Preconditioners for Inexact Two-Sided Inverse and Rayleigh Quotient Iteration

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Inexact two-sided methods

Two-sided Rayleigh quotient iteration (inexact)

Input: : Matrices A, M, shift $\sigma \approx \lambda$, initial vectors x_0, y_0 with $\|x_0\| = \|y_0\| = 1$, $x_0^H M y_0 \neq 0$.
Output: : Approximate eigentriple (λ, x, y)

for $k = 1$ to $\ldots$ **do**

Choose shift σ.

Run j steps of a Krylov subspace method to obtain $\hat{x}$, $\hat{y}$ such that

$$0 \leq \|(A - \sigma M)\hat{x} - Mx\| \leq \xi^R \|Mx\|$$
$$0 \leq \|(A - \sigma M)^H \hat{y} - M^H y\| \leq \xi^L \|M^H y\|,$$

Rescale $x = \frac{\hat{x}}{\|\hat{x}\|}$, $y = \frac{\hat{y}}{\|\hat{y}\|}$.

Update $\lambda = \rho(x, y) = \frac{y^H A x}{y^H M x}$ (two-sided Rayleigh quotient)

Test for convergence (using eigenvalue residuals ($r_x = (A - \lambda M)x$, $r_y = (A - \lambda M)^H y$).

end for

Inexact two-sided methods

Convergence rates of two-sided methods - exact solves

- fixed shift: linear
- two-sided RQ shift: cubic convergence even for *nonnormal* A [Parlett '74, F./Kürschner '14]

Convergence rates of two-sided methods - inexact solves

- fixed shift: linear convergence if decreasing solve tolerance
- two-sided RQ shift:
  - fixed tolerances: quadratic convergence [Hochstenbach/Sleijpen '03, F./Kürschner '14]
  - decreasing tolerances: cubic convergence [F./Kürschner '14]

Inexact two-sided methods

Convergence rates of two-sided methods - exact solves

rlett '74,

Main requirement: decreasing accuracy of the inner solves:

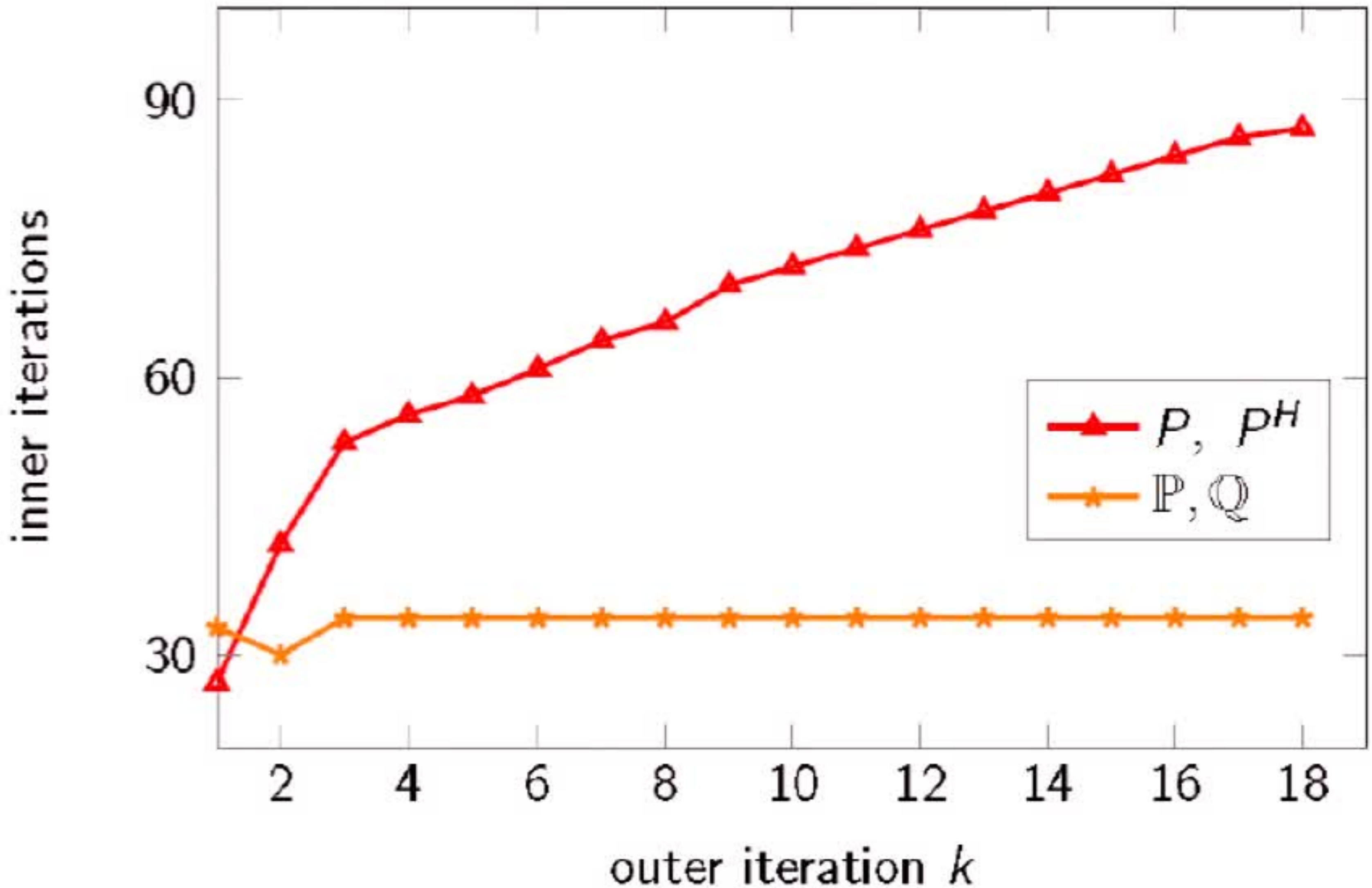
$$\xi_k^R \leq \xi_{k-1}^R, \quad \xi_k^L \leq \xi_{k-1}^L$$

Then: The convergence speed of the exact methods can be re-established.

- fixed tolerances: quadratic convergence [Hochstenbach/Sleijpen '03, F./Kürschner '14]
- decreasing tolerances: cubic convergence [F./Kürschner '14]

Inexact two-sided methods - dilemma resolved!

Example: matrix `sherman5`, $n = 3312$, fixed shift $\sigma = 0$.



Tuned preconditioners

Observation

The number of inner iterations remains ≈ constant, (slight) increase for two-sided RQI.

		# MVs	avg.	# precs	time (s)
μ fixed	no prec.	11232	624	0	78.8
	standard prec.	1216	68	1216	6.0
	tuned prec.	**607**	**34**	**643**	**4.6**
RQI	no prec.	2787	1394	0	51.8
	standard prec.	163	82	163	0.79
	tuned prec.	**112**	**56**	**116**	**0.64**

Tuned preconditioners for two-sided methods

For the use of BiCG/QMR/CSBCG we need a tuned preconditioner satisfying

$$\mathbb{S}x = x \quad \text{and} \quad \mathbb{S}^H y = y$$

This is not possible with a rank-one modification of a standard preconditioner!

Case 2: Tuned preconditioners for simultaneous solution

When solving (here $M = I$)

$$(A - \sigma I)\hat{x} = x$$
$$(A - \sigma I)^H \hat{y} = y$$

simultaneously we use

$$\mathbb{S} = P + \left((1 + \frac{\tau}{2})x - Px\right) y^H + x\left((1 + \frac{\tau}{2})y^H - y^H P\right),$$
$$\tau = y^H (Px - x)$$

with the normalisation $y^H x = 1$.

Similarly for

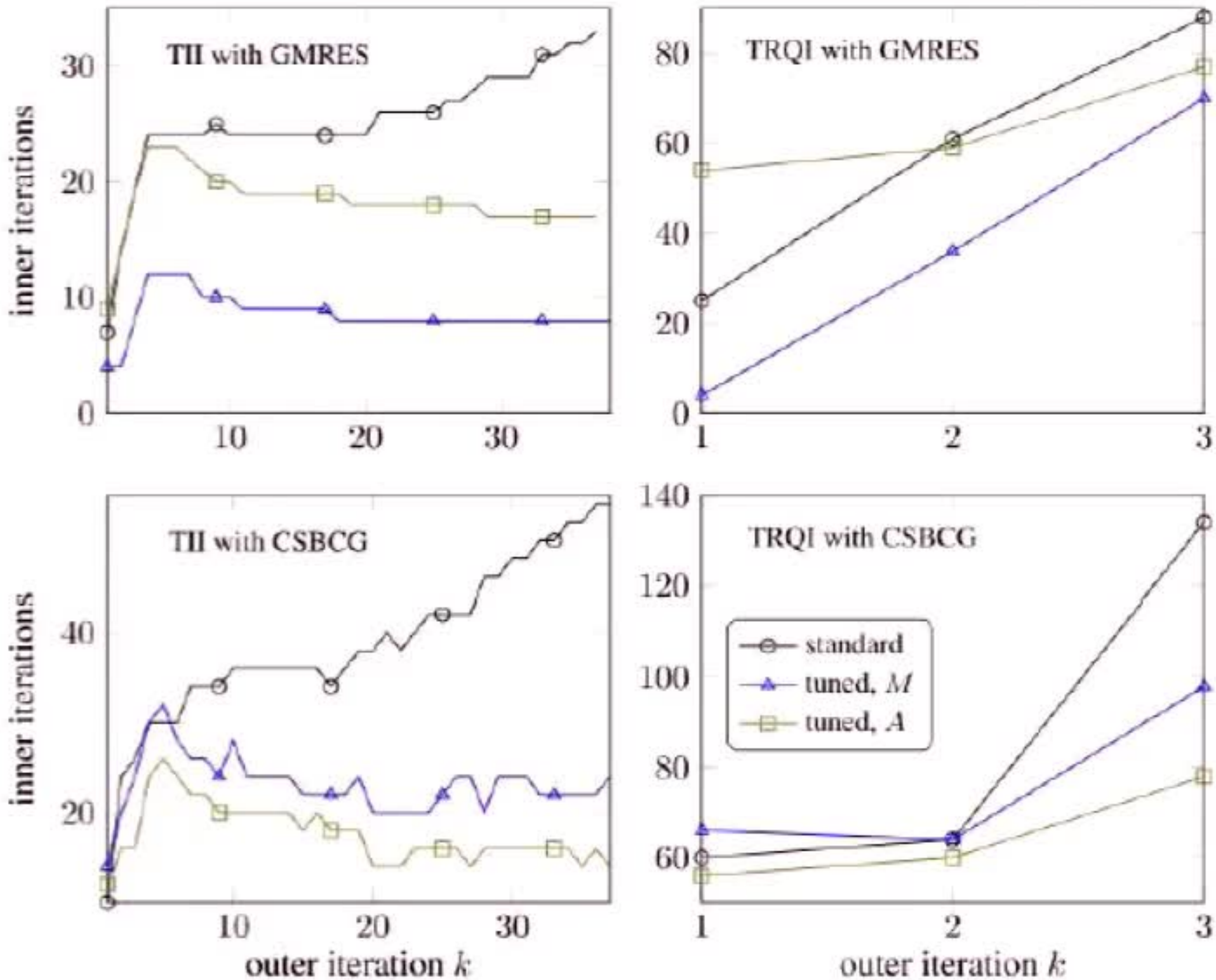
$$\mathbb{S}x = Mx \quad \text{and} \quad \mathbb{S}^H y = M^H y$$

More results

Table: Results for inexact TII and TRQI using standard and tuned preconditioners for an IFISS example, k and i are the required outer and total inner iterations.

Ex.	Methods	Precond.	k	i	aver.	# precs	time
IFISS $n = 66049$	TII - GMRES	standard	38	936	25	936	30.6
		tuned, M	39	325	9	401	16.4
		tuned, A	38	679	18	753	24.2
	TII - CSBCG	standard	38	1436	39	1436	29.1
		tuned, M	38	856	23	930	19.9
		tuned, A	38	650	18	724	15.7
	TRQI - GMRES	standard	4	174	58	174	5.9
		tuned, M	4	110	37	116	3.8
		tuned, A	4	190	63	196	5.8
	TRQI - CSBCG	standard	4	258	86	258	5.2
		tuned, M	4	228	76	234	4.6
		tuned, A	4	194	65	200	4.0

IFISS example



More results

Table: Results for inexact TII and TRQI using standard and tuned preconditioners for anemo example, k and i are the required outer and total inner iterations.

Ex.	Methods	Precond.	k	i	aver.	# precs	time
anemo $n = 29008$	TII - GMRES	standard	8	1685	241	1685	55.6
		tuned, M	stagnation at $\max(\|r_{u_k}\|, \|r_{v_k}\|) \approx 7.6$				
		tuned, A	10	1398	155	1416	42.3
	TII - CSBCG	standard	6	1530	306	1530	15.8
		tuned, M	6	1116	223	1126	9.9
		tuned, A	7	1120	187	1132	10.4
	TRQI - GMRES	standard	no convergence				
		tuned, M	stagnation at $\max(\|r_{u_k}\|, \|r_{v_k}\|) \approx 7.6$				
		tuned, A	stagnation at $\max(\|r_{u_k}\|, \|r_{v_k}\|) \approx 10^{-4}$				
	TRQI - CSBCG	standard	3	484	242	484	4.6
		tuned, M	3	466	233	470	4.8
		tuned, A	3	430	215	434	3.9