High frequency, weakly nonlinear limit of NLS on the 2-torus

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Outline

NLS on the 2-torus

The equation

$$(NLS) \qquad \begin{cases} i\partial_t u - \Delta u = \pm |u|^2 u \\ u(t=0) = u_0. \end{cases}$$

where

$$(t,x) \in [0,\infty) \times \mathbb{T}^2$$
$$u(t,x) \in \mathbb{C}$$

and $\mathbb{T}^2 = [0,1]^2$ with periodic boundary conditions.

Local and global well-posedness

Local well posedness in H^s, $s > 0$ due to [Bourgain]

Conserved quantities Mass $M(u) = \int_{\mathbb{T}^2} |u|^2$

and Energy $E(u) = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u|^2 \mp \frac{1}{4} \int_{\mathbb{T}^2} |\nabla u|^4$

Global well-posedness in H^s, $s > 0$ follows for small mass.

Large-time behavior

Question: What is the qualitative behavior of u as $t \to \infty$?

Expectation: energy transfered to high frequencies
("weak turbulence").

Is it generic? Chaotic behavior?

Sobolev norm growth

Recall that $\|f\|_{H^s}^2 = \|\nabla^s f\|_{L^2}^2$.

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Lower bounds Given ϵ, M, there exists T and a solution u s.t.

$$\|u(t=0)\|_{H^s} < \epsilon \quad \text{but} \quad \|u(t=T)\|_{H^s} > M$$

[Colliander-Keel-Staffilani-Takaoka-Tao], see also [Kuksin], [Guardia-Kaloshin], [Hani-Pausader-Tzvetkov-Visciglia]

Weak turbulence

Statistical theory favored by physicists to understand u as $t \to \infty$. Consider

$$(NLS) \qquad i\partial_t u - \Delta u = \epsilon^2 |u|^2 u \quad \text{on } L\mathbb{T}^2.$$

and expand

$$u(t,x) = \sum_{k \in \mathbb{Z}^2/L} \widehat{u}_k(t) e^{ikx}.$$

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Consider the regime characterized by

- $\epsilon \to 0$ (weak non-linearity)
- $L \to \infty$ (big box / high frequency)
- $\widehat{u}_k = r_k e^{i\theta_k}$ with $\theta_k \sim$ uniform on $[0, 2\pi]$, iid (random phase approximation)

The kinetic wave equation

In the (statistical) regime above,

$$\rho_k(t) = \mathbb{E}|\widehat{u}_k(t)|^2$$

solves

$$\partial_t \rho_k = \int_{(\mathbb{R}^2)^3} \delta(k+l=m+n)\delta(|k|^2+|l|^2=|m|^2+|n|^2)$$
$$(\rho_l\rho_m\rho_n + \rho_k\rho_m\rho_n - \rho_k\rho_l\rho_m - \rho_k\rho_l\rho_n)\, dl\, dm\, dn$$

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- No rigorous derivation but [Lukkarinen-Spohn], [Escobedo-Velazquez]

Derivation of the continuous resonant equation

Main result

Let u solve $i\partial_t u - \Delta u = \epsilon^2 |u|^2 u \quad$ on $L\mathbb{T}^2$ with data $u(t=0) = g$ and expand $u(t,x) = \sum_{k \in \mathbb{Z}^2/L} \widehat{f}_k(t) e^{i(kx - t|k|^2)}$.

Theorem (E. Faou - PG - Z. Hani - simplified statement!)

As $\epsilon \to 0$ and $L \to \infty$, f solves - up to a time rescaling -

$$(CR) \qquad i\partial_t f(t,k) = \mathcal{T}(f,f,f)$$

with data $f(t=0) = g$ and

$$\mathcal{T}(f,f,f) = \int_{(\mathbb{R}^2)^3} \delta(k + l = m + n)\delta(|k|^2 + |l|^2 = |m|^2 + |n|^2) \overline{f(t,l)} f(t,m) f(t,n)\, dl\, dm\, dn.$$

Preprocessing

Start from

$$i\partial_t u - \Delta u = \epsilon^2 |u|^2 u \quad \text{on } L\mathbb{T}^2,$$

expand in Fourier series

$$u(t,x) = \sum_{k \in \mathbb{Z}^2/L} \widehat{u}_k(t) e^{ikx},$$

and filter by the linear group

$$f(t,x) = e^{it|k|^2} \widehat{u}_k(t).$$

Then f solves

$$i\partial_t f_k(t) = \epsilon^2 \sum_{\substack{l,m,n \in \mathbb{Z}^2/L \\ k+l=m+n}} \overline{f_l}(t) f_m(t) f_n(t) e^{it(|k|^2+|l|^2+|m|^2+|n|^2)}$$

Resonances

Split into

$$i\partial_t f_k = \underbrace{\epsilon^2 \sum_{\substack{k+l=m+n \\ |k|^2+|l|^2=|m|^2+|n|^2}} \overline{f_l} f_m f_n}_{\text{resonant interactions}} + \underbrace{\epsilon^2 \sum_{\substack{k+l=m+n \\ |k|^2+|l|^2\neq|m|^2+|n|^2}} \overline{f_l} f_m f_n e^{it(|k|^2+|l|^2-|m|^2-|n|^2)}}_{\text{non-resonant interactions}}$$

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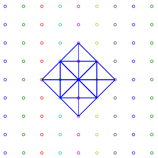
$$\begin{cases} k, l, m, n \in \mathbb{Z}^2/L \\ k + l = m + n \\ |k|^2 + |l|^2 = |m|^2 + |n|^2 \end{cases} \Leftrightarrow (k, l, m, n) \text{ rectangle}$$



The limit $\epsilon \to 0$

The dynamics is determined by resonant modes

$$i\partial_t f_k = \epsilon^2 \sum_{\substack{k+l=m+n \\ |k|^2+|l|^2=|m|^2+|n|^2}} \overline{f_l} f_m f_n$$

(theory of Hamiltonian systems ; normal form transformation)

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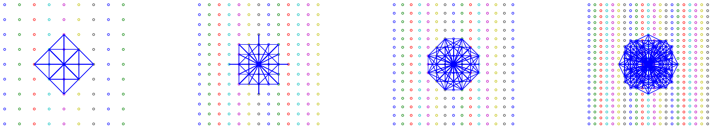

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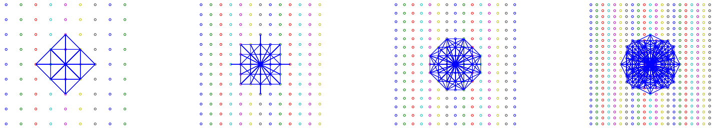
As $L \to \infty$, $\qquad \mathbb{Z}^2/L \longrightarrow \mathbb{R}^2$



$$\sum_{l,m,n \in \mathbb{Z}^2/L} \delta(k+l=m+n)\delta(|k|^2+|l|^2=|m|^2+|n|^2)\overline{f(l)}f(m)f(n)$$

$$\longrightarrow \int_{(\mathbb{R}^2)^3} \delta(k+l=m+n)\delta(|k|^2+|l|^2=|m|^2+|n|^2)$$

$$\overline{f(l)}f(m)f(n)\,dl\,dm\,dn = \mathcal{T}(f,f,f)$$

if f smooth (number theory)

Summarizing

$$i\partial_t f_k = \epsilon^2 \sum_{\substack{l,m,n\in\mathbb{Z}^2/L \\ k+l=m+n}} \overline{f_l} f_m f_n e^{it(|k|^2+|l|^2+|m|^2+|n|^2)} \qquad (NLS)$$

$$\downarrow \qquad \epsilon \to 0 \quad \text{(weak nonlinearity)}$$

$$i\partial_t f_k = \sum_{\substack{k+l=m+n \\ k^2+l^2+m^2+n^2}} \overline{f_l} f_m f_n$$

$$= \sum_{l,m,n\in\mathbb{Z}^2/L} \delta(k+l=m+n)\delta(|k|^2+|l|^2=|m|^2+|n|^2)\overline{f_l} f_m f_n$$

$$\downarrow \qquad L \to \infty \quad \text{(big box)}$$

$$i\partial_t f(k) = \int_{(\mathbb{R}^2)^3} \delta(k+l=m+n)\delta(|k|^2+|l|^2=|m|^2+|n|^2) \overline{f(l)} f(m) f(n)\, dl\, dm\, dn = \mathcal{T}(f,f,f) \qquad (CR)$$

Properties of the continuous resonant equation

Hamiltonian structure

The equation

$$(CR) \qquad i\partial_t f = \mathcal{T}(f,f,f)$$

is a Hamiltonian evolution equation, with Hamiltonian function

$$H(f) = \int_{\mathbb{R}\times\mathbb{R}^2} |e^{it\Delta} f|^4(t,x)\,dx\,dt. \qquad (L^4 \text{ Strichartz norm})$$

Other conserved quantity: the mass

$$M(f) = \int_{\mathbb{R}^2} |f|^2(x)\,dx\,dt.$$

Gaussians are thus stationary waves since

$$\mathrm{Argmin}_{M(f)=M_0} H(f) = \text{Gaussians}.$$

[Hundertmark-Zharnistky, Foschi]

The role of special Hermite functions

Special Hermite functions $(\phi_{n,m})_{n\in\mathbb{N}\,,\, m\in\{-n,2-n,\dots,n-2,n\}}$ give a Hilbertian basis of $L^2(\mathbb{R}^2)$ such that

$$\begin{cases} (-\Delta + |x|^2)\phi_{n,m} = 2(n+1)\phi_{n,m} \\ (x\partial_y - y\partial_x)\phi_{n,m} = m\phi_{n,m}. \end{cases}$$

Theorem (PG - Z. Hani - L. Thomann)

Special Hermite functions "diagonalize" the operator J: namely,

$$\mathcal{T}(\phi_{n_1,m_1}, \phi_{n_2,m_2}, \phi_{n_3,m_3}) = c_{n_1 n_2 n_3 m_1 m_2 m_3} \phi_{-n_1+n_2+n_3, -m_1+m_2+m_3}.$$

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- The equation (CR) provides the normal form for the Hermite-NLS equation (set on $\mathbb{R}^2$)

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$$i\partial_t u - \Delta u + |x|^2 u = |u|^2 u.$$

- The Bargmann-Fock space $L^2 \cap \{e^{-|z|^2} f(z) \text{ with } \partial_{\bar{z}} f = 0\}$ is invariant by (CR), which becomes there

$$(LLL) \qquad i\partial_t f = \Pi(|f|^2 f).$$

Open questions

- Complete integrability of (CR)?
- Other dimensions? [Buckmaster - Germain - Hani - Shatah]
- Other equations?
- Other domains?
- Derivation of the Kinetic Wave equation?
  [Gallagher - Germain - Hani]

Thank you for your attention!