MS142: Symmetry, Asymmetry, and Network Synchronization

Organizer: Adilson E. Motter and Takashi Nishikawa, Northwestern University, USA

8:30-8:55 Symmetric States Requiring System Asymmetry Takashi Nishikawa and Adilson E. Motter, Northwestern University, USA

States Become Unstable

Michael Rosenblum, University of Potsdam, Germany

9:30-9:55 Resynchronization of Circadian Oscillators and the East-West Asymmetry of Jet-Lag Zhixin Lu, University of Maryland, USA; Kevin Klein-Cardena, DePaul University, USA; Steven Lee, City University of New York, Brooklyn, USA; Thomas Antonsen Jr., Michelle Girvan, and Edward Ott, University of Maryland, USA

10:00-10:25 Chimera and Chimera-Like States in Populations of Nonlocally Coupled **Homogeneous and Heterogeneous Chemical Oscillators** Kenneth Showalter, West Virginia University, USA

9:00-9:25 Two Types of Quasiperiodic Partial Synchrony: What Happens When Symmetric

Symmetric States Requiring System Asymmetry

<u>Takashi Nishikawa</u>, Yuanzhao Zhang, and Adilson E. Motter Department of Physics and Astronomy

> NORTHWESTERN UNIVERSITY

Funding: ARO, Simons Foundation

TN & AEM, Symmetric states requiring system asymmetry, Phys. Rev. Lett. **117**, 114101 (2016) YZ, TN, & AEM, Asymmetry-induced synchronization in oscillator networks, to appear in Phys. Rev. E, arXiv:1705.07907

> SIAM Conference on Applications of Dynamical Systems MS142: Symmetry, Asymmetry, and Network Synchronization - May 25, 2017





Symmetry

Complex networks

MacArthur, Sánchez-García, & Anderson, Discrete Appl. Math. 156, 3525 (2008)

Synchronization

Strogatz, Sync: The Emerging Science of Spontaneous Order (2003) Pikovsky, Rosenblum, & Kurths, Synchronization: A Universal Concept in Nonlinear Sciences (2003)

Symmetry

Network symmetry \iff dynamical symmetry

Golubitsky & Stewart, *The Symmetry Perspective* (2002) Pecora, Sorrentino, Hagerstrom, Murphy, & Roy, Nat. Commun. 5, 4079 (2014)

Symmetry

Symmetry

Symmetry breaking

Weyl, *Symmetry* (1952) Golubitsky & Stewart, *The Symmetry Perspective* (2002)

Chimera states

Kuramoto & Battogtokh, Nonlinear Phenom. Complex Syst. 5, 380 (2002) Abrams & Strogatz, Phys. Rev. Lett. 93,174102 (2004) Tinsley, Nkomo, & Showalter, Nat. Phys. 8, 662 (2012) Hagerstrom, Murphy, Roy, Hovel, Omelchenko, & Schöll, Nat. Phys. 8, 658 (2012)

$$\hat{\theta}_{i} = \omega + r_{i} - 1 - \gamma r_{i} \sum_{j=1}^{n} \sin(\theta_{j} - \theta_{i})$$
$$\dot{r}_{i} = b_{i} r_{i} (1 - r_{i}) + \varepsilon r_{i} \sum_{j=1}^{n} A_{ij} \sin(\theta_{j} - \theta_{i})$$

Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

TN & AEM, Symmetric states requiring system asymmetry, Phys. Rev. Lett. **117**, 114101 (2016)



$$\dot{\theta}_i = \omega + r_i - 1$$
$$\dot{r}_i = b_i r_i (1 - r_i)$$

Parameters: ω

$$\int \theta_i(t) \equiv \theta_0 \, \cdot$$

TN & AEM, Symmetric states requiring system asymmetry, Phys. Rev. Lett. **117**, 114101 (2016)

= 1,
$$\gamma = 0.1$$
, $\varepsilon = 2$

mit cycle -+ ωt , $r_i(t) \equiv 1$



$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^n \sin(\theta_j - \theta_i)$$
$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i)$$

Parameters: ω

Synchronous state - $\theta_1(t) = \cdots = \theta_n(t) \equiv \theta_0 + \omega t, \quad r_1(t) = \cdots = r_n(t) \equiv 1$ uniform and symmetric

TN & AEM, Symmetric states requiring system asymmetry, Phys. Rev. Lett. **117**, 114101 (2016)

$$\gamma = 1, \ \gamma = 0.1, \ \varepsilon = 2$$



$$\dot{\theta}_{i} = \omega + r_{i} - 1 - \gamma r_{i} \sum_{j=1}^{n} \sin(\theta_{j} - \theta_{i})$$

$$\dot{r}_{i} = \underbrace{b_{i}}_{i} (1 - r_{i}) + \varepsilon r_{i} \sum_{j=1}^{n} A_{ij} \sin(\theta_{j} - \theta_{i})$$
Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

$$\underbrace{tunable}_{tunable} \text{ oscillator parameters}$$

$$\Theta_{1}(t) = \dots = \Theta_{n}(t) \equiv \Theta_{0} + \omega t, \ r_{1}(t) = \dots = r_{n}(t) \equiv 1$$

$$\underbrace{uniform \ and \ symmetric}$$



Symmetric network structure



Coupling strength $A_{ij} = 1 + \delta \text{ or } 1 - \delta$ 1.3 or 0.7 $(\delta = 0.3)$



$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^n \sin(\theta_j - \theta_i)$$
$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i)$$

Best homogeneous b_i value















Synchronization dynamics

Complete synchronization with nonidentical oscillators Complete synchronization only with nonidentical oscillators symmetric state system asymmetry

"Symmetric states requiring system asymmetry"

We have a converse of symmetry breaking

Symmetric stable state requiring system to be asymmetric

Symmetric **system** *requiring* **stable state** to be asymmetric (symmetry breaking)

















How often does this occur?

YZ, TN, & AEM, Asymmetry-induced synchronization in oscillator networks, to appear in Phys. Rev. E, arXiv:1705.07907



Networks with multiple link types Adjacency matrices $A^{(\alpha)}, \ \alpha = 1, \dots, K$



Networks with multiple link types $Adjacency matrices A^{(1)}$



Networks with multiple link types Adjacency matrices $A^{(1)}$, $A^{(2)}$



Networks with multiple link types Adjacency matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$



Symmetric network structures

- Symmetric network: every node can be mapped to any other node by some permutation of nodes without changing any $A^{(\alpha)}$.
- For undirected networks with a single link type, they are called *vertex-transitive graphs*.
- Includes *circulant graphs*, defined as a network whose nodes can be arranged in a ring so that the network is invariant under rotations.



Example of symmetric network (circulant graph)

$\dot{oldsymbol{X}}_i = oldsymbol{F}_i(oldsymbol{X}_i)$: dynamics of isolated node i







Network of non-identical oscillators K = N $\dot{\boldsymbol{X}}_{i} = \boldsymbol{F}_{i}(\boldsymbol{X}_{i}) + \sum \sum A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_{i}, \boldsymbol{X}_{i'})$ $\begin{array}{c} \alpha = 1 \ i' = 1 \\ i' \neq i \end{array}$

 $A_{ii'}^{(\alpha)}$: directed link of type α from node i' to node i $H^{(lpha)}(X_i,X_{i'})$: coupling function





Network of non-identical oscillators K = N $\dot{\boldsymbol{X}}_{i} = \boldsymbol{F}_{i}(\boldsymbol{X}_{i}) + \sum \sum A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_{i}, \boldsymbol{X}_{i'})$ $\begin{array}{c} \alpha = 1 \ i' = 1 \\ i' \neq i \end{array}$

 $A_{ii'}^{(\alpha)}$: directed link of type α from node i' to node i $H^{(lpha)}(X_i,X_{i'})$: coupling function





Defining asymmetry-induced synchronization

For a symmetric network

$$\dot{\boldsymbol{X}}_{i} = \boldsymbol{F}_{i}(\boldsymbol{X}_{i}) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1\\i'\neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_{i}, \boldsymbol{X}_{i'})$$

with completely synchronous state,

2. Synchronous state is stable for some heterogeneous system.

 F_3 F_2 $A^{(1)}, A^{(2)}, A^{(3)}$ $H^{(1)}, H^{(2)}, H^{(3)}$

- Conditions
- 1. Synchronous state is unstable for any homogeneous system. $F_1 = \cdots = F_N$

 F_4

 $F_i \neq F_{i'}$ for some $i \neq i'$



 F_1

K = N $\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum \sum A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$ $\alpha = 1 i' = 1$ $i' \neq i$

Class of multilayer systems



















2 subnodes for each node



















Internal links with strength a











Internal link pattern defines node type





External links?





Pattern of sublinks for each link type

1

1 /

()

· **1**/

2′





















Multilayer network



\mathcal{U} $F^{(i)}$

termines the dynamics of every isolated subnode, and hMultilayet and the interaction function common to all sublinks. Here Multilayet and the sublinks of the sublinks of the sublinks of the sublinks of the sublinks. Here corresponding coupling matrix $\widetilde{A}^{(ii')} := (\widetilde{A}^{(ii')}_{\ell\ell'}), i \neq i',$ is the same and encodes the subnode connection pattern for that link type In contrast, the subnode connection pattern within each hode is encoded in the matrix tote that the node to node interactions are not necessarily diffusive, but the submode-to-subnode interactions are diffusive. This guarantees the existence of ial^{i} synchrotions state of EA(ii'(2)) by m(i)(i) = s(t), $\forall i, \ell \text{ with } \dot{s} = f(s_i) + hich \text{ corresponds to a synchronous}$ state of Eq. (1). Thus, we have a general class of multiGompletely synchronous state is quaranteed admit complete • Stahility fazily on sure of stability frages in Stability Fragencies. To facilitate the stability canalysis, we ffatten the 19981-• Vailay for a bitwark f approximation into a single layer (see Fig. 1(c) for an example). By defining a single index



Network of nodes and links



$$\dot{\boldsymbol{X}}_{i} = \boldsymbol{F}_{i}(\boldsymbol{X}_{i}) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1\\i'\neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_{i}, \boldsymbol{X}_{i'})$$

- Synchronous state is guaranteed
- Valid for arbitrary f and h

node dynamics is different internal link pattern is different

 Stability readily computed using Master Stability Function L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. 80, 2109 (1998)



Lyapunov exponents

works if N is prime). Sampling uniformly emitting this class (see SM [26], Sec. ??, for details [30]), we observe that significant fraction of external sublink structures are Alsync-favoring over a range of sublink densities [Fig. 3(a)] and network sizes [Fig. 3(b)]. We also observe that sparse and dense structures favor AISync more of-that the effect a internal sublink heterogeneity would be smaller why meet the heterogeneity would be Given a symmetric network of identical oscillators, it Heterogeneous is instructive to compare our results above in which the symmetry is broken by making the oscillators nonidentical with the alternative scenario in which the symmetry is broken by making the network structure asymmetric. For diversed and hetworks of diffusively-coupled Widentical oscillators, it can be shown that: 1) with the exception of the complete graphs, all topologies that op-20 timize sy**BO** hronizability (i.e., thoseO with $\sigma = 0$) are asymmetric [31]; 2) any network topology that can be spanned



What about other symmetric networks?

- AlSync strength r quantifies the degree to which a network structure favors AlSync.
- $r = 0 \Rightarrow No AlSync$
- Larger $r \Rightarrow$ Favors AlSync more strongly • $r = 1 \Rightarrow$ There is an optimal heterogenous system.







Fraction of networks with r > 0.05

Within class of circulant-graphs (= all symmetric networks, if N is prime)



Summary

Symmetric states requiring system asymmetry (converse of symmetry breaking)

- In network synchronization: fully synchronous state stable only when the oscillators are non-identical
- Observed quite often in the class of multilayer networks we considered

TN & AEM, Symmetric states requiring system asymmetry, Phys. Rev. Lett. **117**, 114101 (2016)

YZ, TN, & AEM, Asymmetry-induced synchronization in oscillator networks, to appear in Phys. Rev. E, arXiv:1705.07907



More generally: states with more symmetry requiring system to have less symmetry

- Curie's principle
- Convergent vs divergent pattern formation

TN & AEM, Symmetric states requiring system asymmetry, Phys. Rev. Lett. **117**, 114101 (2016) YZ, TN, & AEM, Asymmetry-induced synchronization in oscillator networks, to appear in Phys. Rev. E, arXiv:1705.07907

Final remarks







node = *L* identical subnodes

node = *L identical* subnodes



 $2^4 - 1 = 15$ possibilities



node = *L identical* subnodes

node types = patterns of *internal* sublinks



binary

link types = patterns of *external* sublinks





 $2^4 - 1 = 15$ possibilities







Multilayer networks of identical oscillators







nodes and links

 $\dot{X}_{i} = F_{i}(X_{i}) + \sum_{i=1}^{K} \sum_{\substack{i=1 \ \text{subinode} \ \ell \ \text{(i)} \\ \text{subinode} \ \ell \ \text{(i)}}}^{N} = \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}(t) \text{ is the } m \text{-dimensional state vector f } \mathbf{x}_{\ell}^{(i)}($ $\alpha = 1$ i' = 1 termines the dynamics of every isolated subnode, and $i' \neq i$ is the interaction function common to all sublinks. Here for all links of a givenbood and the set of a give by a corresponding coupling matrix $\widetilde{A}^{(ii')} := (\widetilde{A}^{(ii')}_{\ell\ell'}) i \neq i$ is the same and encodes the subrode connection particle is the subrode connectis tern før that link type. In contrast, the submore / conne tion pattern within each node i is encoded in the matr $F^{(i)} := (\widetilde{A}_{\mu})^{(i)}$ Note that the node-to-node interaction are not necessarily diffusive, but the subnode-to-subno interactions are diffusive. This guarantees the existen of $i \in \mathcal{I}$ synchronication is state of $E_{\ell\ell'}^{(ii')}(2)$ here $i \in \mathcal{I}_{\ell\ell'}^{(ii')}(2)$ by $\mathcal{H}_{\ell}^{(i)}(x_{\ell'}) = s(x_{\ell'})$ $\forall i, \ell \text{ with } \dot{s} = f(s_{\ell'}) = w$ and corresponds to a synchronication of $i \in \mathcal{I}$. state of Eq. (1). Thus, we have a general class of multiple of Eq. (1).

 $i' = 1 \ell' = 1$



tern for that link type. In contrast, the subnode connec- $Masie A_{\mu\nu}$ is encoded in the matrix $Masie A_{\mu\nu}$. The matrix $Masie A_{\mu\nu}$ is encoded in the matrix is encoded in the matrix is encoded in the matrix is encoded. The matrix is encoded in t are not necessarily diffusive, But the submode-to-subnode interactions are diffusive. This guarantees the existence of ia_{ℓ}^{i} synchron ia_{ℓ}^{i} of $ia_{\ell}^{i}(2)$ how $ia_{\ell}^{i}(2)$ by $\mathbf{n}_{\ell}^{i}(\mathbf{x}_{\ell}^{i}(t)) \neq \mathbf{s}(t)$, $\forall i, \ell \text{ with } \mathbf{\dot{s}} = \mathbf{f}(\mathbf{s}_{i}) = \mathbf{w}_{\ell} \mathbf{hich} \text{ corresponds to a synchronous state of Eq. (1). Thus, we have a general class of multi$ lagerigates symmetric networks that admit complete synchronization (see Ste gale horization or an or details). To facilitate the stated for analyzing for a fatten the multilayerinettoringkexperiesentalinealizatoleasystegie) layer (see Fig. 1(c) for an example). By defining a single index

