MS142: Symmetry, Asymmetry, and Network Synchronization

Organizer: *Adilson E. Motter and Takashi Nishikawa*, Northwestern University, USA

8:30-8:55 Symmetric States Requiring System Asymmetry

Takashi Nishikawa and Adilson E. Motter, Northwestern University, USA

9:00-9:25 Two Types of Quasiperiodic Partial Synchrony: What Happens When Symmetric States Become Unstable

Michael Rosenblum, University of Potsdam, Germany

9:30-9:55 Resynchronization of Circadian Oscillators and the East-West Asymmetry of Jet-Lag

Zhixin Lu, University of Maryland, USA; Kevin Klein-Cardena, DePaul University, USA; Steven Lee, City University of New York, Brooklyn, USA; Thomas Antonsen Jr., *Michelle Girvan*, and Edward Ott, University of Maryland, USA

10:00-10:25 Chimera and Chimera-Like States in Populations of Nonlocally Coupled Homogeneous and Heterogeneous Chemical Oscillators

Kenneth Showalter, West Virginia University, USA

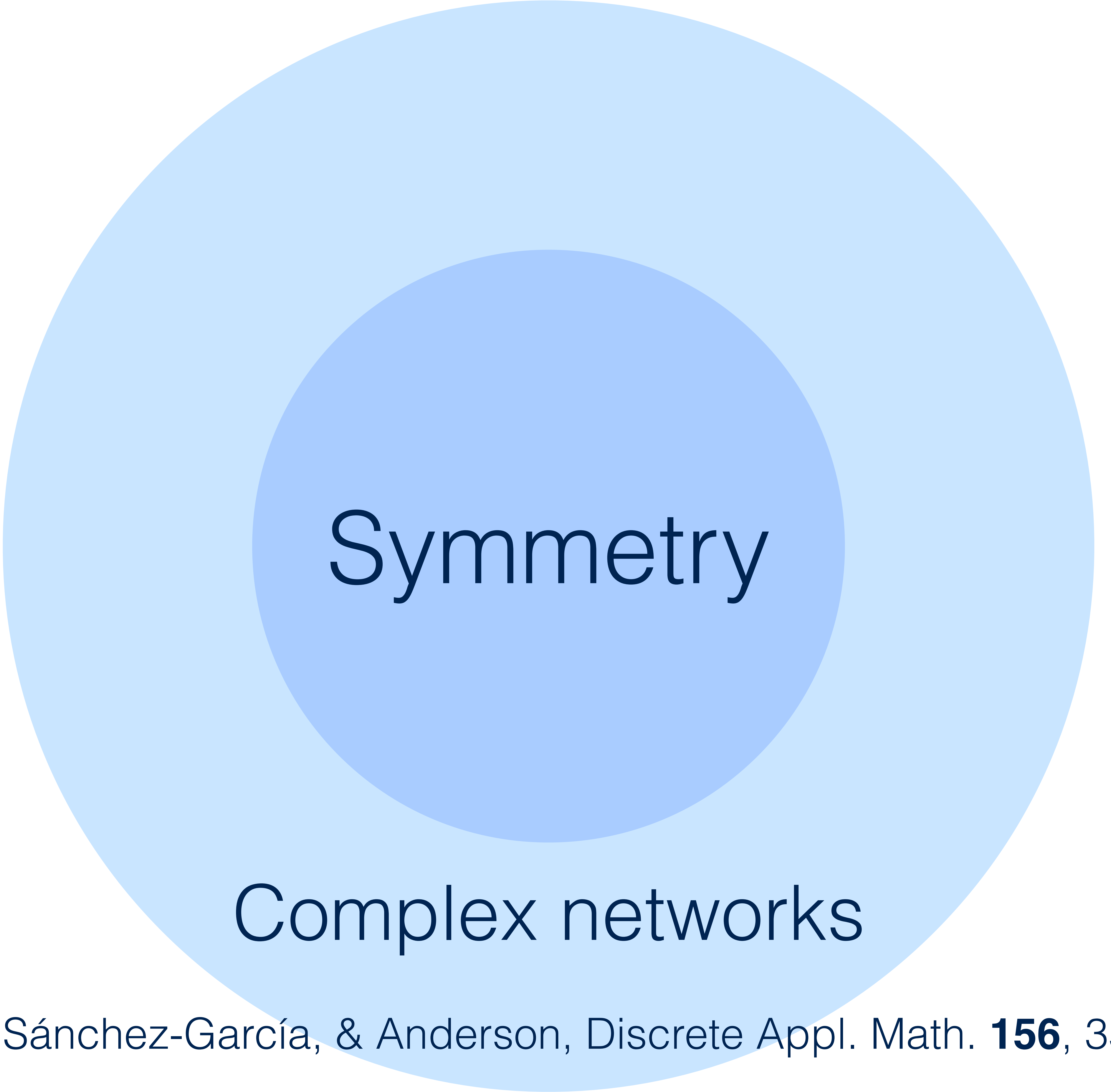
Symmetric States Requiring System Asymmetry

Takashi Nishikawa, Yuanzhao Zhang, and Adilson E. Motter
Department of Physics and Astronomy
NORTHWESTERN UNIVERSITY

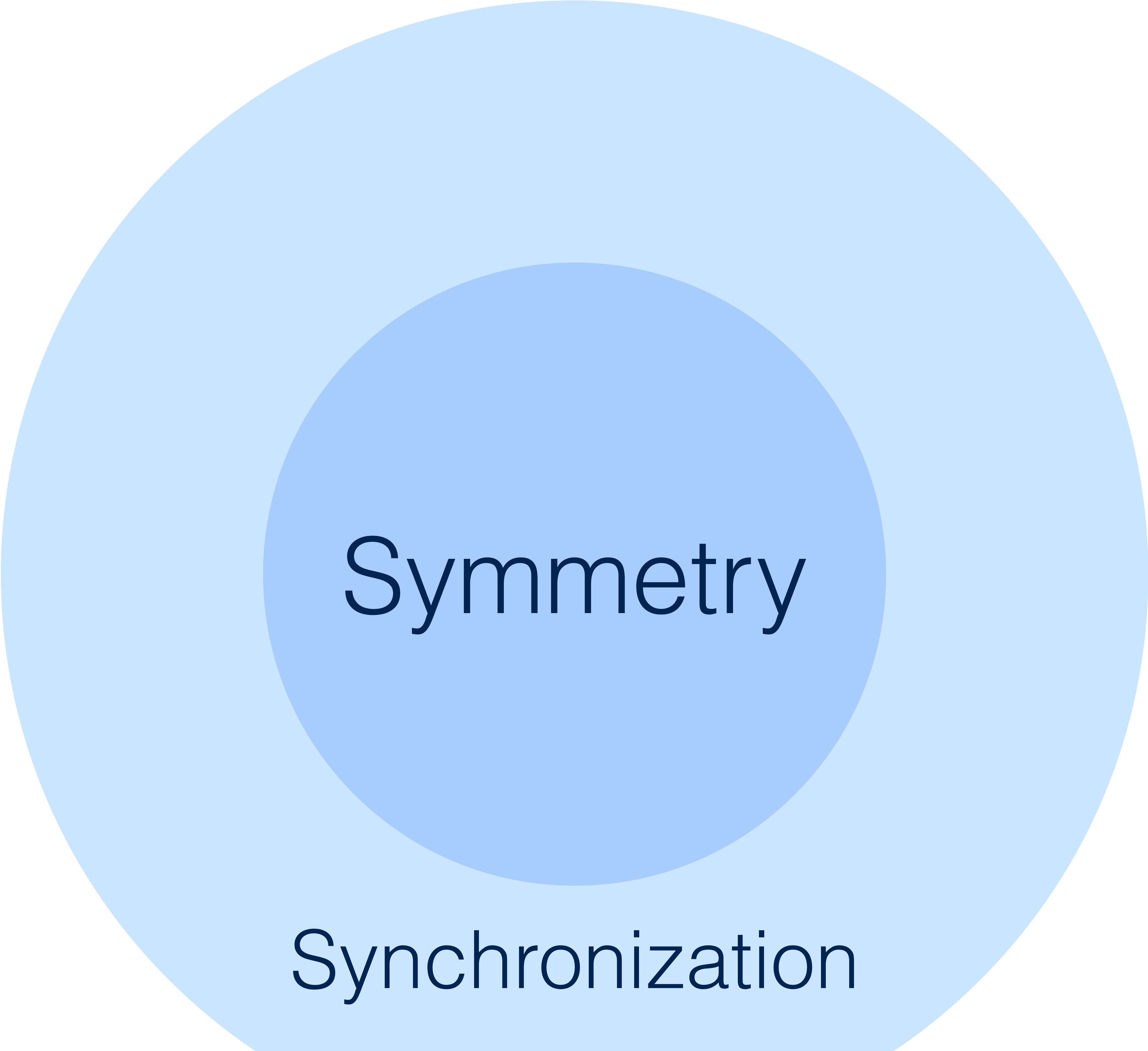
Funding: ARO, Simons Foundation

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)
YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907

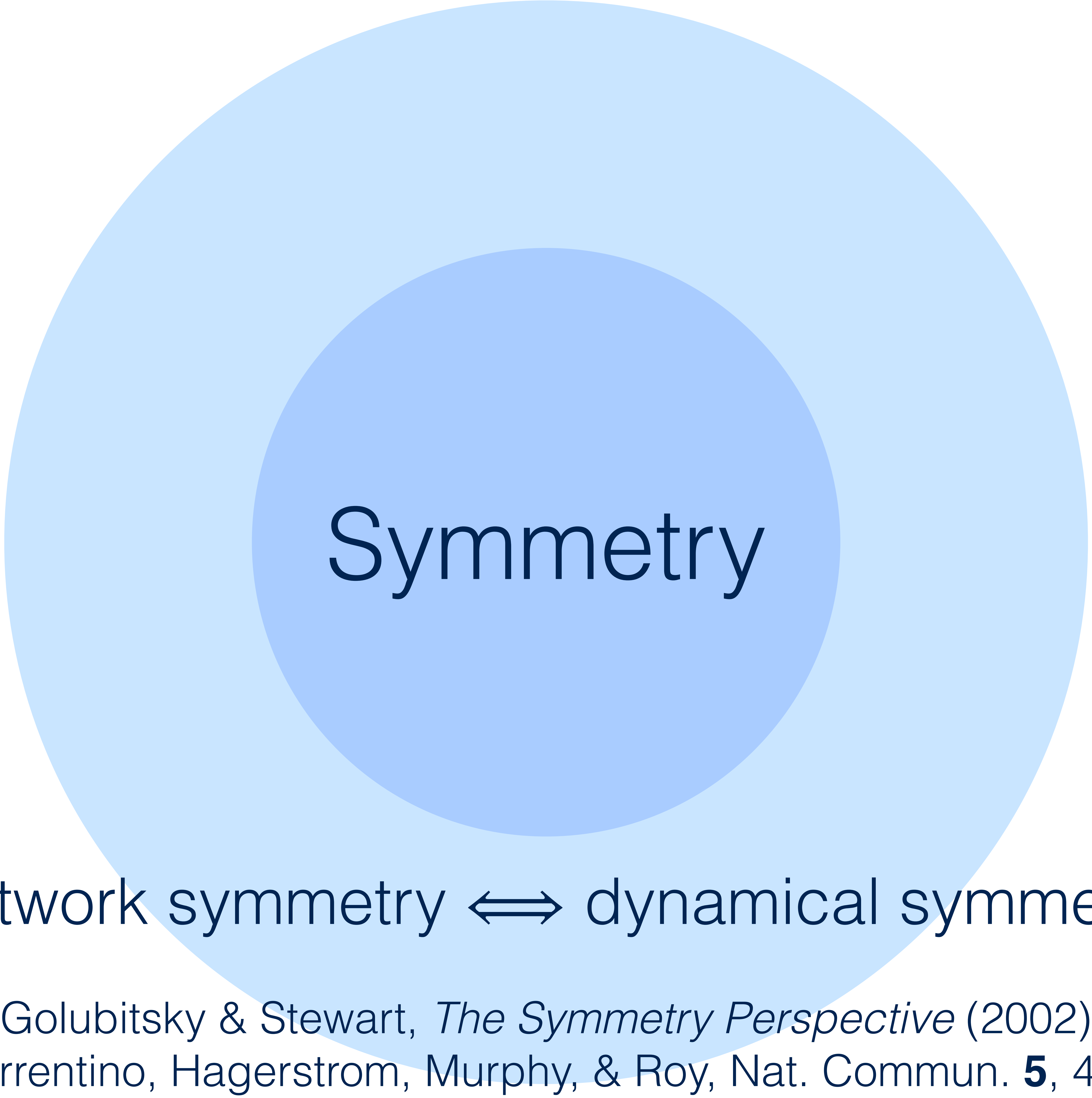
SIAM Conference on Applications of Dynamical Systems
MS142: Symmetry, Asymmetry, and Network Synchronization - May 25, 2017



MacArthur, Sánchez-García, & Anderson, Discrete Appl. Math. **156**, 3525 (2008)



Strogatz, *Sync: The Emerging Science of Spontaneous Order* (2003)
Pikovsky, Rosenblum, & Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (2003)



Network symmetry $\Longleftrightarrow$ dynamical symmetry

Golubitsky & Stewart, *The Symmetry Perspective* (2002)
Pecora, Sorrentino, Hagerstrom, Murphy, & Roy, Nat. Commun. **5**, 4079 (2014)

Symmetry

Symmetry breaking

Weyl, *Symmetry* (1952)
Golubitsky & Stewart, *The Symmetry Perspective* (2002)

Chimera states

Kuramoto & Battogtokh, Nonlinear Phenom. Complex Syst. **5**, 380 (2002)
Abrams & Strogatz, Phys. Rev. Lett. **93**,174102 (2004)
Tinsley, Nkomo, & Showalter, Nat. Phys. **8**, 662 (2012)
Hagerstrom, Murphy, Roy, Hovel, Omelchenko, & Schöll, Nat. Phys. **8**, 658 (2012)

Example: Network of n phase-amplitude oscillators

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i)$$

Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)

Example: Network of n phase-amplitude oscillators

$$\dot{\theta}_i = \omega + r_i - 1$$

$$\dot{r}_i = b_i r_i (1 - r_i)$$

Parameters: $\omega = 1,\ \gamma = 0.1,\ \varepsilon = 2$

Limit cycle

$$\theta_i(t) \equiv \theta_0 + \omega t, \quad r_i(t) \equiv 1$$

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)

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Synchronous state

$$\theta_1(t) = \cdots = \theta_n(t) \equiv \theta_0 + \omega t, \quad r_1(t) = \cdots = r_n(t) \equiv 1$$

uniform and symmetric

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)

Example: Network of n phase-amplitude oscillators

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Parameters: $\omega = 1,\ \gamma = 0.1,\ \varepsilon = 2$

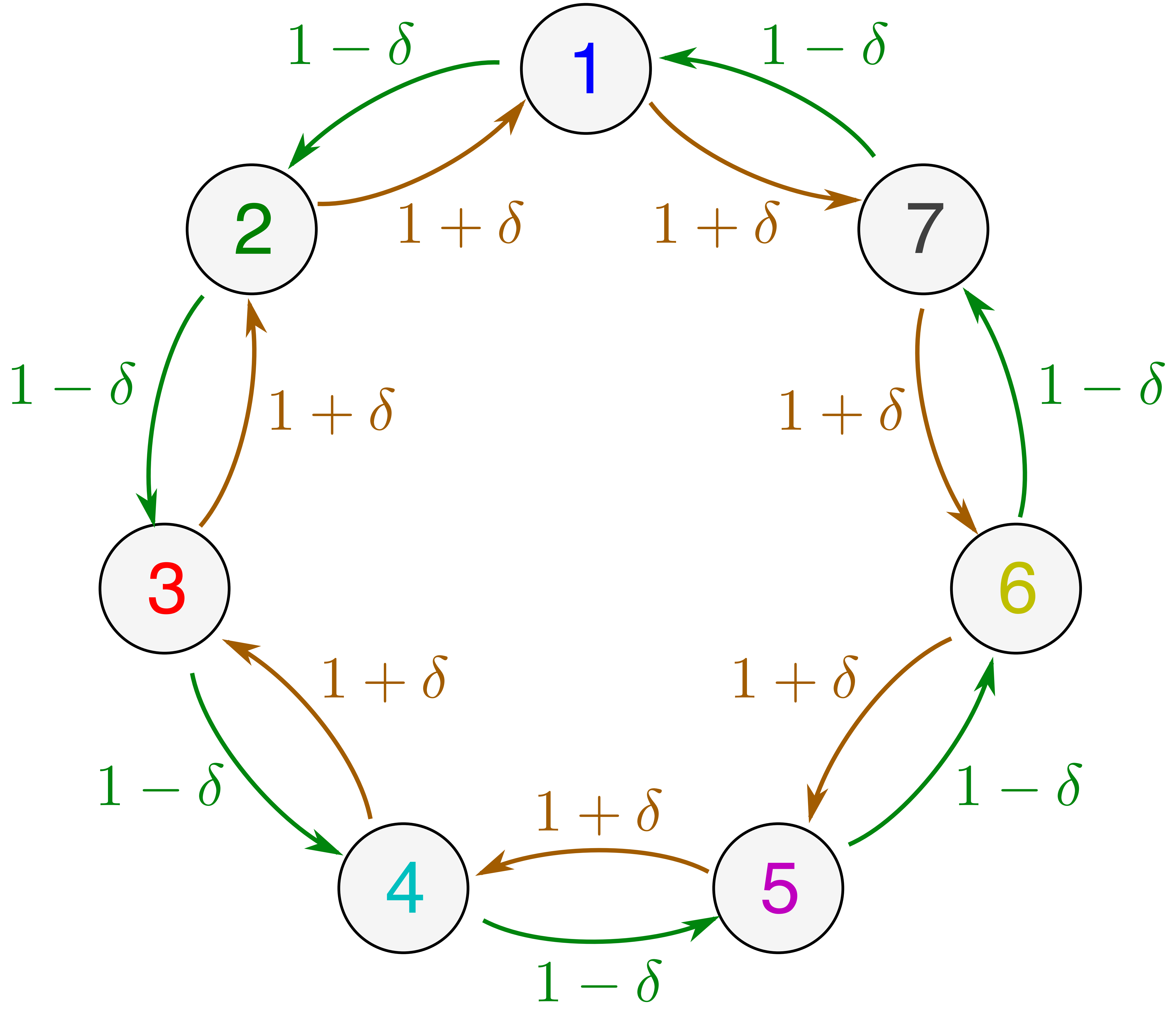
tunable oscillator parameters

Synchronous state

$$\theta_1(t) = \cdots = \theta_n(t) \equiv \theta_0 + \omega t, \quad r_1(t) = \cdots = r_n(t) \equiv 1$$

uniform and symmetric

Symmetric network structure



Coupling strength

$A_{ij} = 1 + \delta$ or $1 - \delta$

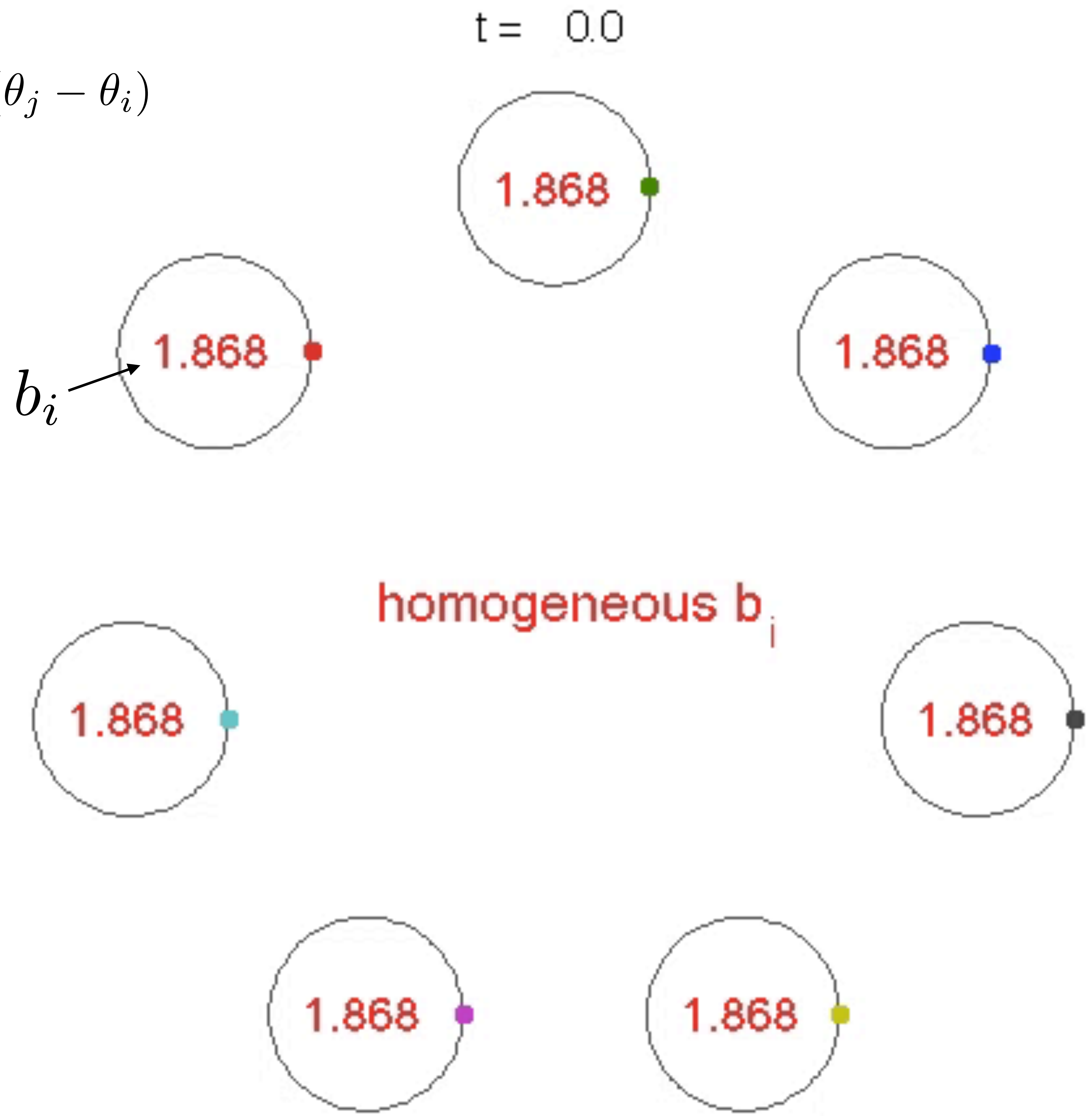
1.3 or 0.7

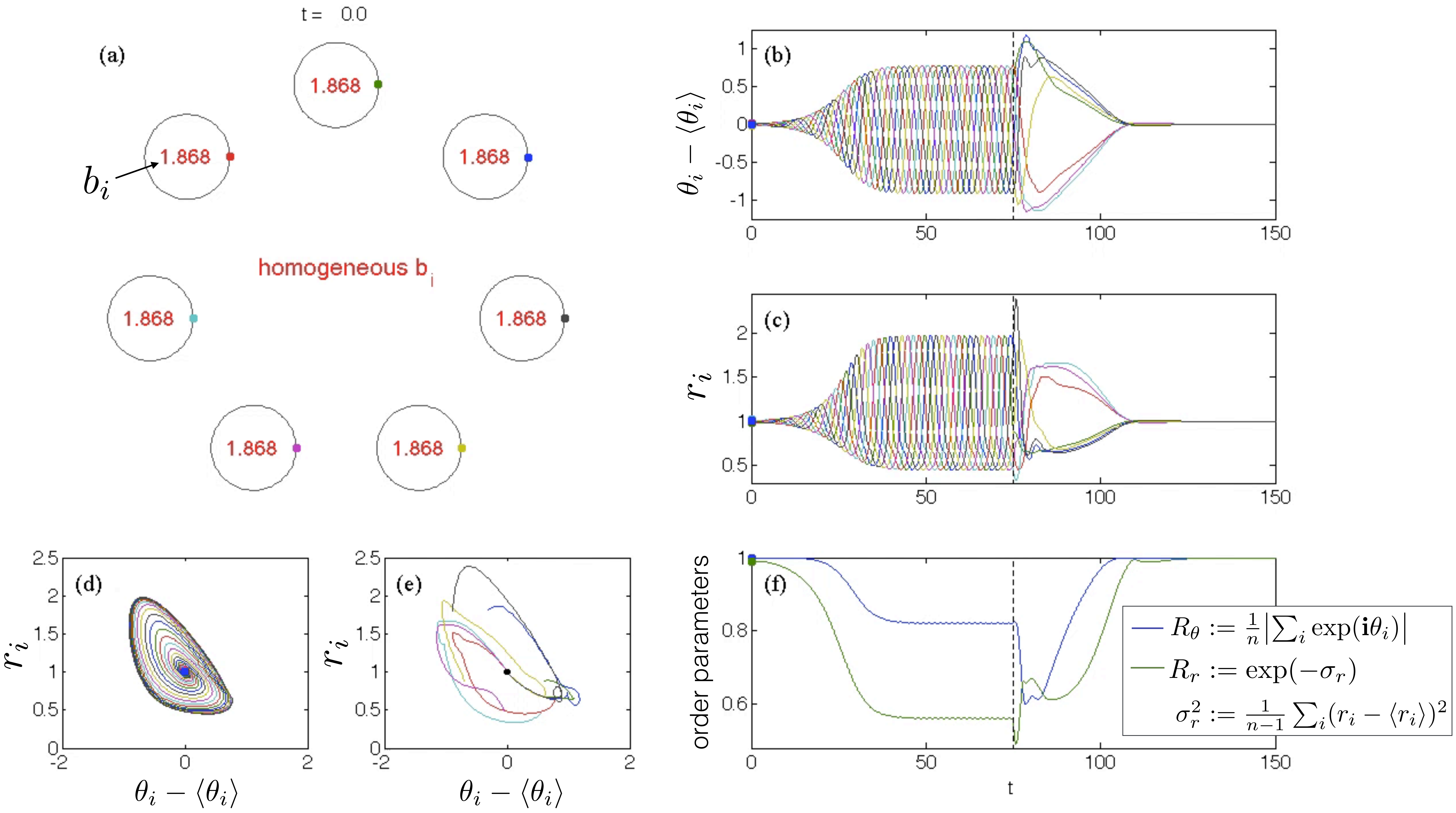
($\delta = 0.3$)

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i)$$

Best homogeneous b_i value





t = 0.0
(a)
b_i
1.868
homogeneous b_i
(b)
$\theta_i - \langle\theta_i\rangle$
(c)
r_i
(d)
(e)
(f)
order parameters
t
$R_\theta := \frac{1}{n}\left|\sum_i \exp(\mathbf{i}\theta_i)\right|$
$R_r := \exp(-\sigma_r)$
$\sigma_r^2 := \frac{1}{n-1}\sum_i (r_i - \langle r_i\rangle)^2$

Synchronization dynamics

Complete synchronization with nonidentical oscillators

Complete synchronization only with nonidentical oscillators

symmetric state

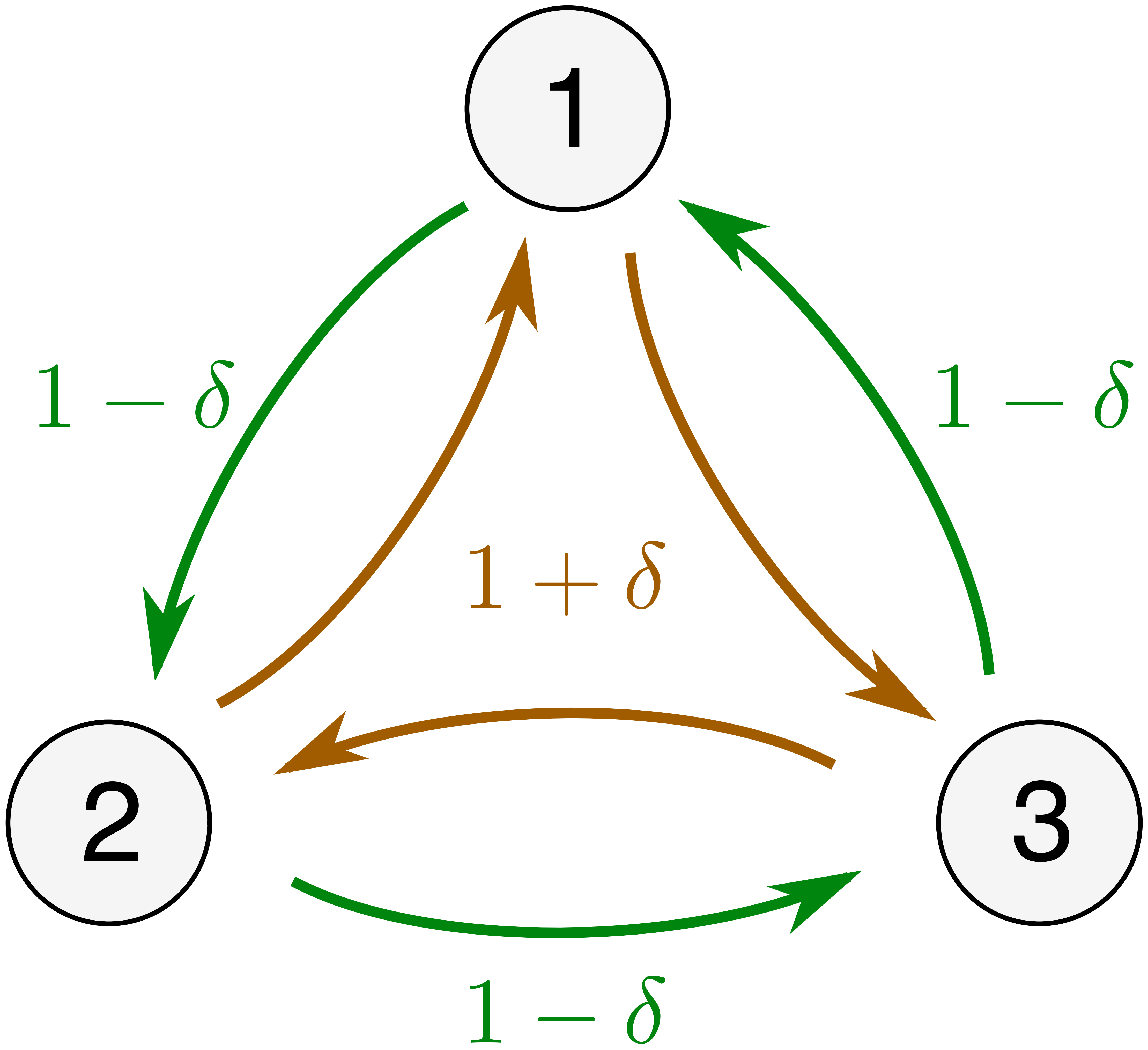
system asymmetry

"Symmetric states *requiring* system asymmetry"

We have a converse of symmetry breaking

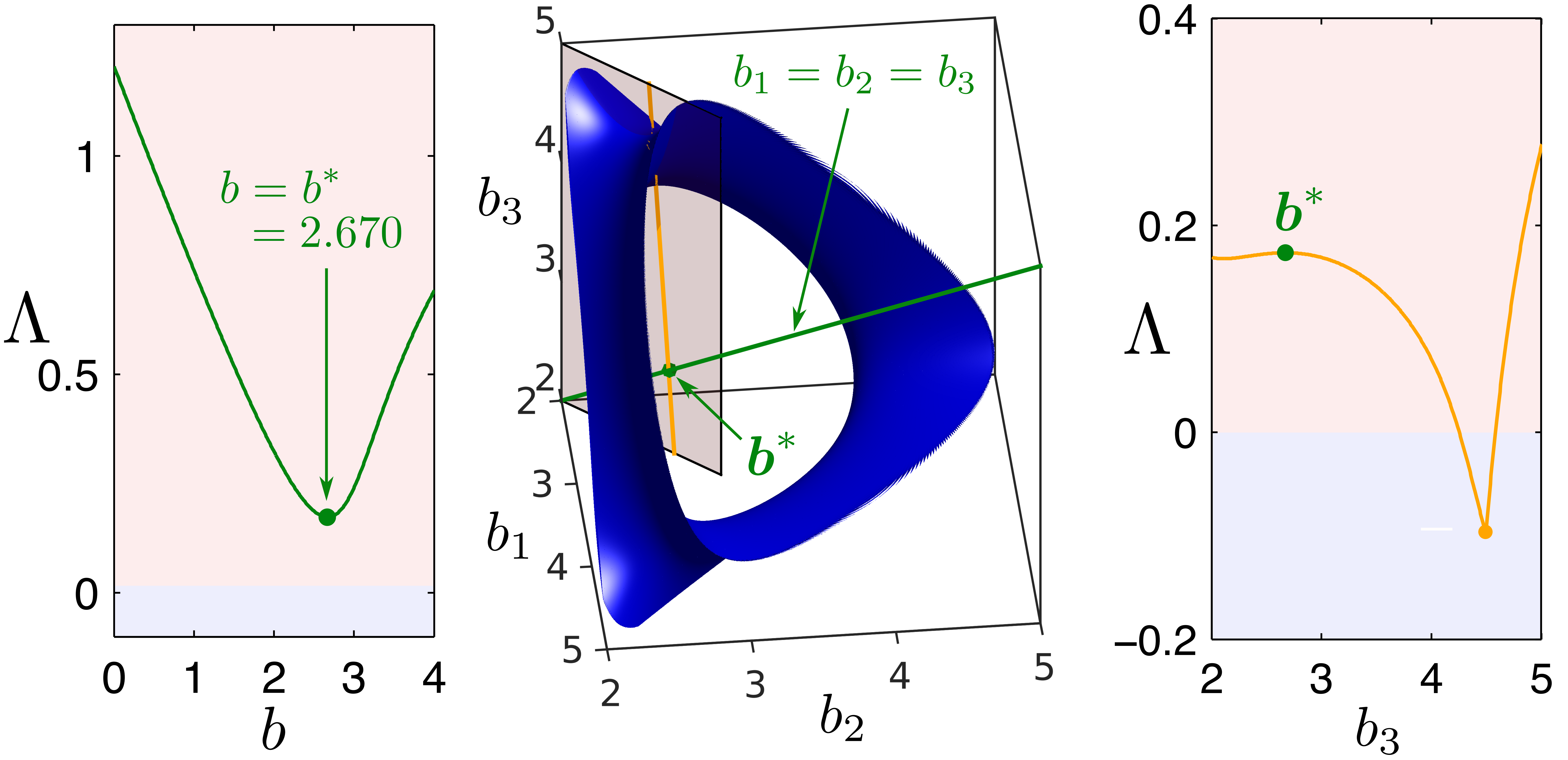
Symmetric **stable state** *requiring* **system** to be asymmetric

Symmetric **system** *requiring* **stable state** to be asymmetric
(symmetry breaking)


1
2
3
$1-\delta$
$1-\delta$
$1+\delta$
$1-\delta$

Stability landscape

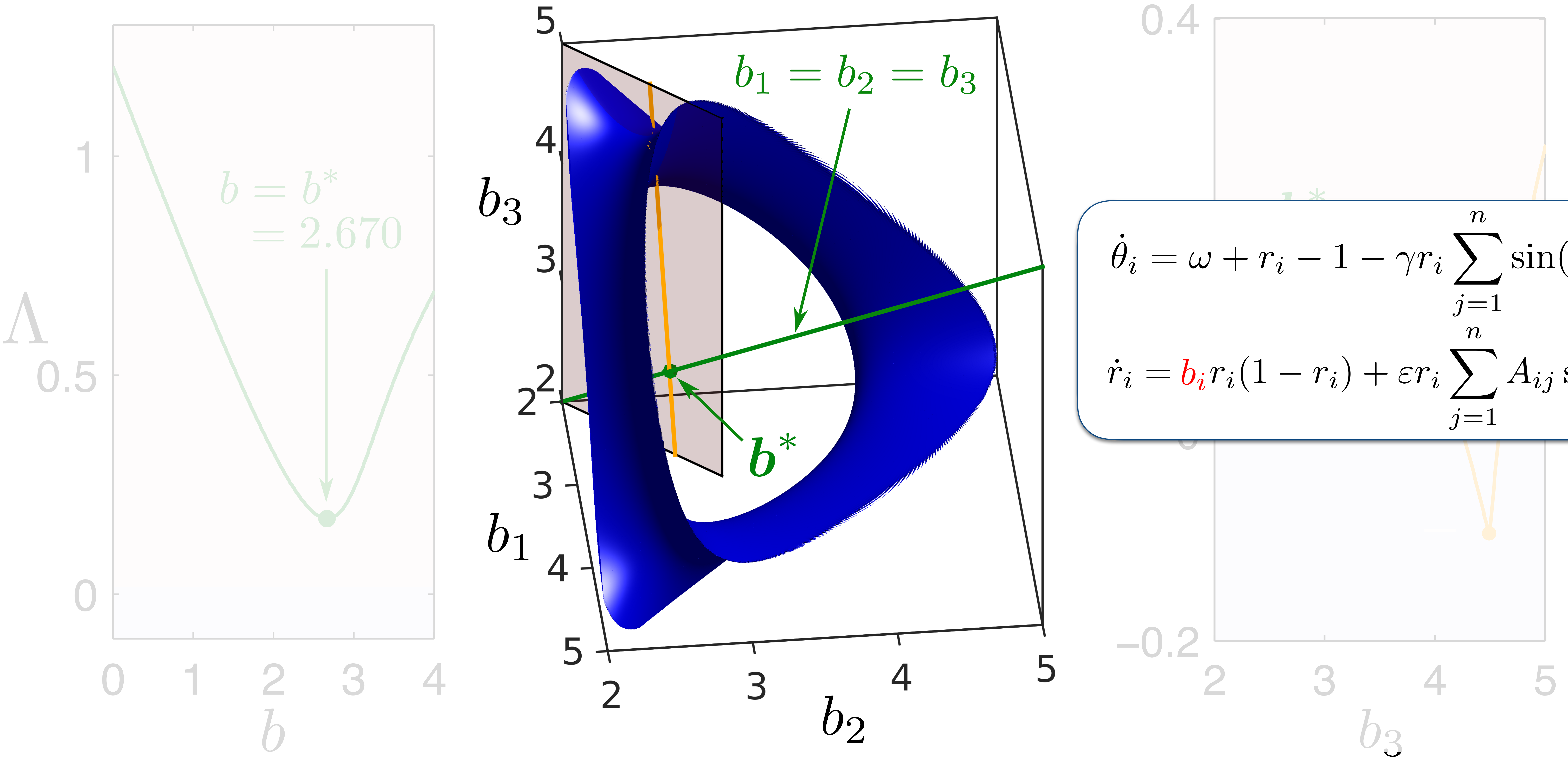
Λ = maximum transverse Lyapunov exponent



$$\gamma = 0.65,\ \varepsilon = 2,\ \delta = 0.3$$

Stability landscape

Λ = maximum transverse Lyapunov exponent



$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i)$$

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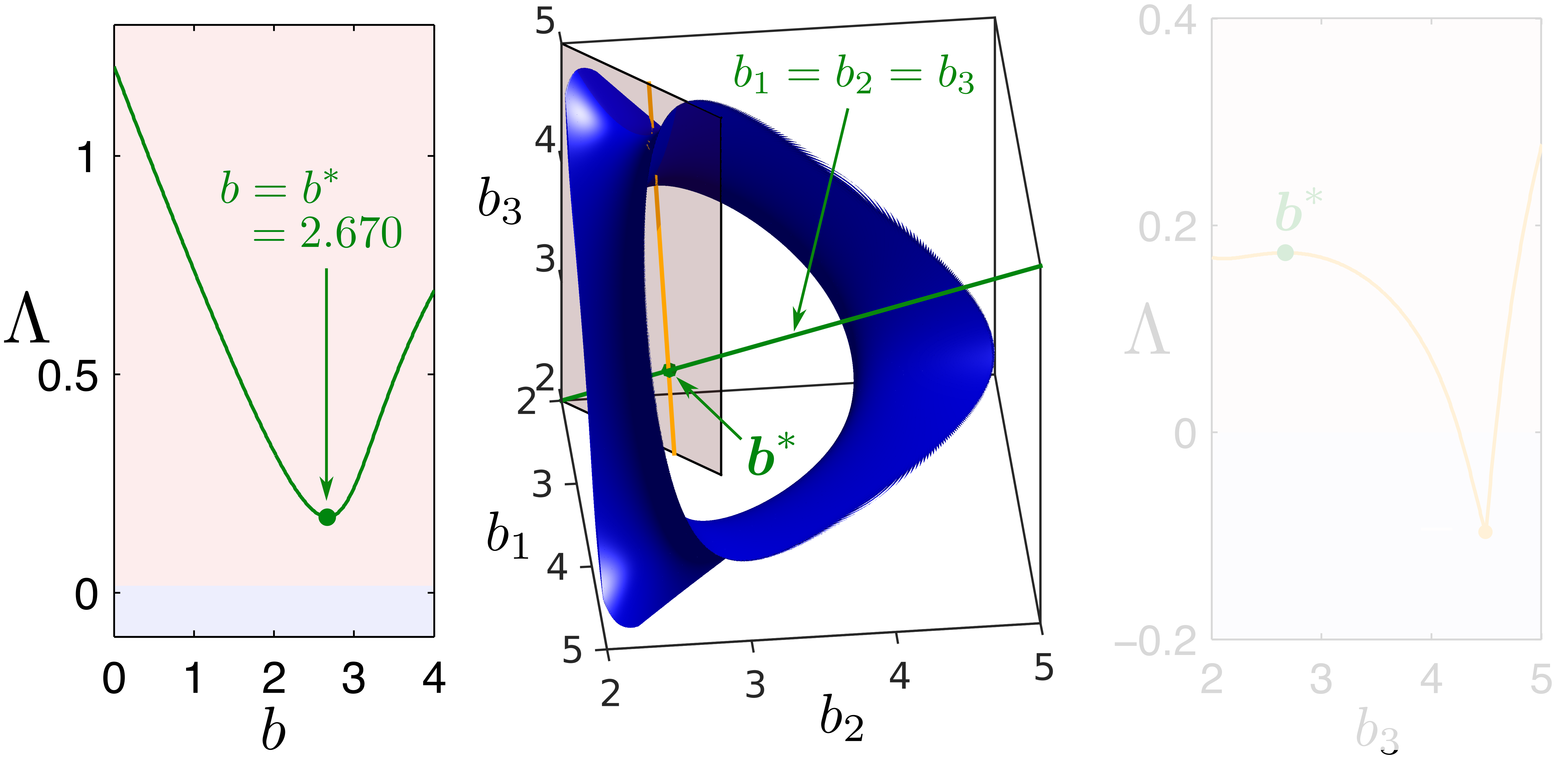
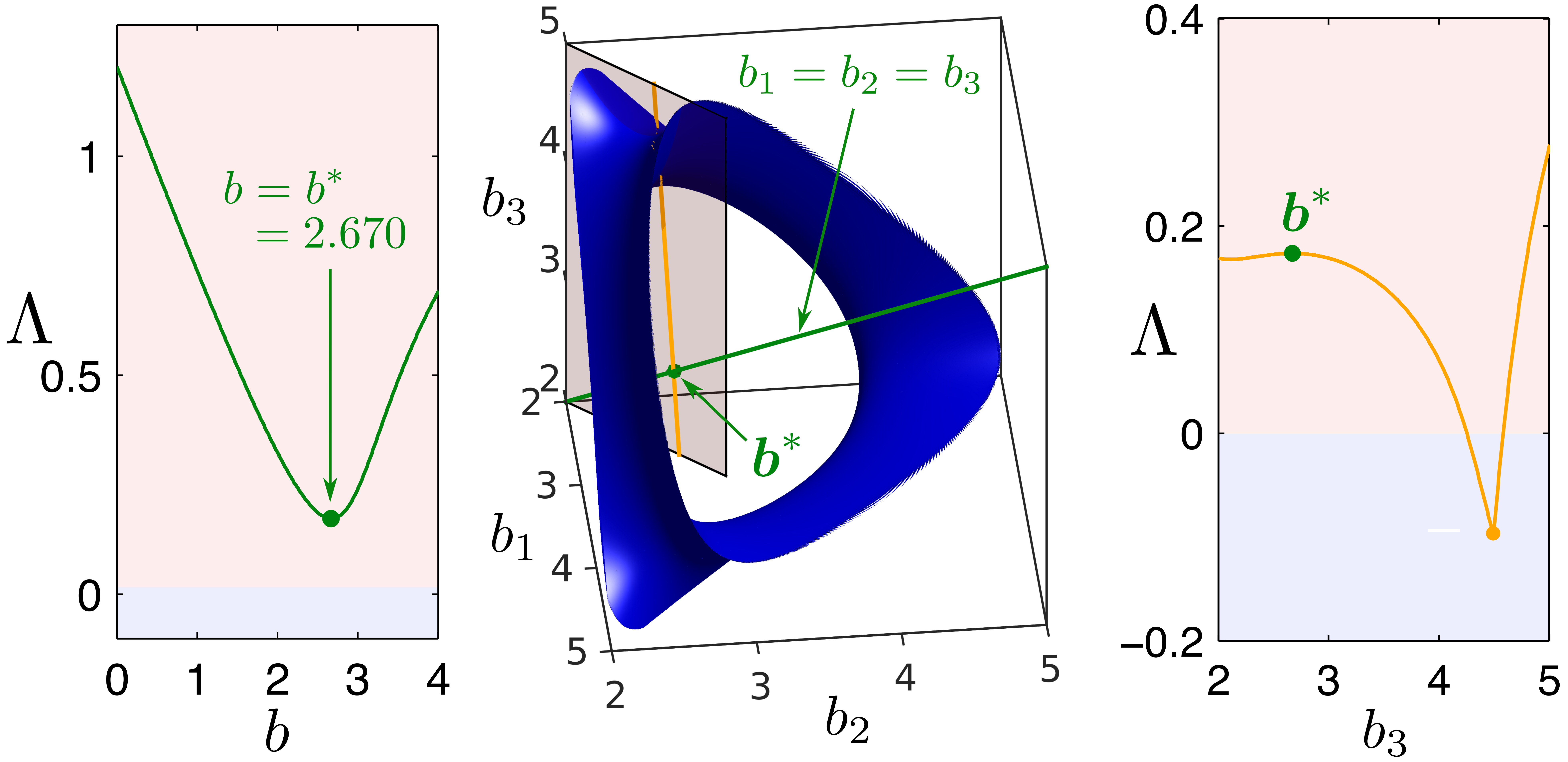


$\gamma = 0.65,\ \varepsilon = 2,\ \delta = 0.3$

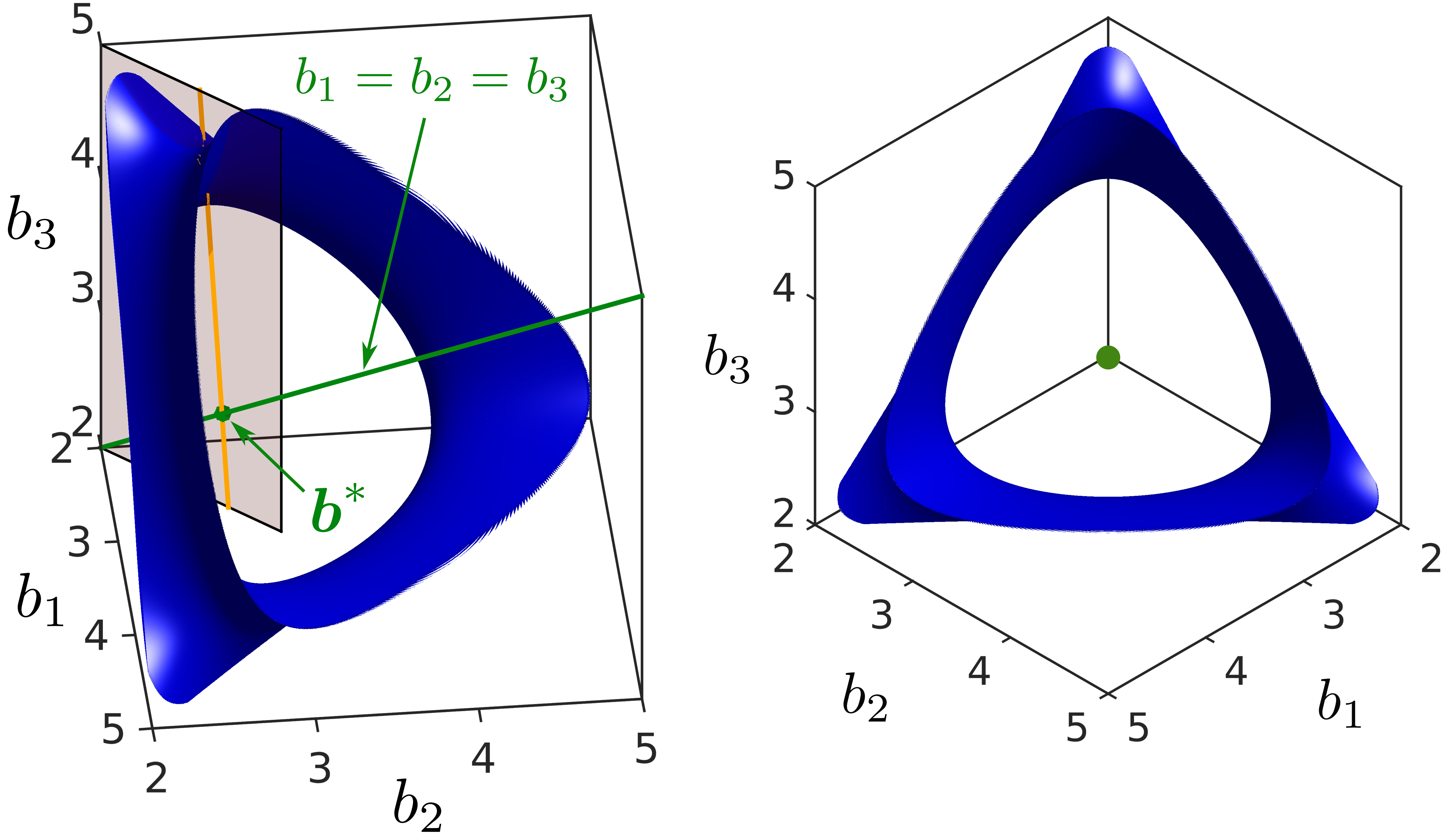
Stability landscape

Λ = maximum transverse Lyapunov exponent



$$\gamma = 0.65,\ \varepsilon = 2,\ \delta = 0.3$$

Symmetry of stability landscape

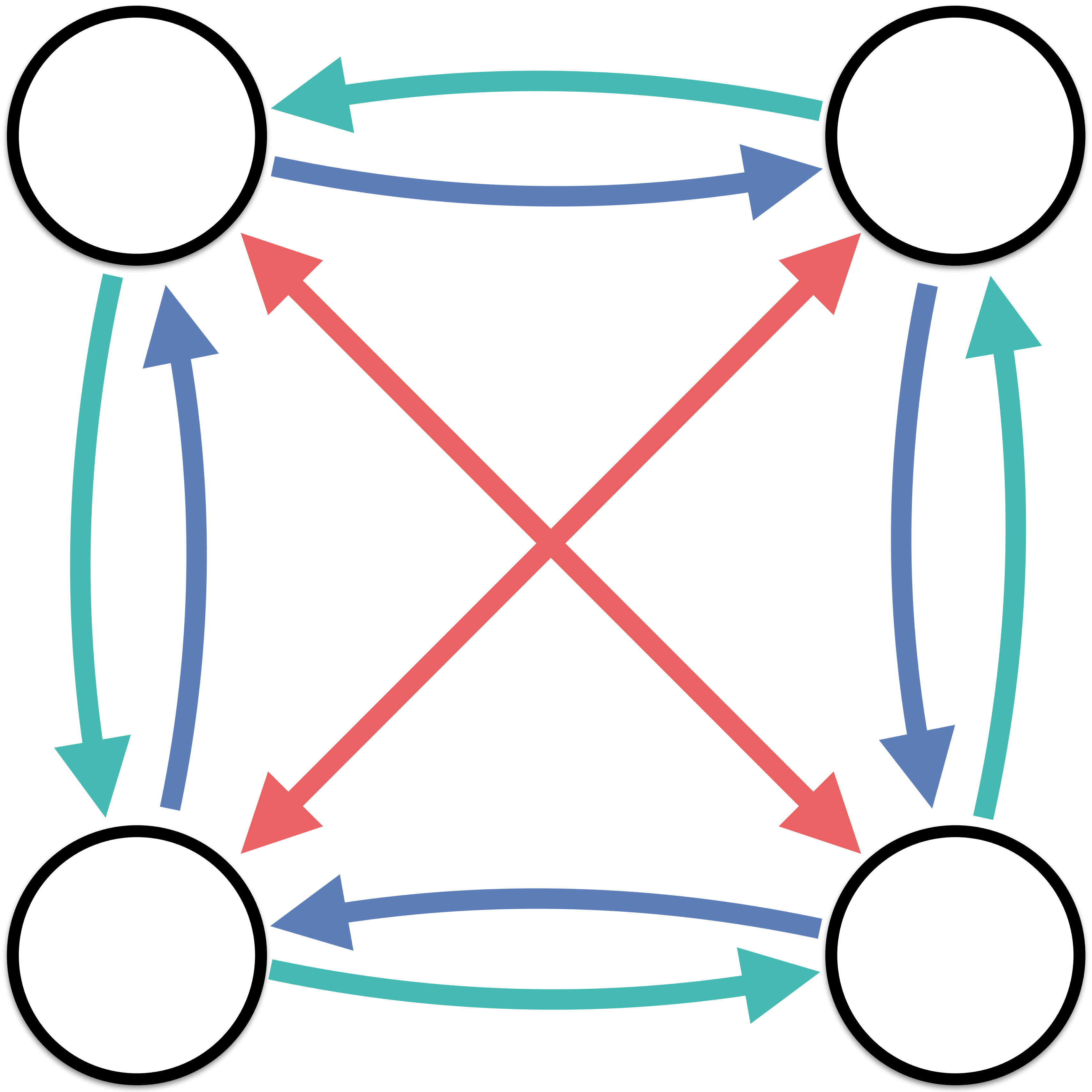


How often does this occur?

YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907

Networks with multiple link types

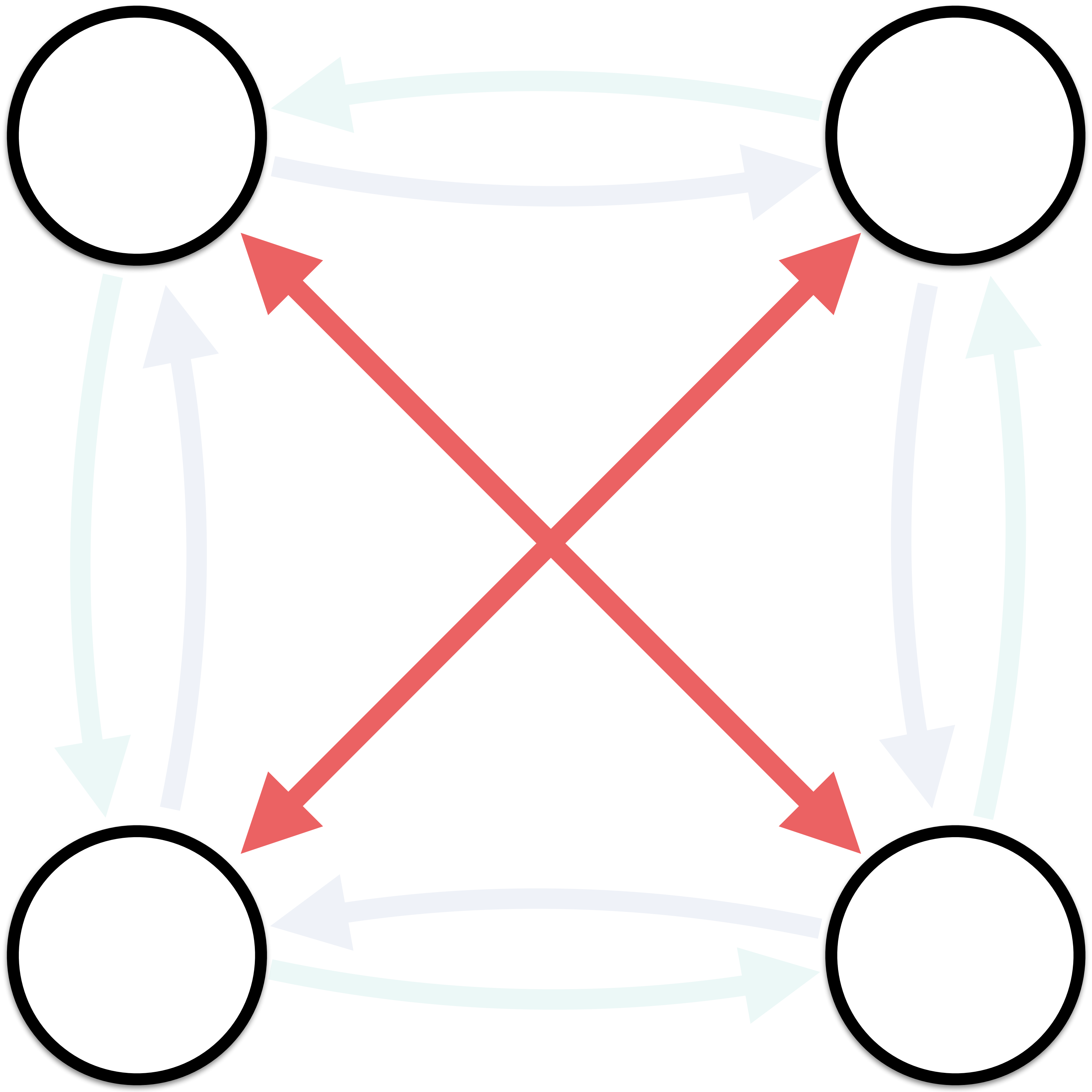
Adjacency matrices $A^{(\alpha)}, \quad \alpha = 1, \ldots, K$



$K = 3$

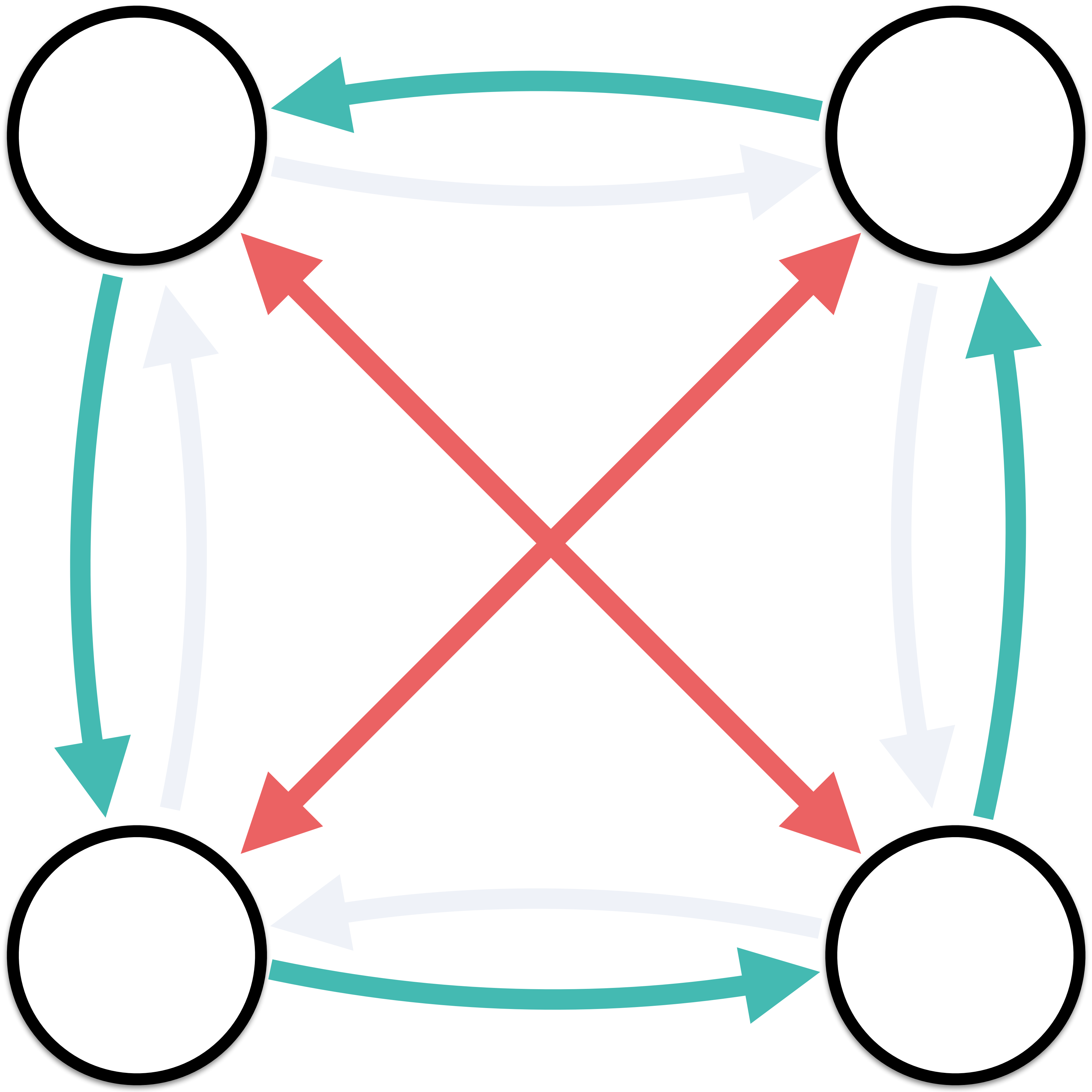
Networks with multiple link types

Adjacency matrices $A^{(1)}$



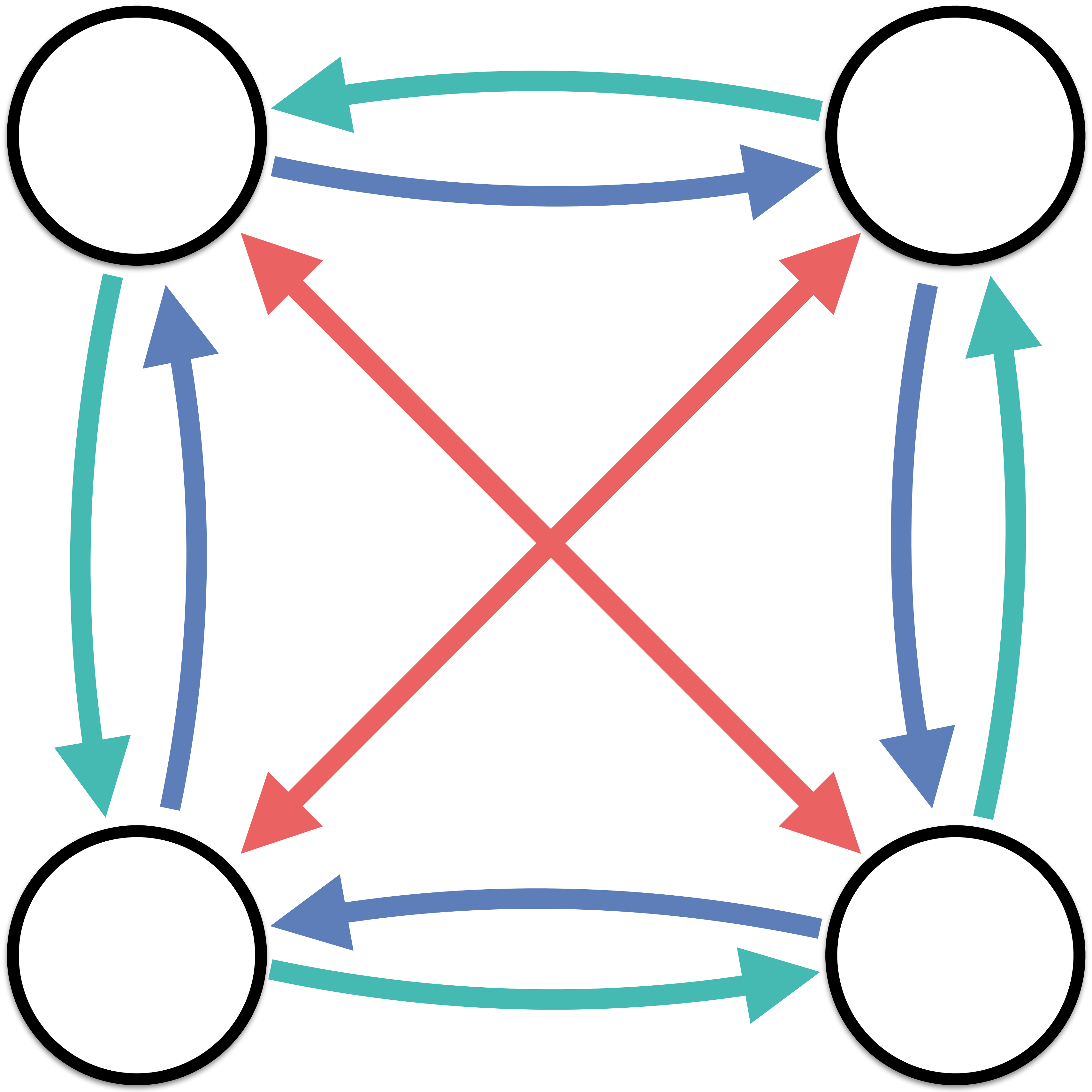
Networks with multiple link types

Adjacency matrices $A^{(1)}$, $A^{(2)}$



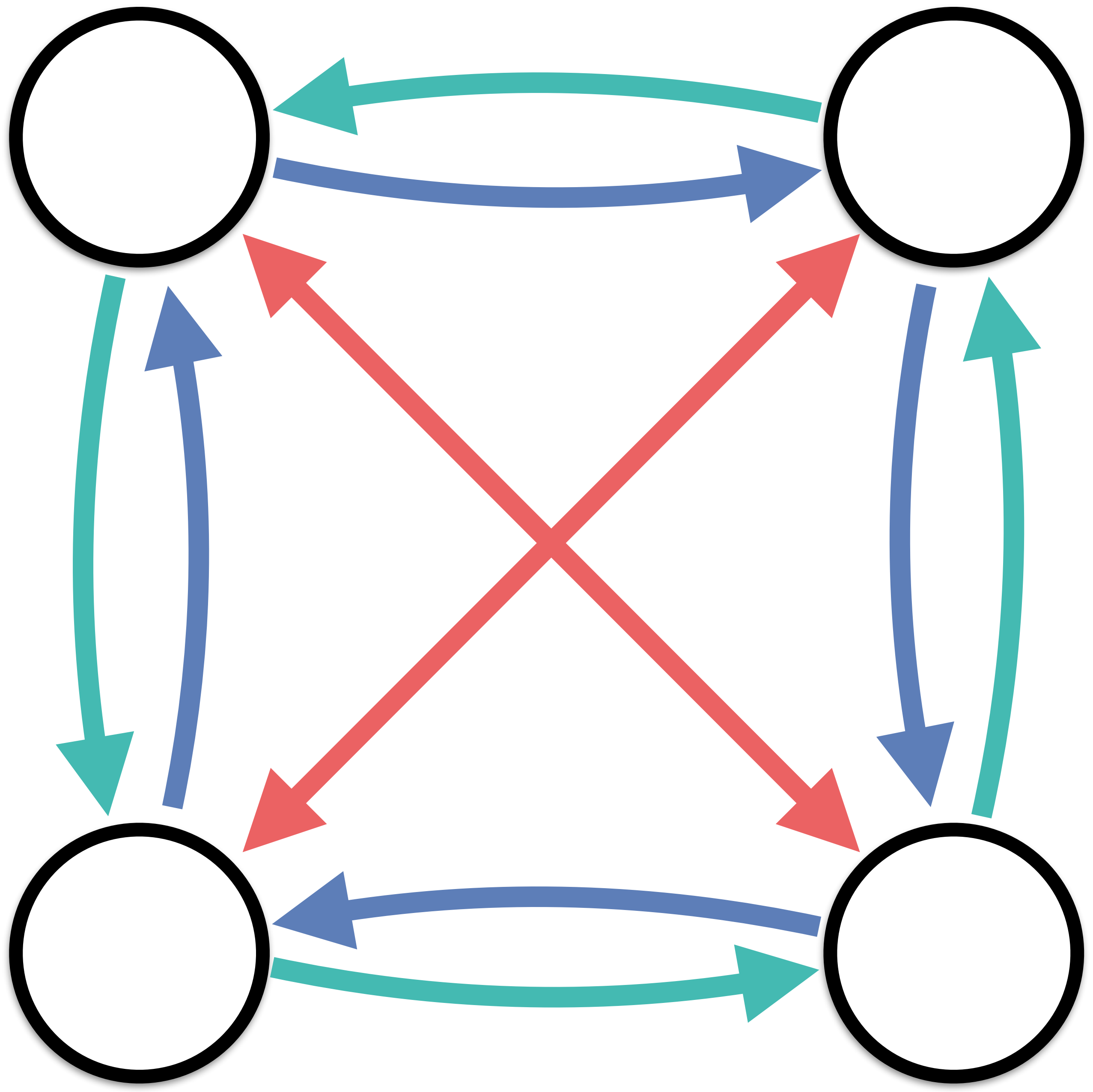
Networks with multiple link types

Adjacency matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$



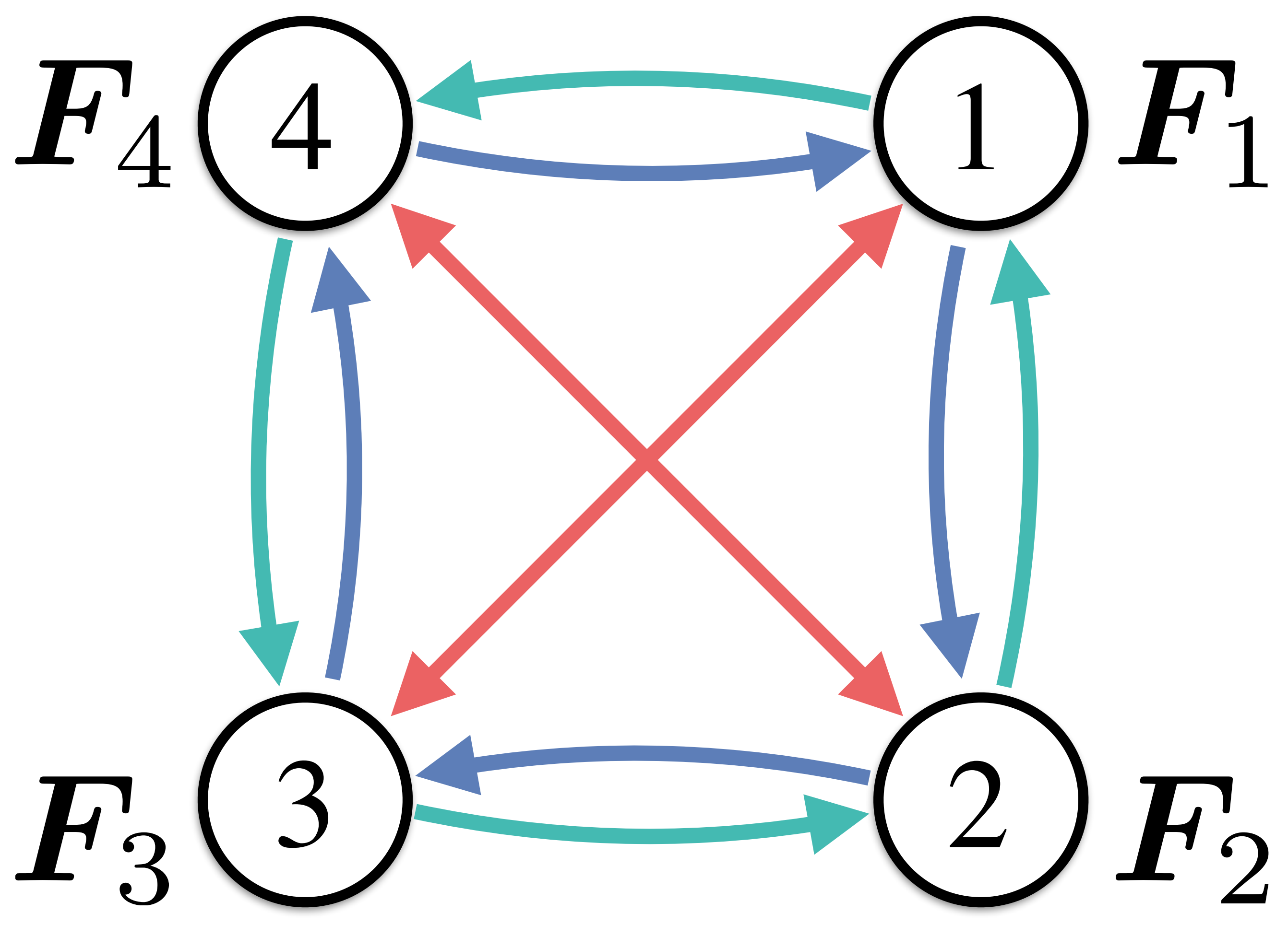
Symmetric network structures

- *Symmetric network*: every node can be mapped to any other node by some permutation of nodes without changing any $A^{(\alpha)}$.
- For undirected networks with a single link type, they are called *vertex-transitive graphs*.
- Includes *circulant graphs*, defined as a network whose nodes can be arranged in a ring so that the network is invariant under rotations.



Example of symmetric network (circulant graph)

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i)$$: dynamics of isolated node i

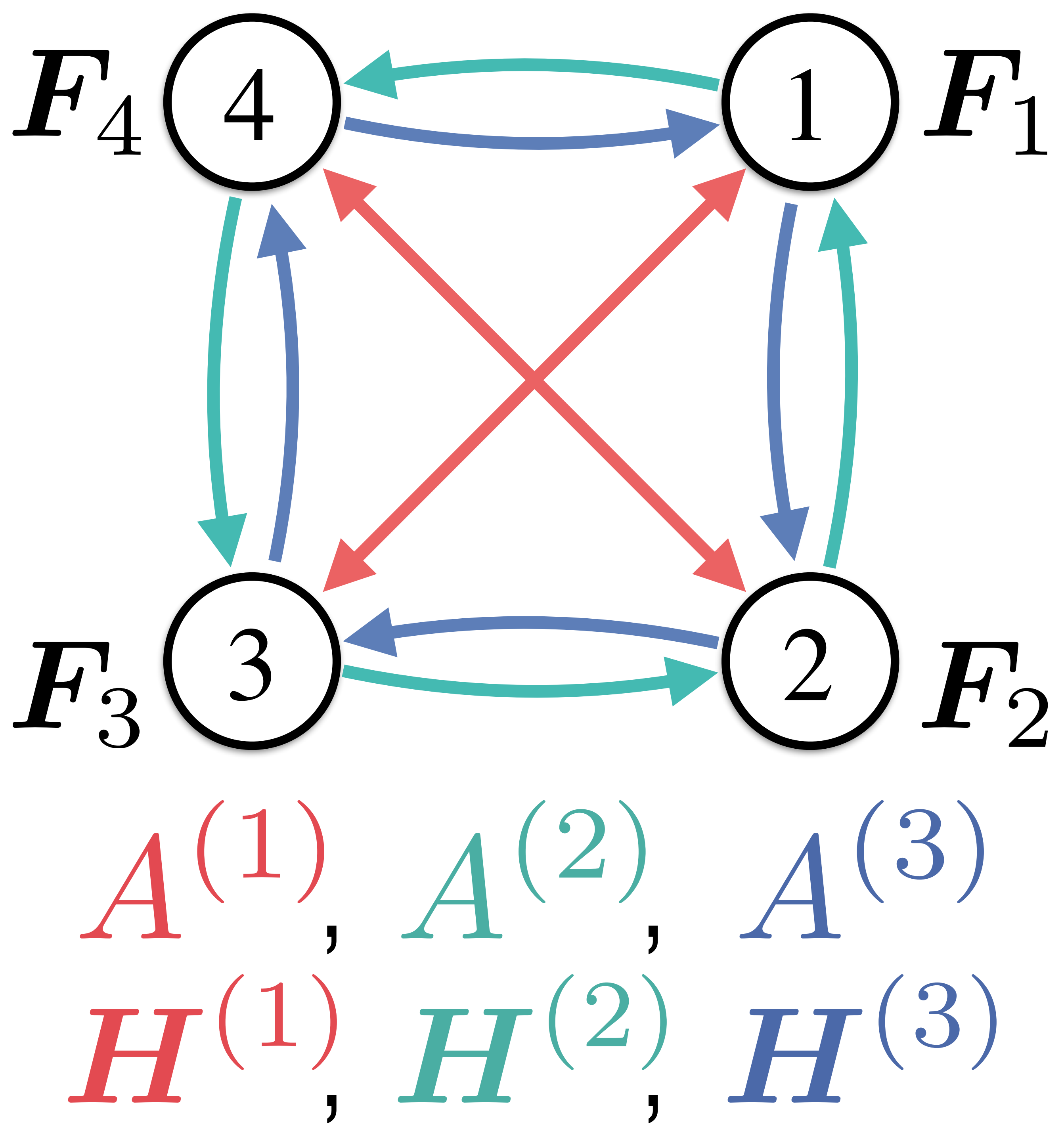


Network of non-identical oscillators

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$

$A_{ii'}^{(\alpha)}$: directed link of type α from node i' to node i

$\boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$: coupling function



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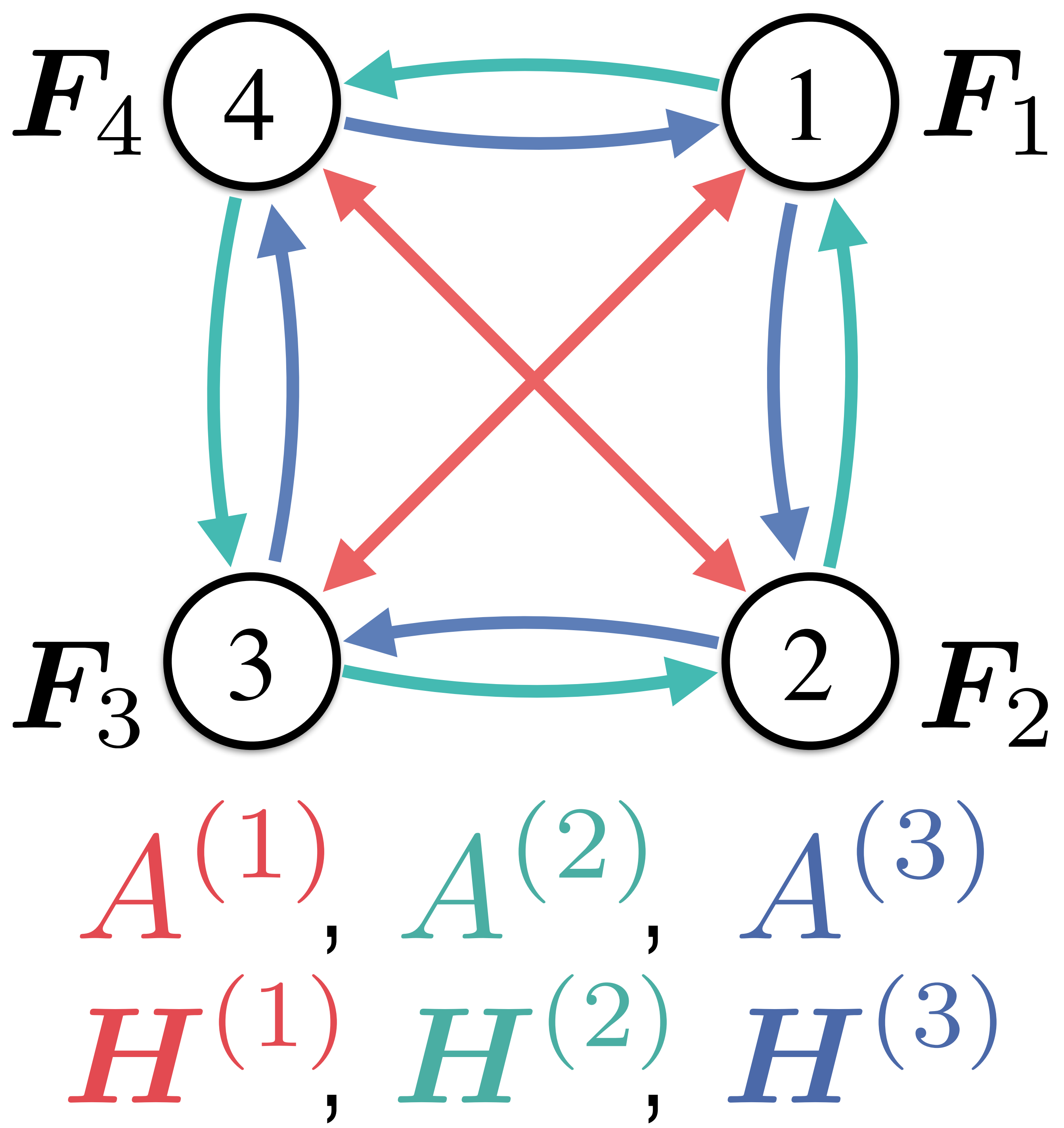
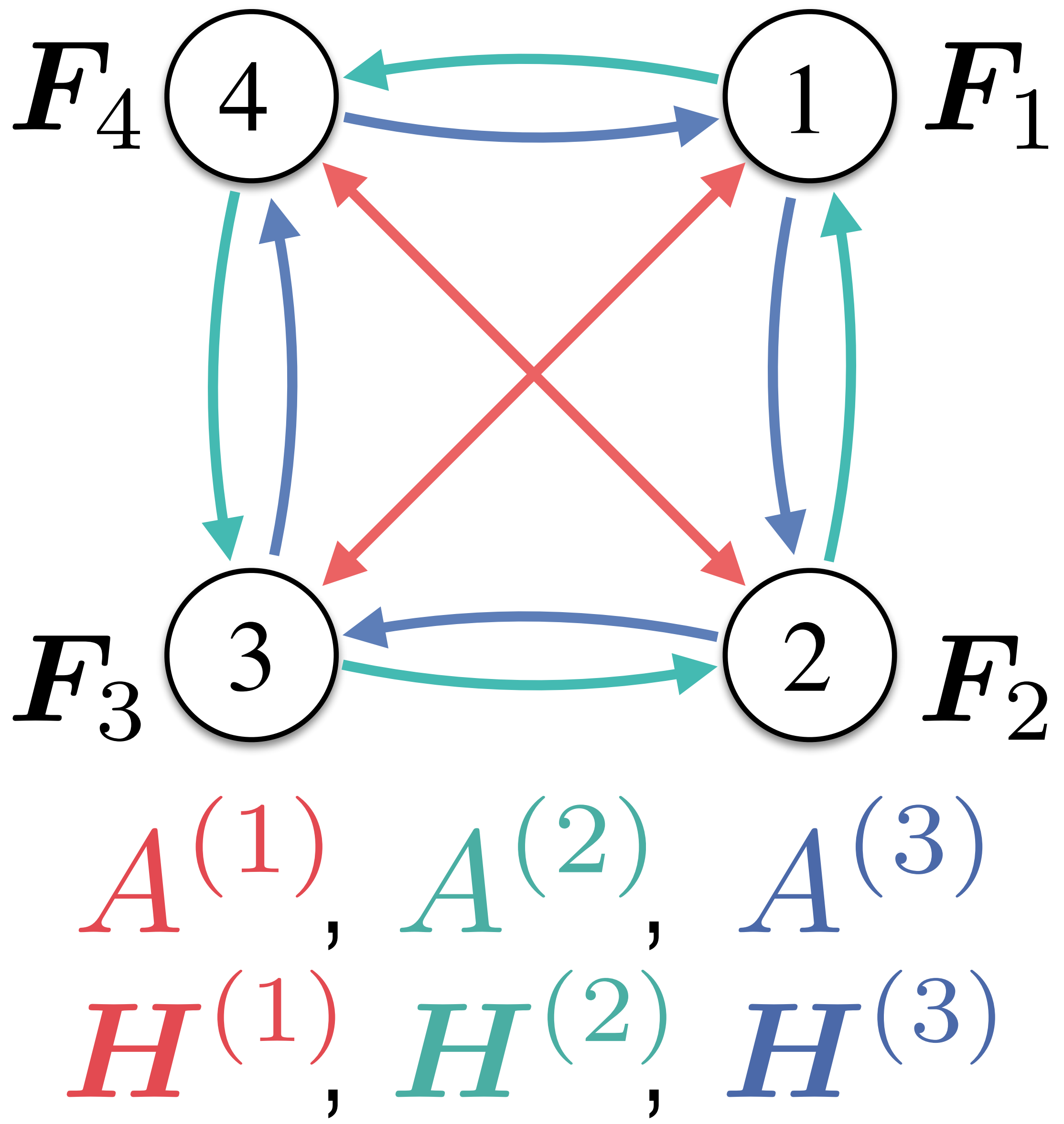


Defining asymmetry-induced synchronization

For a symmetric network

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$



with completely synchronous state,

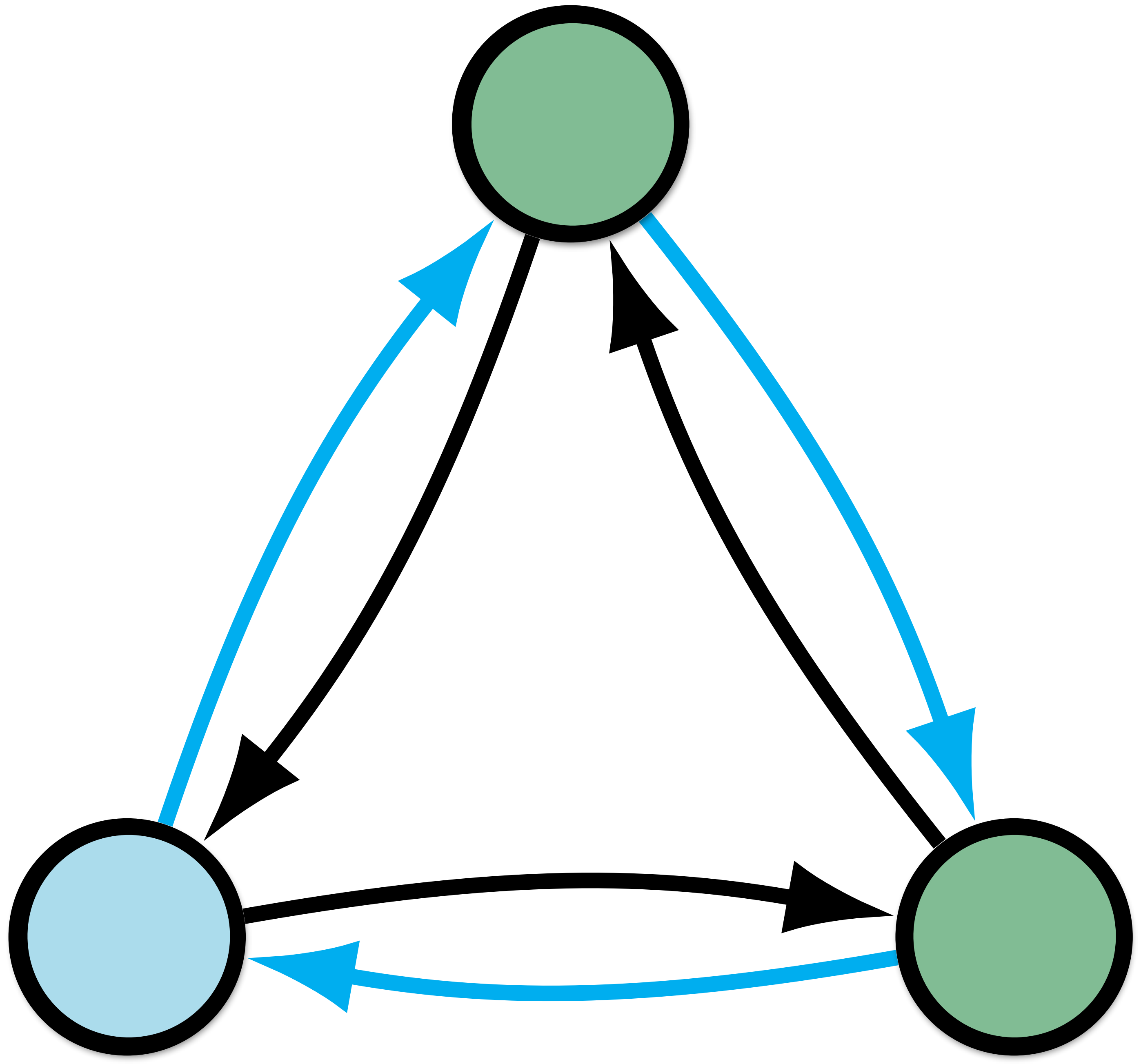
Conditions

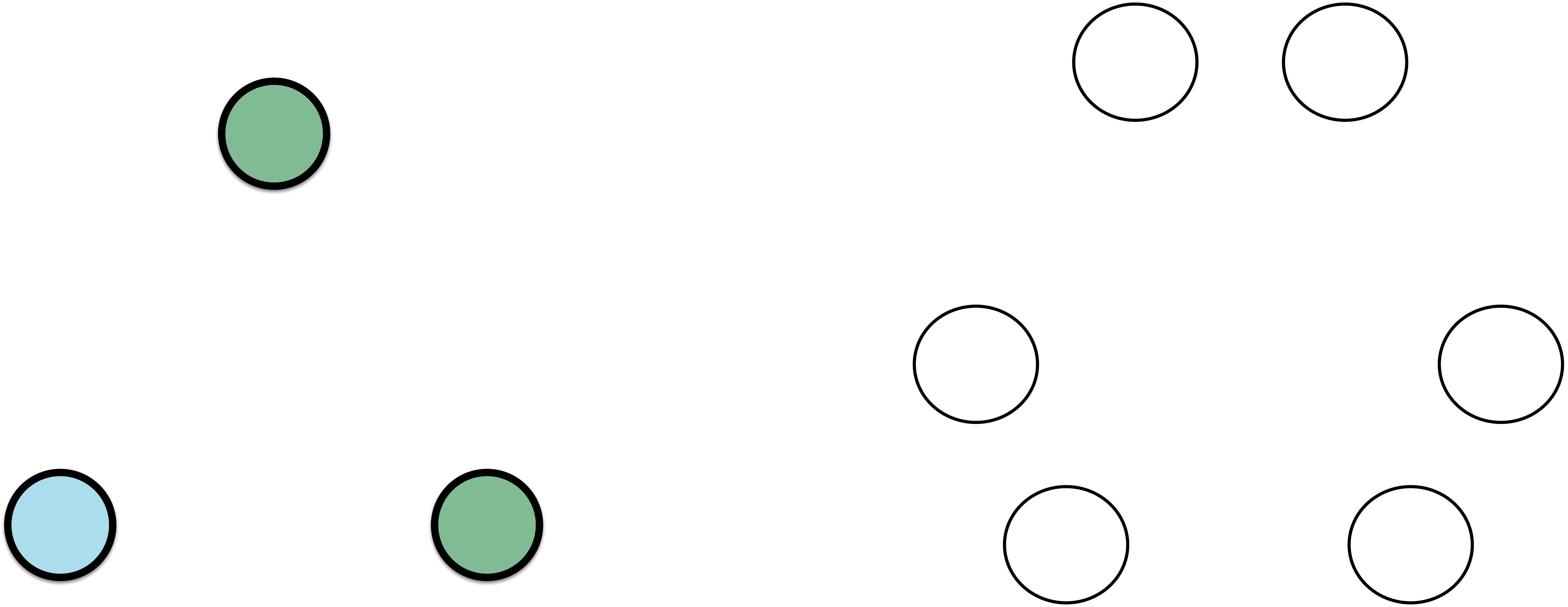
1. Synchronous state is unstable for any homogeneous system.
   $F_1 = \cdots = F_N$

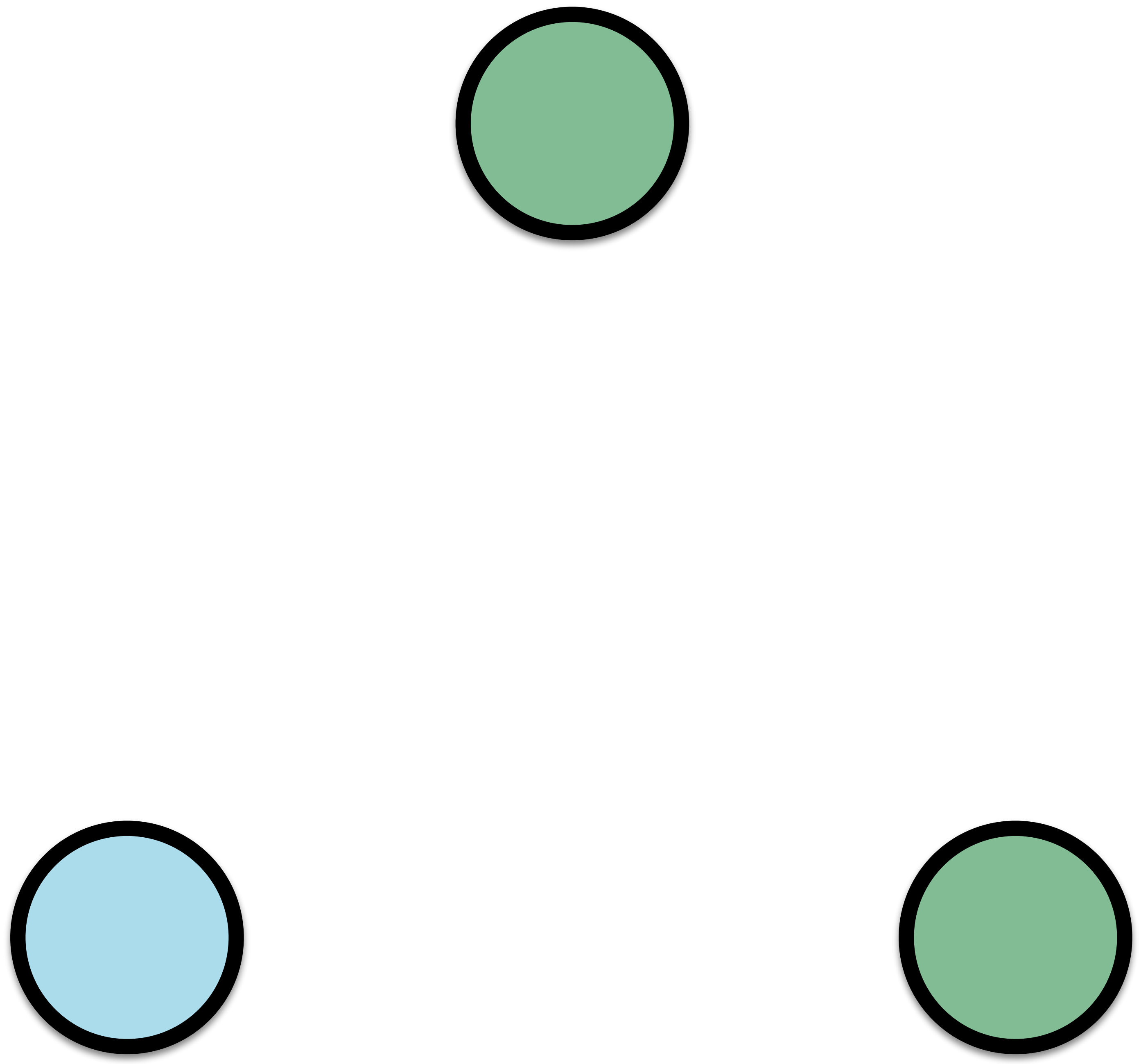
2. Synchronous state is stable for some heterogeneous system.
   $F_i \neq F_{i'}$ for some $i \neq i'$

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$

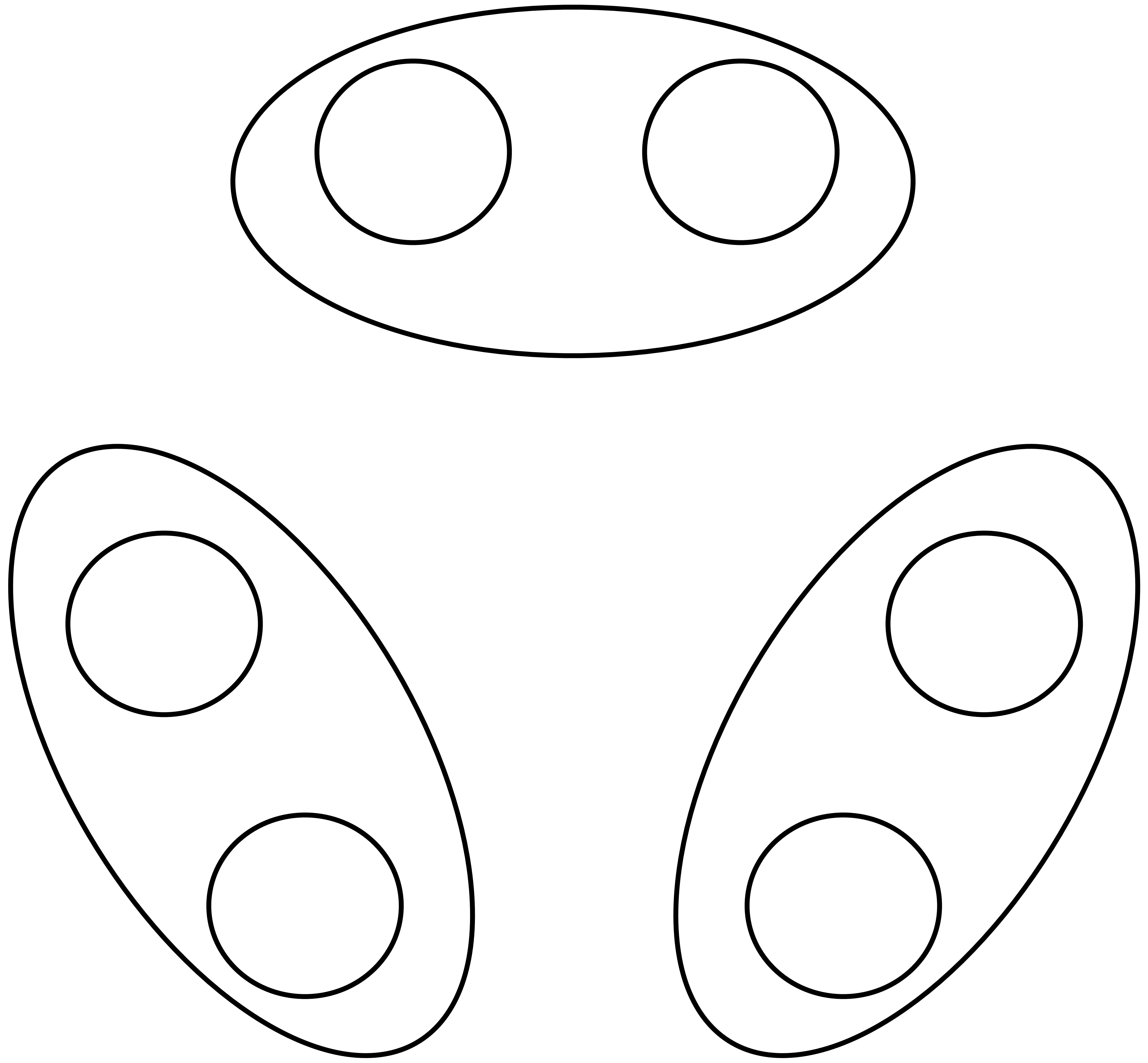
Class of multilayer systems

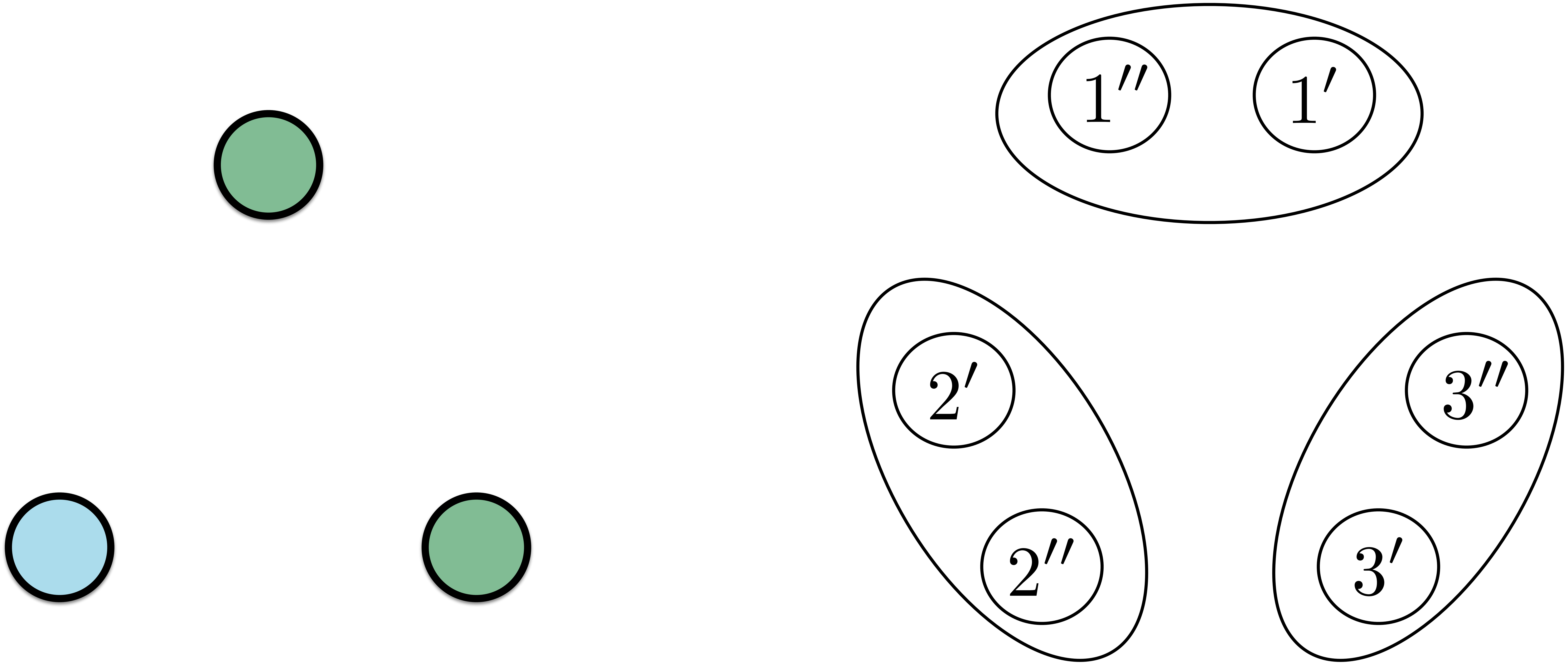




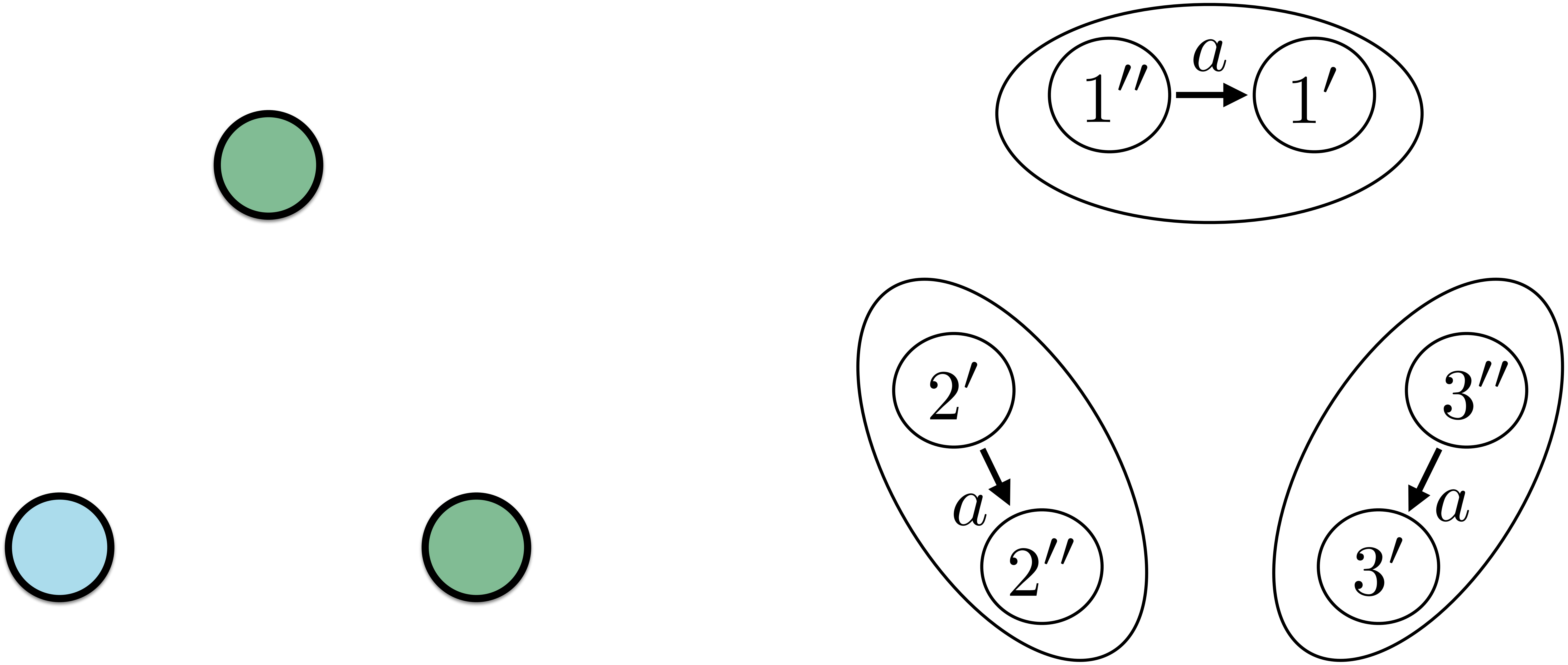


2 subnodes for each node

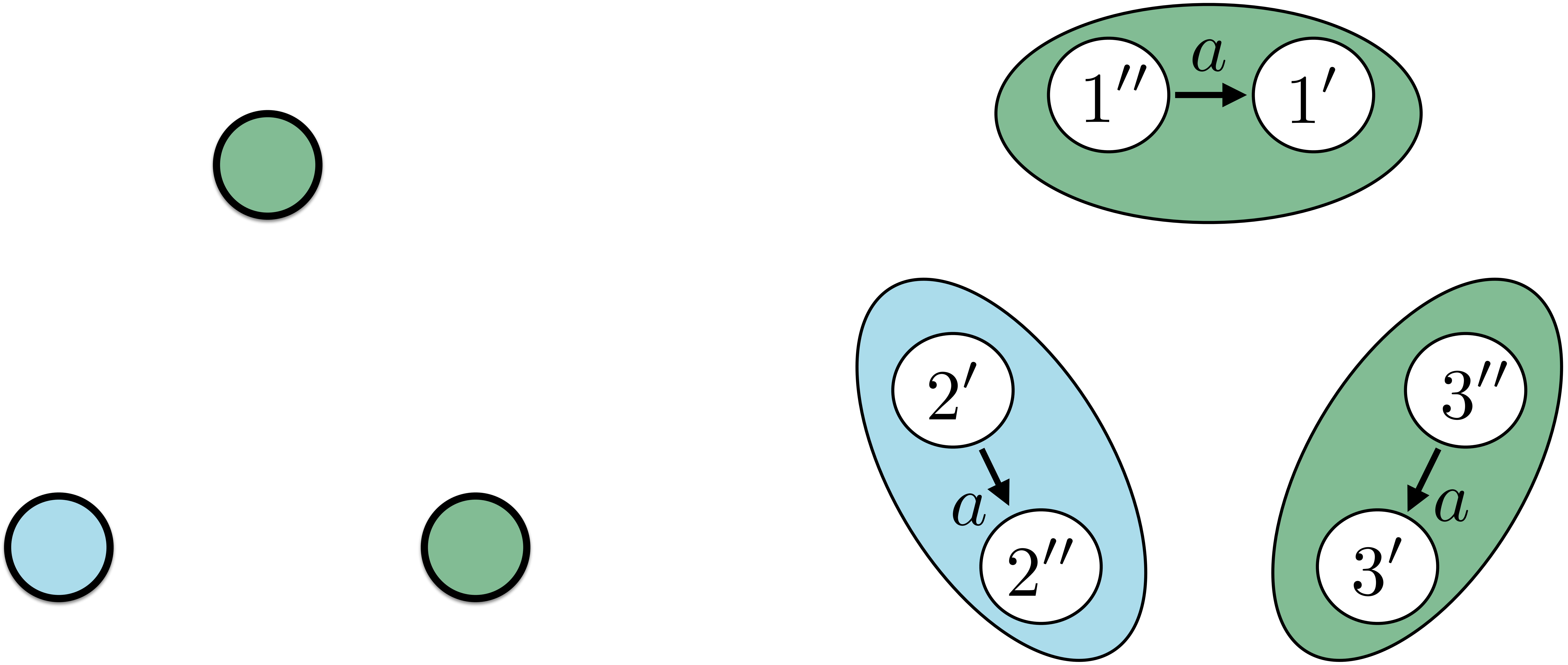



$1''$
$1'$
$2'$
$2''$
$3''$
$3'$

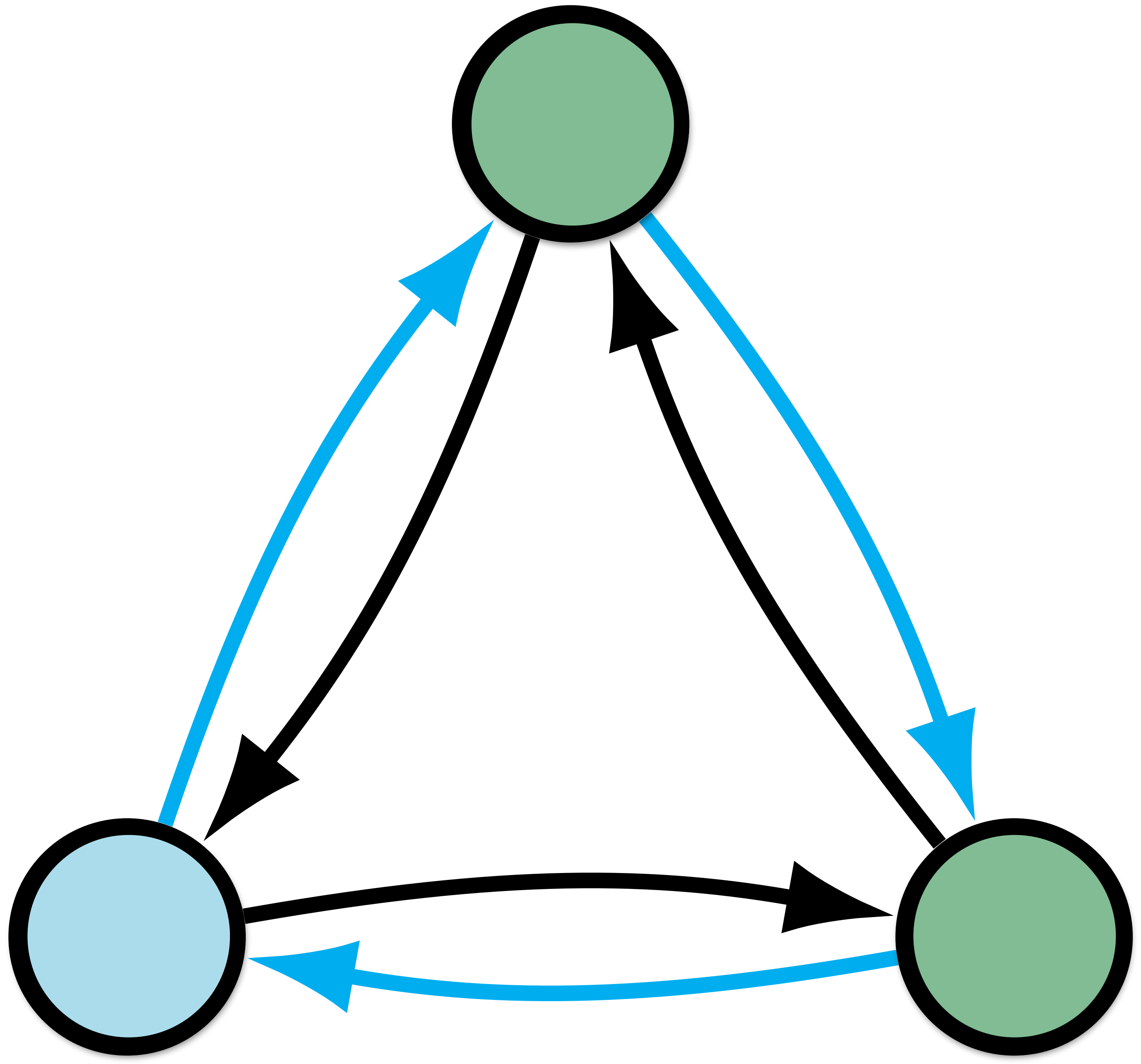
Internal links with strength a

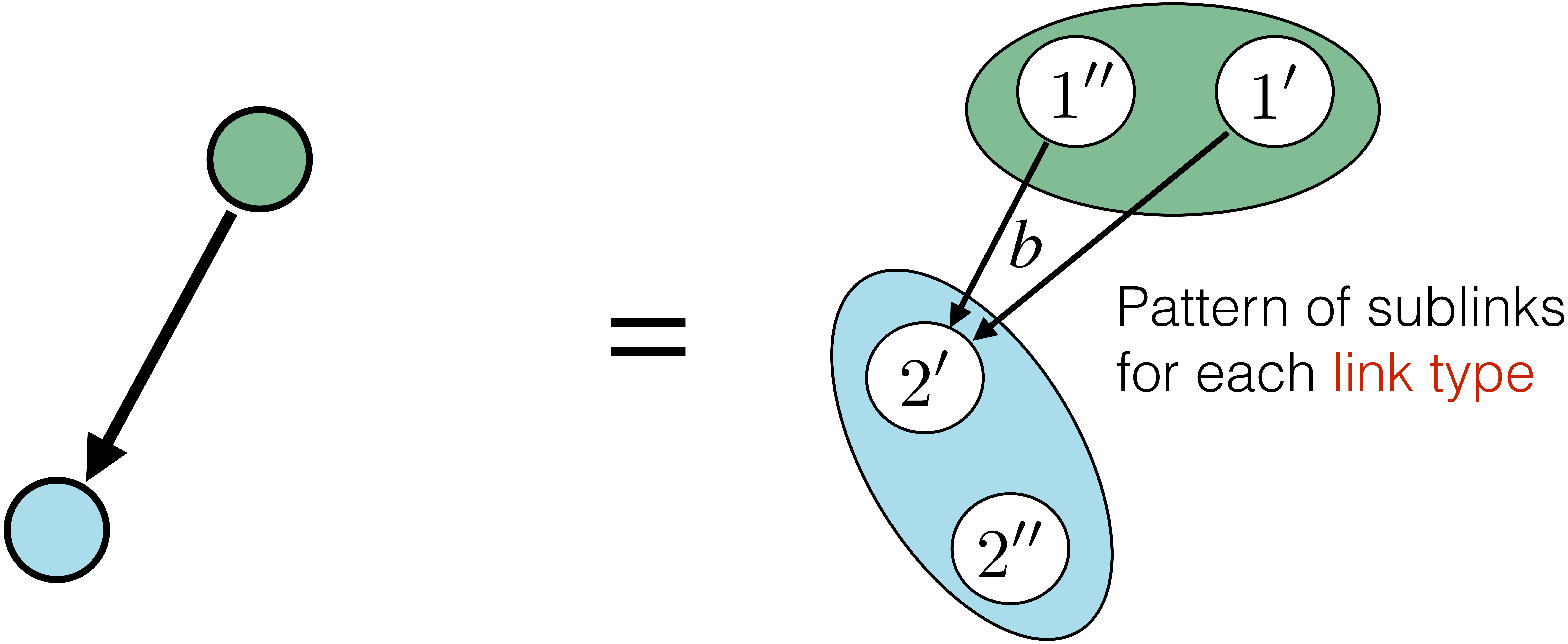


Internal link pattern defines node type

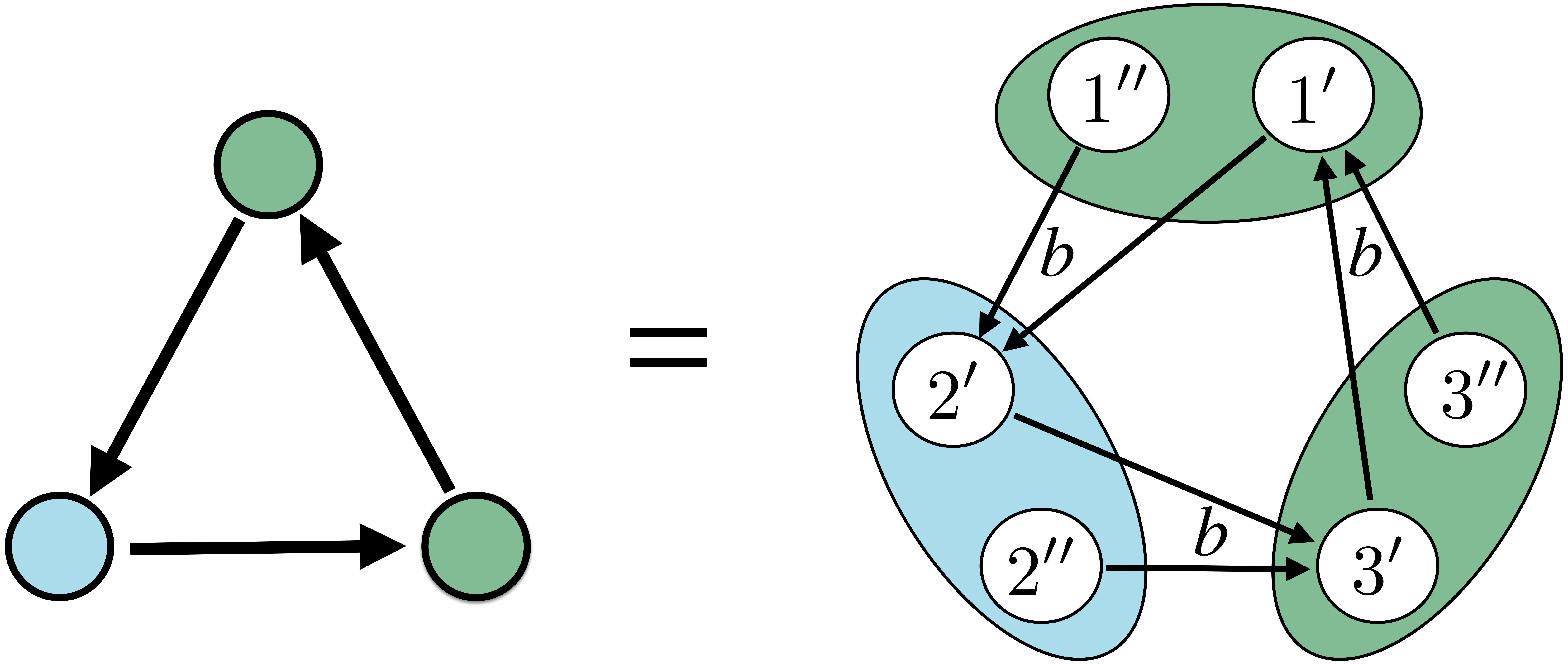


External links?

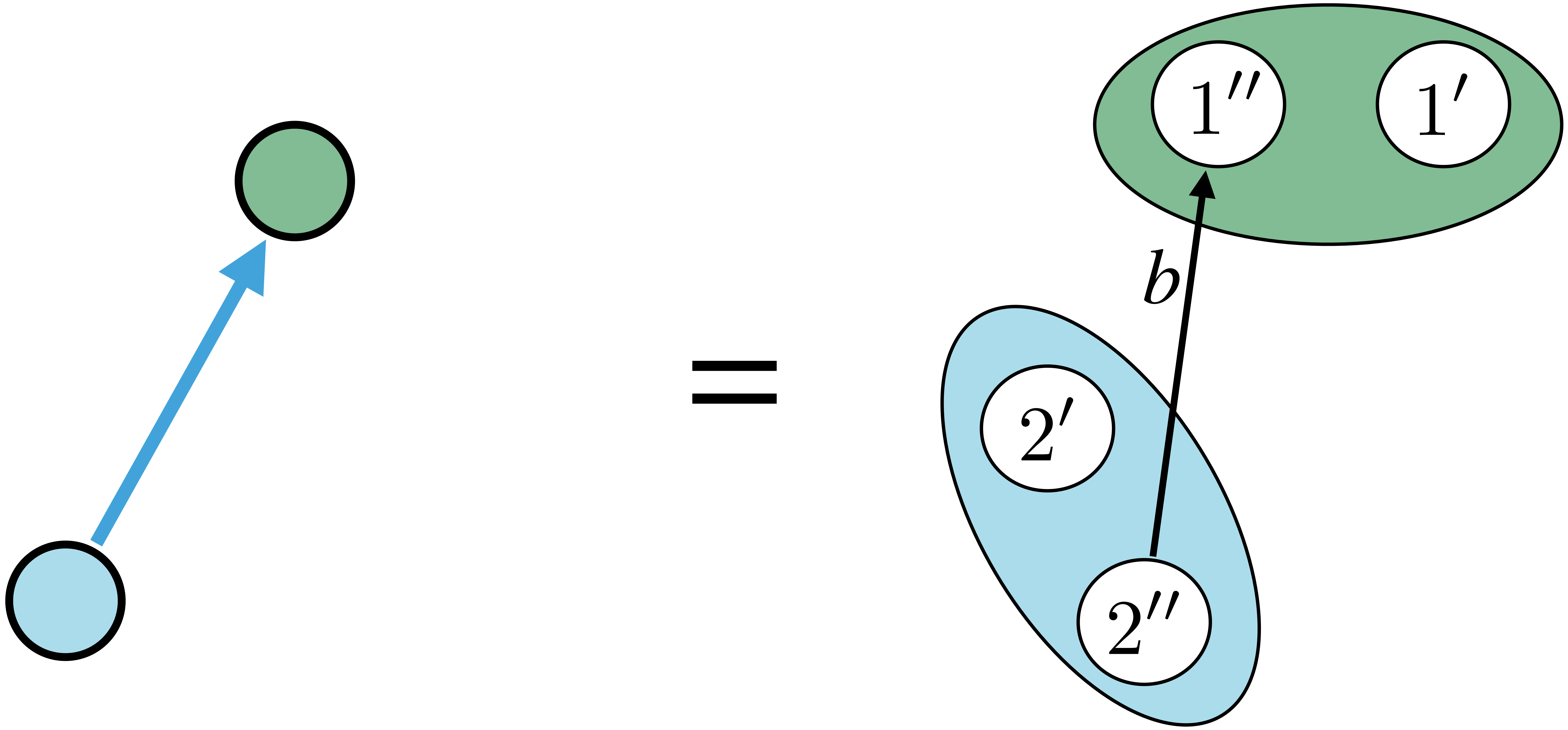




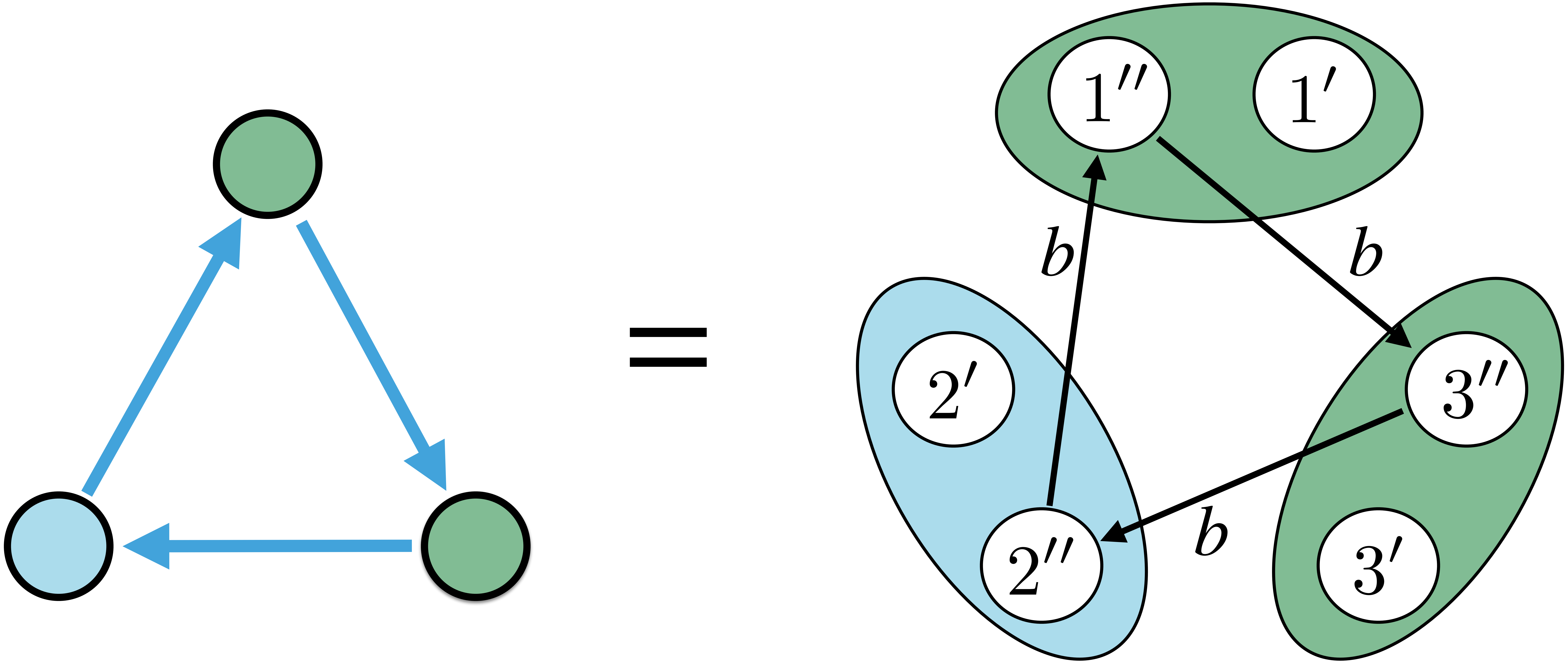
$1''$
$1'$
b
$2'$
$2''$
Pattern of sublinks
for each link type



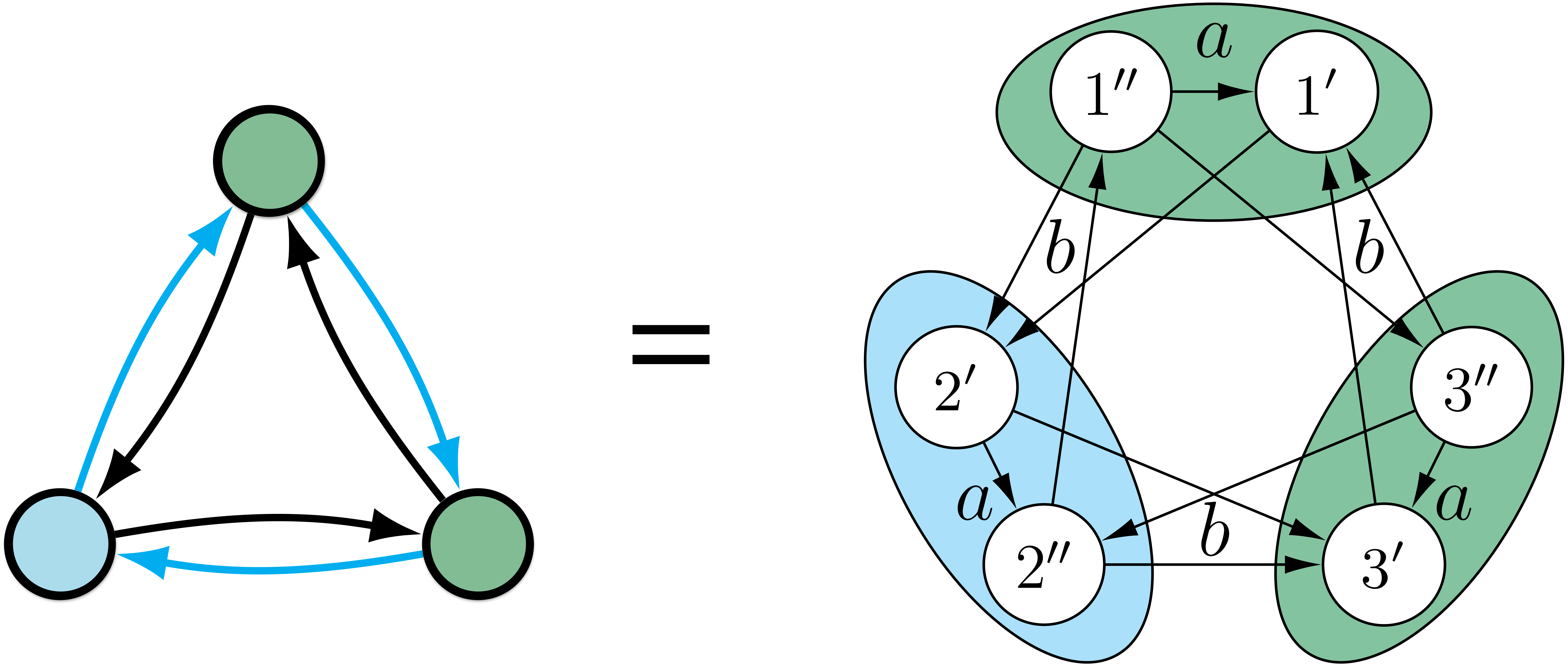
=
1''
1'
2'
2''
3'
3''
b
b
b



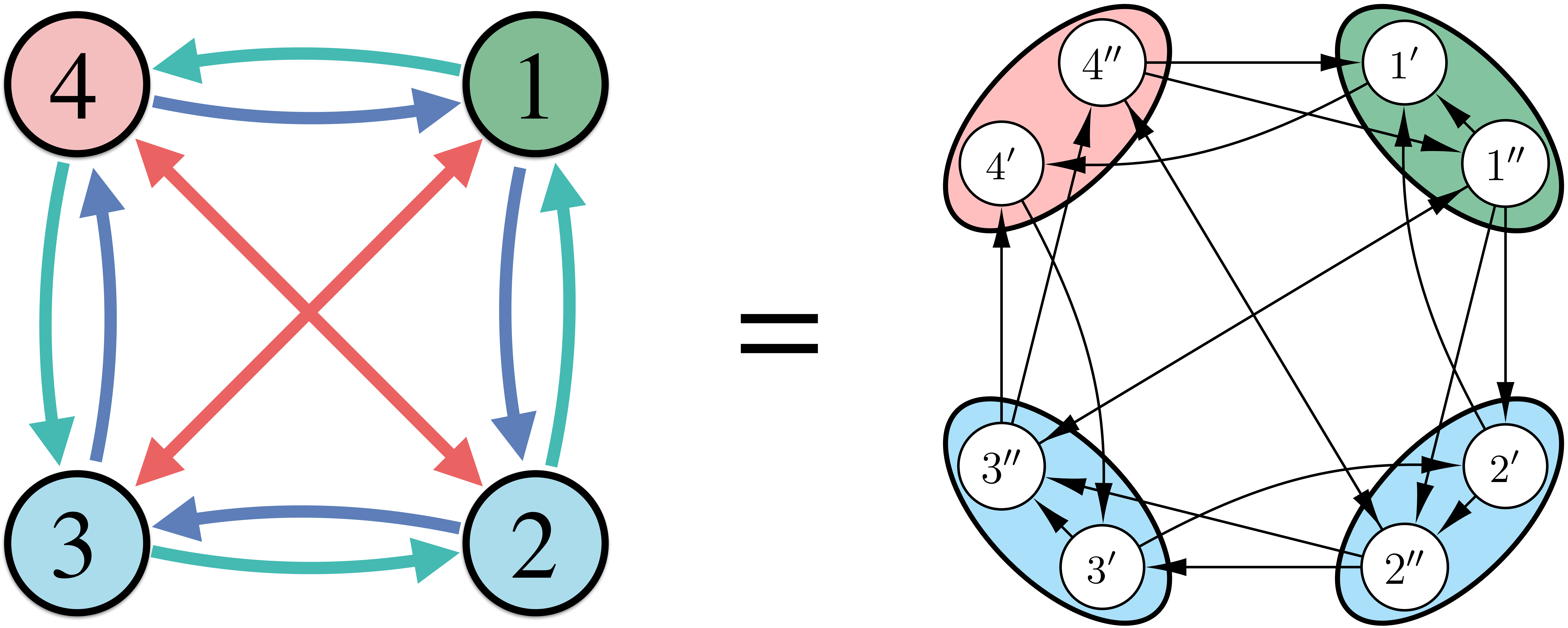
=
1′′
1′
b
2′
2′′

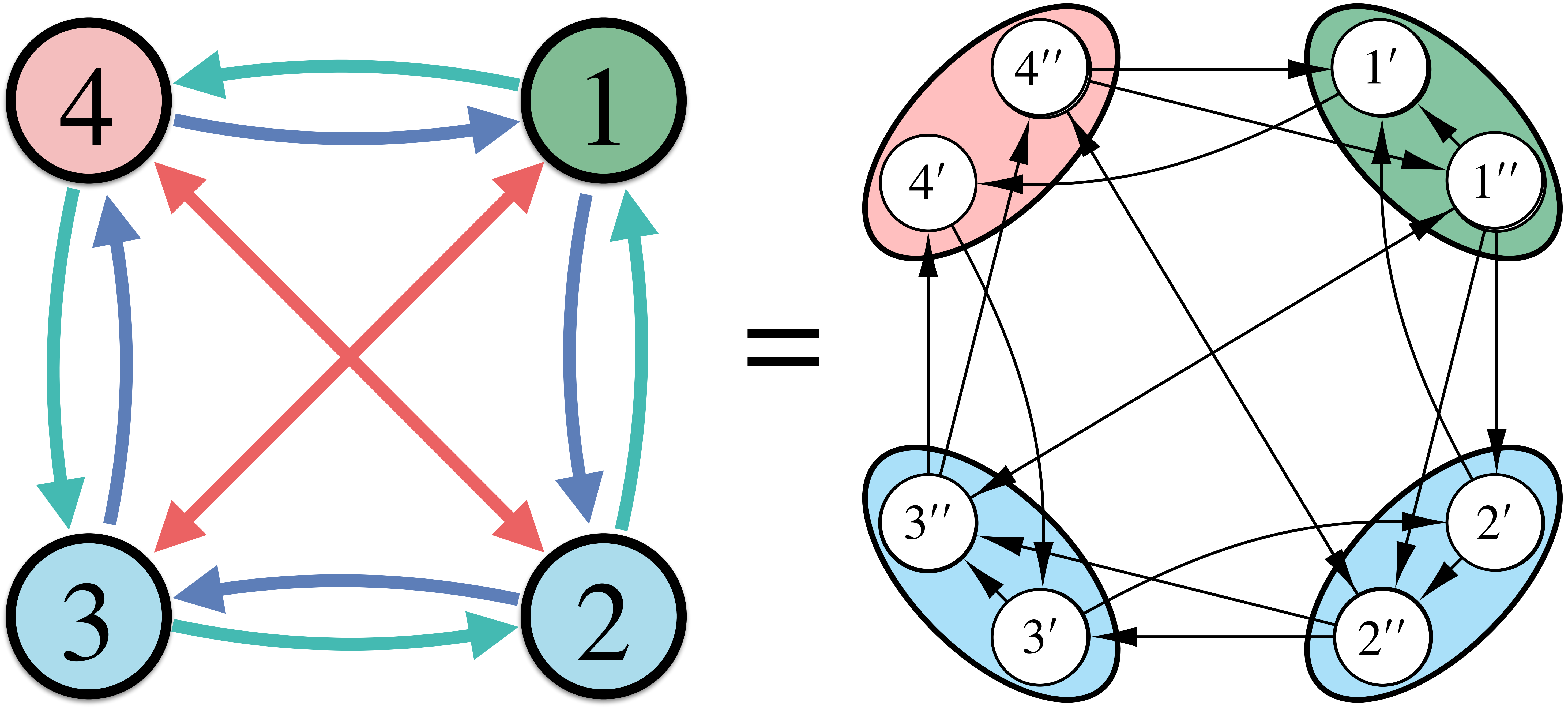


=
$1''$
$1'$
b
b
$2'$
$3''$
$2''$
b
$3'$

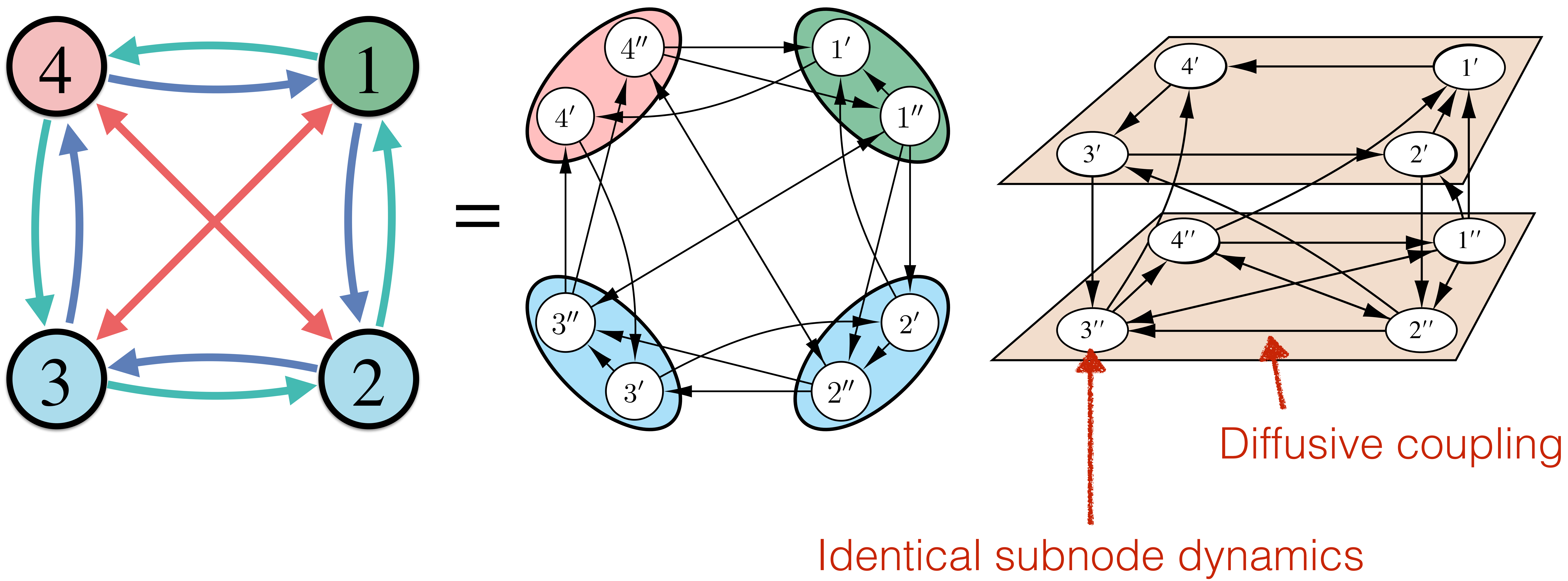


=
a
1″
1′
b
b
2′
3″
a
a
b
2″
3′

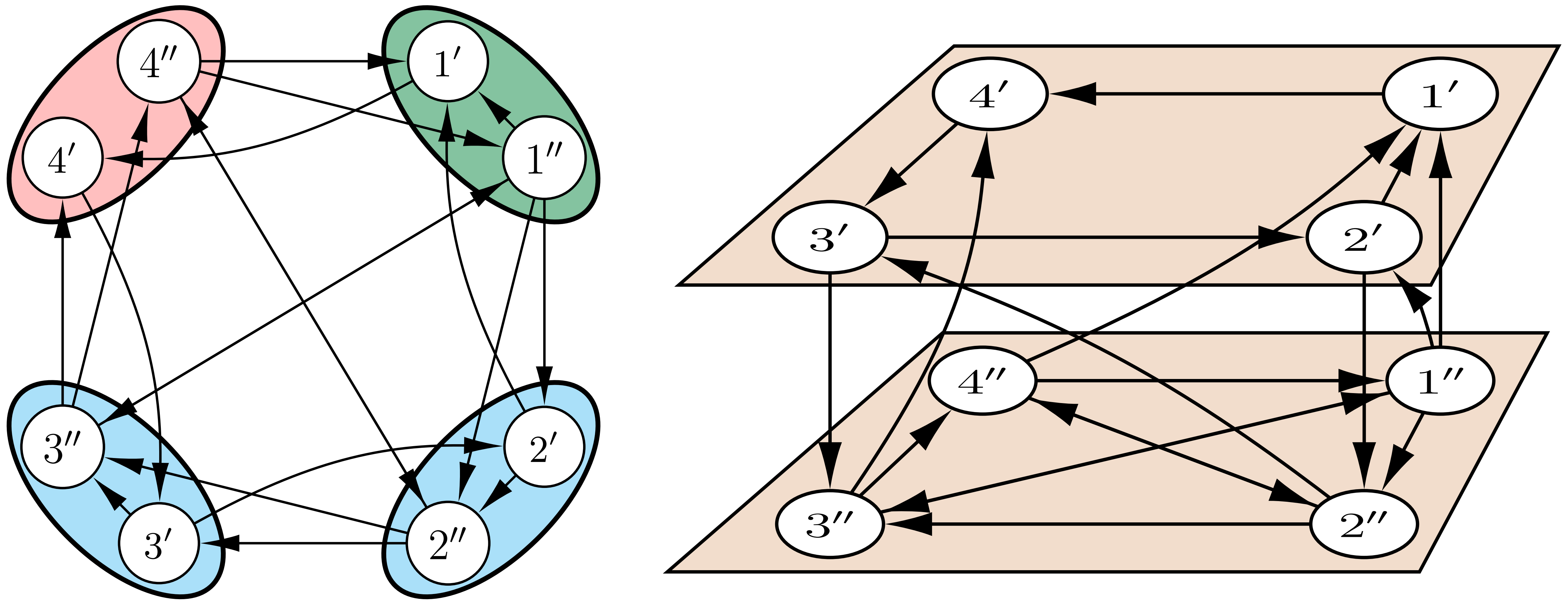

4
1
3
2
4″
1′
4′
1″
3″
2′
3′
2″


4
1
3
2
=
4′′
1′
4′
1′′
3′′
2′
3′
2′′

Multilayer network



Multilayer network of subnodes and sublinks



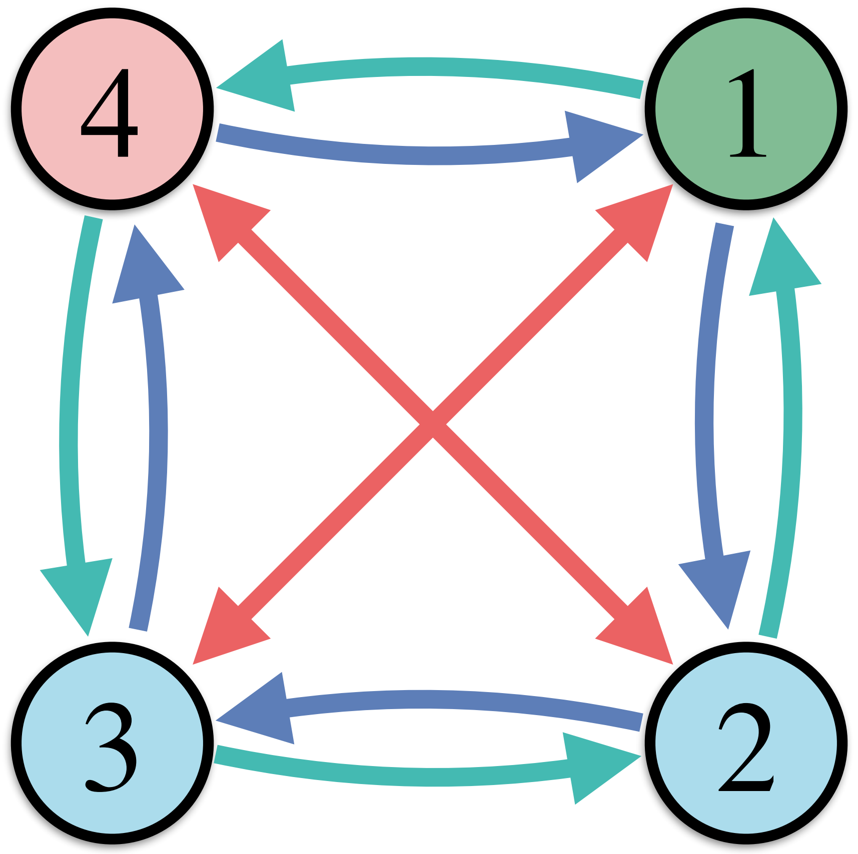
$$\dot{\boldsymbol{x}}_{\ell}^{(i)} = \boldsymbol{f}(\boldsymbol{x}_{\ell}^{(i)}) + \sum_{i'=1}^{N} \sum_{\ell'=1}^{L} \widetilde{A}_{\ell\ell'}^{(ii')} \left[\boldsymbol{h}(\boldsymbol{x}_{\ell'}^{(i')}) - \boldsymbol{h}(\boldsymbol{x}_{\ell}^{(i)})\right]$$

- Completely synchronous state is guaranteed
- Stability readily computed using Master Stability Function

L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998)

- Valid for arbitrary $\boldsymbol{f}$ and $\boldsymbol{h}$

Network of nodes and links



node dynamics is different
if
internal link pattern is different

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$

- Synchronous state is guaranteed
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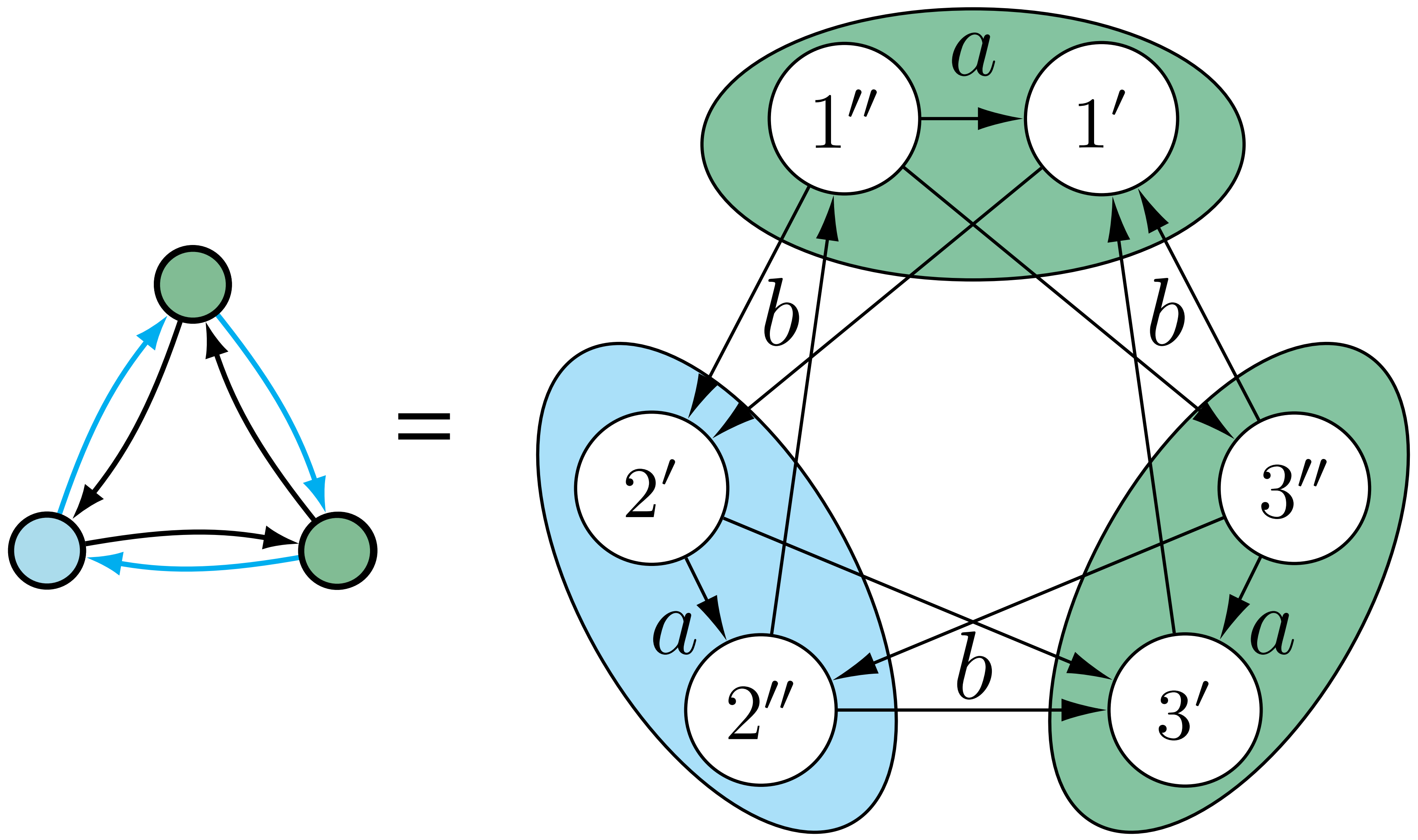
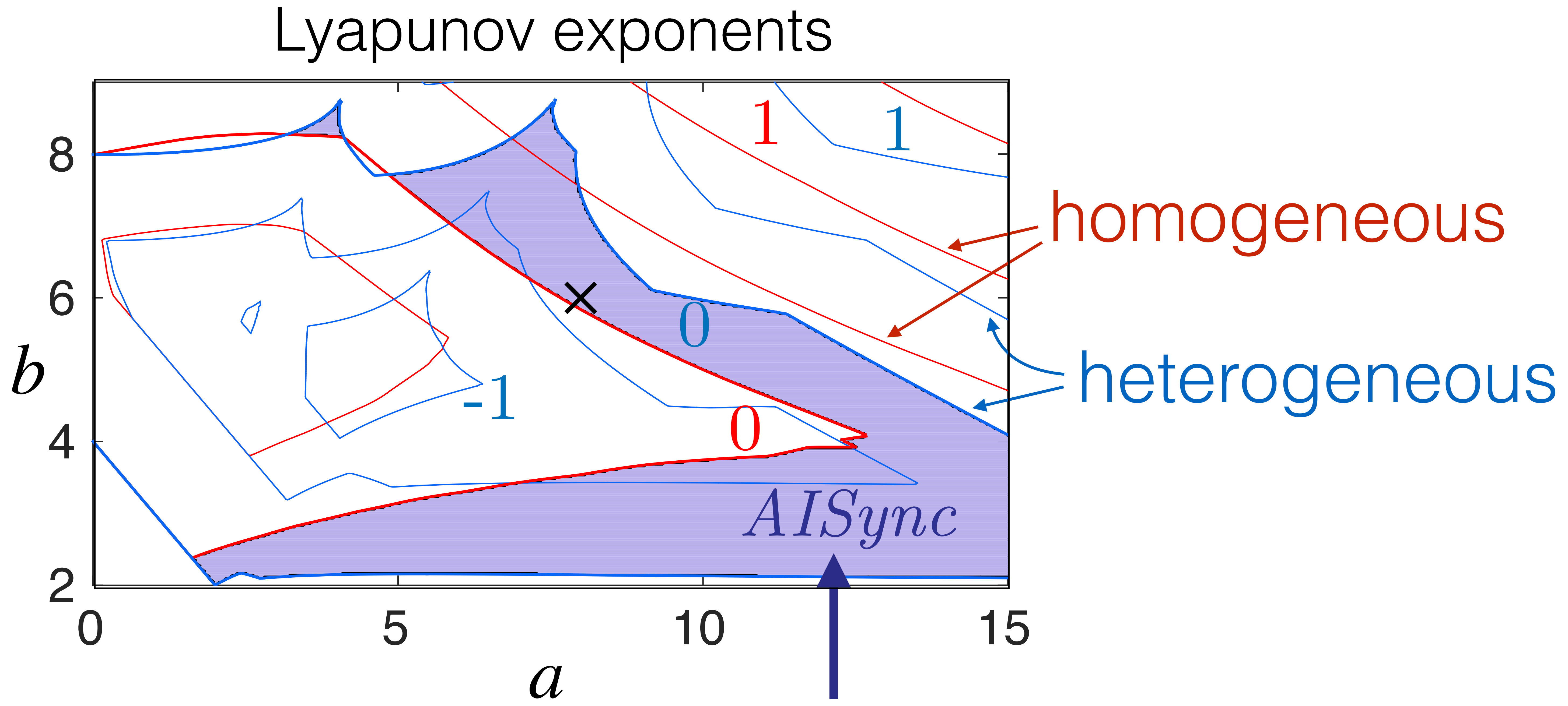
L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998)

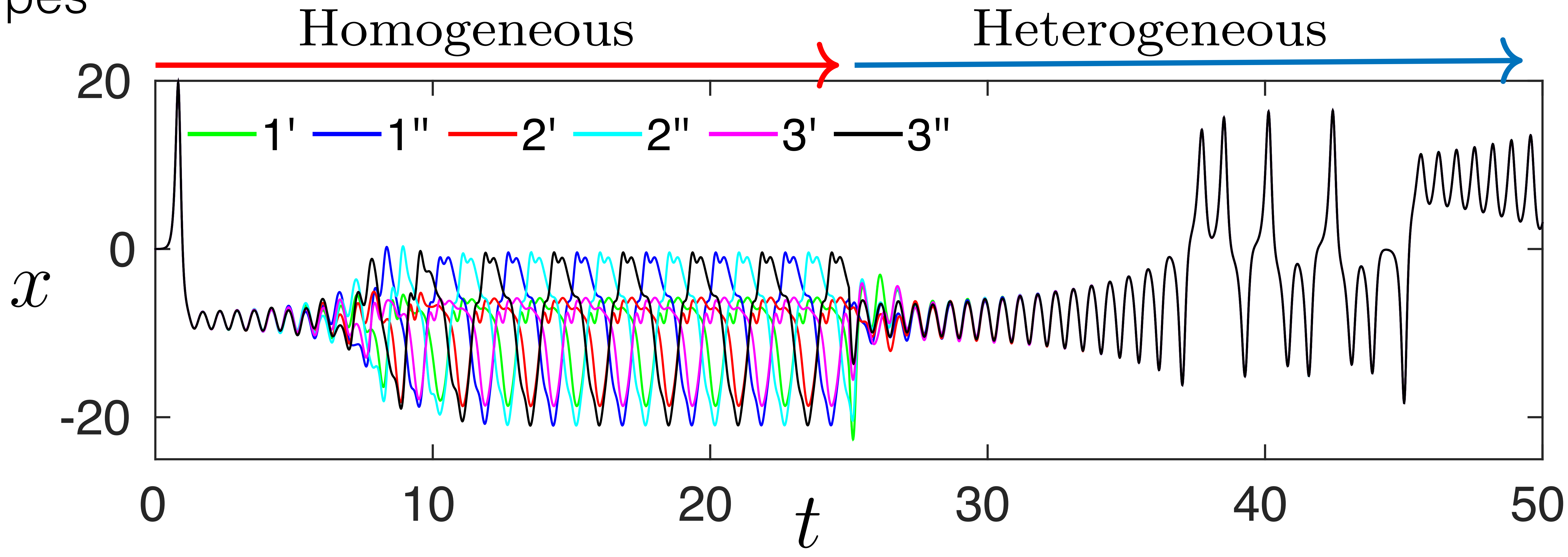
- Valid for arbitrary $\boldsymbol{f}$ and $\boldsymbol{h}$





Asymmetry-**I**nduced **Sync**hronization

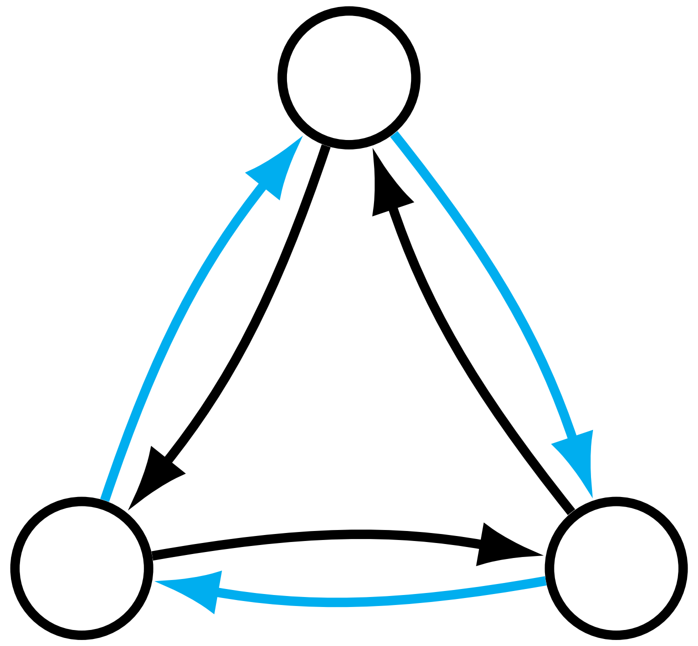
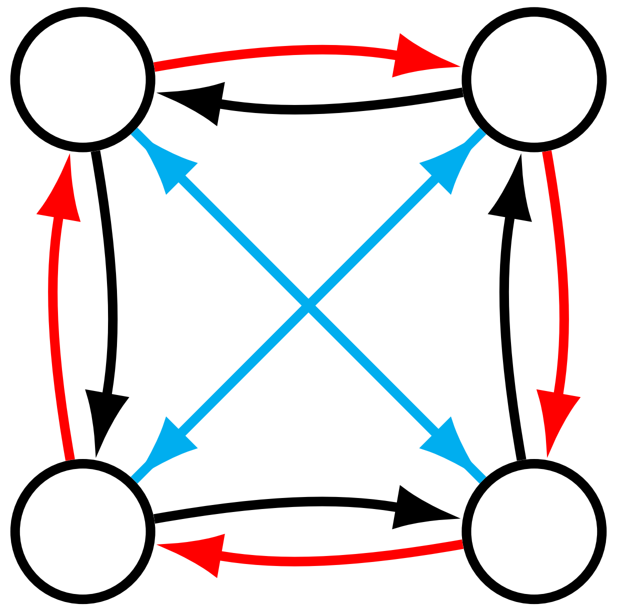
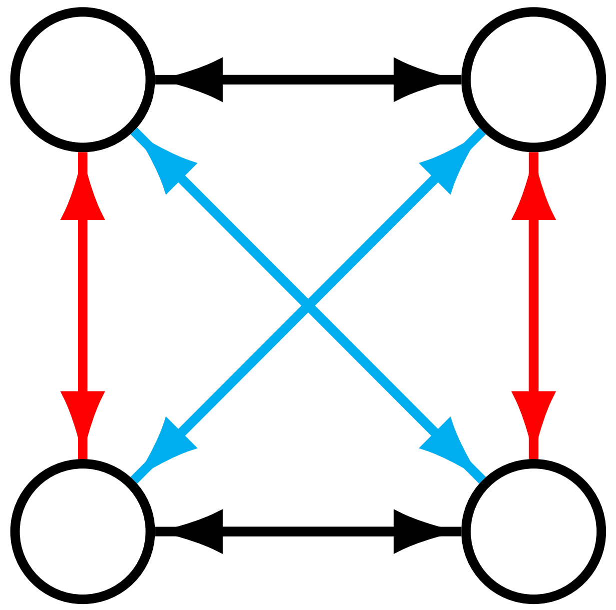
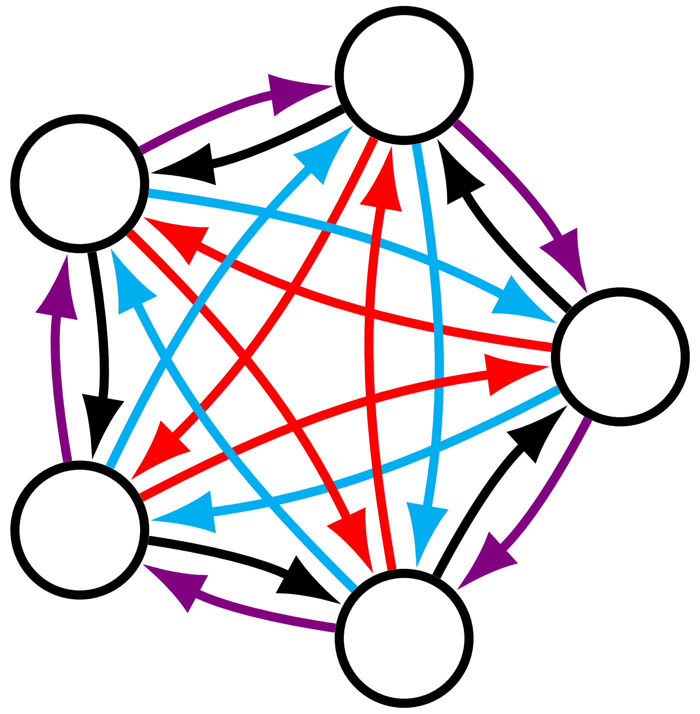
subnode = chaotic Lorenz oscillator
fixed external sublink pattern (strength b)
binary node types



What about other symmetric networks?

AISync strength r quantifies the degree to which a network structure favors AISync.

- $r = 0 \Rightarrow$ No AISync
- Larger $r \Rightarrow$ Favors AISync more strongly
- $r = 1 \Rightarrow$ There is an optimal heterogenous system.

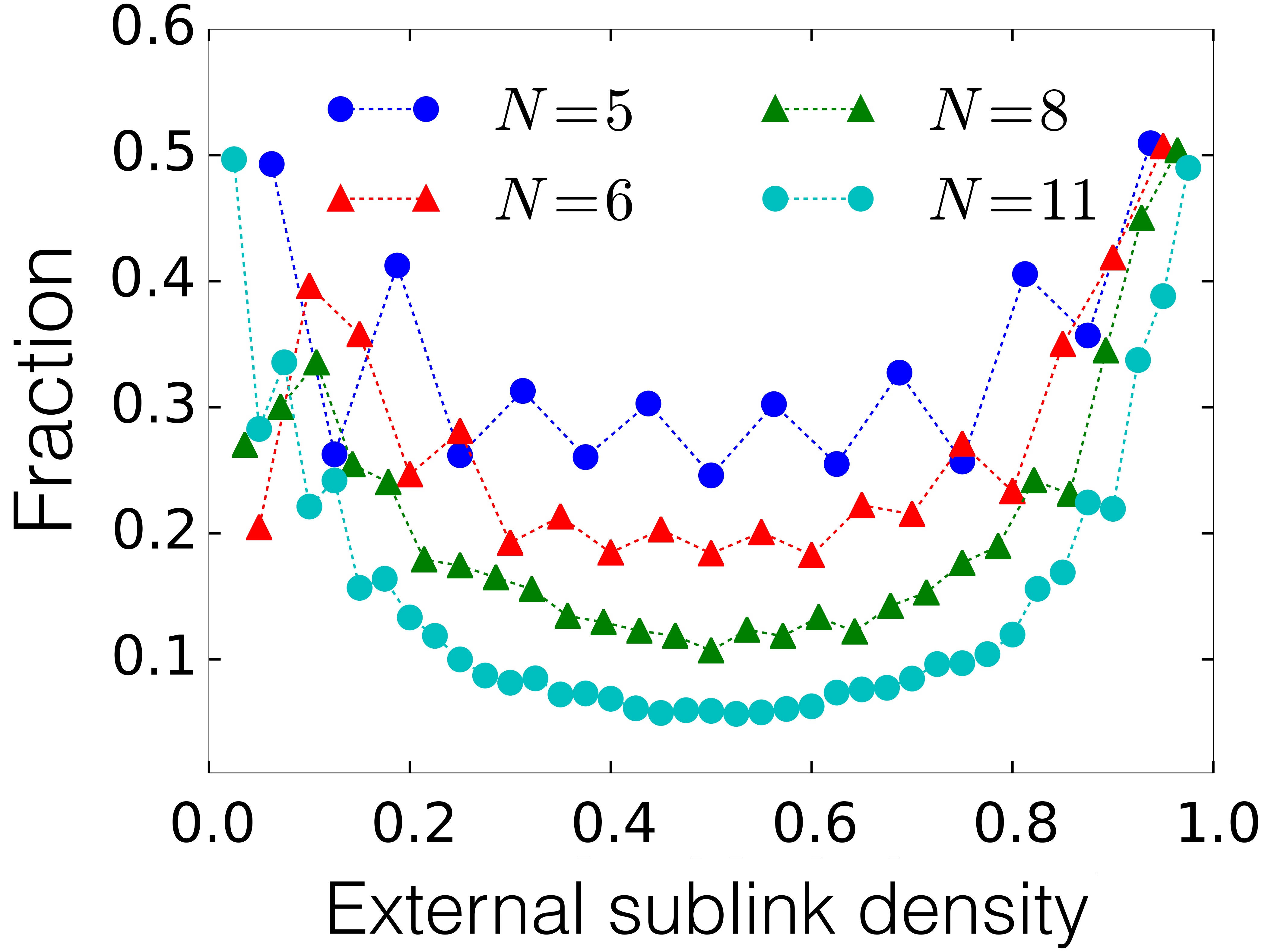
	$N = 3$	$N = 4$	$N = 5$
symmetric networks		 	
All 4 types (optimal)	9	14	21
All 4 types ($r > 0.2$)	11	81	254
All 4 types ($r > 0.05$)	29	318	2154
Binary ($r > 0.2$)	11	101	204
Binary ($r > 0.05$)	31	400	2406

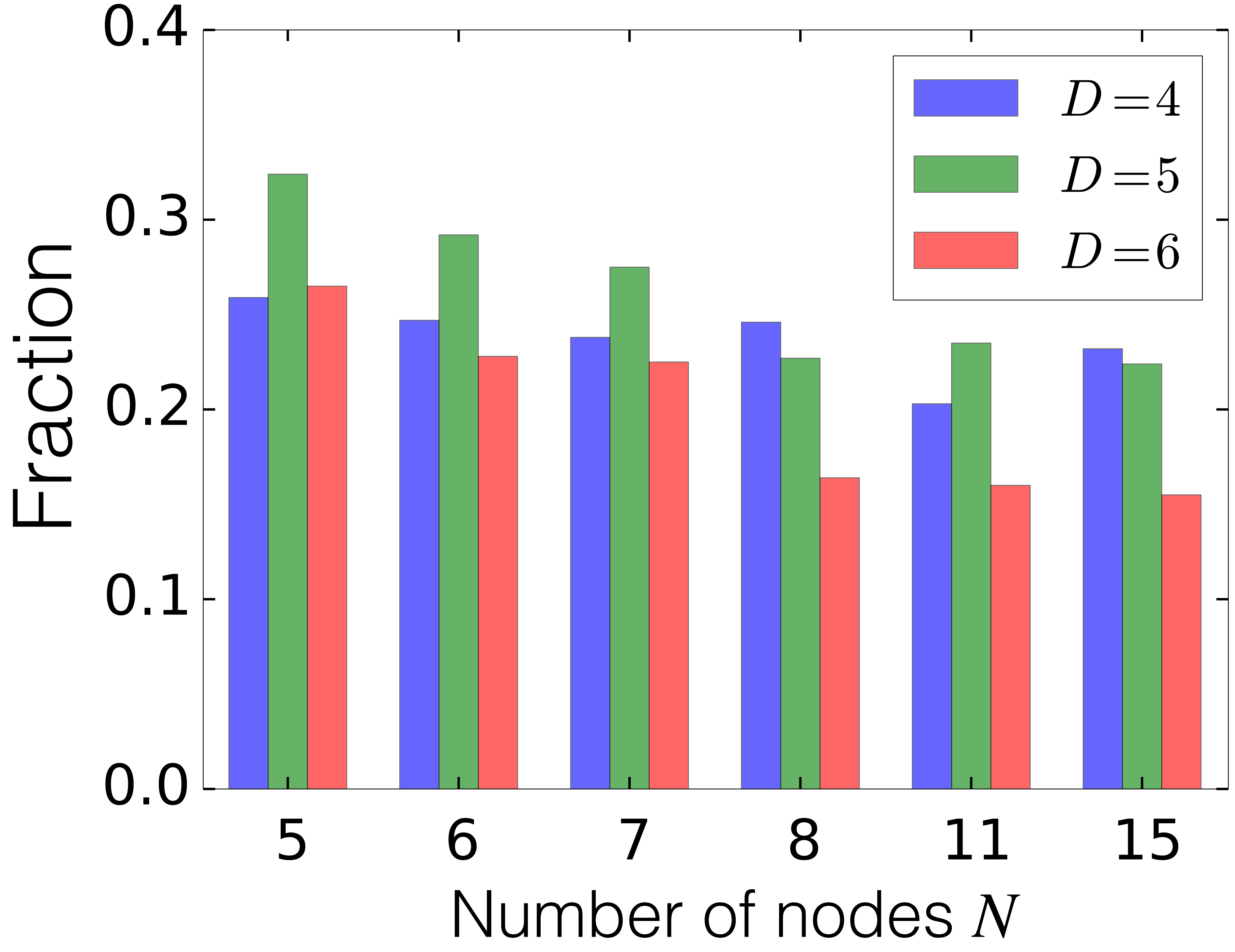
↑

Node types

Fraction of networks with $r > 0.05$

Within class of circulant-graphs (= all symmetric networks, if N is prime)





Significant fraction of systems are AISync-favoring for a range of system parameters

Summary

Symmetric states requiring system asymmetry
(converse of symmetry breaking)

- In network synchronization: fully synchronous state stable only when the oscillators are non-identical
- Observed quite often in the class of multilayer networks we considered

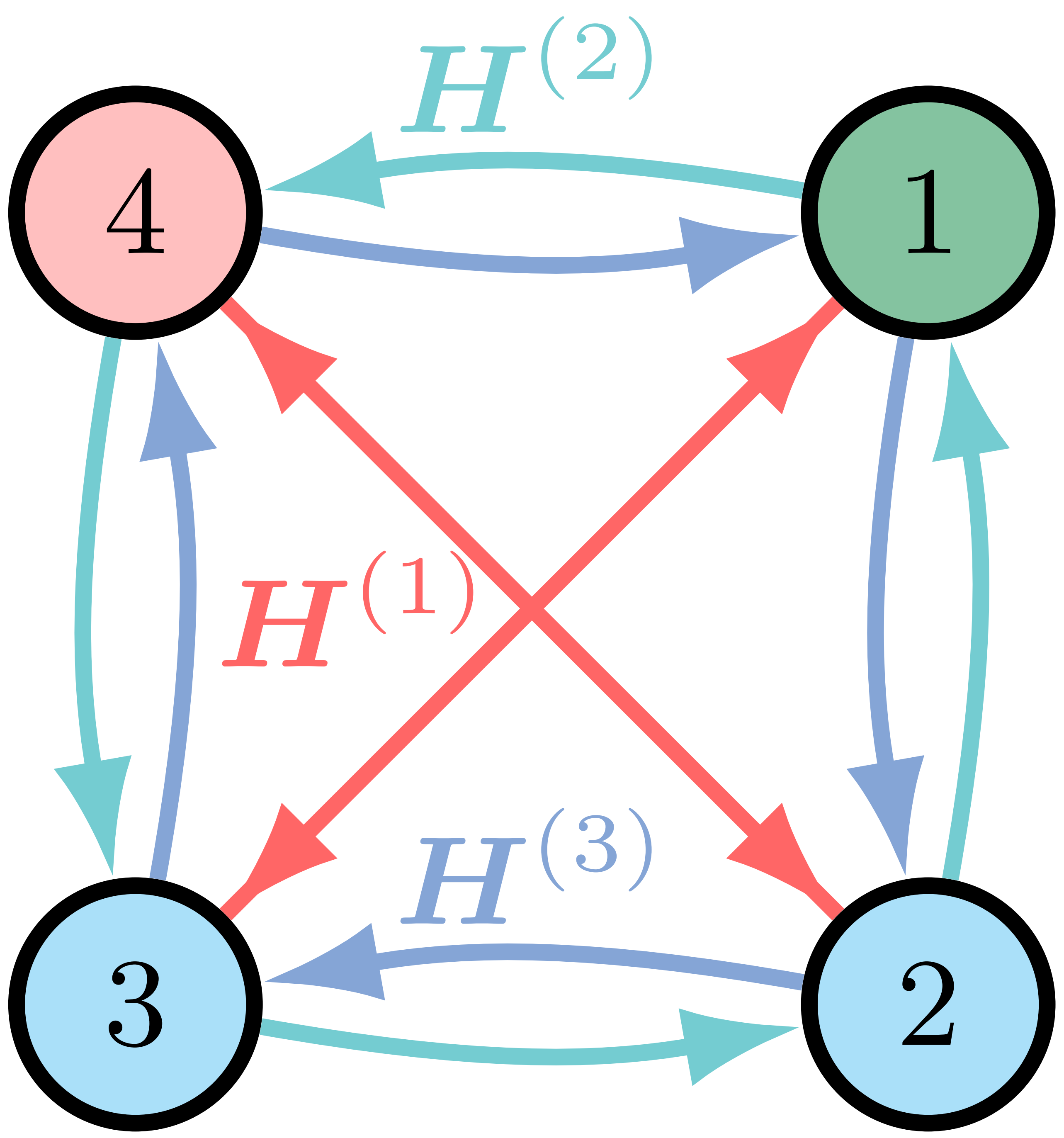
TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)
YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907

Final remarks

More generally: states with more symmetry requiring system to have less symmetry

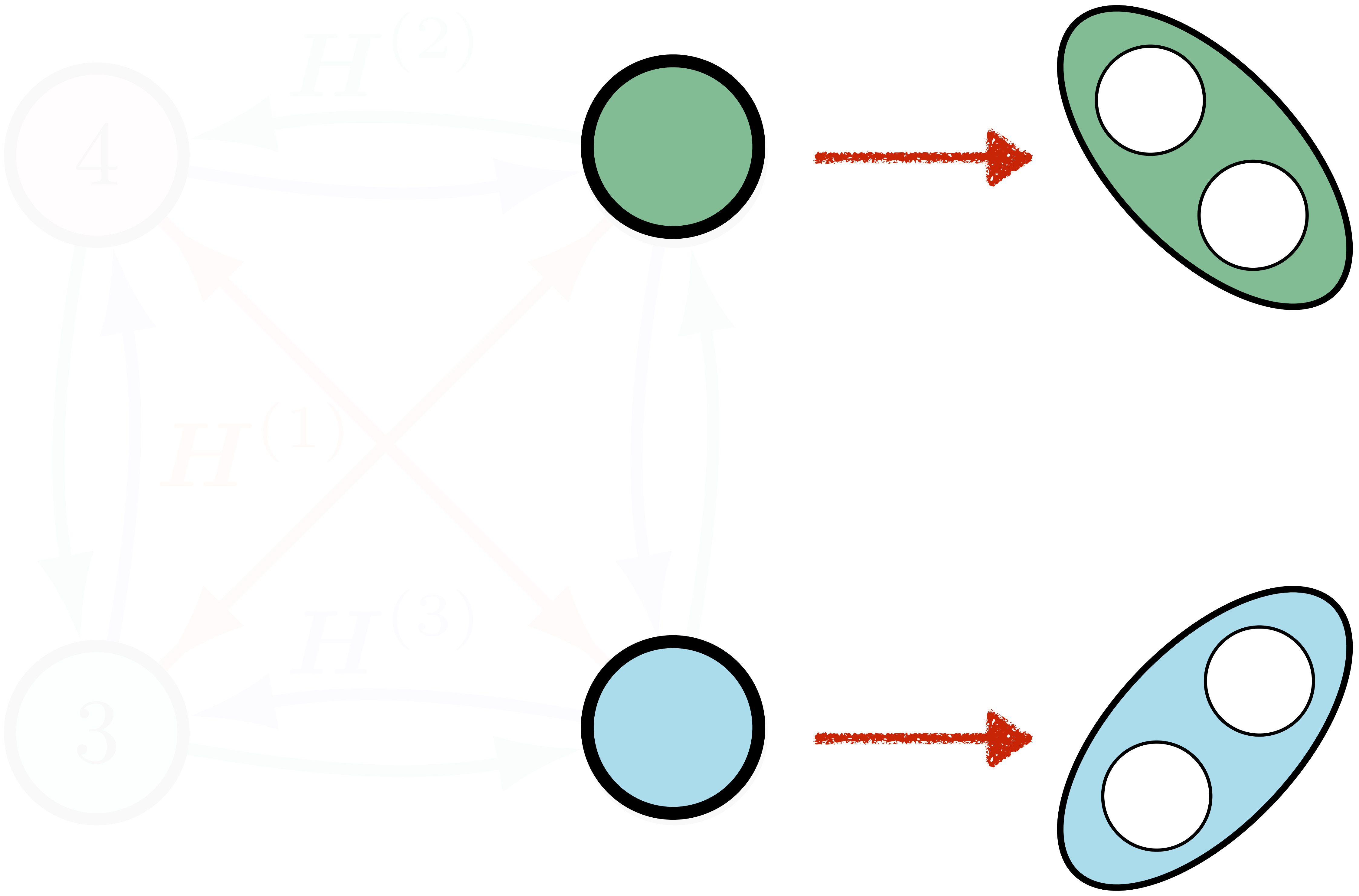
- Curie's principle
- Convergent vs divergent pattern formation

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)
YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907

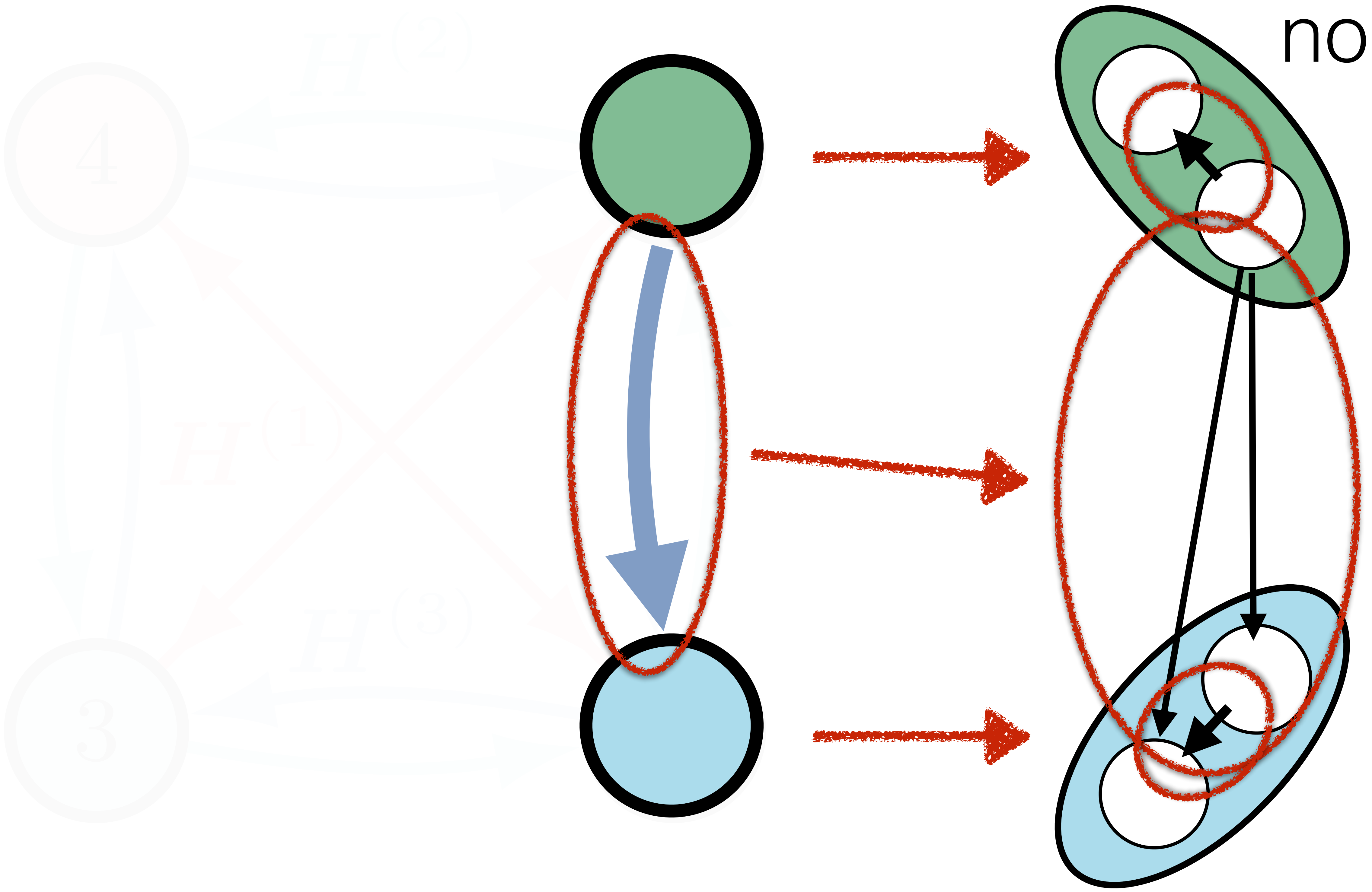


$H^{(2)}$
4
1
$H^{(1)}$
$H^{(3)}$
3
2

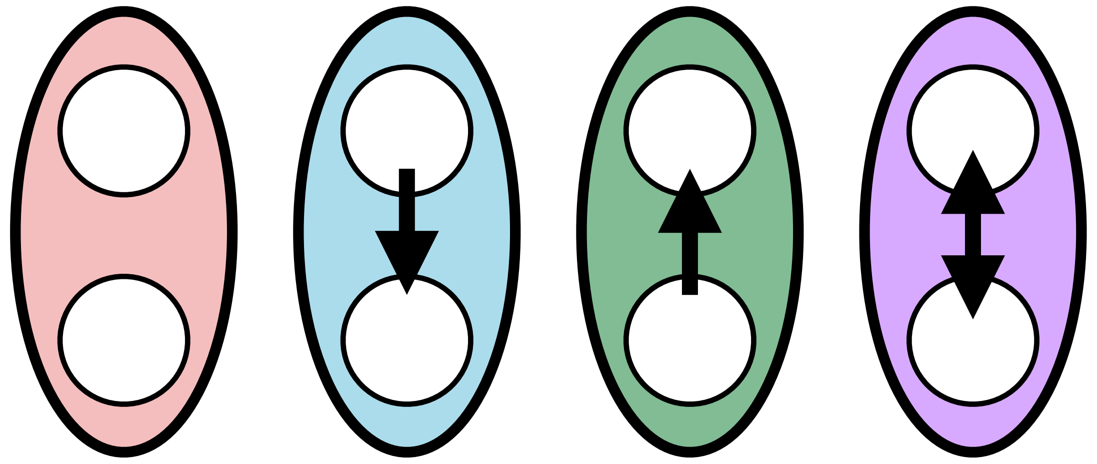
node = L *identical* subnodes



node = L *identical* subnodes

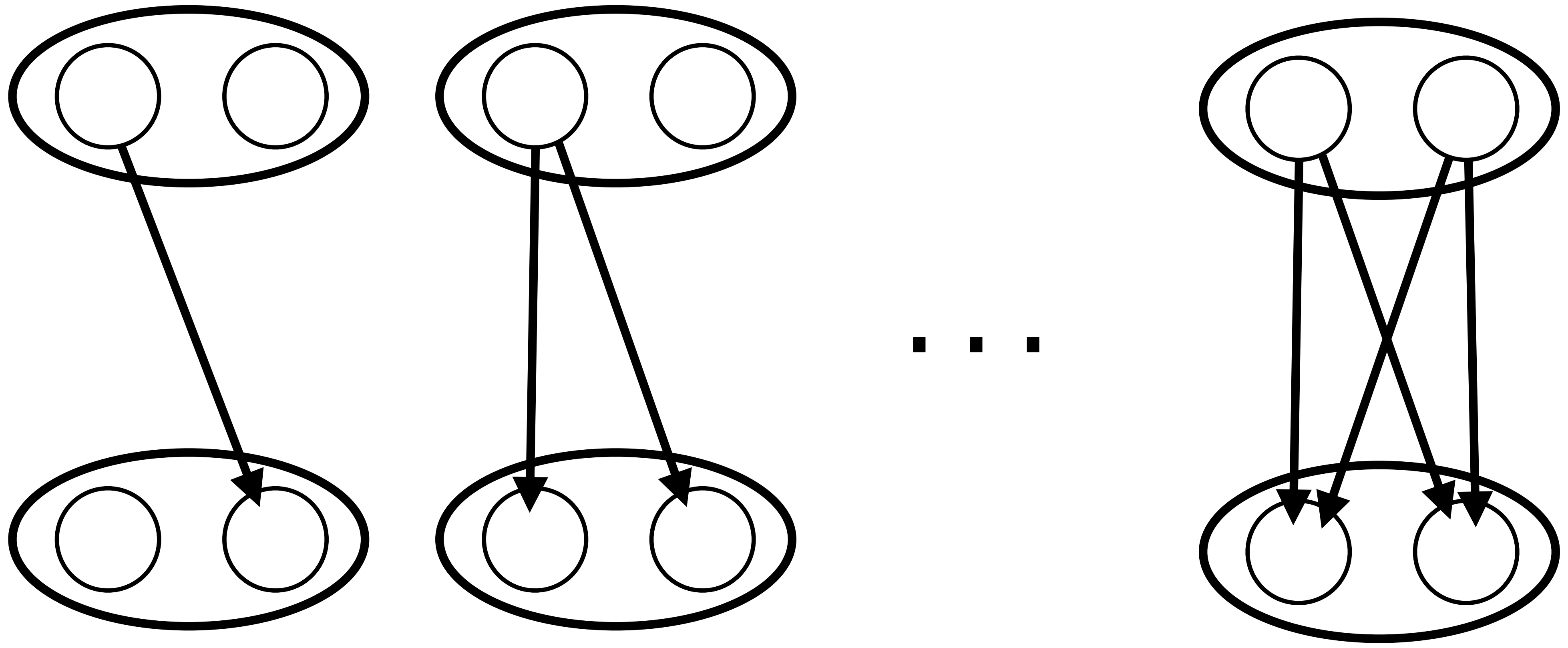


node types = patterns of *internal* sublinks



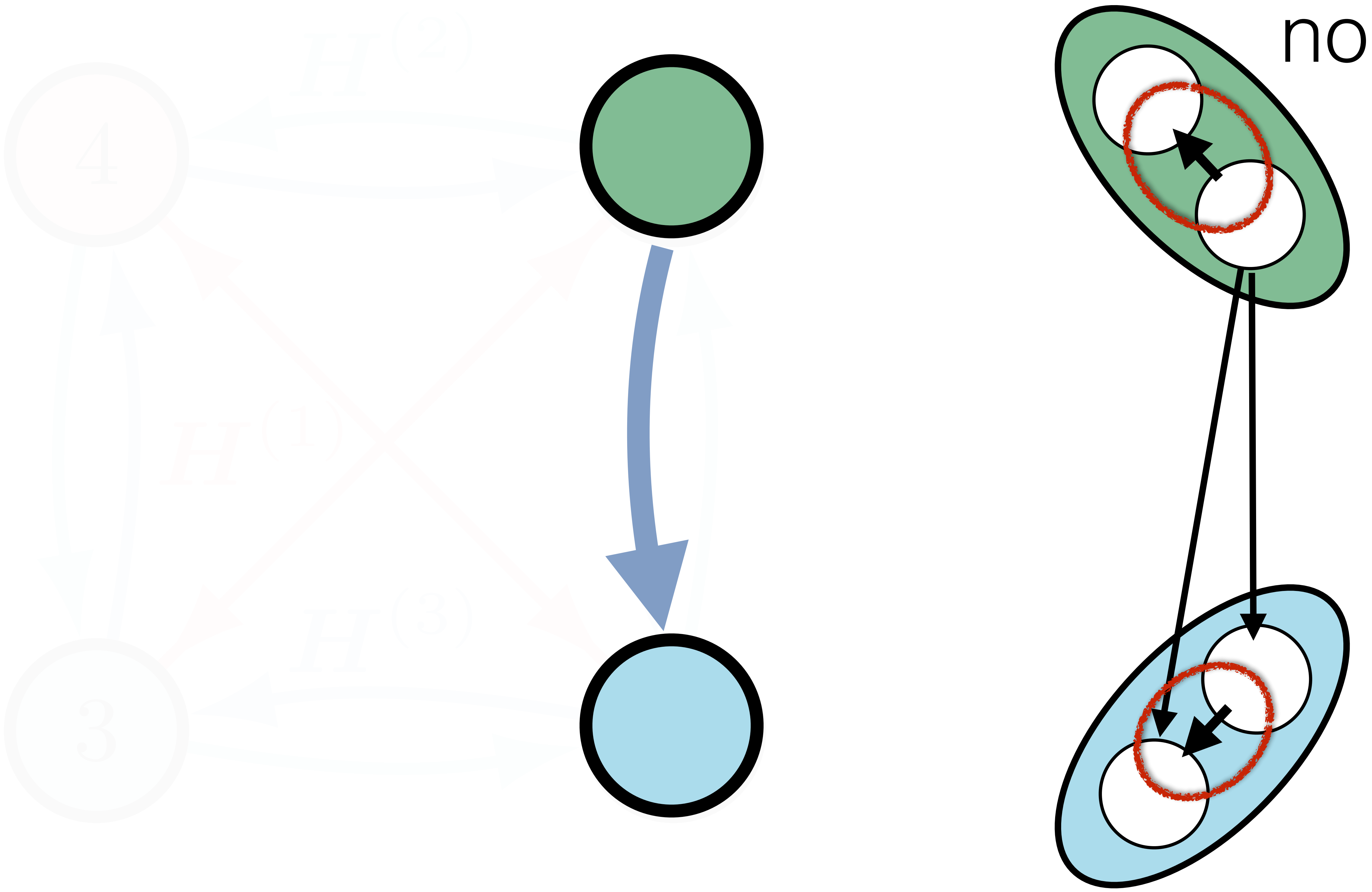
quaternary

link types = patterns of *external* sublinks

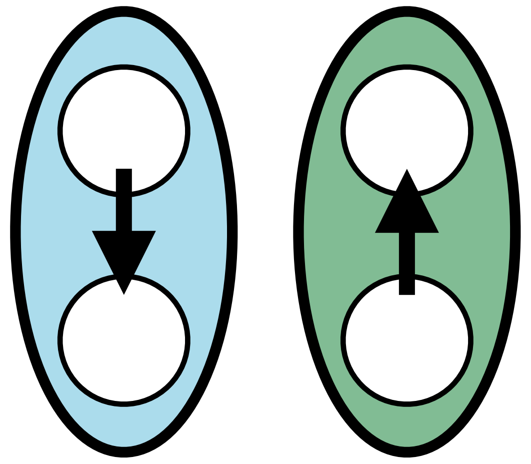


$2^4 - 1 = 15$ possibilities

node = L *identical* subnodes

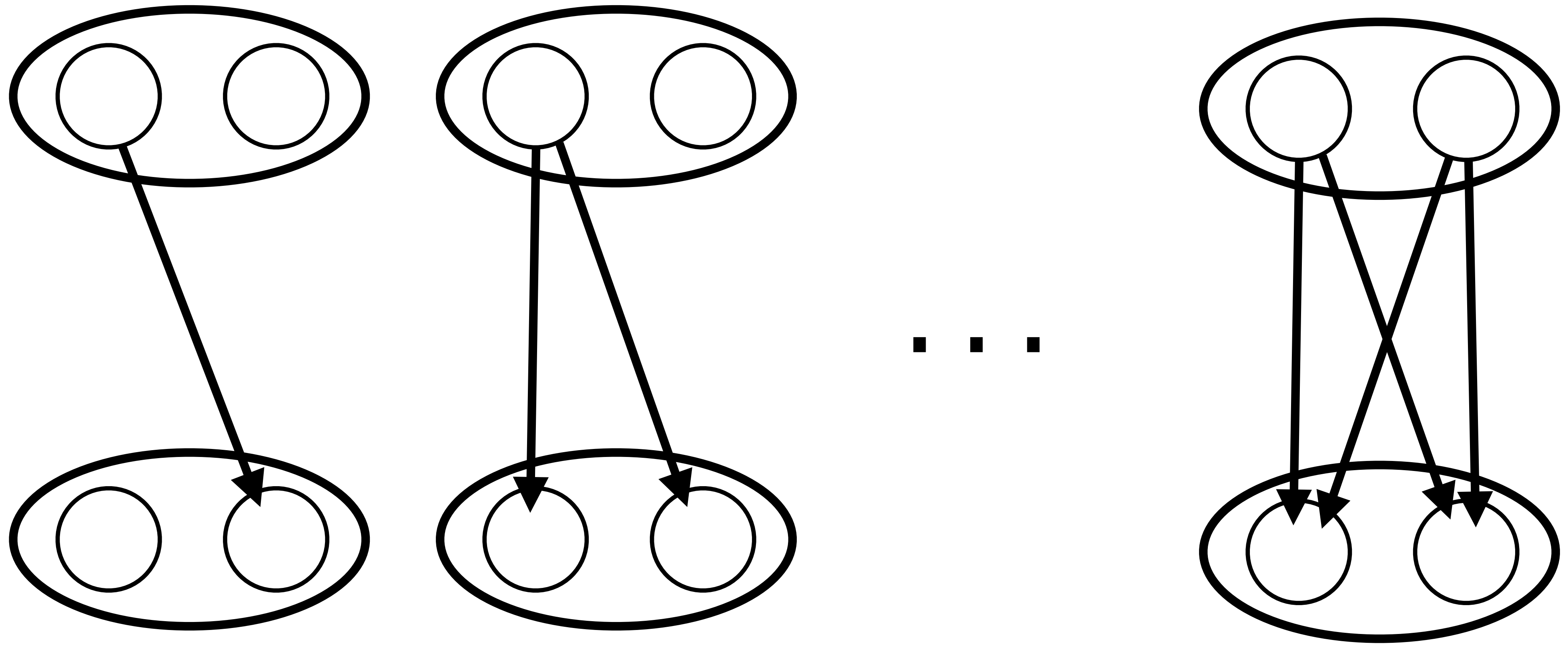


node types = patterns of *internal* sublinks

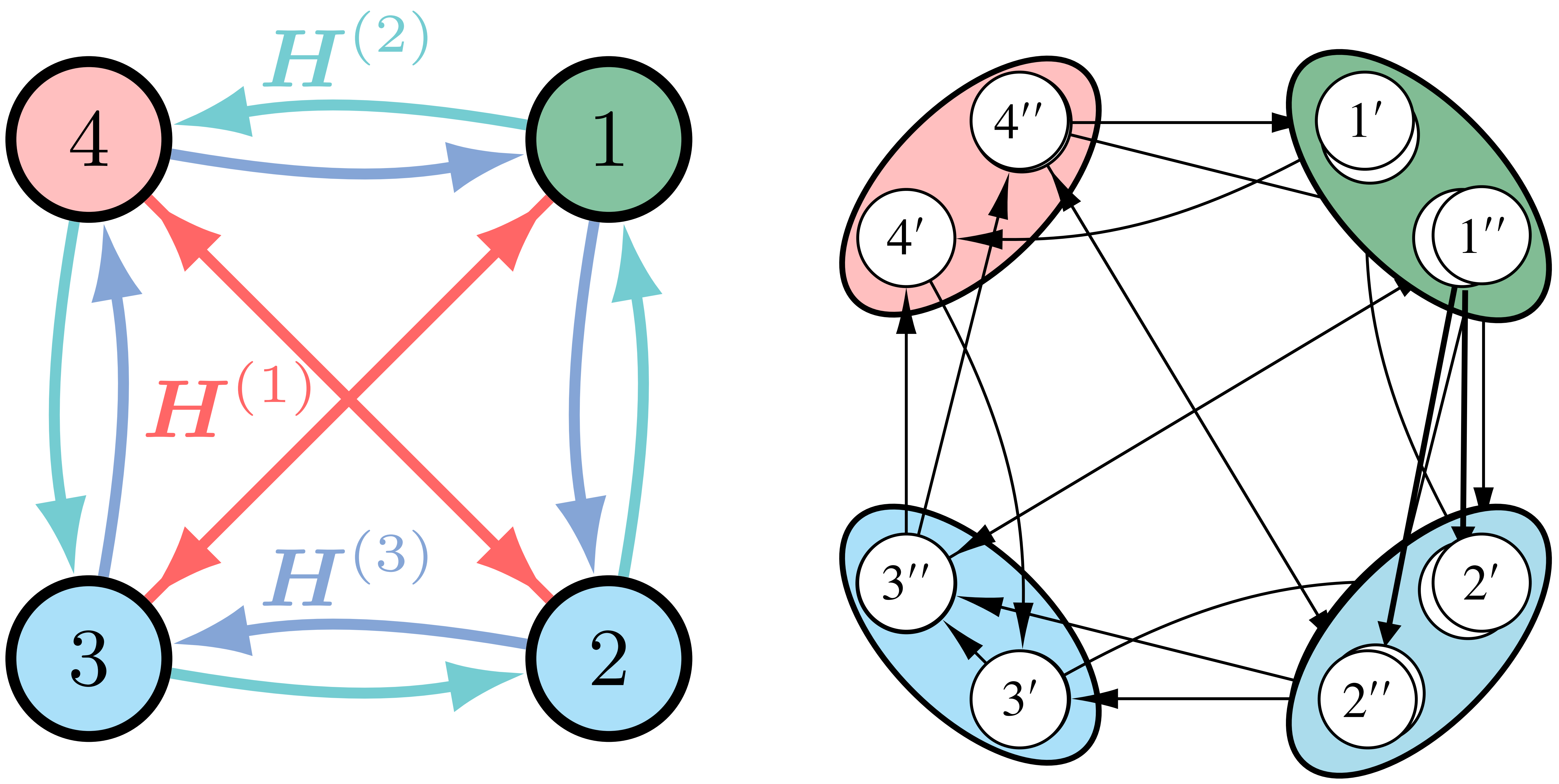


binary

link types = patterns of *external* sublinks

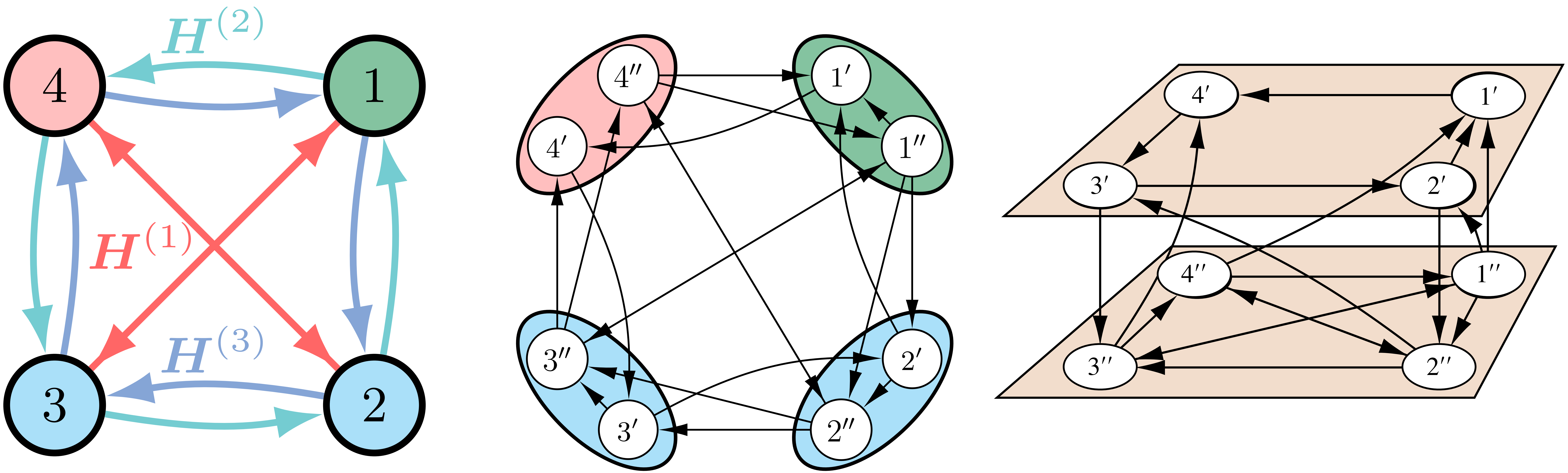


$2^4 - 1 = 15$ possibilities

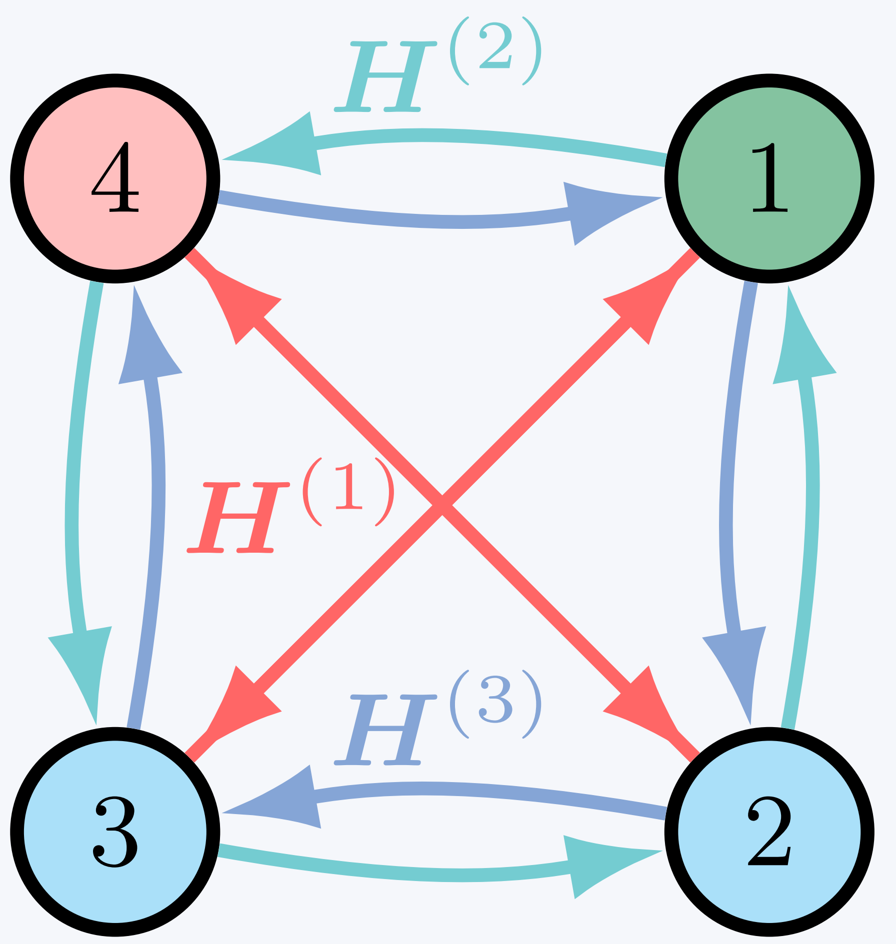


$H^{(2)}$
$H^{(1)}$
$H^{(3)}$
4
1
3
2
$4''$
$1'$
$4'$
$1''$
$3''$
$2'$
$3'$
$2''$

Multilayer networks of identical oscillators

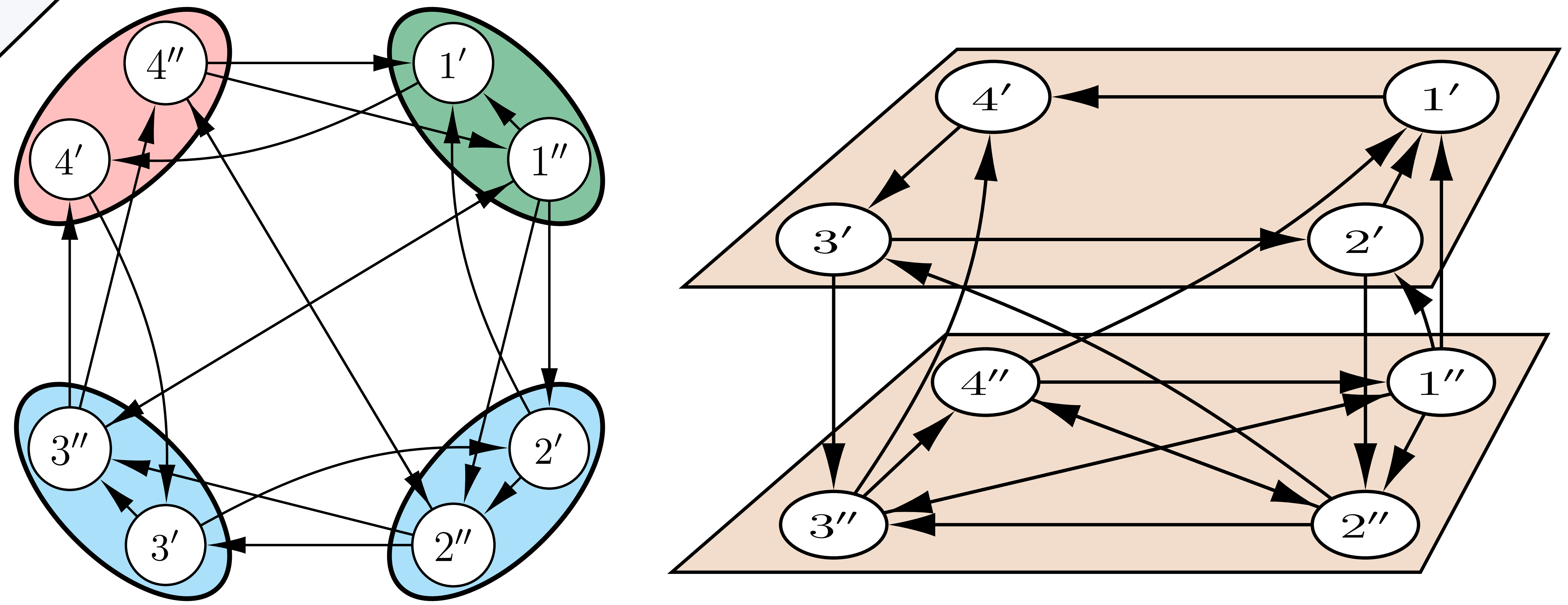


$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K}\sum_{\substack{i'=1\\ i'\neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$



nodes and links

subnodes and sublinks



$$\dot{\boldsymbol{x}}_\ell^{(i)} = \boldsymbol{f}(\boldsymbol{x}_\ell^{(i)}) + \sum_{i'=1}^{N}\sum_{\ell'=1}^{L} \widetilde{A}_{\ell\ell'}^{(ii')}\left[\boldsymbol{h}(\boldsymbol{x}_{\ell'}^{(i')}) - \boldsymbol{h}(\boldsymbol{x}_\ell^{(i)})\right]$$

Master stability function analysis

L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998)

$$\dot{\boldsymbol{x}}_\ell^{(i)} = \boldsymbol{f}(\boldsymbol{x}_\ell^{(i)}) + \sum_{i'=1}^{N} \sum_{\ell'=1}^{L} \widetilde{A}_{\ell\ell'}^{(ii')} \left[\boldsymbol{h}(\boldsymbol{x}_{\ell'}^{(i')}) - \boldsymbol{h}(\boldsymbol{x}_\ell^{(i)}) \right]$$

Diffusive coupling

Identical subnodes

Stability of complete synchronization can be readily computed for arbitrary $\boldsymbol{f}$ and $\boldsymbol{h}$ (including experimentally realizable systems)