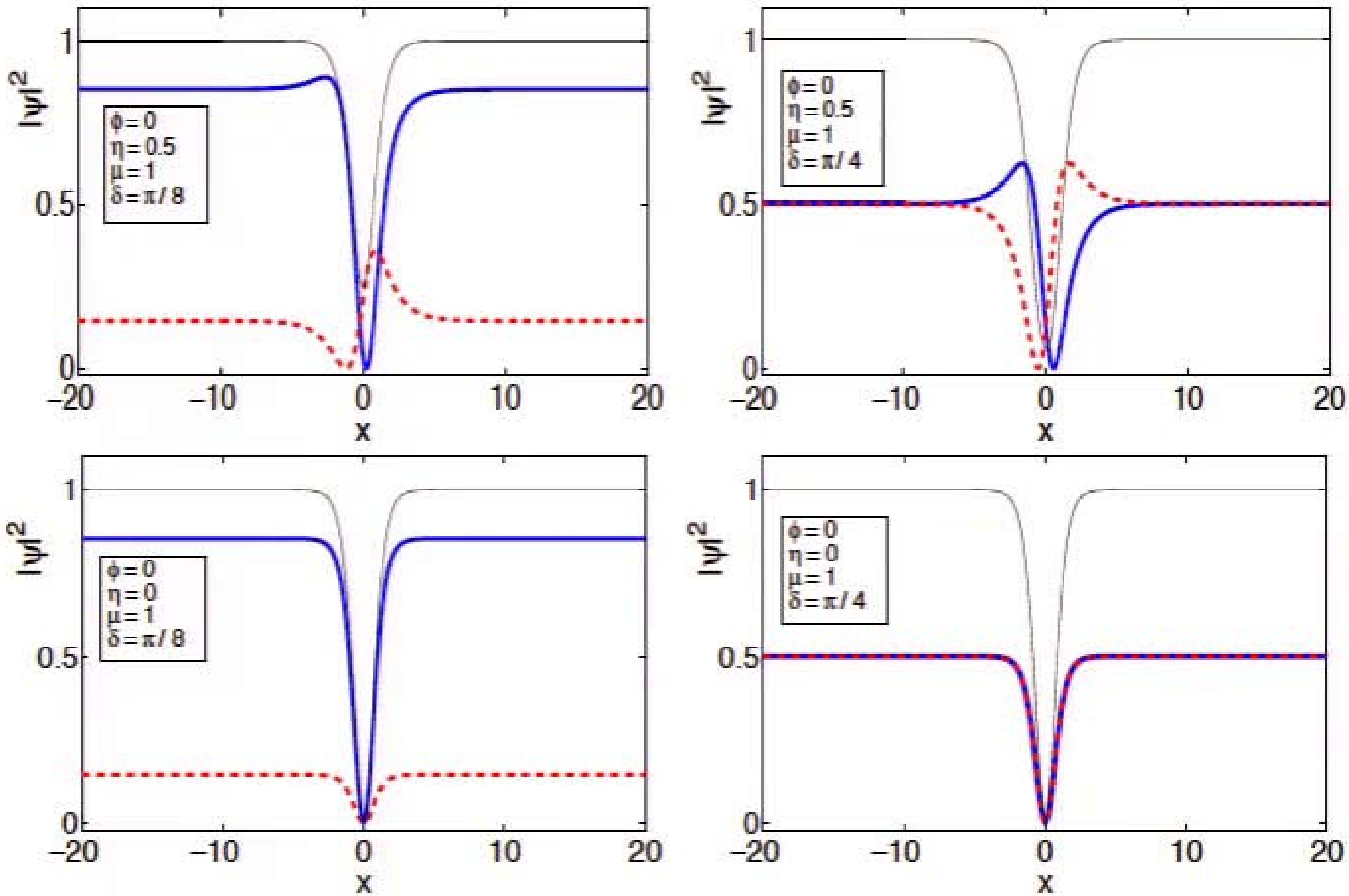
$$n_1 \equiv |\psi_1|^2 = \mu\cos^2(\delta) - (\mu\cos^2(\delta)\cos^2\phi - \eta^2\sin^2(\delta))\operatorname{sech}^2\xi - \sqrt{\mu}\eta\sin(2\delta) \times \{\sin\phi\sin[kx+\theta(t)] + \cos\phi\cos[kx+\theta(t)]\tanh\xi\}\operatorname{sech}\xi, \quad (15)$$

$$n_2 \equiv |\psi_2|^2 = \mu\sin^2(\delta) - (\mu\sin^2(\delta)\cos^2\phi - \eta^2\cos^2(\delta))\operatorname{sech}^2\xi + \sqrt{\mu}\eta\sin(2\delta) \times \{\sin\phi\sin[kx+\theta(t)] + \cos\phi\cos[kx+\theta(t)]\tanh\xi\}\operatorname{sech}\xi, \quad (16)$$

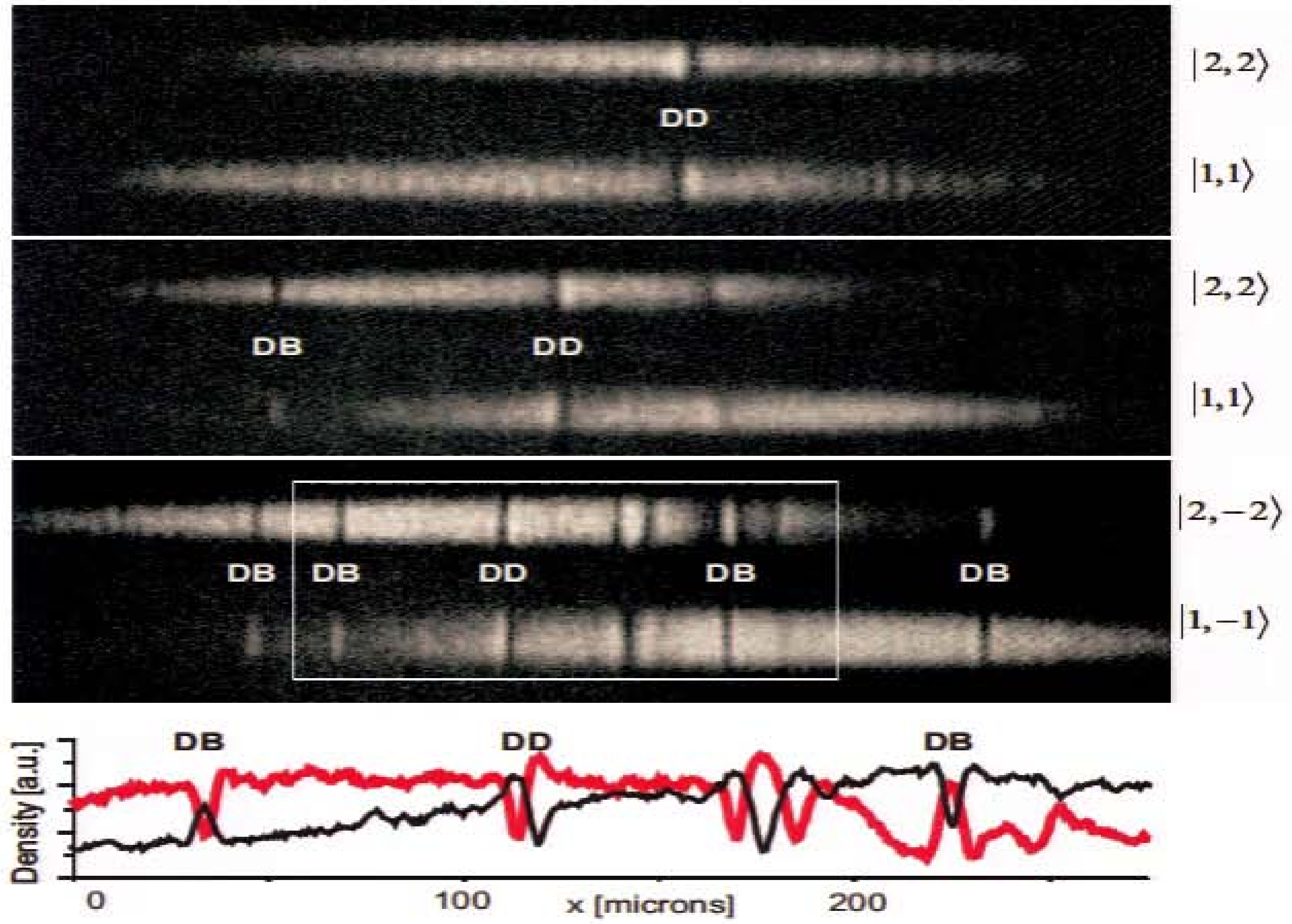
- The Frequency of DD Soliton Oscillation for $x_0(t) = kt$ is given by $(\frac{1}{2}k^2 < \omega_0 < \frac{1}{2}\mu)$:

$$\omega_0 = \frac{1}{2}(k^2 + D^2) = \frac{1}{2}(\mu - \eta^2\sec^2\phi), \quad (17)$$

The Single Dark-Dark Soliton: Spatial Representation in Different Cases



Emergence of DB and DD Solitons in Experiments: Check !



DB Solitons in a Trap

- Rewrite the problem with the Trap in a Perturbed Form (using $u_d \to u_{TF} u_d$ with $|u_{d,\mathrm{TF}}|^2 = \max\{\mu - V(x), 0\}$, $t \to \mu t$, $x \to \sqrt{\mu} x$, $|u_b|^2 \to \mu^{-1}|u_b|^2$):

$$i\partial_t u_d + \frac{1}{2}\partial_x^2 u_d - \left(|u_d|^2 + |u_b|^2 - 1\right) u_d = R_d, \quad (18)$$

$$i\partial_t u_b + \frac{1}{2}\partial_x^2 u_b - \left(|u_b|^2 + |u_d|^2 - \tilde{\mu}\right) u_b = R_b, \quad (19)$$

where $\tilde{\mu} = 1 + \Delta/\mu$ and the functional perturbations R_d and R_b are given by:

$$R_d \equiv (2\mu^2)^{-1}[2(1 - |u_d|^2)V(x)u_d + V'(x)\partial_x u_d], \quad (20)$$

$$R_b \equiv \mu^{-2}[(1 - |u_d|^2)V(x)u_b]. \quad (21)$$

- Consider the Model's Energy:

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} |\partial_x u_d|^2 + |\partial_x u_b|^2 + (|u_d|^2 + |u_b|^2 - 1)^2 - 2(\tilde{\mu} - 1)|u_b|^2 \, dx. \quad (22)$$

- For a DB, the Energy is:

$$E = \frac{4}{3}D^3 + \chi\left(\frac{1}{2}D^2 \sec^2\phi - \frac{\Delta}{\mu}\right) \quad \text{with} \quad \chi \equiv \frac{N_b}{\sqrt{\mu}}. \quad (23)$$

DB Solitons in a Trap (Contd.)

- We Assume a Slowly Varrying DB with $\phi \to \phi(t)$ and $D \to D(t)$

$$D^2(t) = \cos^2\phi(t) - \frac{\chi}{2}D(t), \quad \dot{x}_0(t) = D(t)\tan\phi(t), \tag{24}$$

- Then, the Energy Evolution (for the Soliton Parameters) is:

$$\frac{dE}{dt} = 4\dot{D}D^2 + \chi D\sec^2\phi(\dot{D} + D\dot{\phi}\tan\phi). \tag{25}$$

- Calculating the dE/dt from the Equation of Motion:

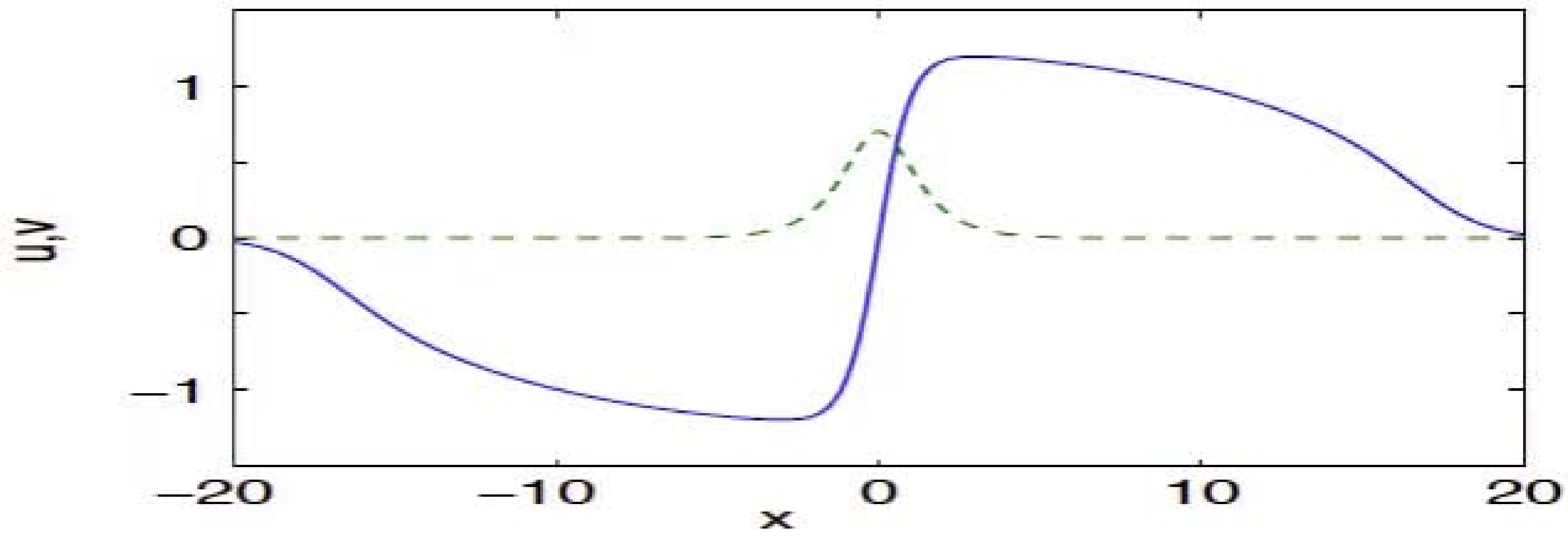
$$\frac{dE}{dt} = -2\mathrm{Re}\left\{\int_{-\infty}^{+\infty}(R_d^*\partial_t u_d + R_b^*\partial_t u_b)dx\right\}, \tag{26}$$

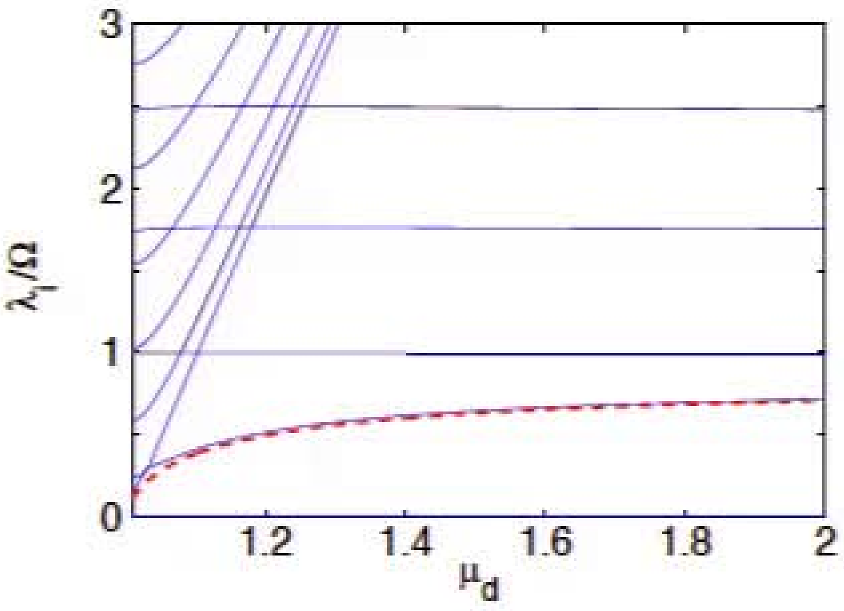
- Evaluating this RHS and Setting it Equal to Eq. (25)

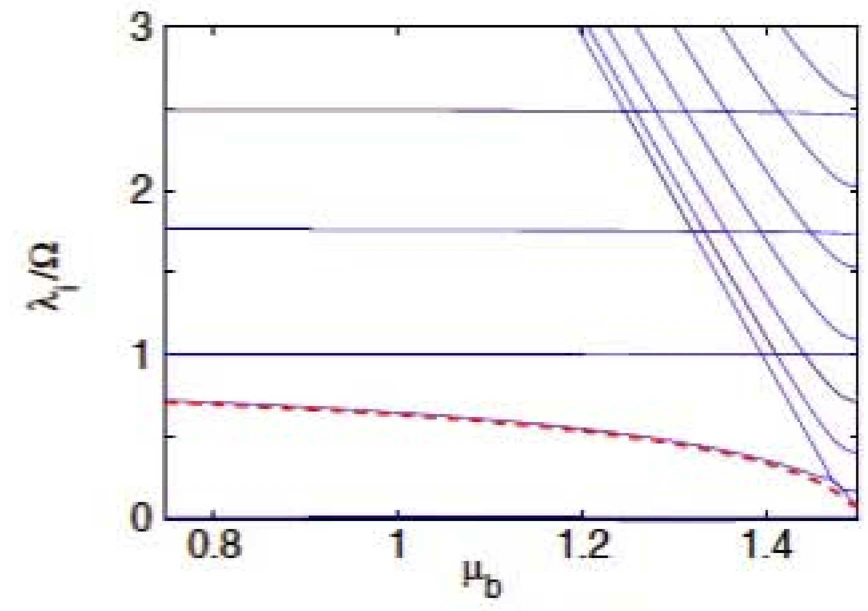
$$\begin{gathered}4\dot{D}D^2 + \chi D\sec^2\phi(\dot{D} + D\dot{\phi}\tan\phi) = \\ \frac{1}{\mu^2}\left(2\cos^3\phi\sin\phi - \chi D\sin\phi\cos\phi\right)V'(x_0).\end{gathered} \tag{27}$$

- Then the Fixed Point is: $x_{0,\text{eq}} = 0$, $\phi_{\text{eq}} = 0$, $D_{\text{eq}} = \sqrt{1 + \left(\frac{\chi}{4}\right)^2} - \frac{\chi}{4}$ and the Linearization around it yields (with $\chi_0 \equiv 8\sqrt{1 + (\chi/4)^2}$, $\chi = N_b/\sqrt{\mu}$):
$\ddot{x}_0 = -\left(\frac{1}{2} - \frac{\chi}{\chi_0}\right)V'(x_0) \Rightarrow \omega_{\text{osc}}^2 = \Omega^2\left(\frac{1}{2} - \frac{\chi}{\chi_0}\right)$.

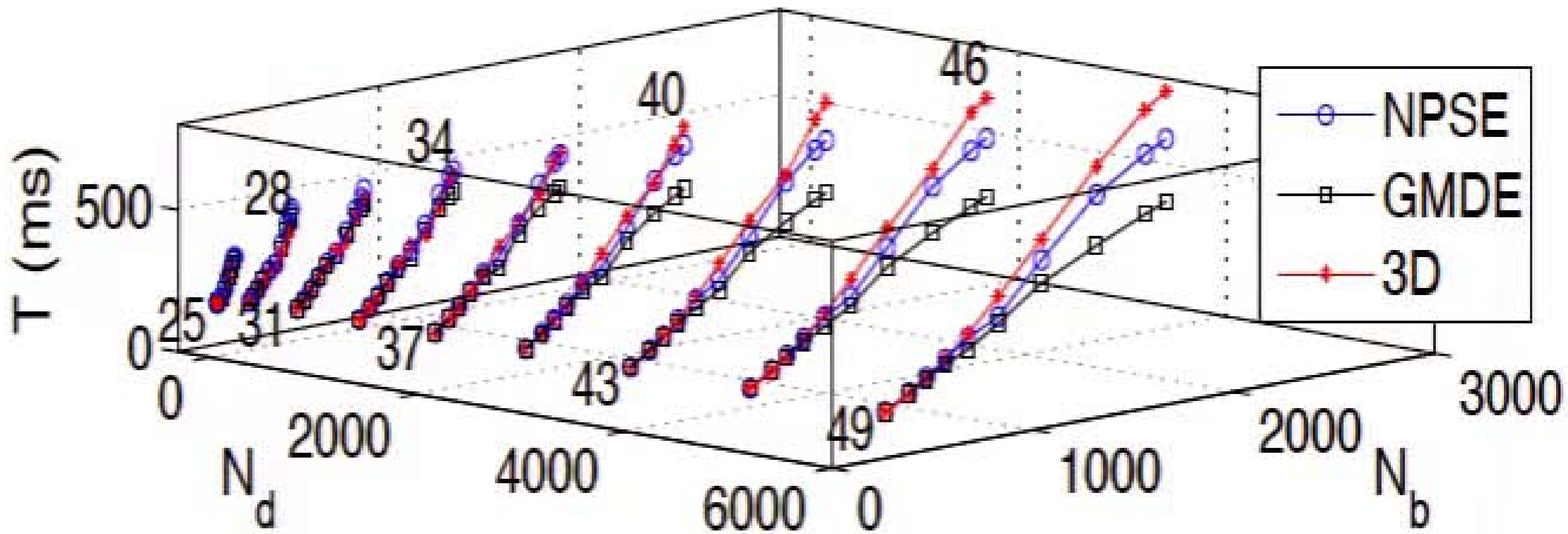
DB Soliton in a Trap: Checking the Theoretical Prediction (in 1d)

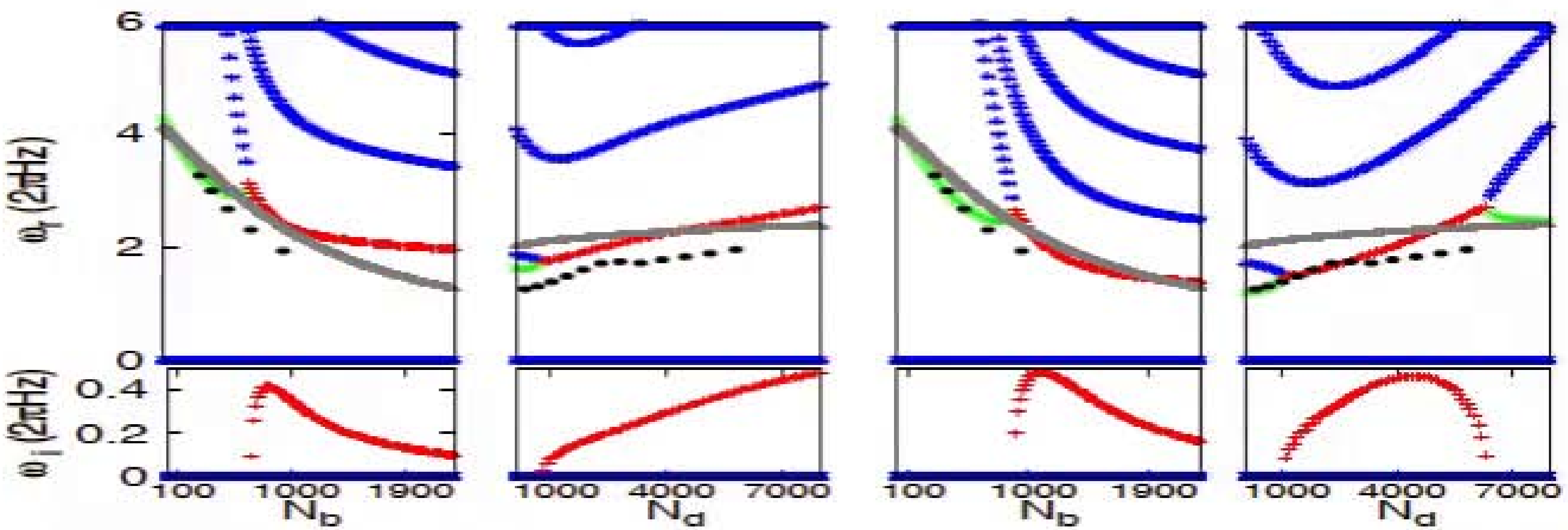




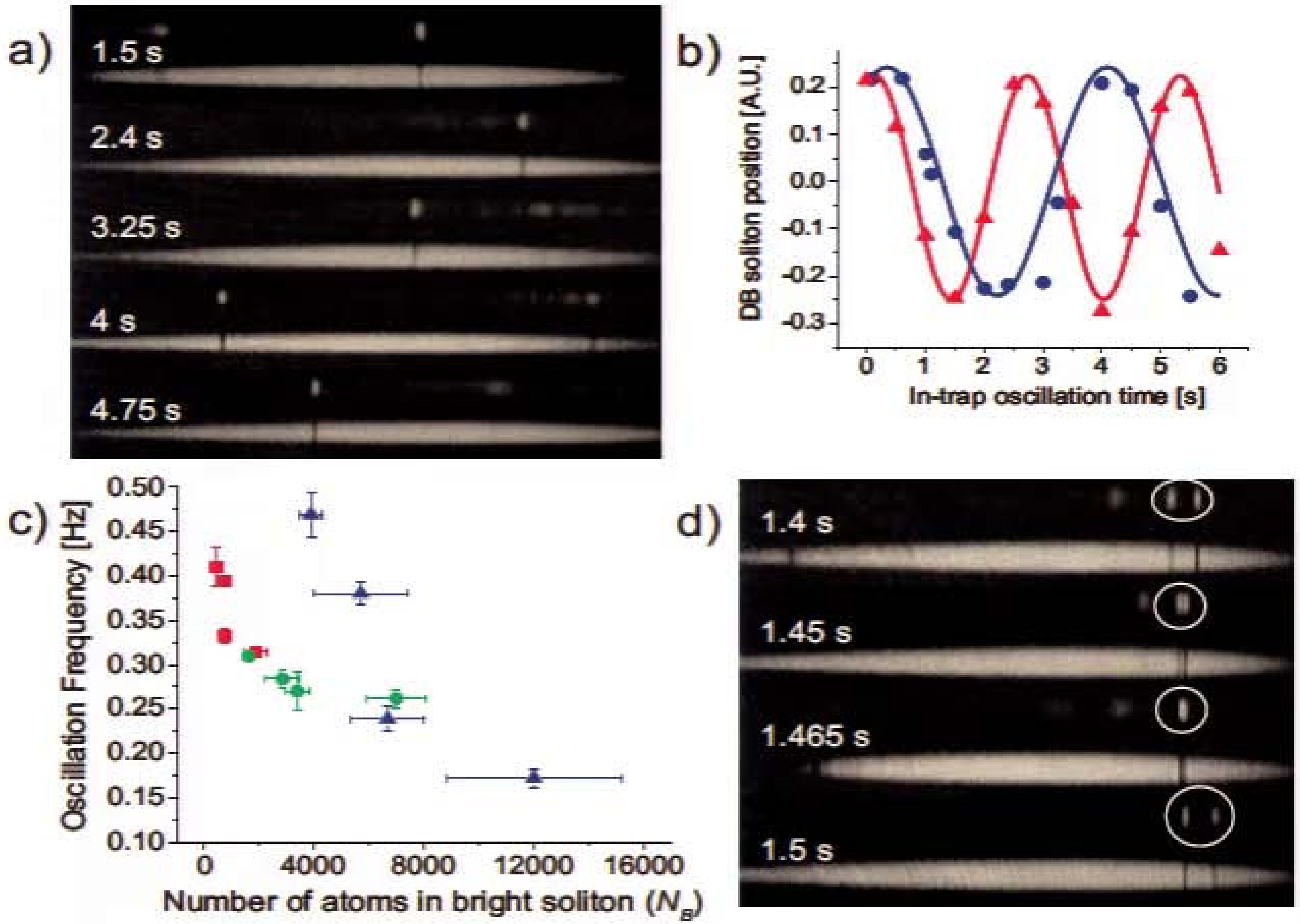


DB Soliton in a Trap: Experimentally Realistic 3d Setup

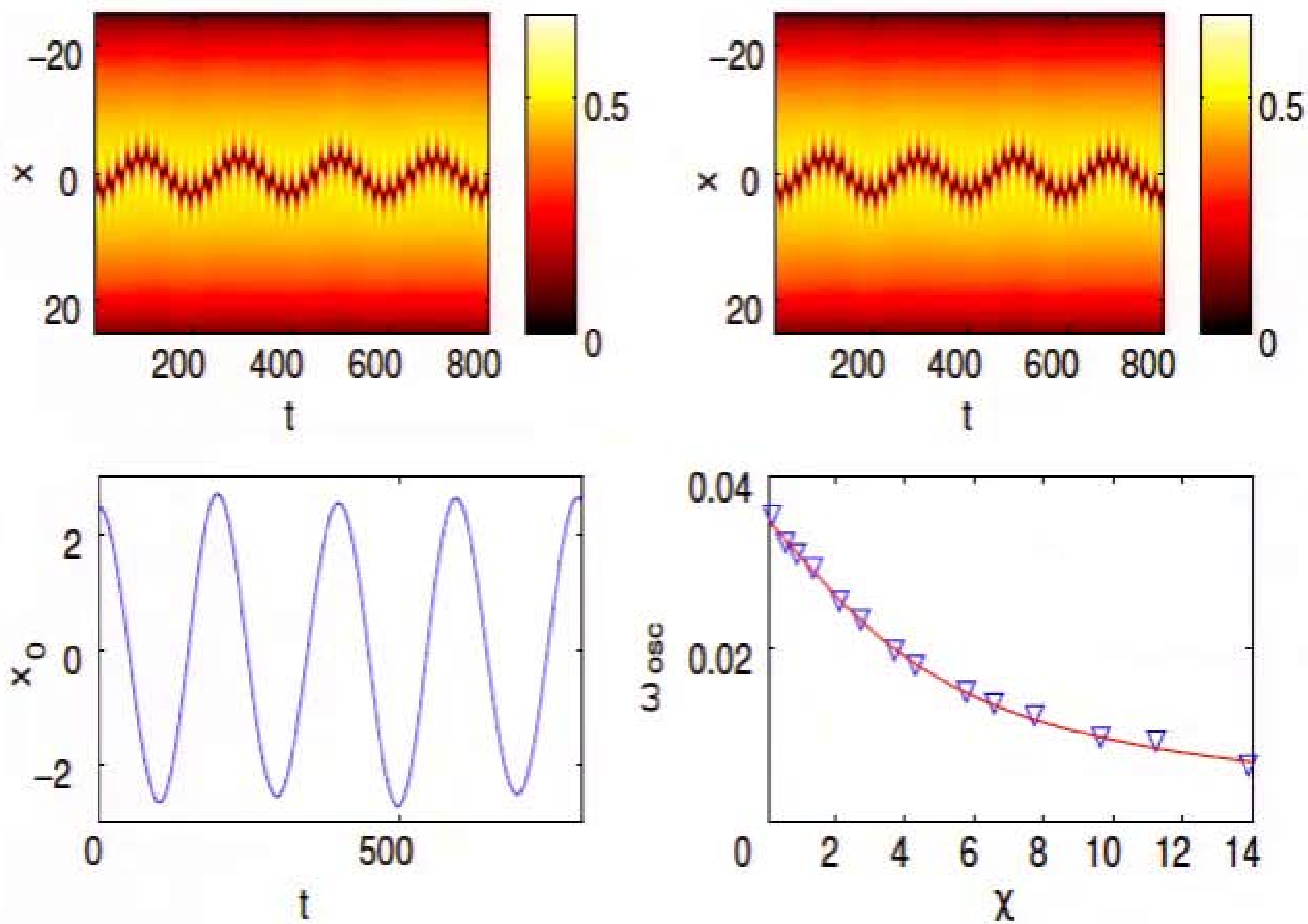




Experimental Confirmation of DB Soliton In-Trap Oscillation: Check !



DD Soliton in a Trap: SO(2) Leaves the Oscillation Intact



Multiple DBs: Bound States or *Solitonic Gluons*

- Consider the Ansatz of 2 DB Solitons with $X_\pm = D\left(x \pm x_0(t)\right)$:

$$u_d(x,t) = \left(\cos\phi \tanh X_- + i\sin\phi\right)\left(\cos\phi \tanh X_+ - i\sin\phi\right), \quad (28)$$

$$u_b(x,t) = \eta\, \mathrm{sech} X_-\, e^{i[kx+\theta)+(\tilde{\mu}-1)t]} + \eta\, \mathrm{sech} X_+\, e^{i[-kx+\theta)+(\tilde{\mu}-1)t]}\, e^{i\Delta\theta}, \quad (29)$$

- Derive again the Multi DB Energy & Equation of Motion:

$$E = 2E_1 + E_{\mathrm{DD}} + E_{\mathrm{BB}} + 2E_{\mathrm{DB}} \Rightarrow \ddot{x}_0 = F_{\mathrm{DD}} + F_{\mathrm{BB}} + 2F_{\mathrm{DB}}. \quad (30)$$

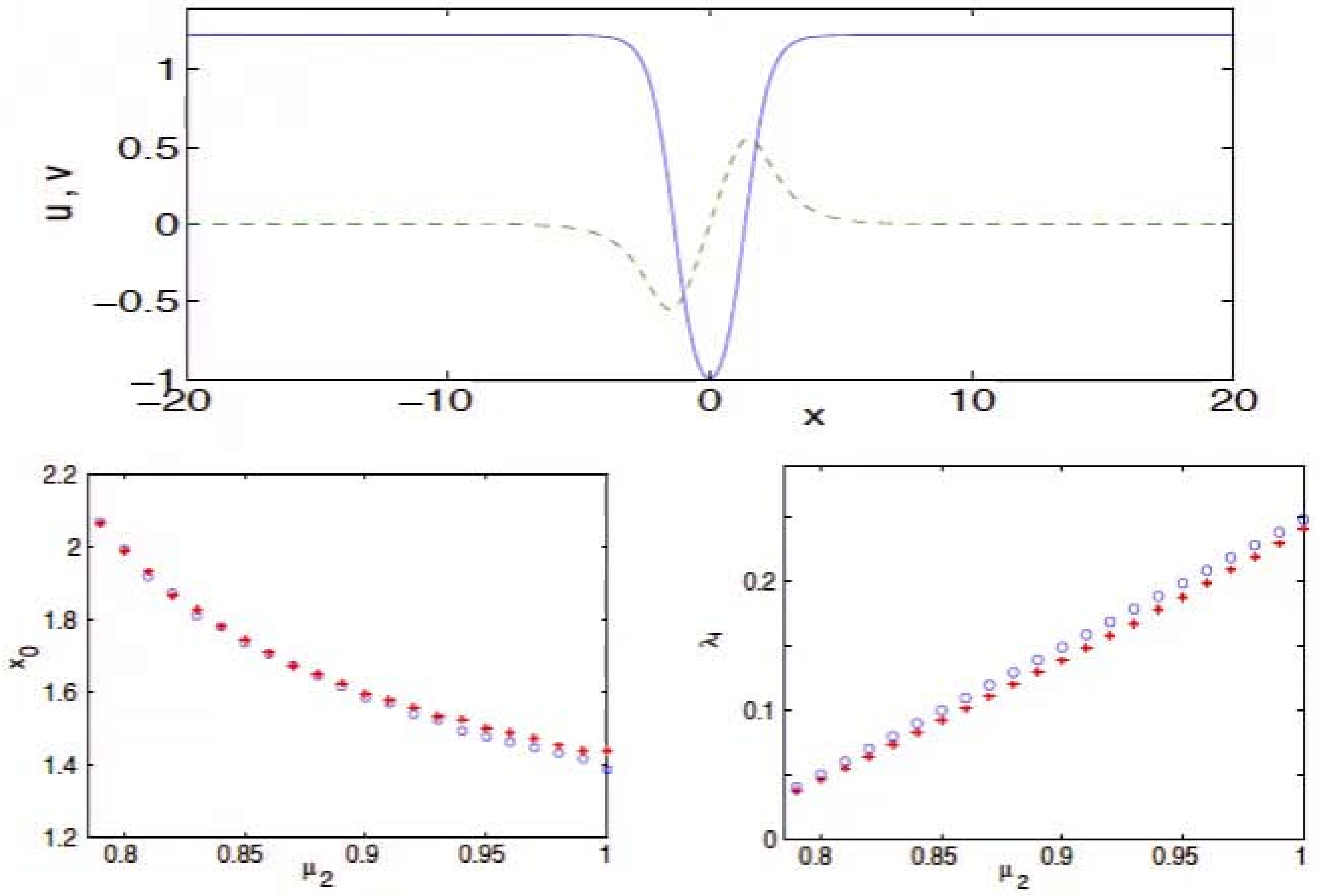
where:

$$F_{\mathrm{DD}} = \frac{1}{\chi_0}\left[\frac{1}{3}(544 - 352D_0^2) + 128D_0\left(D_0^2 - 1\right)x_0\right]\mathrm{e}^{-4D_0x_0},$$

$$F_{\mathrm{BB}} = \frac{\chi}{\chi_0}\left[\left(4 - 2\chi D_0 - 6D_0^2\right)D_0 + 4D_0^2\left(D_0^2 + 1\right)x_0\right]\cos\Delta\theta\mathrm{e}^{-2D_0x_0} - 8\frac{\chi^2}{\chi_0}D_0^3x_0\cos^2\Delta\theta\mathrm{e}$$

$$F_{\mathrm{DB}} = \frac{\chi}{2\chi_0}\left(6\chi D_0^2 + 12\chi D_0^2\cos\Delta\theta - \frac{214}{3}D_0 + 8\left(8D_0^2 - \chi D_0^3\right)x_0\right)\mathrm{e}^{-4D_0x_0},$$

Confirming the Prediction: Existence of Bound States



Existence of *Solitonic Gluons*: Check !

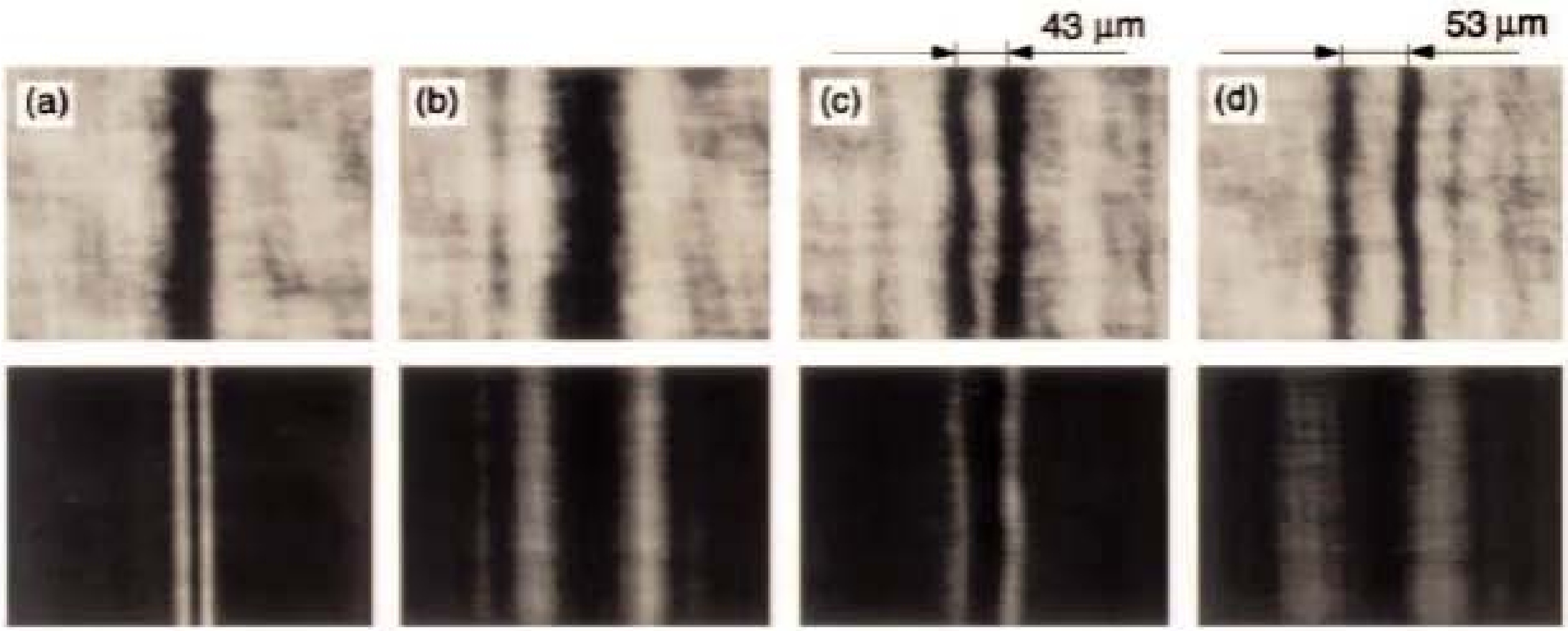


Fig. 3

Citation
Elena A. Ostrovskaya, Yuri S. Kivshar, Zhigang Chen, Mordechai Segev, "Interaction between vector solitons and solitonic gluons," Opt. Lett. **24**, 327-329 (1999);
http://www.opticsinfobase.org/ol/abstract.cfm?URI=ol-24-5-327

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Multiple DBs in a Trap: Theoretical Considerations

- Without a Trap, an Out Of Phase Internal Mode can be found for $x_0(t) = x_{\text{eq}} + \delta(t)$ and $\omega_0^2 = -\frac{\partial F_{\text{int}}}{\partial x_0}\Big|_{x_0 = x_{\text{eq}}}$ as:

$$\ddot{\delta} + \omega_0^2 \delta = 0, \tag{31}$$

- With the Trap, the Dynamics is described with $F_{Trap} = -\Omega_{DB}^2 x_0 = -\Omega^2 \left(\frac{1}{2} - \frac{\chi}{\chi_0}\right) x_0$ by:

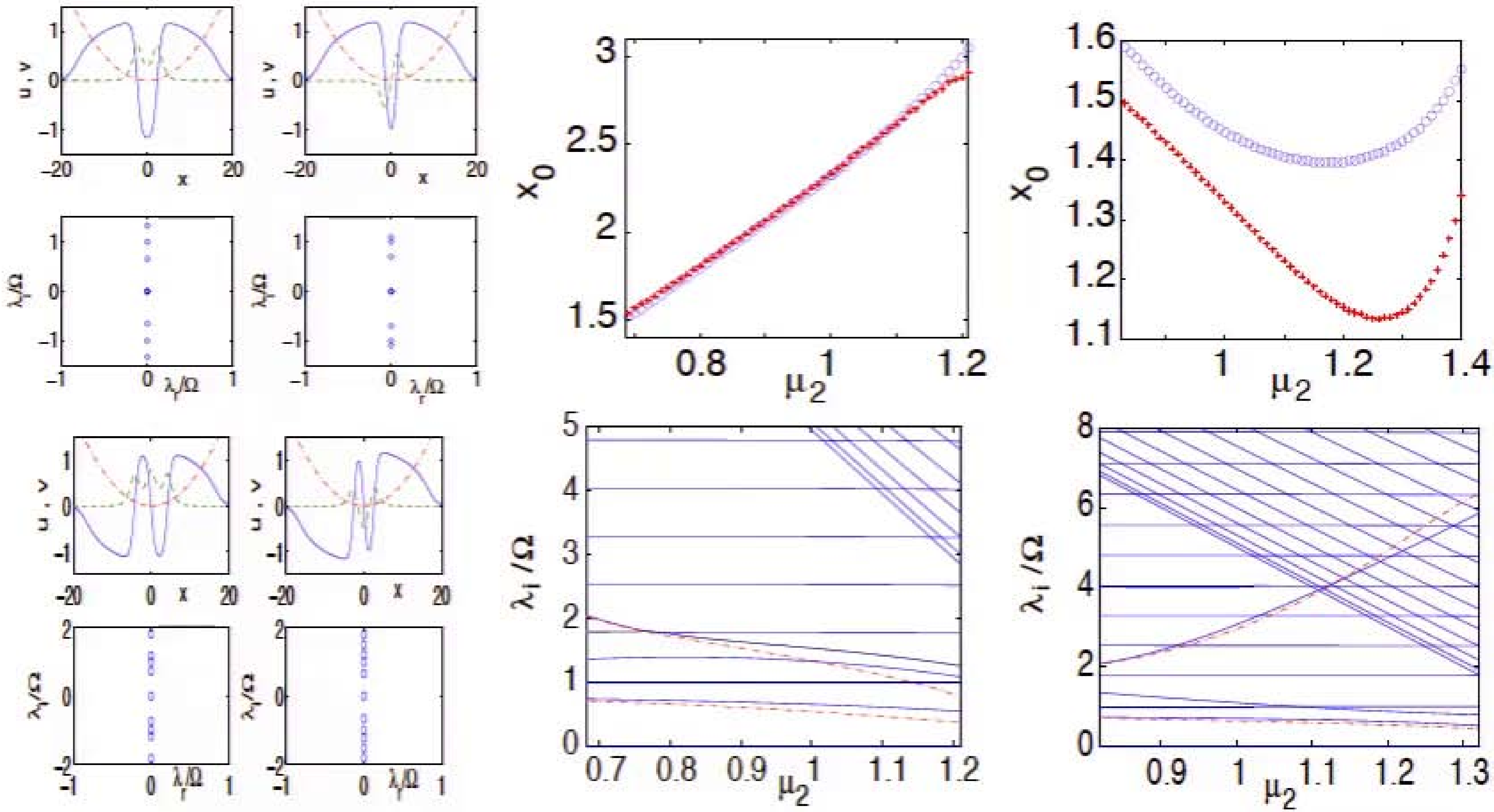
$$\ddot{x}_0 = F_{\text{tr}} + F_{\text{int}}. \tag{32}$$

- Now, there are Two Collective Oscillation Modes:

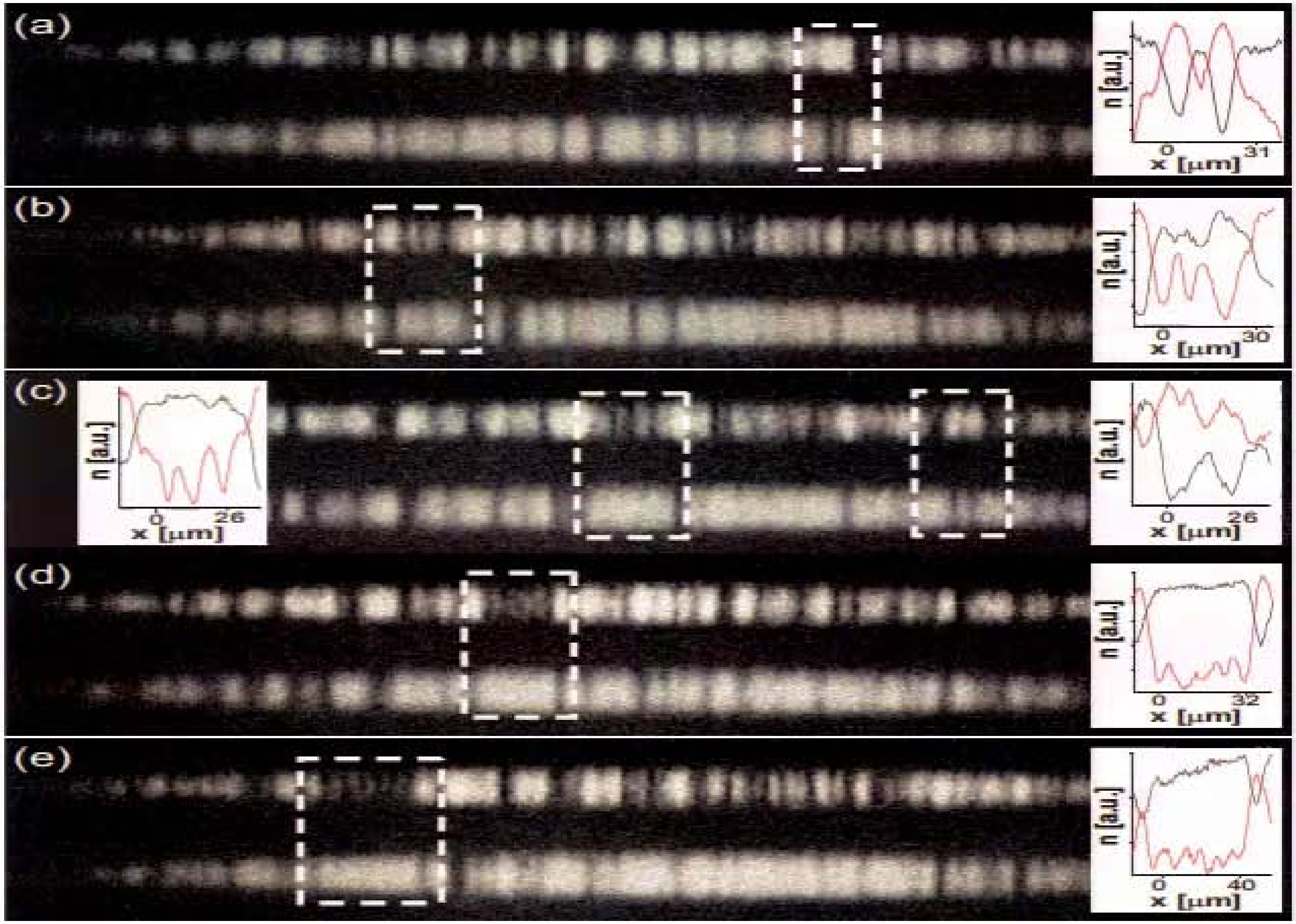
$$\text{Out} - \text{Of} - \text{Phase} \quad \omega_1^2 = \omega_{\text{osc}}^2 + \omega_0^2, \tag{33}$$

$$\text{In} - \text{Phase} \quad \omega_2 = \omega_{\text{osc}}. \tag{34}$$

Multiple DB Solitons: Capturing Equilibria & Normal Modes



Experimental Observations of Multi-DB Equilibria: Check !



Many DBs: DB Lattices & Cnoidal Functions

- Now, Inspired by the Experiment, consider Lattices of OOP Solitons

$$u_1 = A_1 \,\mathrm{sn}(bx,k), \quad u_2 = A_2 \,\mathrm{cn}(bx,k), \tag{35}$$

- Conditions for their Existence:

$$A_1^2 = \frac{2k^2(g_{12}-g_{22})\mu_1}{(g_{12}^2 - g_{11}g_{22}) + k^2(2g_{11}g_{12} - g_{12}^2 - g_{11}g_{22})}, \tag{36}$$

$$A_2^2 = \frac{2k^2(g_{11}-g_{12})\mu_1}{(g_{12}^2 - g_{11}g_{22}) + k^2(2g_{11}g_{12} - g_{12}^2 - g_{11}g_{22})}, \tag{37}$$

$$b^2 = \frac{2(g_{11}g_{22}-g_{12}^2)\mu_1}{(g_{12}^2 - g_{11}g_{22}) + k^2(2g_{11}g_{12} - g_{12}^2 - g_{11}g_{22})}. \tag{38}$$

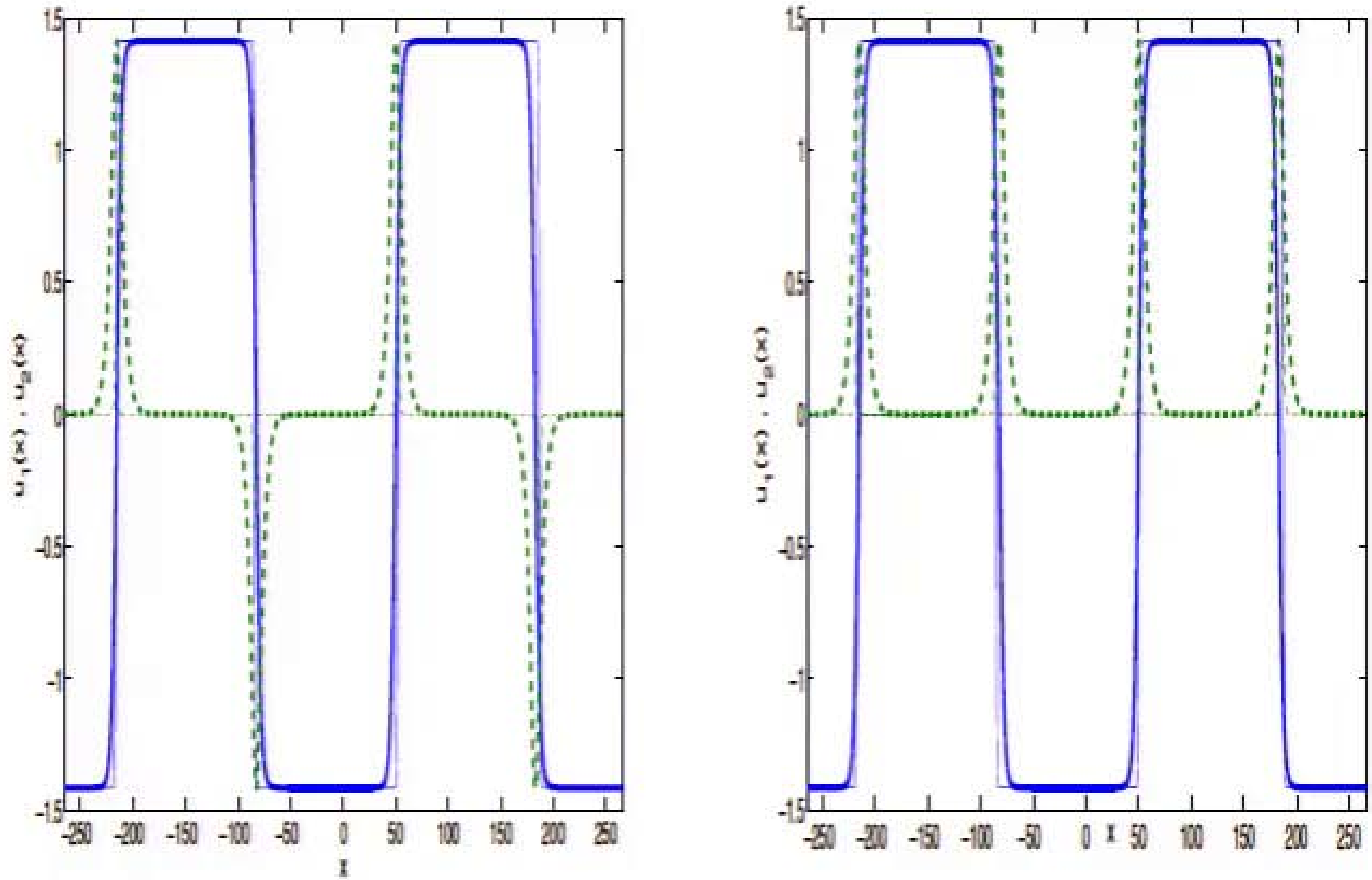
- Also for IP DB Solitons: $u_1 = A_1 \,\mathrm{sn}(bx,k), \quad u_2 = A_2 \,\mathrm{dn}(bx,k)$ with:

$$A_1^2 = \frac{2k^2(g_{12}-g_{22})\mu_1}{2g_{11}g_{12} - g_{12}^2 - g_{11}g_{22} + k^2(g_{12}^2 - g_{11}g_{22})}, \tag{39}$$

$$A_2^2 = \frac{2k^2(g_{11}-g_{12})\mu_1}{2g_{11}g_{12} - g_{12}^2 - g_{11}g_{22} + k^2(g_{12}^2 - g_{11}g_{22})}, \tag{40}$$

$$b^2 = \frac{2(g_{12}^2-g_{11}g_{22})\mu_1}{2g_{11}g_{12} - g_{12}^2 - g_{11}g_{22} + k^2(g_{12}^2 - g_{11}g_{22})}. \tag{41}$$

DB Lattice: Numerical Illustration



Many DBs: From Crystals to Gases

- Consider Many DBs inside a Trap in the form:

$$\psi_1 = \sqrt{\mu_1 - V}\prod_{j=1}^{s}\{\cos\phi_j \tanh\{D_j[\xi - a_j(\tau)]\} + i\sin\phi_j\} \tag{42}$$

$$\psi_2 = \sum_{j=1}^{s}\eta_j \mathrm{sech}\{D_j[\xi - a_j(\tau)]\}e^{ik_j\xi + i\theta_j(\tau)}e^{i\Delta\theta j}. \tag{43}$$

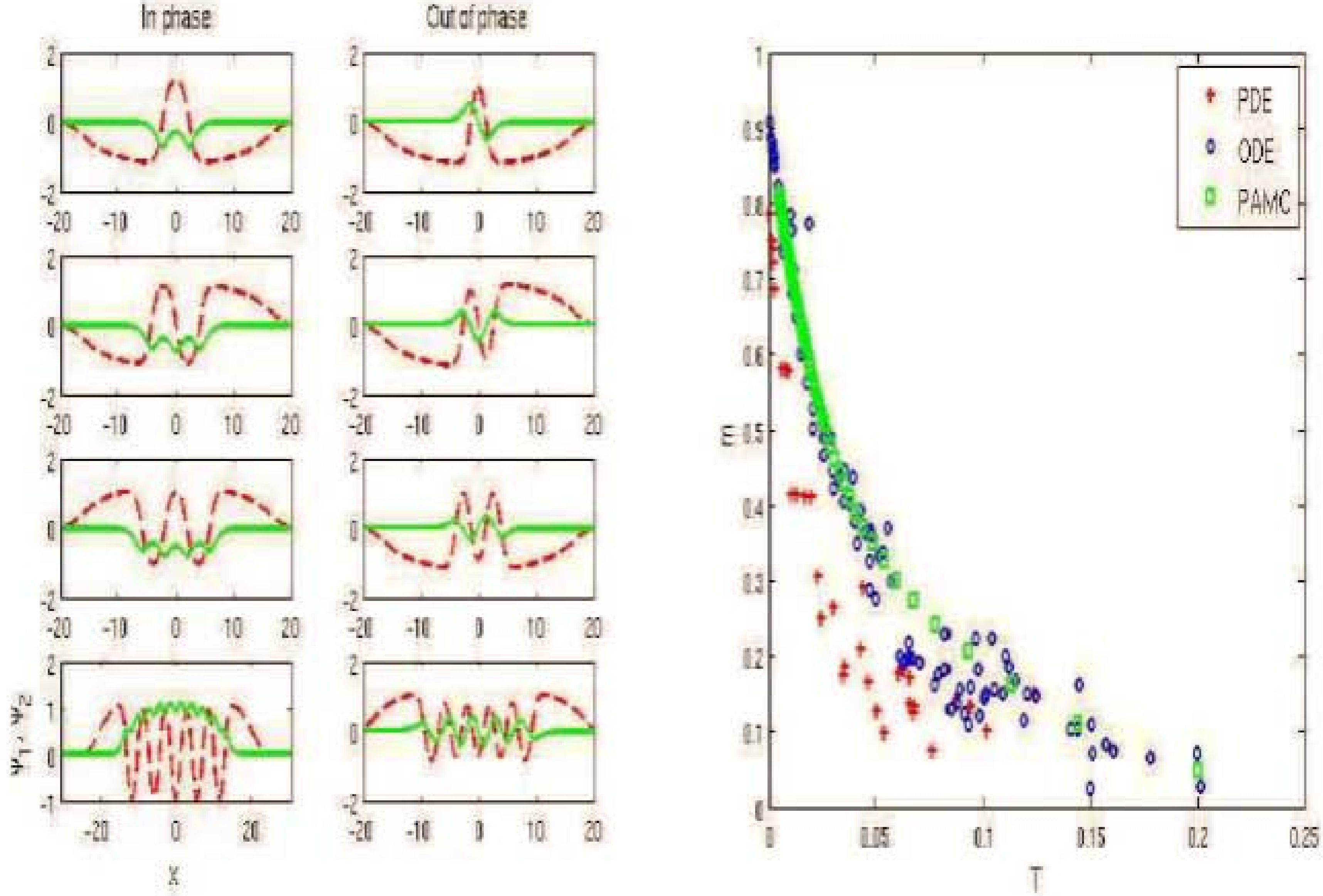
- Define a Kinetic Temperature based on:

$$e_k = \frac{\mu\sum_{i=1}^{N} b_i^2}{2N}. \tag{44}$$

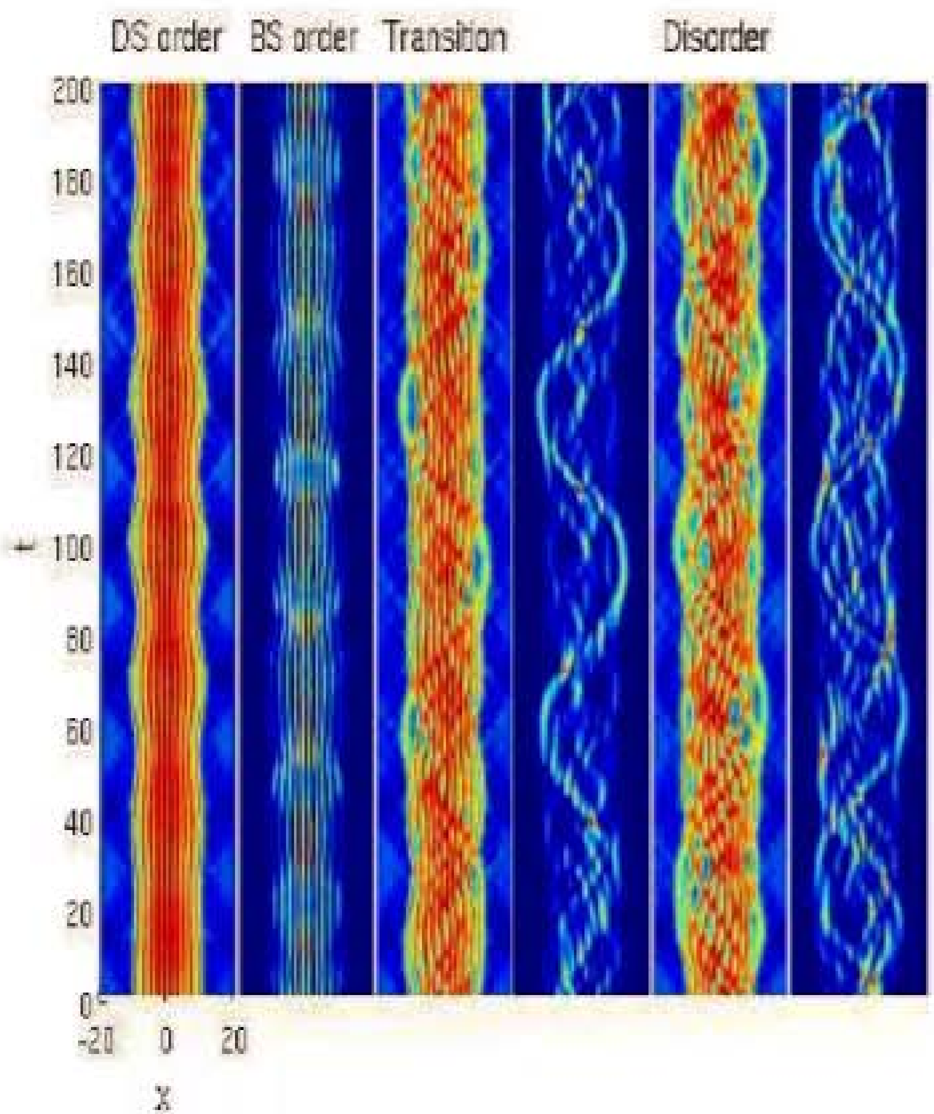
- Define an Order Parameter

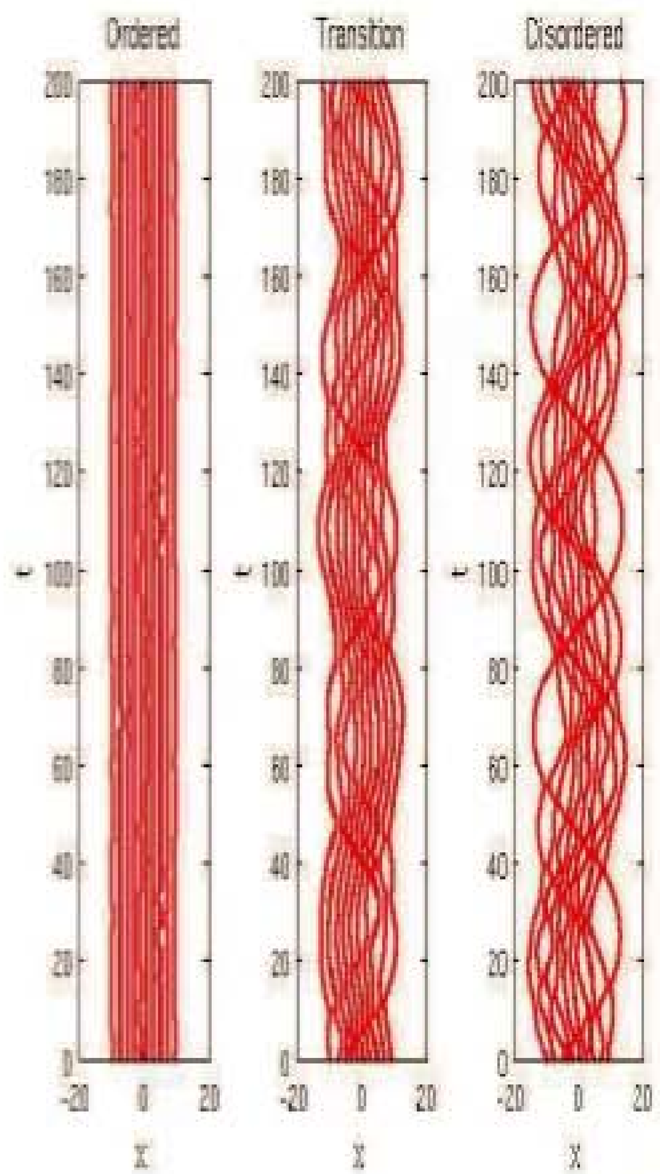
$$m = \frac{\sum_{i=1}^{N-1}\cos(k_{\min}(x_{i+1} - x_i))}{N-1} = \frac{\sum_{i=1}^{N-1}\cos(\frac{2\pi}{a_0}(x_{i+1} - x_i))}{N-1} \tag{45}$$

Many DBs: Transition from Order to Disorder



Many DBs: Transition from Order to Disorder





Variations on the DB Theme: (I) Higher-Excited Trapped States

- Operating Principle: the Dark creates a Potential Well for the Bright
- Idea: Can we Trap Higher Excited States ?

$$\mu_-\phi_- = -\frac{1}{2}(\phi_-)'' + (\phi_-^2 + \phi_+^2)\,\phi_- + V(x)\,\phi_-, \tag{46}$$

$$\mu_+\phi_+ = -\frac{D}{2}(\phi_+)'' + (\phi_-^2 + \phi_+^2)\,\phi_+ + V(x)\,\phi_+, \tag{47}$$

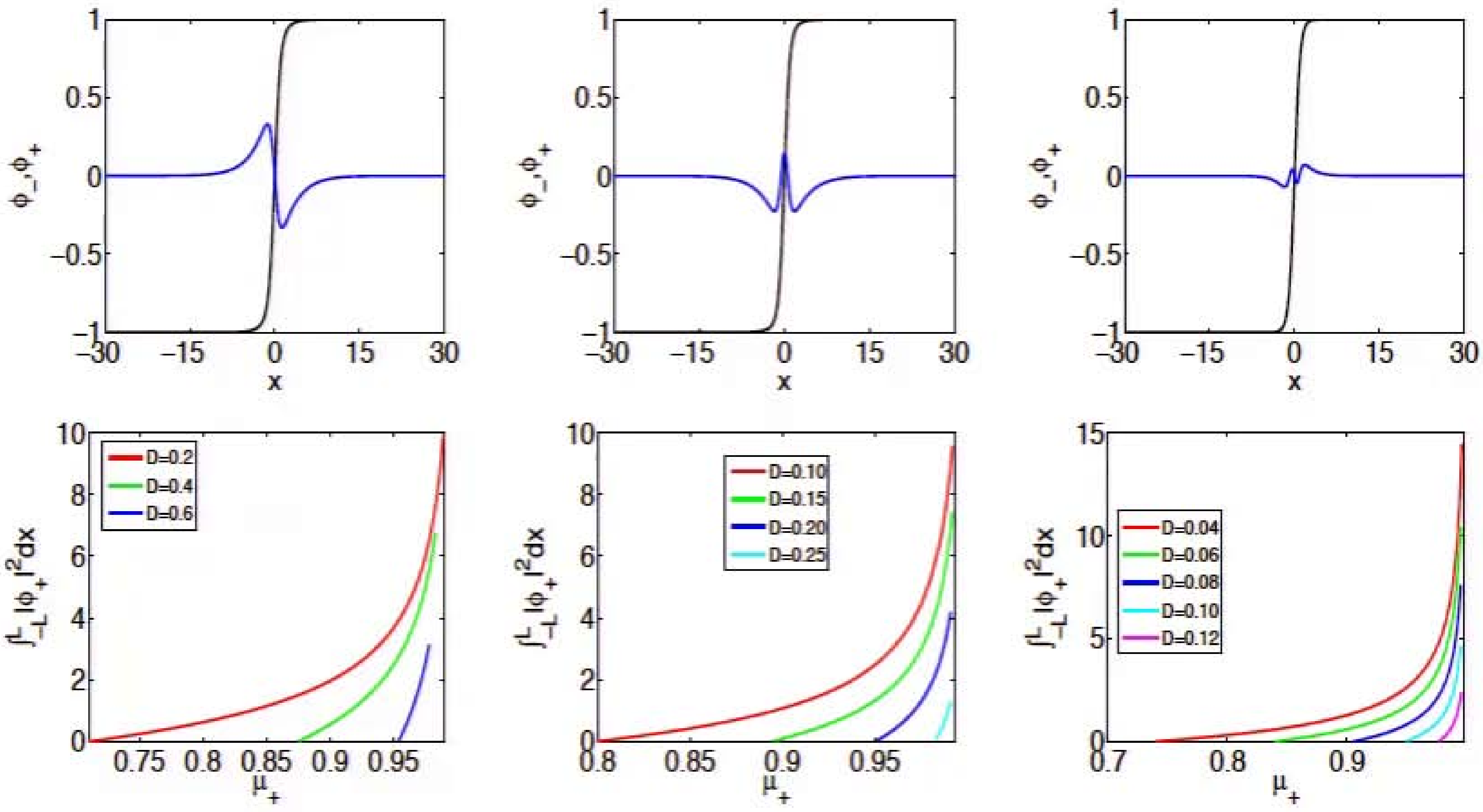
- For a Dark Solution $\phi_-(x) = \sqrt{\mu_-}\tanh\left(\sqrt{\mu_-}x\right)$, we have the Linear Operator:

$$\mathcal{L}\,\phi_+ = \lambda\,\phi_+, \quad \mathcal{L} = \frac{D}{2}\frac{d^2}{dx^2} + \mu_-\mathrm{sech}^2\left(\sqrt{\mu_-}x\right) \tag{48}$$

- Additional Bound States Exist for Pöschl-Teller Potential when $D < D_{\mathrm{crit}}^{(n)} = \frac{2}{n(1+n)}$ and Bifurcate from:

$$\mu_+ = \mu_-\left[1 - \frac{D}{8}\left(\sqrt{1+\frac{8}{D}} - (2n+1)\right)^2\right]. \tag{49}$$

Numerical Illustration: Trapping Higher-Excited States



Variations on the DB Theme: (II) Thermal DB Dynamics

- A Prototypical Model for the (In-Trap) DB Dynamics under Thermal Perturbations is:

$$(i-\gamma_d)\partial_t u_d = -\frac{1}{2}\partial_x^2 u_d + V(x)u_d + (|u_d|^2+|u_b|^2-\mu)u_d, \quad (50)$$

$$(i-\gamma_b)\partial_t u_b = -\frac{1}{2}\partial_x^2 u_b + V(x)u_b + (|u_b|^2+|u_d|^2-\mu-\Delta)u_b, \quad (51)$$

- Considering the Thermal Effect as an Additional Perturbation, we have:

$$R_d \equiv (2\mu^2)^{-1}[2(1-|u_d|^2)V(x)u_d + V'(x)\partial_x u_d] + \gamma_d \mu^{-1}\partial_t u_d, \quad (52)$$

$$R_b \equiv \mu^{-2}[(1-|u_d|^2)V(x)u_b + \mu\gamma_b\partial_t u_b]. \quad (53)$$

- Derive a Dynamical Equation from the Energy:

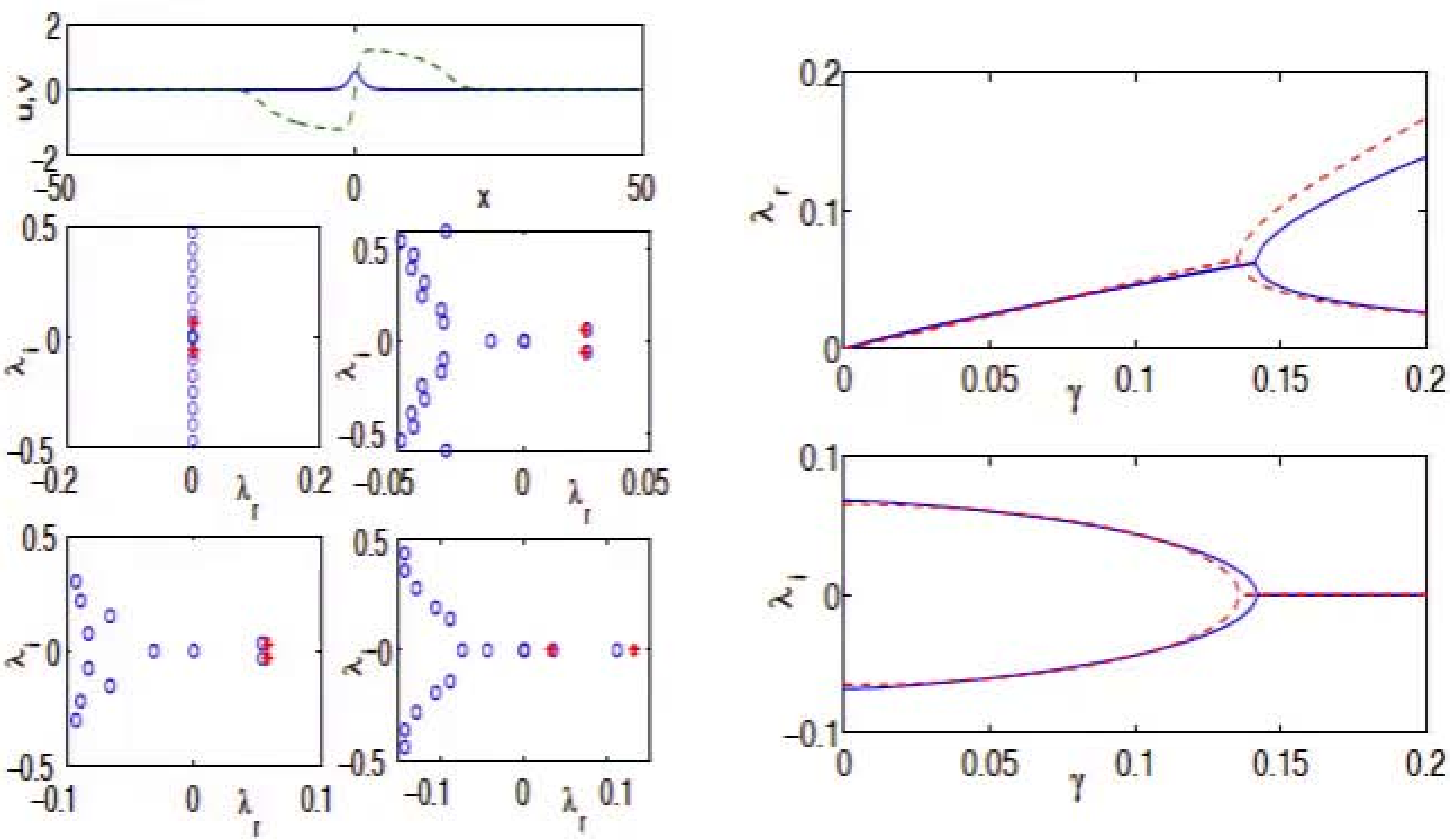
$$4\dot{D}D^2 + \chi D \sec^2\phi(\dot{D} + D\dot{\phi}\tan\phi) = \frac{1}{\mu^2}\left(2\cos^3\phi\sin\phi - \chi D\sin\phi\cos\phi\right)V'(x_0) - \frac{8}{3}\frac{\gamma_d}{\mu}D^3\sin^2\phi - \frac{2}{3}\frac{\gamma_b}{\mu}\chi D^4\tan^2\phi. \quad (54)$$

- Linearizing Around the Equilibrium, we get $\ddot{x}_0 - a\,\dot{x}_0 + \omega_{\text{osc}}^2\,x_0 = 0$, with:

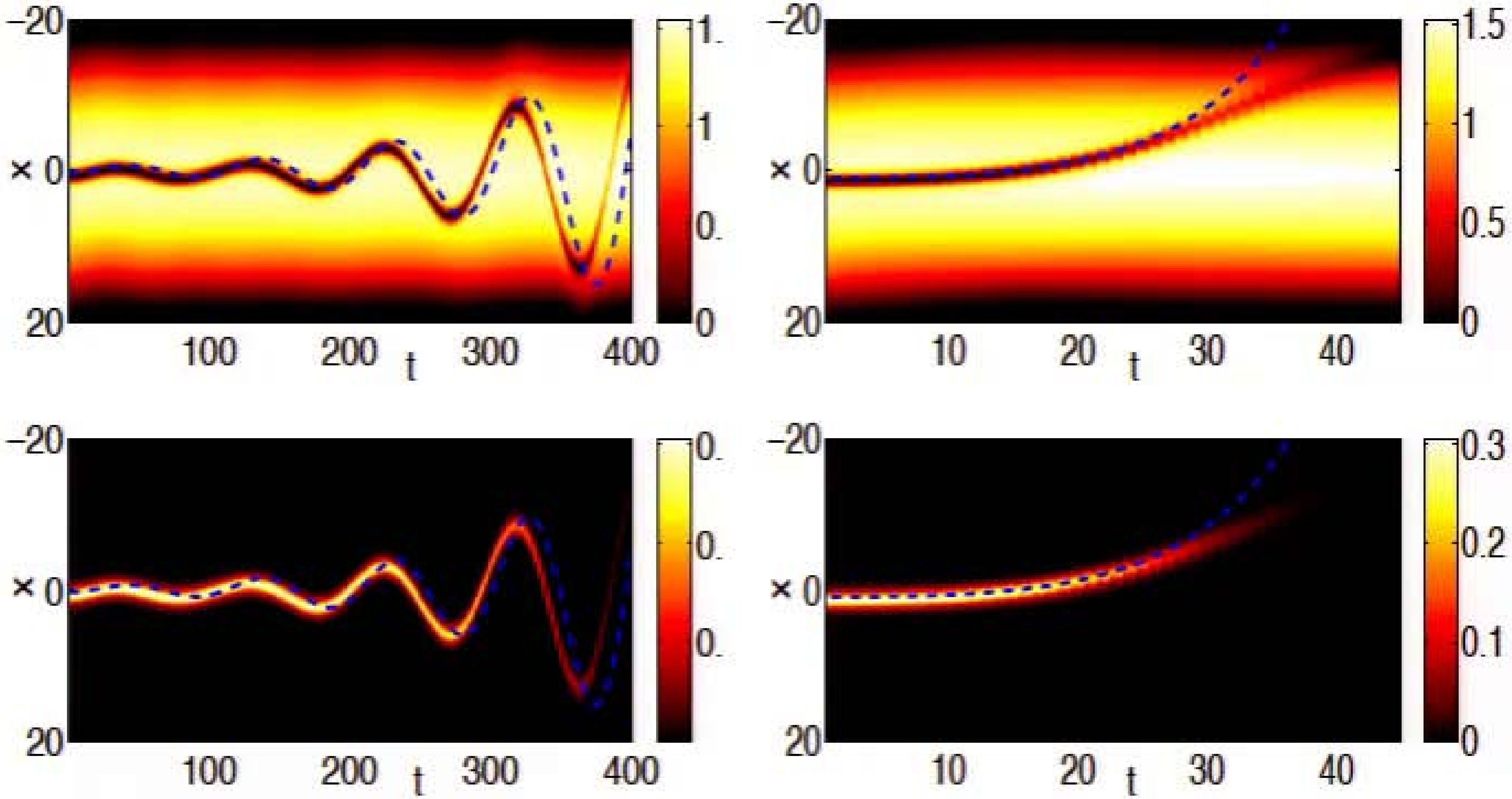
$$a = \frac{2}{3}\mu\left(\gamma_d - \frac{1}{8}\chi^2\gamma_b\right) + \frac{4}{3}\frac{\chi}{\chi_0}\mu\left(\gamma_b - \gamma_d + \frac{1}{8}\chi^2\gamma_b\right). \quad (55)$$

The Single Thermal DB Soliton: Equilibrium & Stability

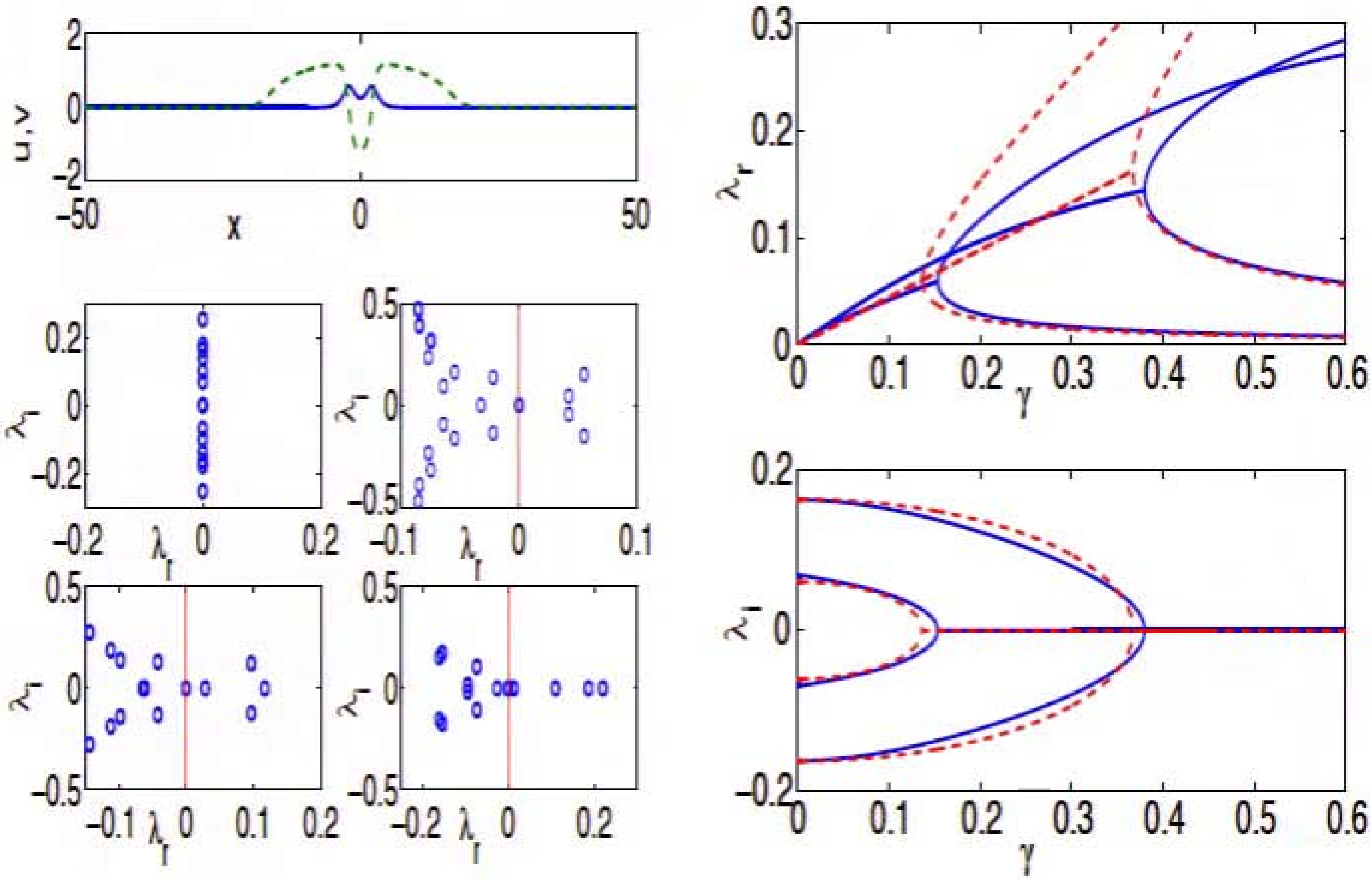
- Depend on Exponential Prefactor $s_{1,2} = \frac{1}{2}\left(a \pm \sqrt{a^2 - a_{\rm cr}^2}\right)$, with $a_{\rm cr} \equiv 2\omega_{\rm osc}$: Either Oscillatory or Exponential Anti-Damped Motion.



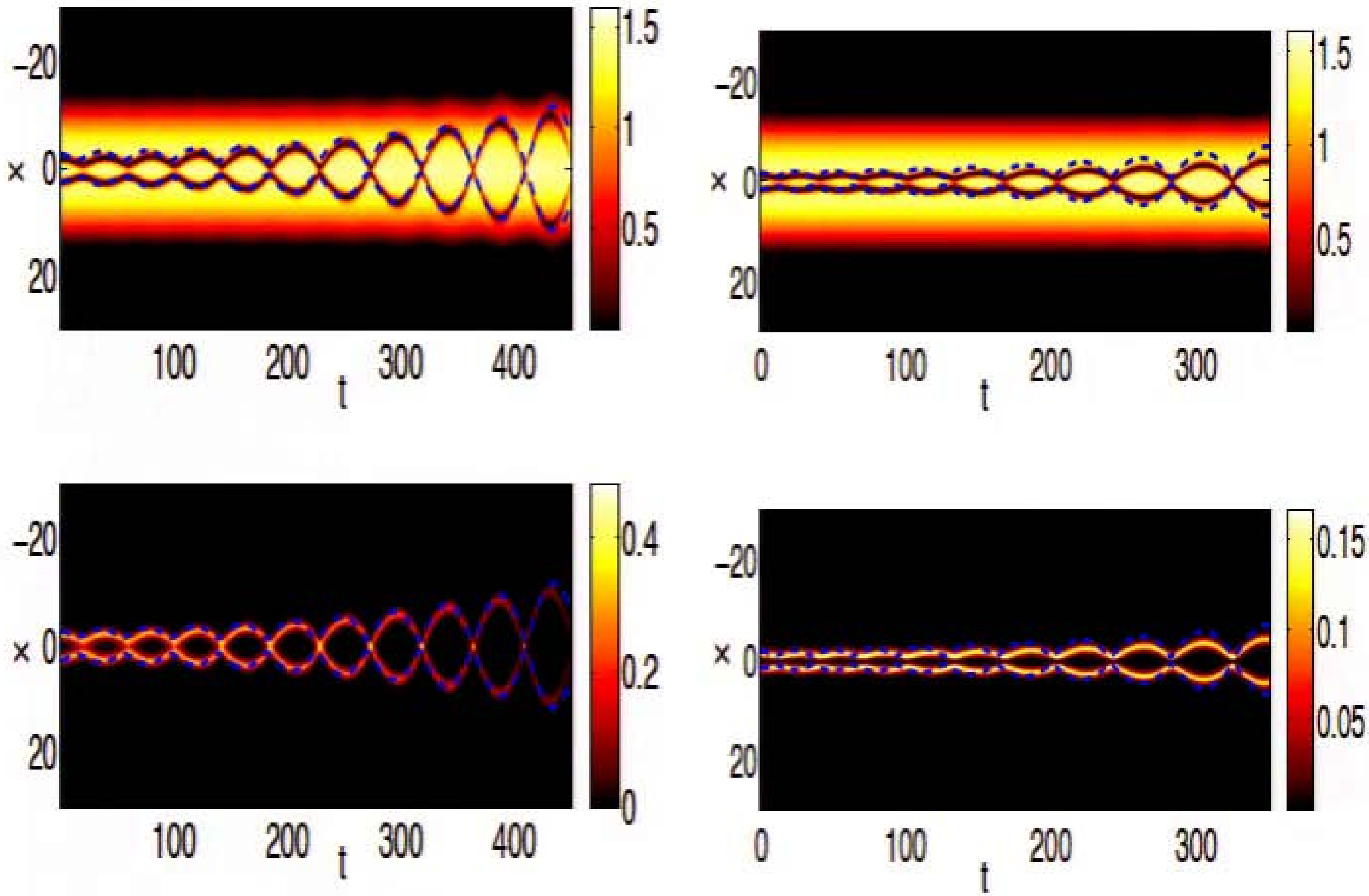
The Single Thermal DB Soliton: Anti-Damped Dynamics



Generalize to 2 IP Thermal DBs: Equilibrium & Stability



IP & OOP Thermal DB Anti-Damped Dynamics: Simulation vs. Theory



Variations on the DB Theme: (III) 2 DBs → Double Well

- Use the 2 Dark Solitons as a Double Well Potential for the 2 Bright Solitons
- We use the Dark Component Wavefunction ψ_D as part of the $V_{\text{eff}}(x,t) = V_{\text{ext}}(x) + |\psi_D(x,t)|^2$ for the bright Effective GPE:

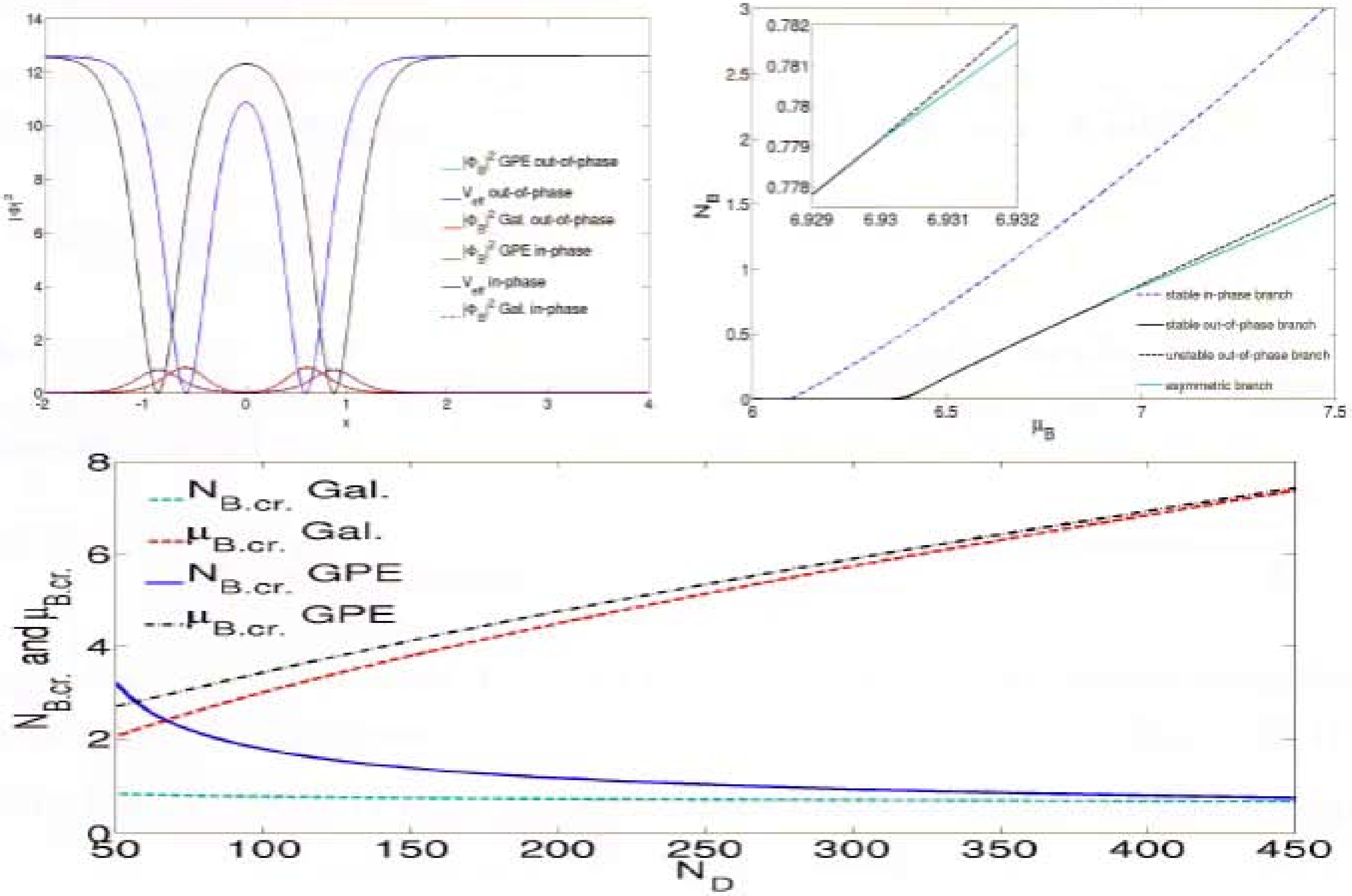
$$i\partial_t \psi_B = \Big[-\frac{1}{2}\partial_x^2 + V_{\text{eff}} + g_B|\psi_B|^2 \Big] \psi_B \tag{56}$$

- In this Effective Double Well, we can do a 2-Mode Analysis $\psi_B(x,t) = c_0(t)\phi_0(x) + c_1(t)\phi_1(x)$ and predict Symmetry-Breaking Bifurcations at

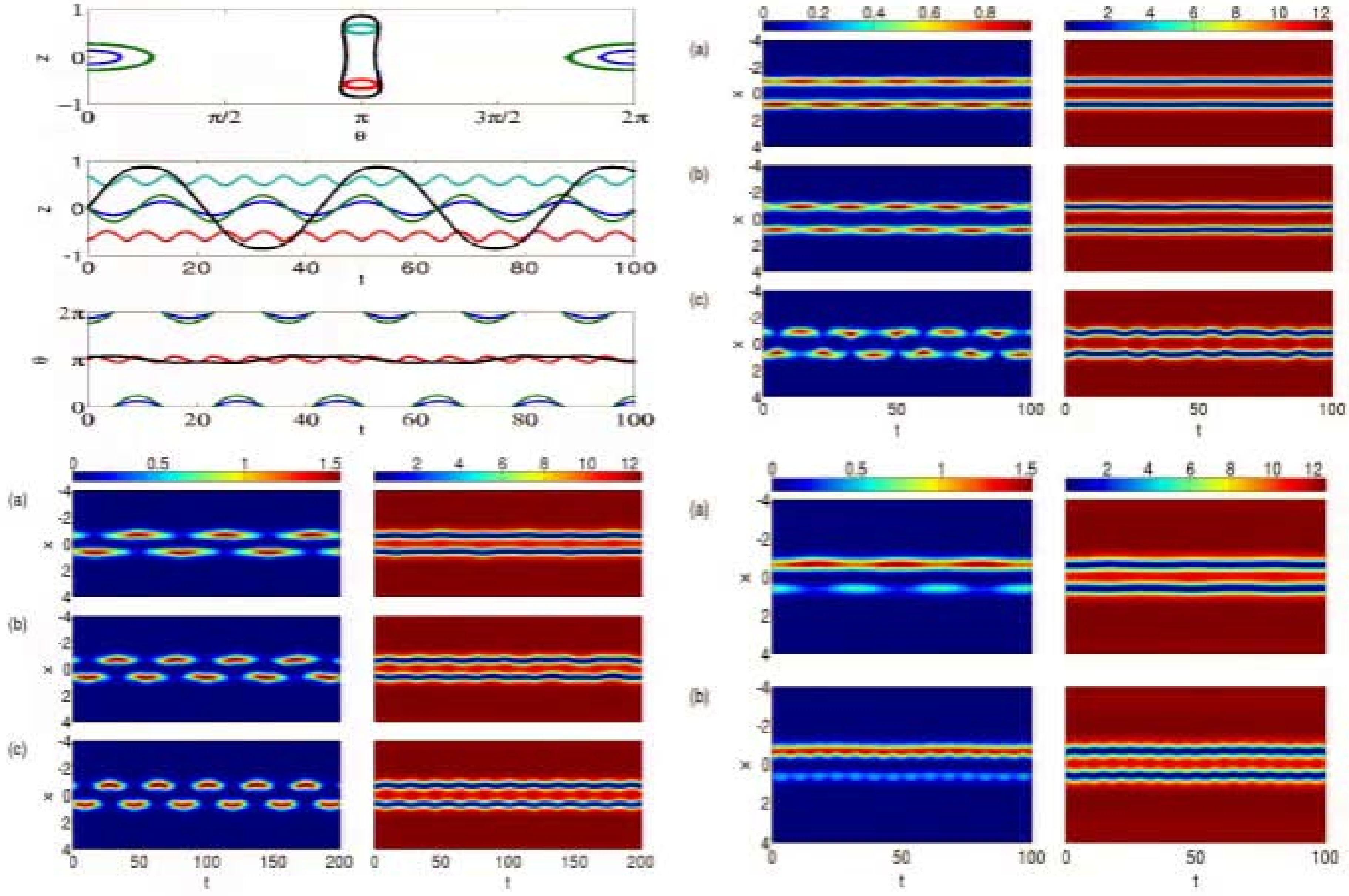
$$N_{\text{cr}} = \frac{\Delta\omega}{g_B(3A_{0011} - A_{1111})}; \quad A_{0011} = \int_{-\infty}^{\infty} \mathrm{d}x\, \phi_0^2 \phi_1^2; \quad A_{1111} = \int_{-\infty}^{\infty} \mathrm{d}x\, \phi_1^4 \tag{57}$$

- Also, Effective Double Well Conjugate Variables can be defined as: (θ, z) with $\theta = \theta_l - \theta_r$ and $z = \frac{N_l - N_r}{N_l + N_r}$ with $N_l = \int_{-\infty}^{0} \mathrm{d}x\, |\psi_B|^2$ and $N_r = \int_0^{\infty} \mathrm{d}x\, |\psi_B|^2$ and the Phases through $\theta_l = \arg(\int_{-\infty}^{0} \mathrm{d}x\, \psi_B)$ and $\theta_r = \arg(\int_0^{\infty} \mathrm{d}x\, \psi_B)$.

Bifurcations for 2 DBs as a Double Well System



Dynamics for 2 DBs as a Double Well System



Variations on the DB Theme: (IV) Spinor BECs, DDB & BBD Solitons

- Consider the $F=1$ Spinor BEC system:

$$i\partial_t\psi_{\pm 1} = H_0\psi_{\pm 1} + \delta\left[(|\psi_{\pm 1}|^2 + |\psi_0|^2 - |\psi_{\mp 1}|^2)\psi_{\pm 1} + \psi_0^2\psi_{\mp 1}^*\right], \tag{58}$$

$$i\partial_t\psi_0 = H_0\psi_0 + \delta\left[(|\psi_{-1}|^2 + |\psi_{+1}|^2)\psi_0 + 2\psi_{-1}\psi_0^*\psi_{+1}\right], \tag{59}$$

where $H_0 \equiv -(1/2)\partial_x^2 + V(x) + n_{\text{tot}}$, the Potential is $V(x) = (1/2)\Omega^2 x^2$, the Trap's strength is $\Omega = \frac{3}{2(a_0+2a_2)n_0}\left(\frac{\omega_z}{\omega_\perp}\right)$, and we define $\delta \equiv \frac{c_2^{(1D)}}{c_0^{(1D)}} = \frac{a_2-a_0}{a_0+2a_2}$.

- $\delta < 0$ and $\delta > 0$ correspond, respectively, to Ferromagnetic and Polar spinor BECs. ^{87}Rb and ^{23}Na atoms have $\delta = -4.66 \times 10^{-3}$ (F), and $\delta = +3.14 \times 10^{-2}$ (P).

- Homogeneous States read:

$$\psi_{-1} = \psi_{+1} = \sqrt{\frac{\mu}{2}}\exp(-i\mu t), \quad \psi_0 = 0; \tag{60}$$

$$\psi_{-1} = \psi_{+1} = 0, \quad \psi_0 = \sqrt{\mu}\exp(-i\mu t). \tag{61}$$

Generalizations to Spinor BECs: DDB and BBD Solitons (Contd.)

- Derive the Yajima-Oikawa system:

$$\partial_T \rho = -(\sqrt{\mu}/2)\partial_X\left(|q|^2\right). \tag{67}$$

$$i\partial_T q + \frac{1}{2}\partial_X^2 q - 2\rho q = 0. \tag{68}$$

- The resulting DDB Soliton Solutions Read:

$$\psi_{\pm 1}(x,t) = \sqrt{(\mu/2) - 2\delta\eta^2\,\mathrm{sech}^2(2\sqrt{\delta}\eta Z)}\,\exp\left[-i\mu t - 2i\eta\sqrt{\delta/\mu}\tanh(2\sqrt{\delta}\eta Z)\right],$$

$$\psi_0(x,t) = 2^{3/2}\delta^{3/4}\eta\mu^{-1/4}\sqrt{\xi}\,\mathrm{sech}(2\sqrt{\delta}\eta Z)\exp\left[-i\mu t + i\sqrt{\mu}x - 2i\sqrt{\delta}\xi Z + 2i\delta(\eta^2 - \xi^2)t\right]$$

- Similarly from the Other Plane Wave, BBD Solutions can be obtained:

$$\psi_{\pm 1}(x,t) = 2\delta^{3/4}\eta\mu^{-1/2}\sqrt{\xi}\,\mathrm{sech}(2\sqrt{\delta}\eta Z)\times\exp\left[-i\mu t + i\sqrt{\mu}x - 2i\sqrt{\delta}\xi Z + 2i\delta(\eta^2 - \xi^2)t\right]$$

$$\psi_0(x,t) = \sqrt{(\mu/2) - 4\delta\eta^2\,\mathrm{sech}^2(2\sqrt{\delta}\eta Z)}\times\exp\left[-i\mu t - 2i\eta\sqrt{\delta/\mu}\tanh(2\sqrt{\delta}\eta Z)\right].$$

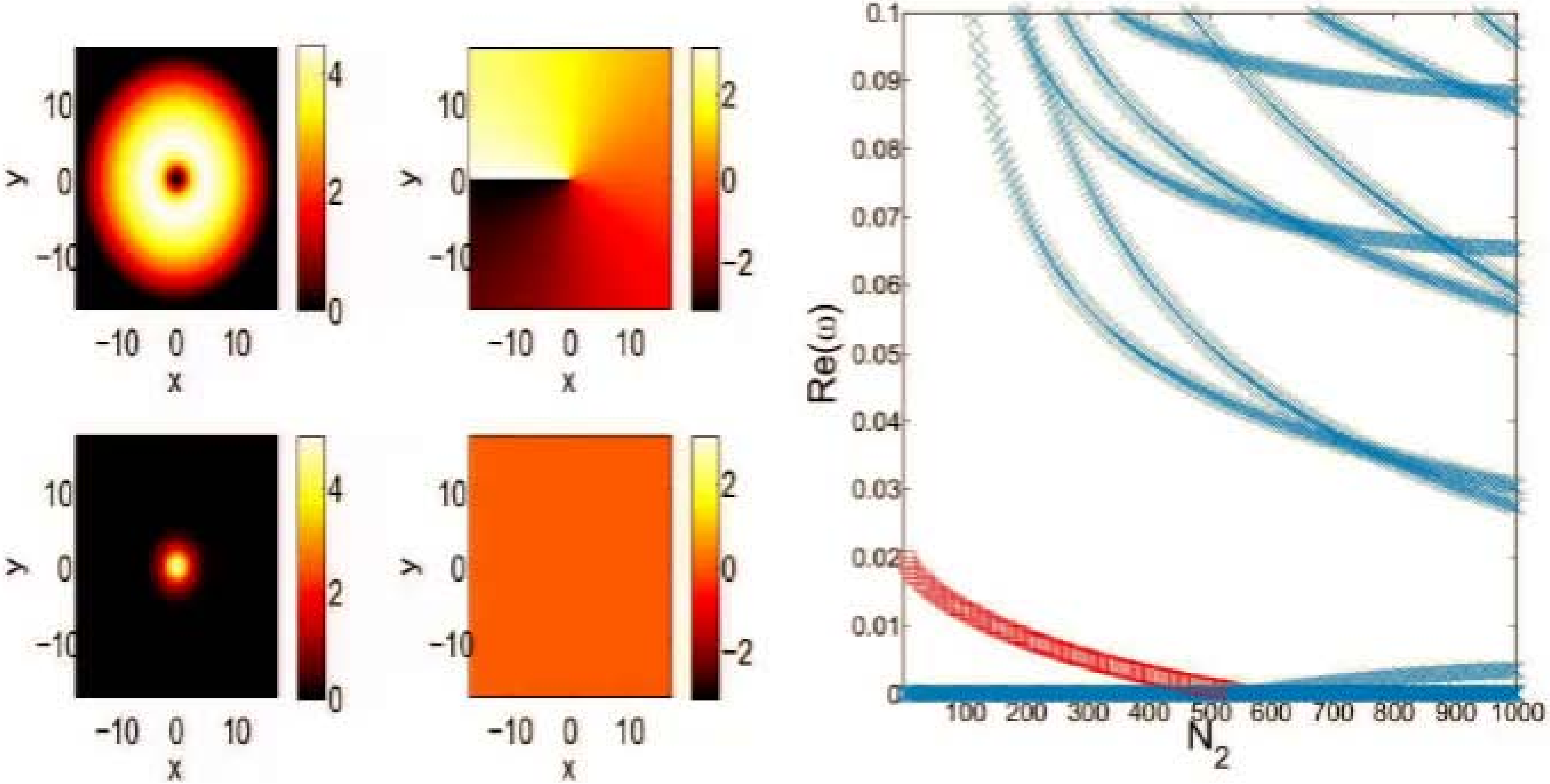
2d Generalizations: Vortex-Bright, Vortex-Vortex, DB-Ring States

- Consider the 2d Generalization:

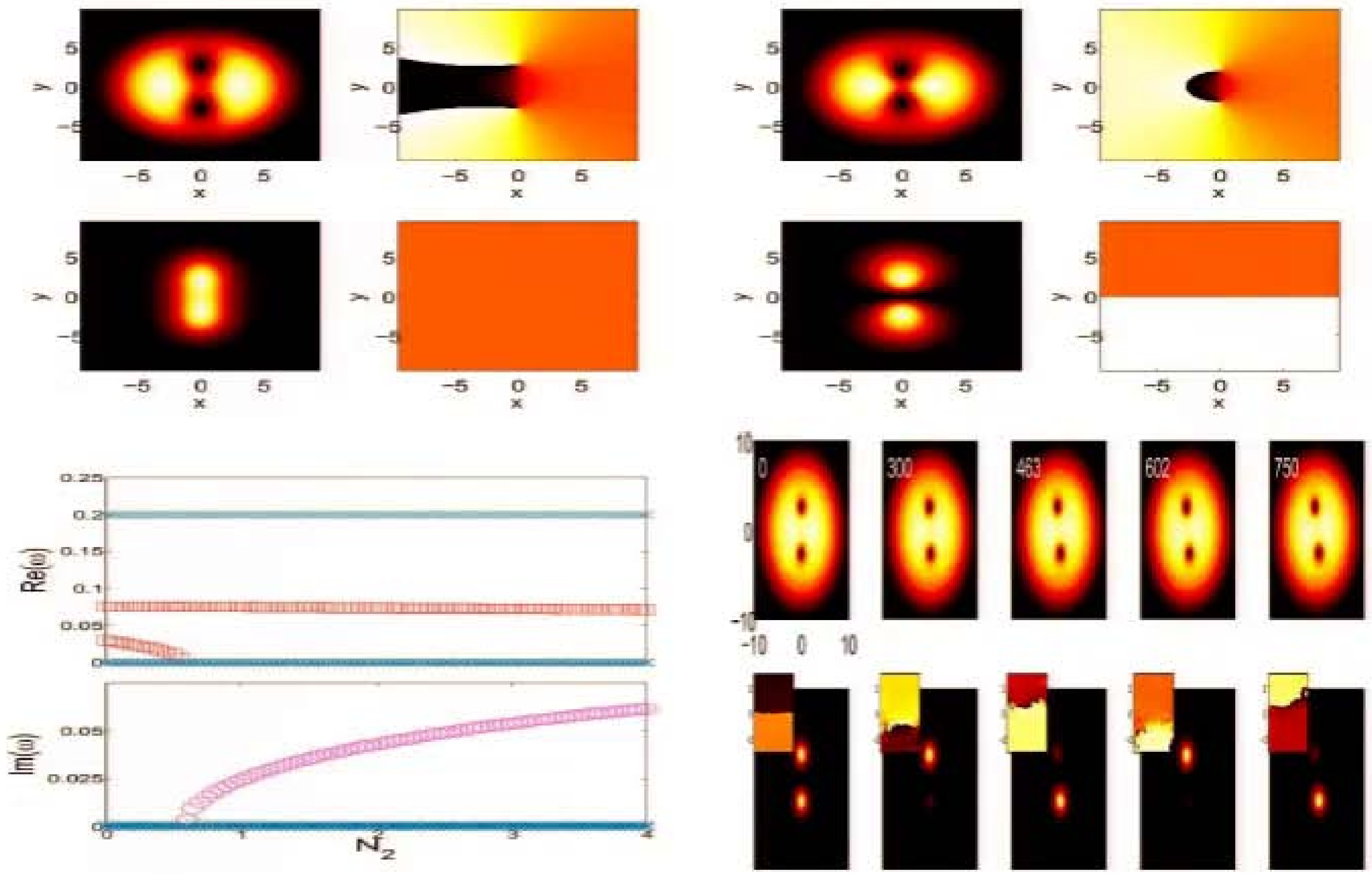
$$i\partial_t \psi_1(x,y,t) = \left[-\frac{1}{2}(\partial_x^2 + \partial_y^2) + V(x,y) + g_1|\psi_1|^2 + \sigma_{12}|\psi_2|^2 \right] \psi_1(x,y,t)$$
$$i\partial_t \psi_2(x,y,t) = \left[-\frac{1}{2}(\partial_x^2 + \partial_y^2) + V(x,y) + g_2|\psi_2|^2 + \sigma_{12}|\psi_1|^2 \right] \psi_2(x,y,t). \tag{73}$$

- Now the Vortex plays the role of the Potential Well that can Trap the Bright Soliton to create a Vortex-Bright State.
- One can use the SO(2)-Rotation of the Vortex-Bright to generate a Vortex-Vortex Time-Periodic State.
- One can use a Ring-Dark-Soliton as the Trapping Well, to form a Dark-Bright Ring.
- The approach can be, in principle, similarly generalized to 3d BECs.

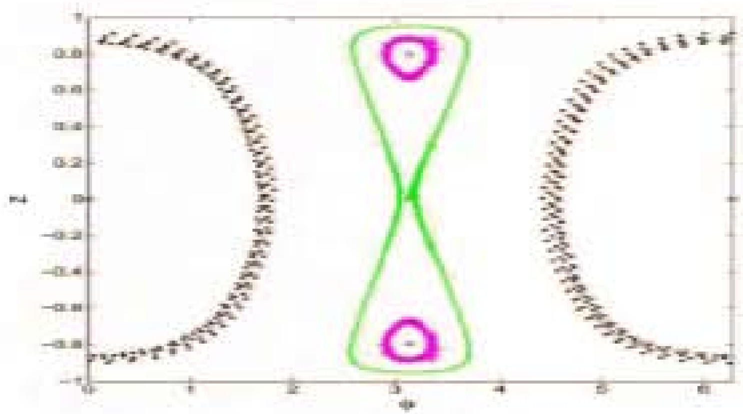
A Single Vortex-Bright Soliton

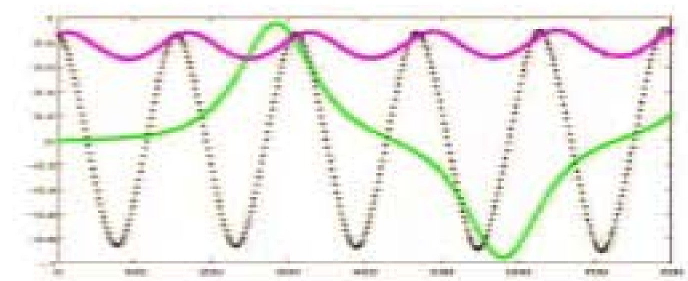


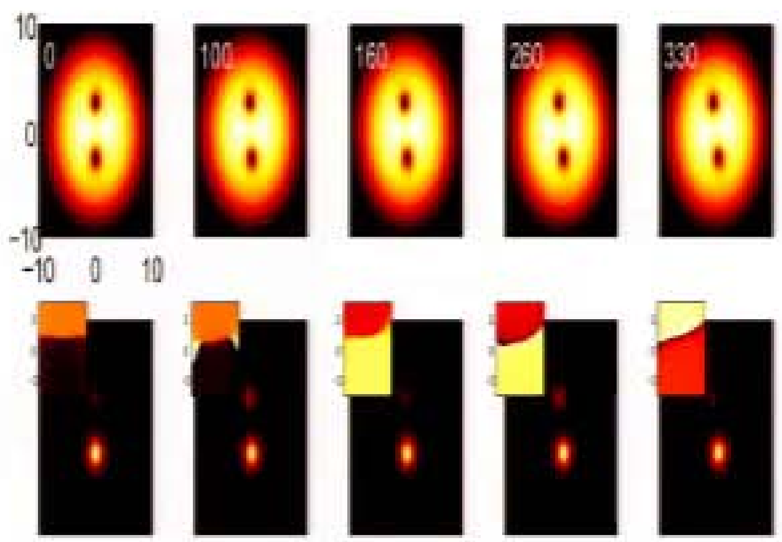
Two Vortex-Bright Solitons & Double Well Bifurcation

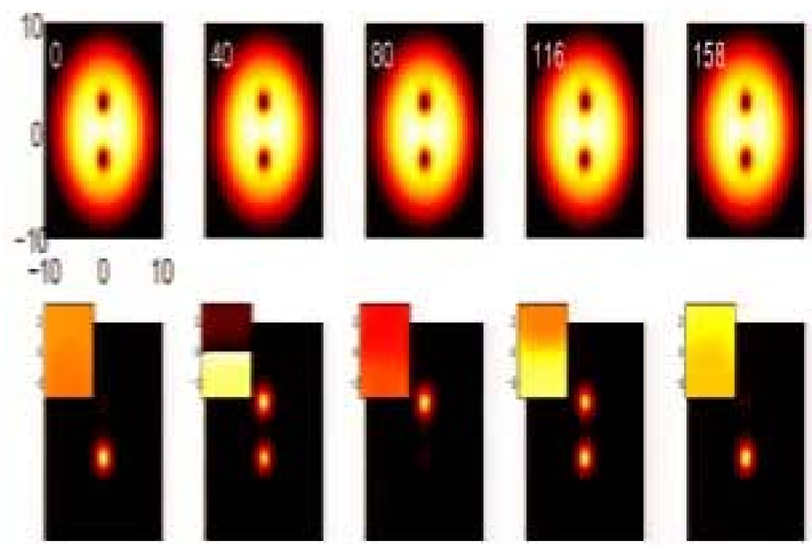


Double Well Dynamics & Symmetry Breaking

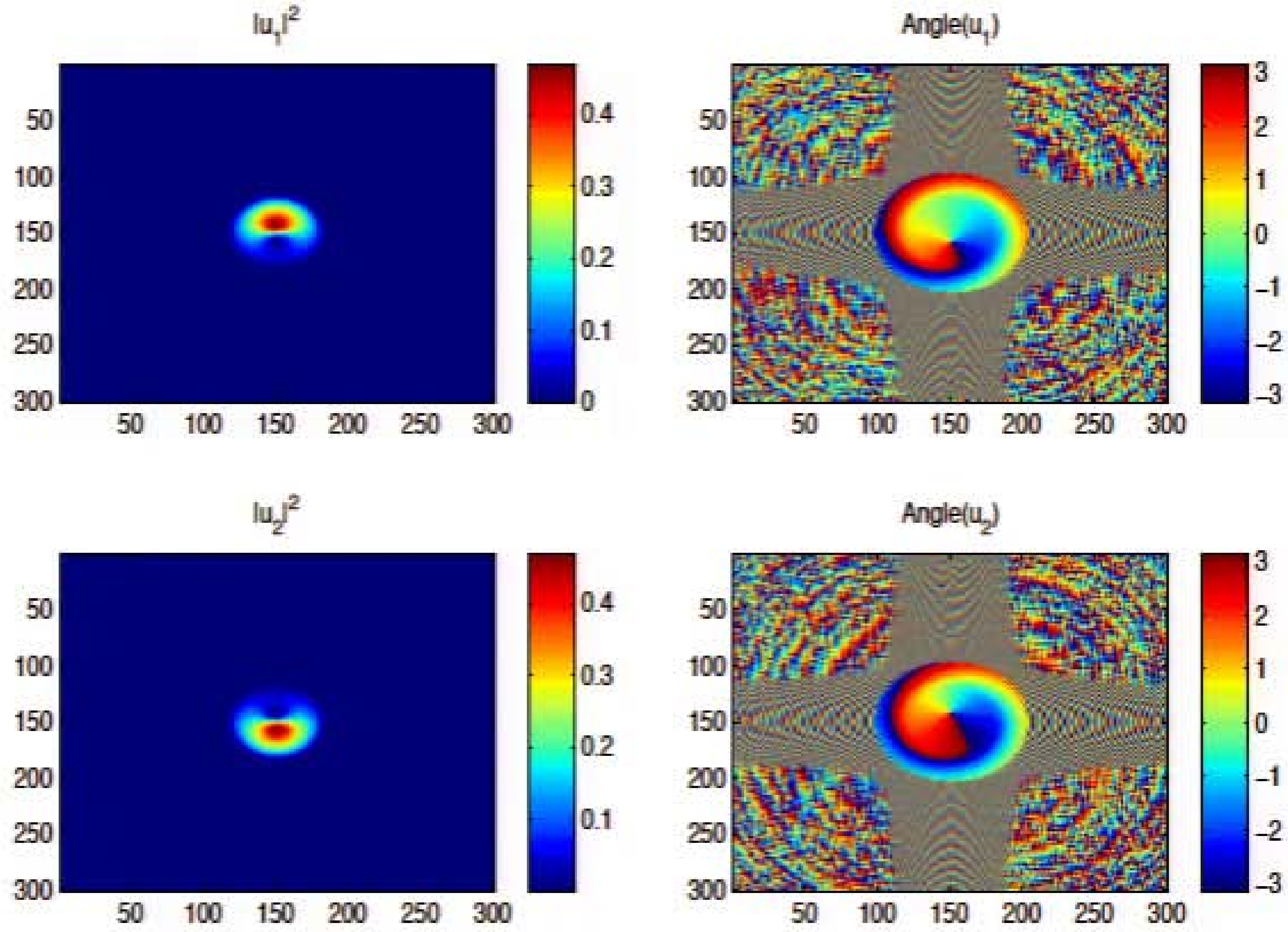




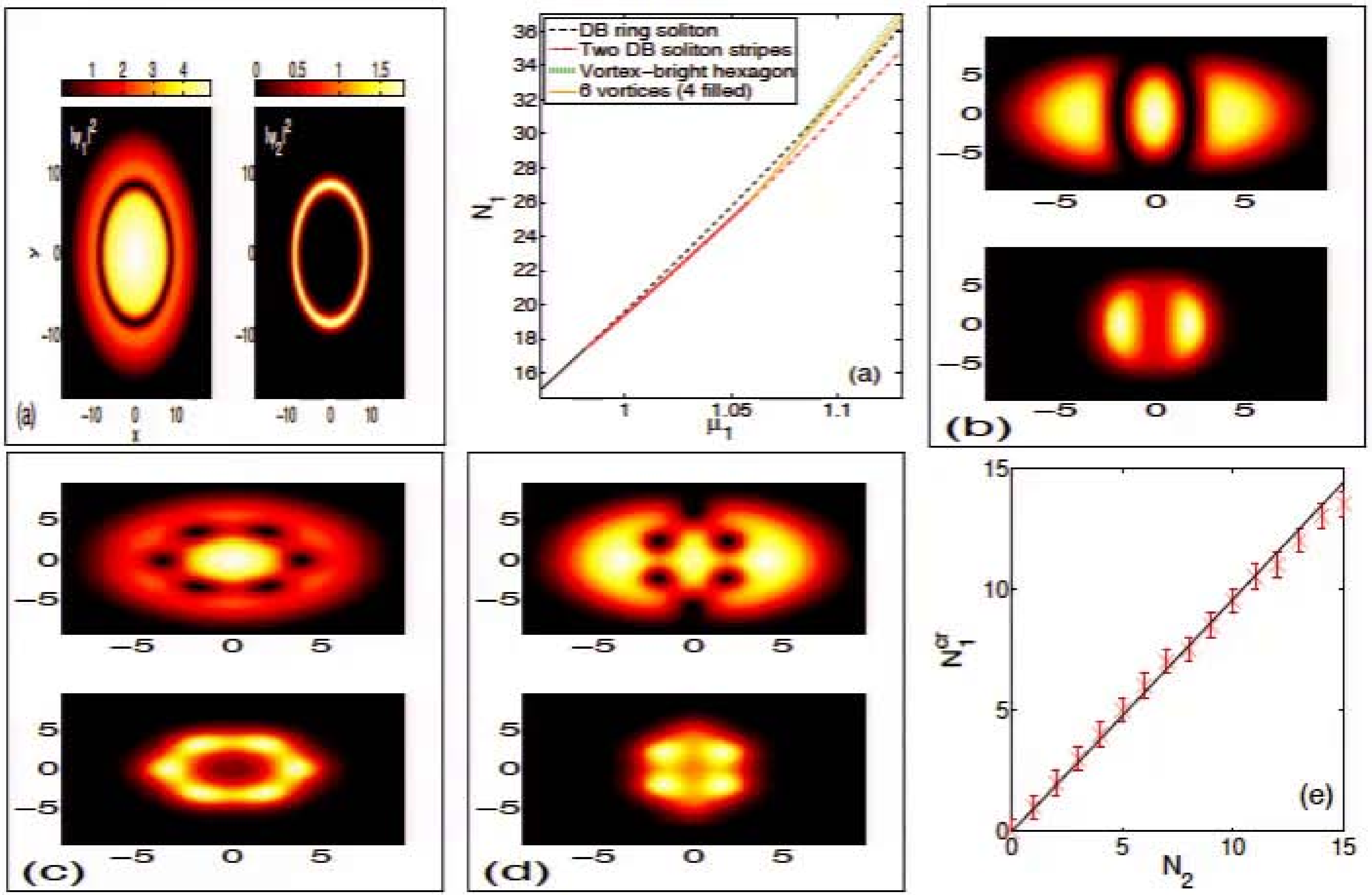




Rotating the Vortex-Bright: the Vortex-Vortex State



Another Twist: Using the Ring Dark Soliton As the Potential Well



Summary: Many Intriguing Aspects in DB Dynamics in Theory & Experiment

- 1 DB Soliton & 1 DD Soliton in Manakov Models
- DB + DD Solitons in a Trap (1 & 3d)
- 1, 2, 3, ... Many DB Solitons: Gluons + States in Trap
- Order to Disorder Transitions
- Variations: Trapping Higher Excited States
- Variations: Thermal DBs (1 & Many)
- Variations: 2 DBs as a Double-Well System
- Variations: Spinor BECs with DDB & BBD Solitons
- Variations: A 2d Prototype: Vortex-Bright Solitons
- Twists on the VB Theme: VBs as a DW, Vortex-Vortex State, Trapping Higher Excited States etc.
- Variations: Also in 2d: Dark-Bright Ring
- Many, Many More