Two Types of Quasiperiodic Partial Synchrony: What Happens When Symmetric States Become Unstable

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Identical globally coupled oscillators

Consider the simplest network:

- Elements are identical and subject to common force
- Phase oscillators, one-harmonic coupling

⟶ The paradigmatic Kuramoto-Sakaguchi model

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta), \ \mathit{with} \ Re^{i\Theta} = \frac{1}{N}\sum_j e^{i\varphi_j}$$

$|\beta| < \pi/2$ ⟶ attractive coupling

$\pi/2 < \beta < 3\pi/2$ ⟶ repulsive coupling

$\beta = \pm\pi/2$ ⟶ neutral coupling

The Kuramoto-Sakaguchi model

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta)$$

$|\beta| < \pi/2$ $\longrightarrow$ synchrony is stable

$\pi/2 < \beta < 3\pi/2$ $\longrightarrow$ splay state is stable

$\beta = \pm\pi/2$ $\longrightarrow$ marginal stability (not interesting)

either full synchrony, $R = 1$
or full asynchrony (splay state), $R = 0$

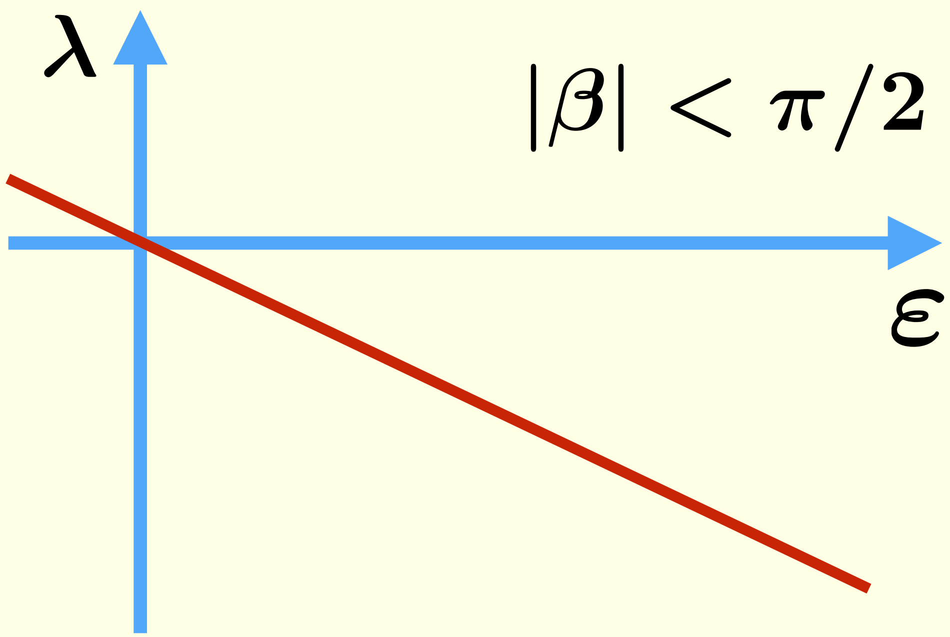
Notice: clusters are not possible, as follows from the Watanabe-Strogatz theory (except for ***N-1,1*** configuration)

Stability of the synchronous state

The Kuramoto-Sakaguchi model, identical oscillators:

Synchronous (one-cluster) state is stable, if $\boldsymbol{\lambda = -\varepsilon \cos\beta < 0}$

eigenvalue



<u>For this model</u>: stability is proportional to coupling
==> tendency to synchrony increases with ε

Specific features of the Kuramoto-Sakaguchi model

1) tendency to synchrony increases with the coupling strength

2) domains of stable synchrony and asynchrony are **complementary**

3) only full synchrony or splay state; no clusters, no chimeras

These properties are typical, but not general!

General phase models: When do we expect complex solutions?

1) tendency to synchrony is not monotonic and/or

2) **both** splay state and synchrony are unstable

The system settles at some intermediate state

We expect: clusters

chimeras

quasiperiodic partially synchronous states

Quasiperiodic partial synchrony in the Kuramoto-Daido model

1) continuous but not uniform distribution of phases

$\longrightarrow$ order parameter $\mathbf{0 < \boldsymbol{R} < 1}$

2) Mean field frequency $\neq$ oscillators frequency

$\longrightarrow$ quasiperiodic dynamics

To be distinguished from the case of ensembles with a frequency distribution, when some oscillators form a synchronous cluster while some are not locked to the mean field

A minimal model

Kuramoto-Daido model with two harmonics, Hansel et al, 1993

$$\dot{\varphi}_k = R_1 \sin(\Theta_1 - \varphi_k + \gamma_1) + aR_2 \sin(\Theta_2 - 2\varphi_k + \gamma_2)$$

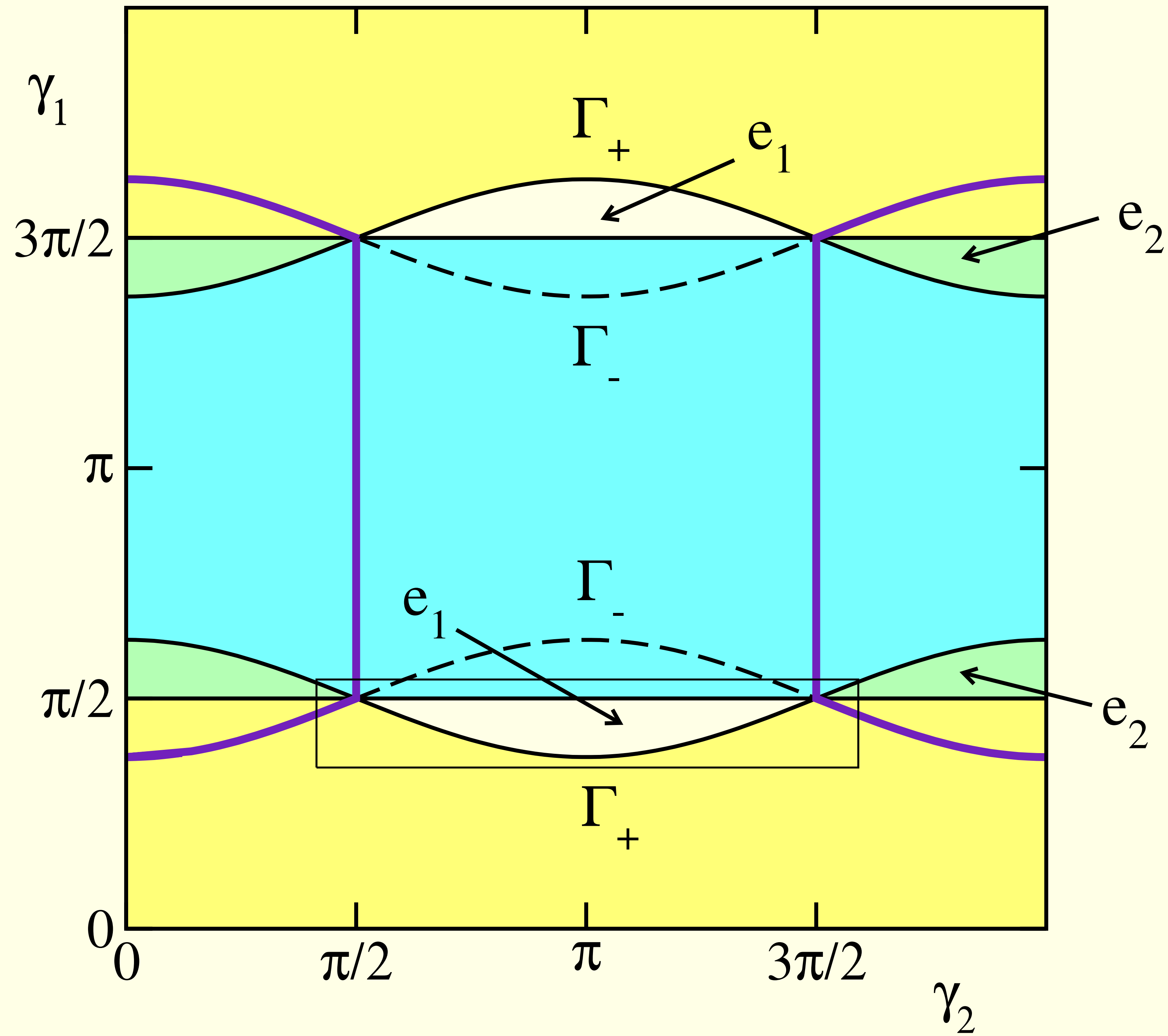
Generalized order parameters $R_m e^{i\Theta_m} = N^{-1} \sum_j e^{im\varphi_j}$

1) frequency can be removed by a transformation to a co-rotating frame

2) coupling strength can be removed by rescaling of time

3) parameter ***a=0.2*** is fixed, parameters $\gamma_{1,2}$ are varied

The biharmonic model: stability analysis



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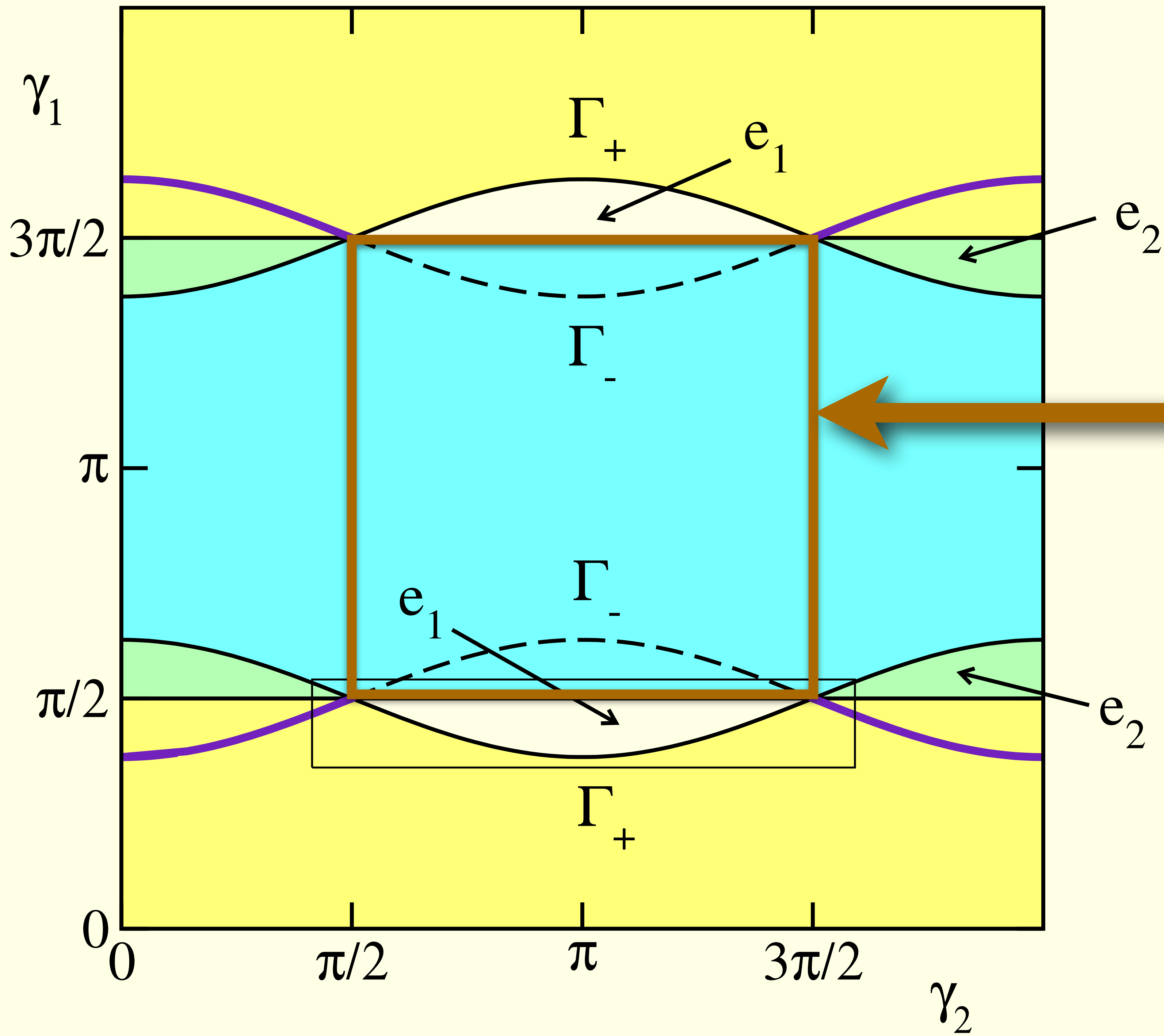
The biharmonic model: stability analysis



Splay state stable here

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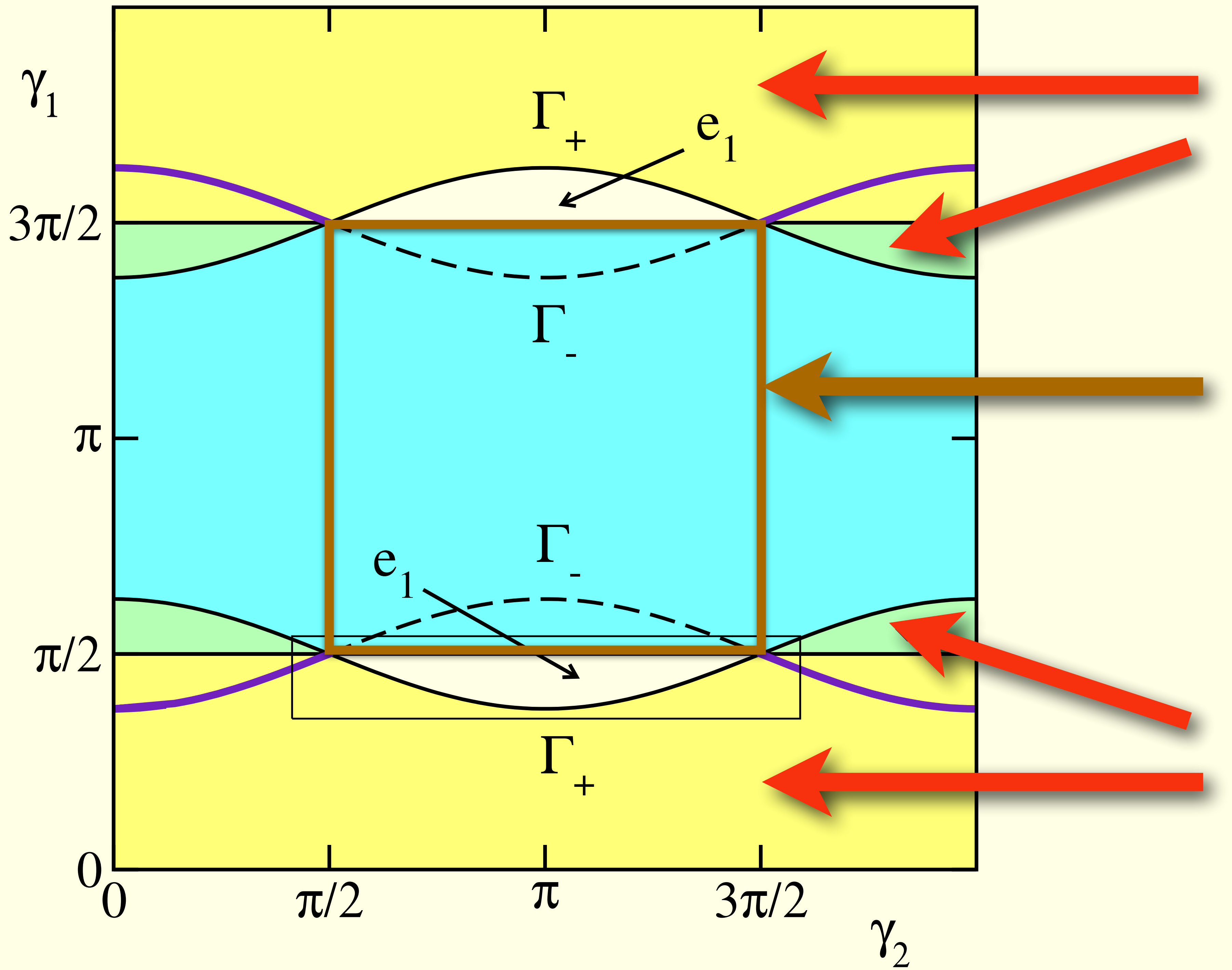

Full synchrony stable here

Splay state stable here

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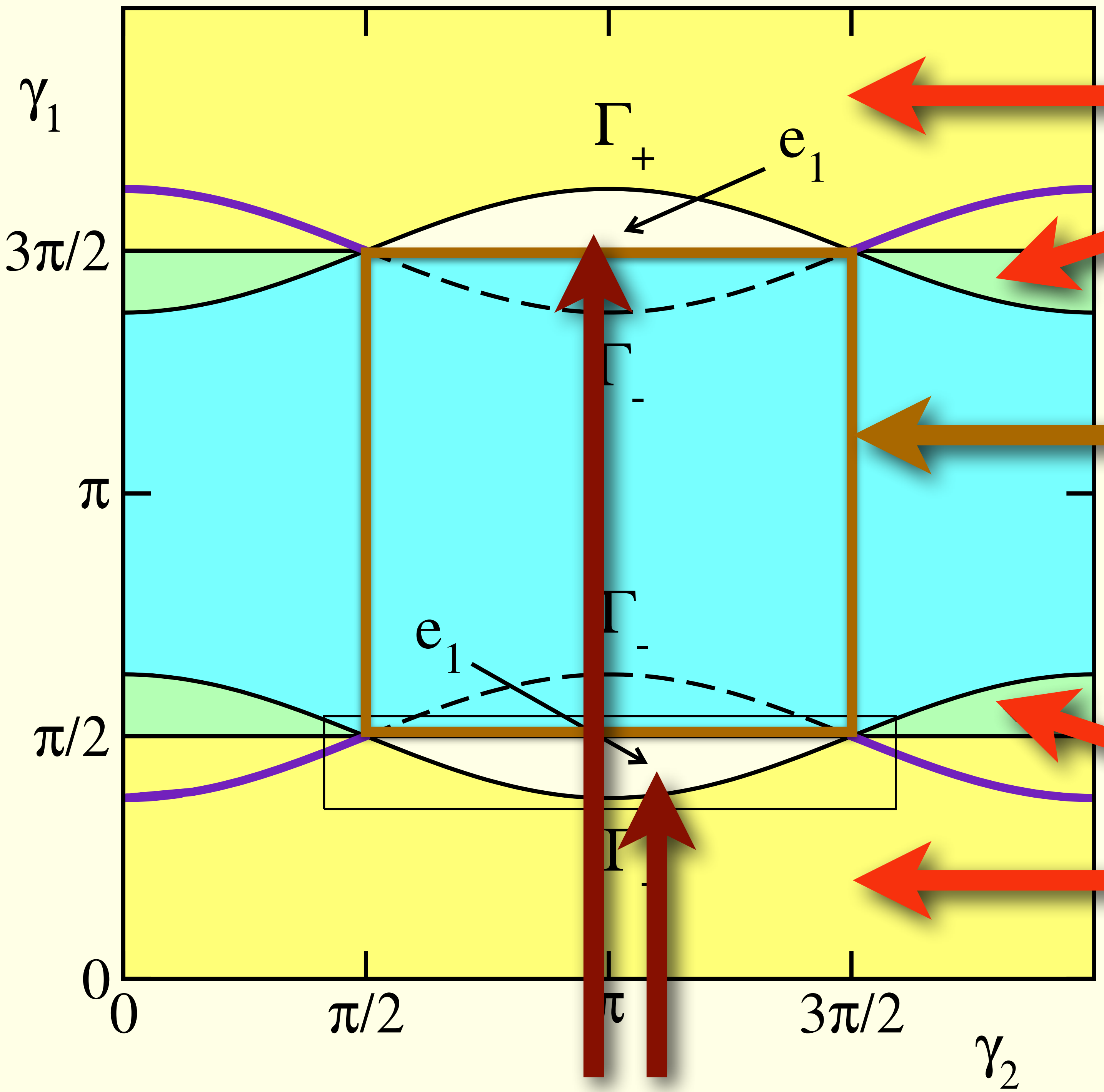
The biharmonic model: stability analysis



Full synchrony stable here

Splay state stable here

Full synchrony stable here

synchrony AND splay unstable here

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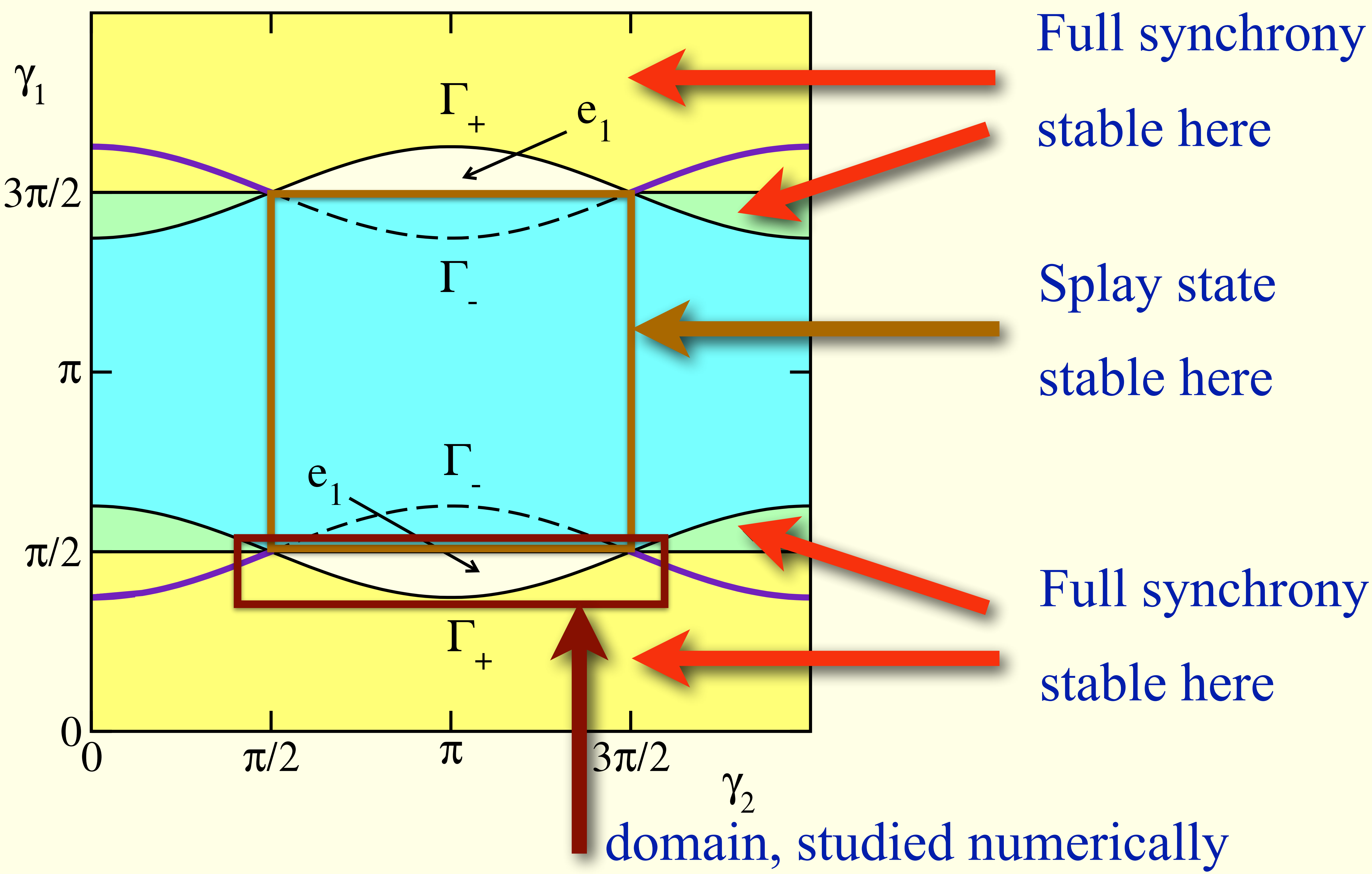
The biharmonic model: stability analysis



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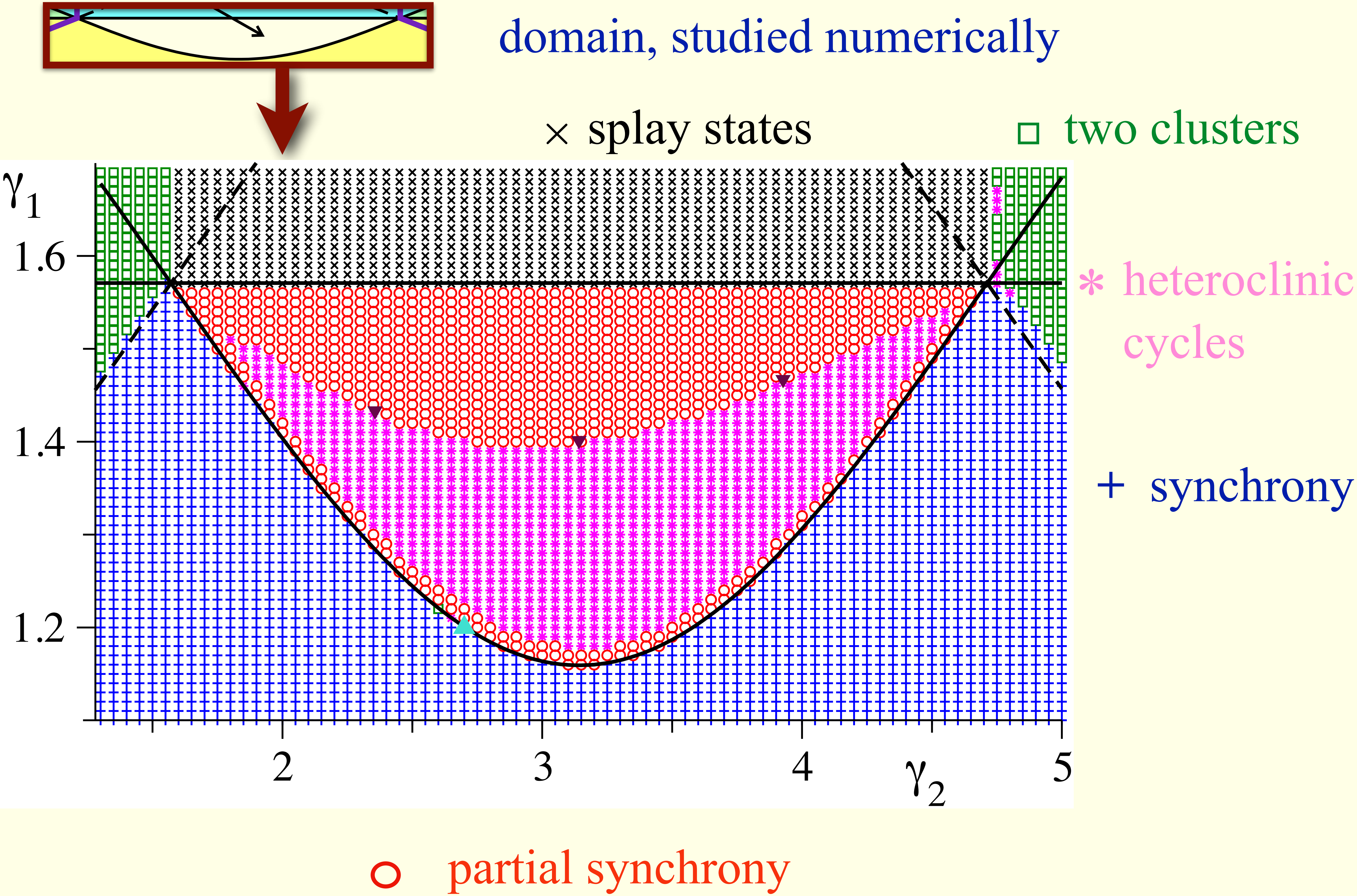
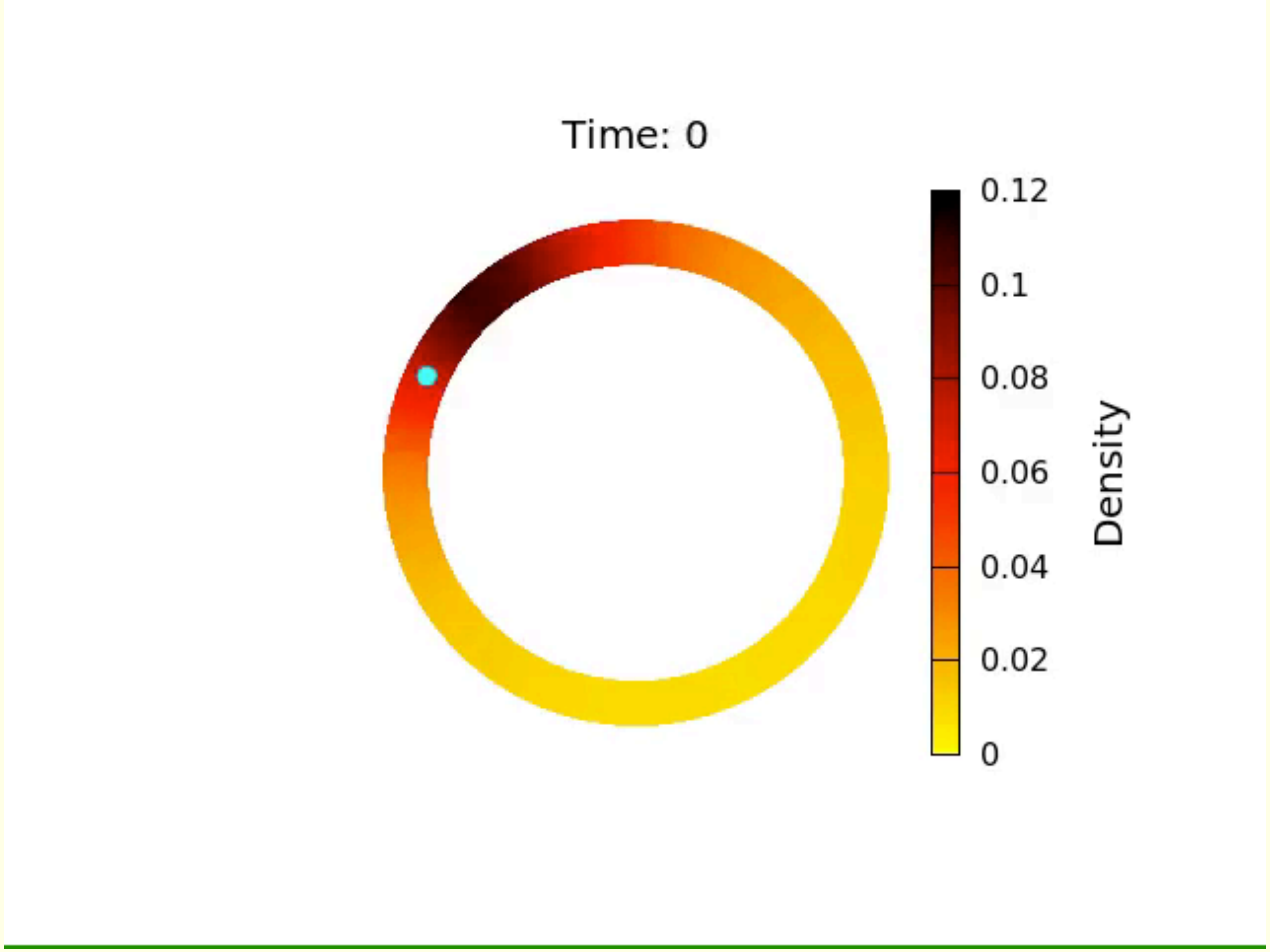
The biharmonic model: numerics

domain, studied numerically

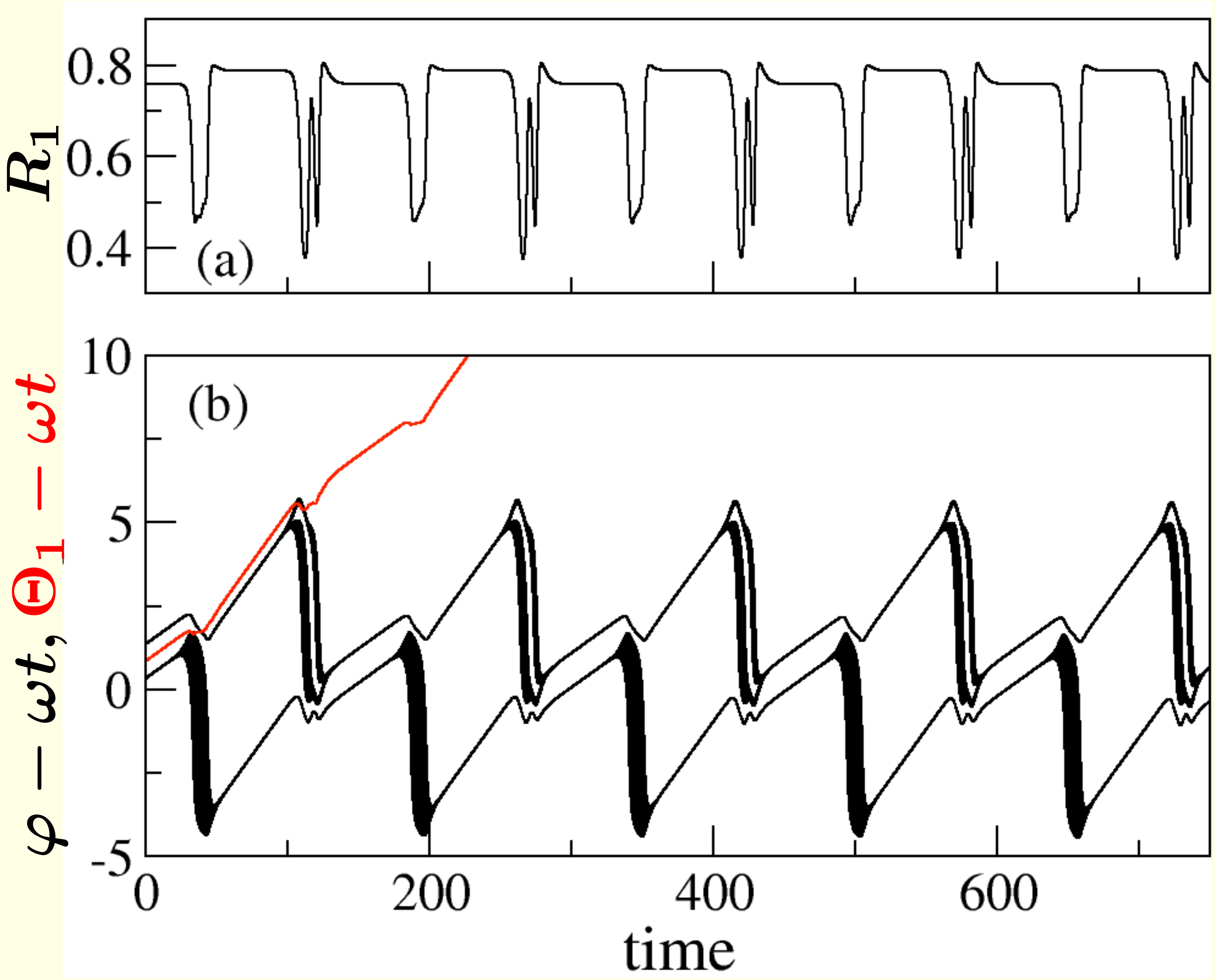


Partial synchrony



Heteroclinic cycles as partially synchronous states

HC in biharmonic model: Hansel et al, 1993; Kori and Kuramoto, 2001



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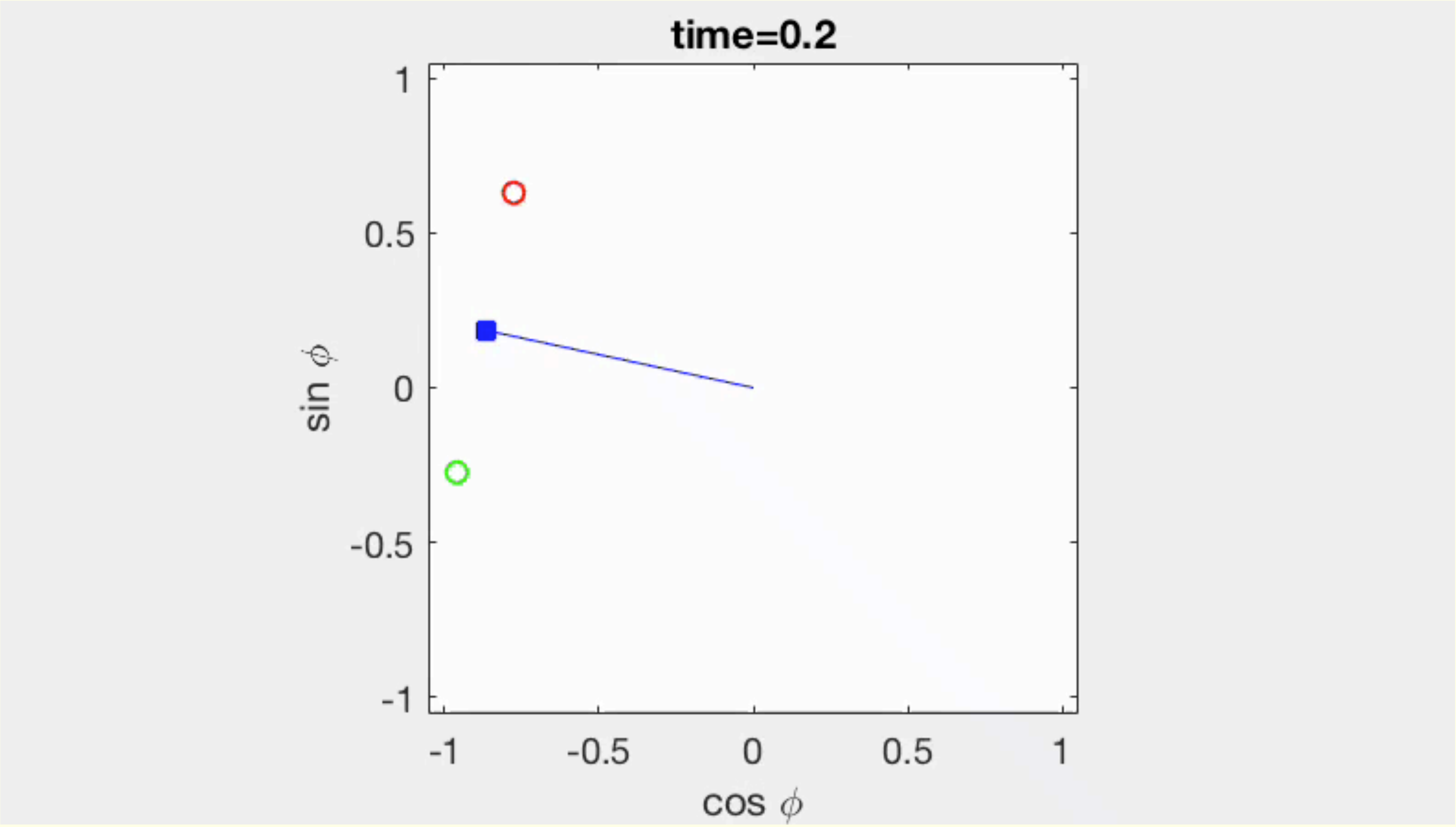

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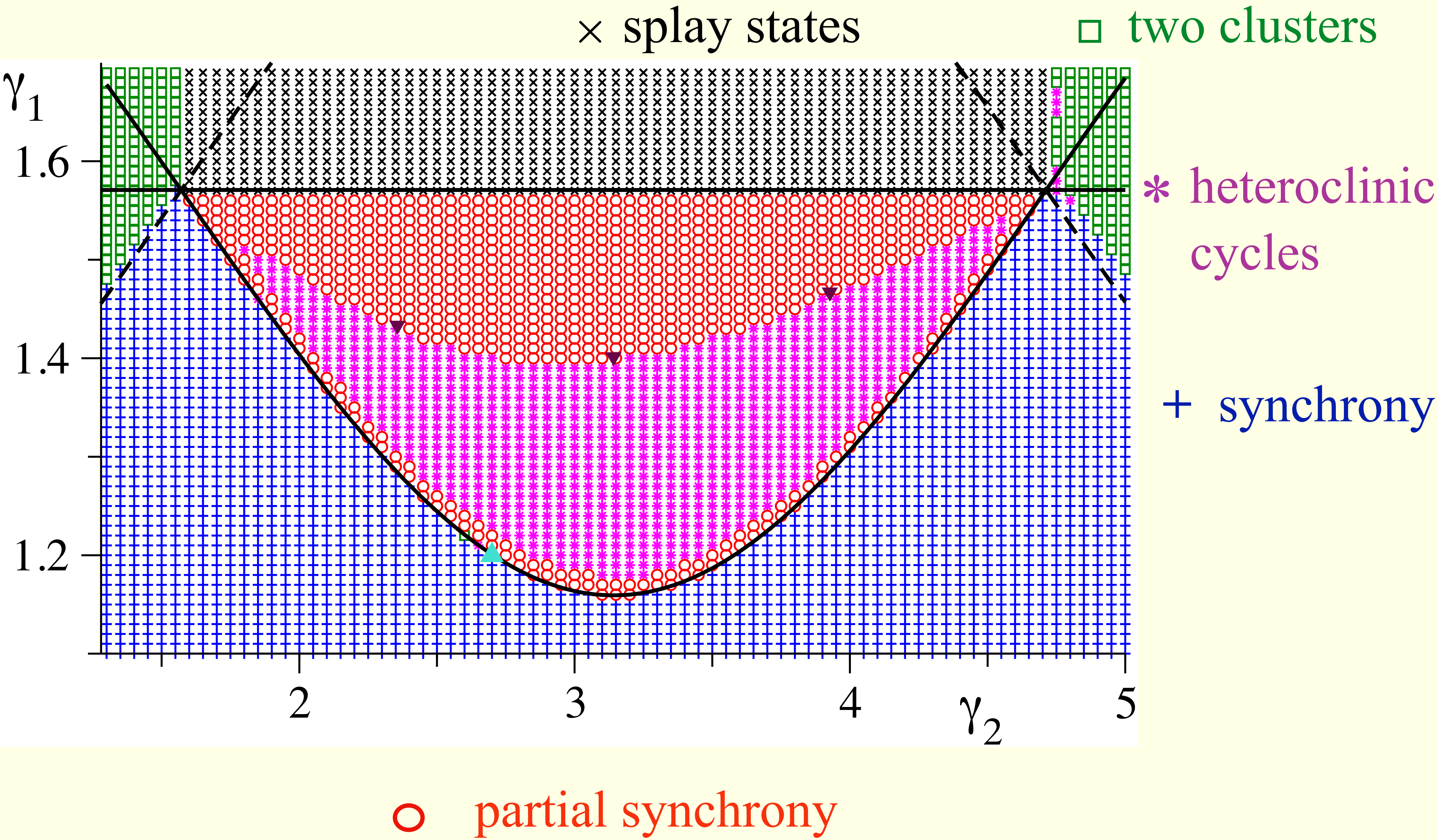

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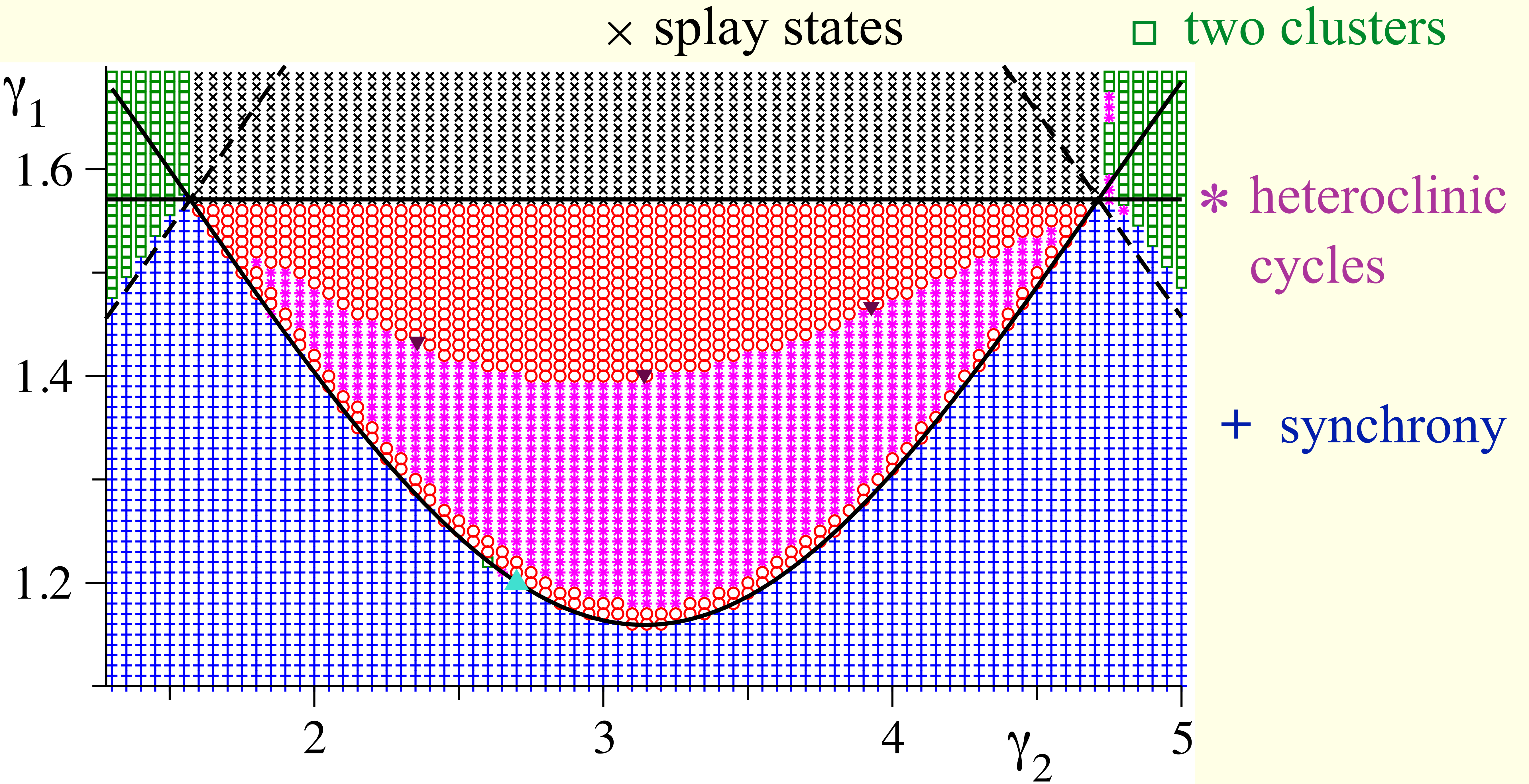
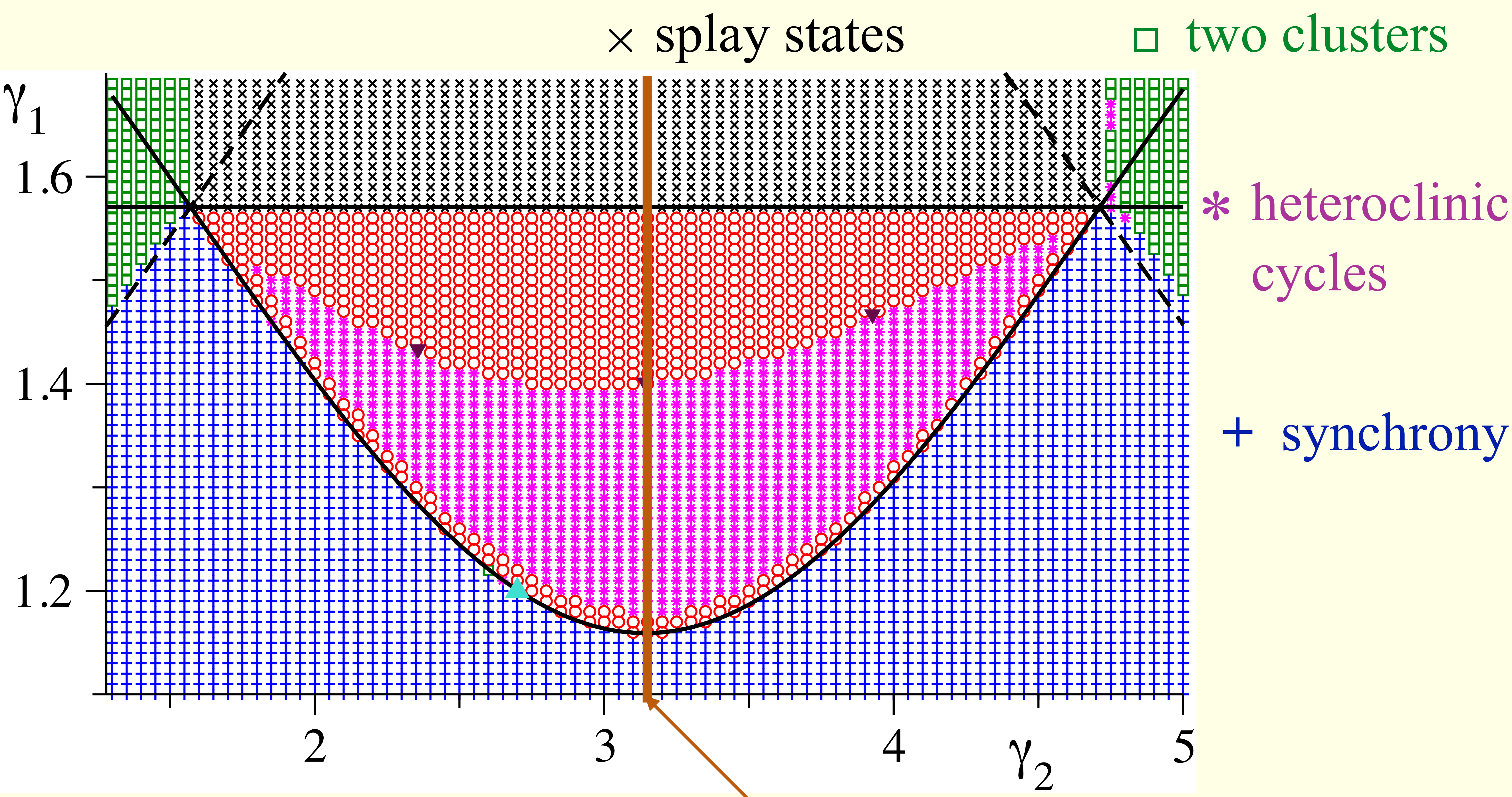


○ partial synchrony

Initial conditions: perturbed splay state!

The biharmonic model: numerics

domain, studied numerically

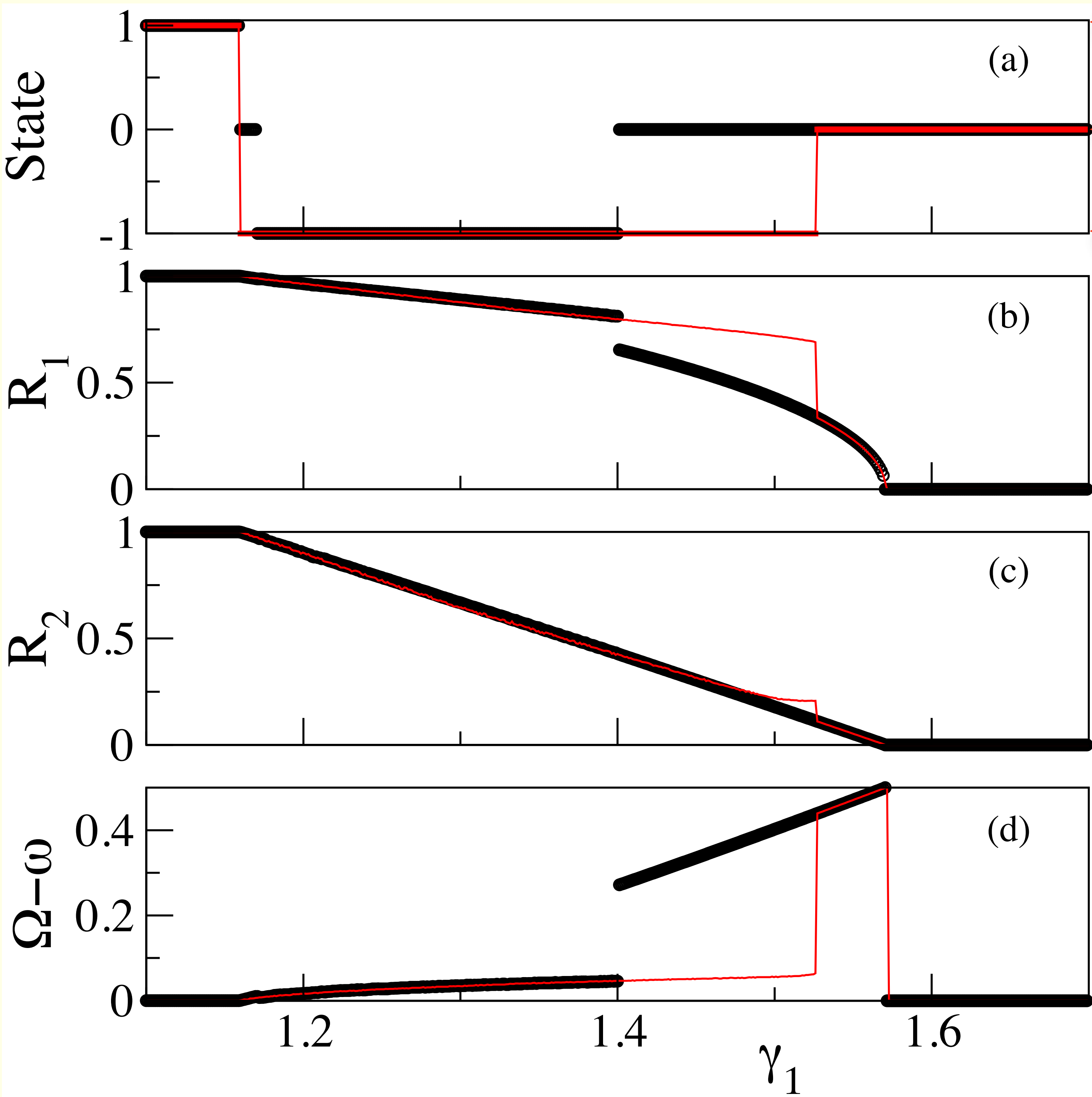


we go along this line $\gamma_2 = \pi$

○ partial synchrony

Different initial conditions!

The biharmonic model: multistability



synchrony

partial synchrony

heteroclinic cycle

red lines: slow increase of γ_1

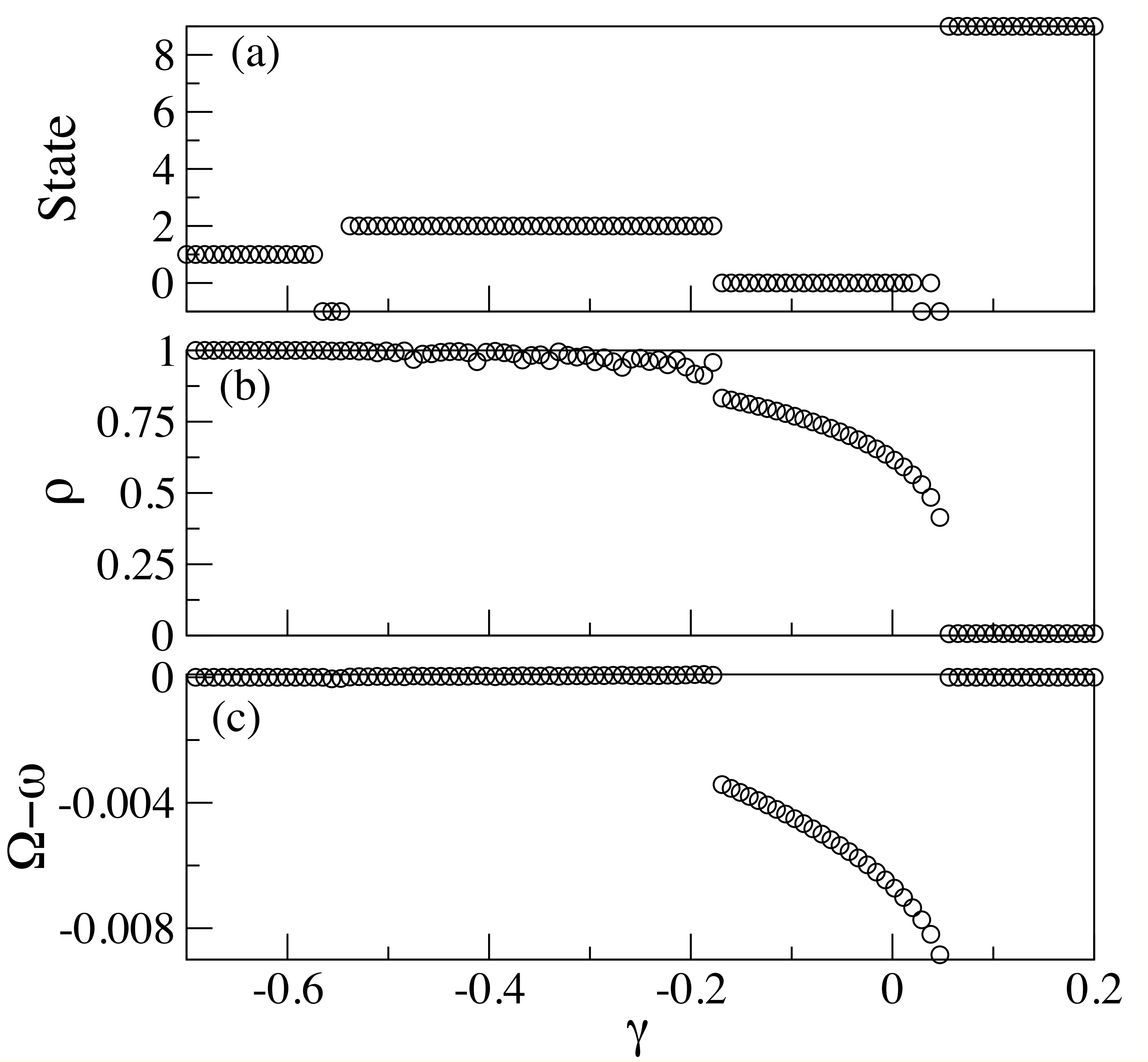
Black symbols:

perturbed splay initial conditions and slow decrease of γ_1

Rayleigh oscillators

$$\ddot{x}_k - \xi(1-\dot{x}_k^2)\dot{x}_k + x_k = \varepsilon \mathrm{Re}\left[e^{i\gamma}(X+iY)\right]$$

mean fields $X = N^{-1}\sum_k x_k, \quad Y = N^{-1}\sum_k \dot{x}_k$



State = number of clusters

(State = 0: intermediate, unclassified states)

Order parameter

$\rho = \mathrm{rms}(X)/\mathrm{rms}(x)$

Rayleigh oscillators

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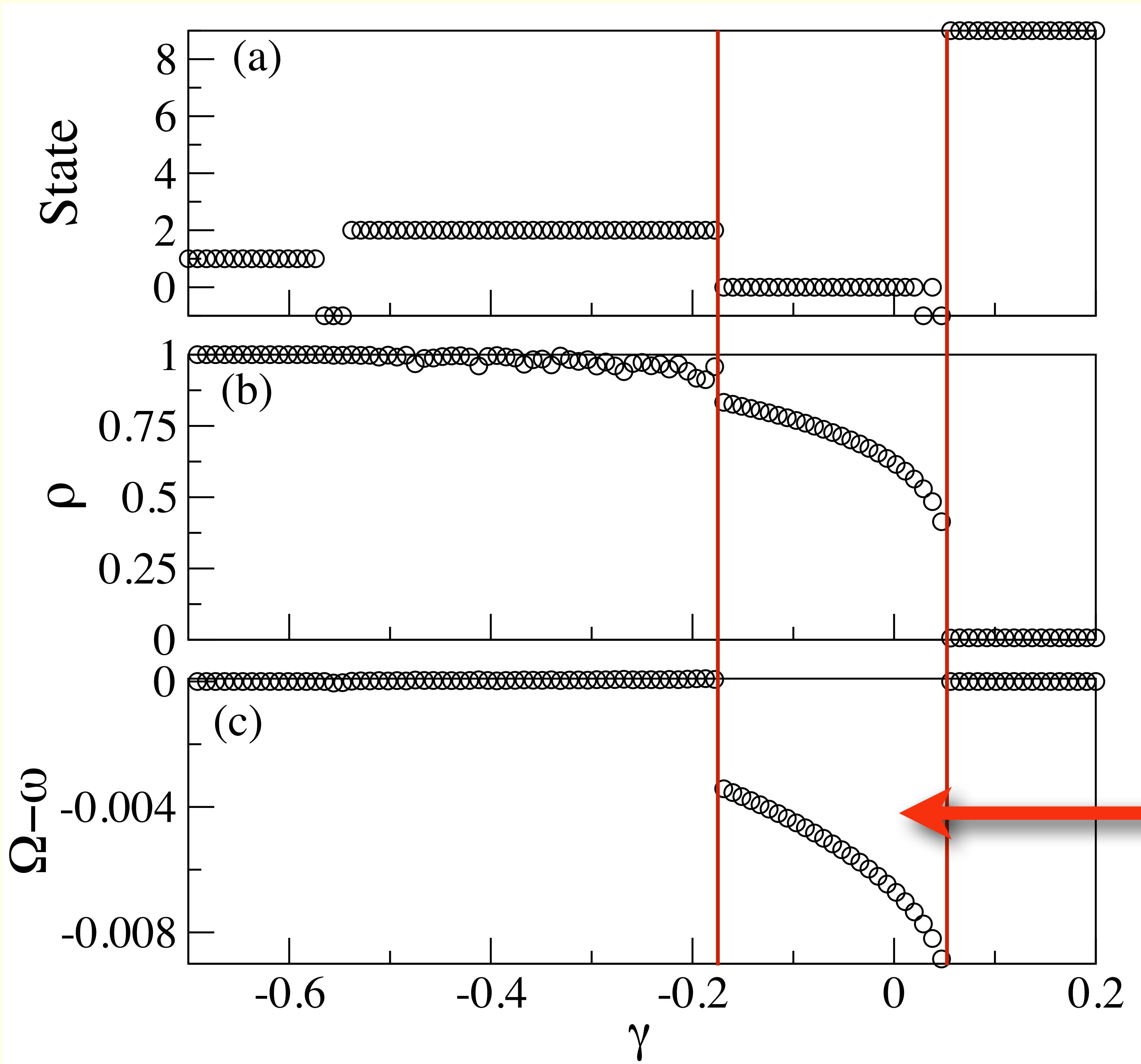

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quasiperiodic partial synchrony

General case:
When do we expect complex solutions?

1) tendency to synchrony is not monotonic and/or

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chimeras

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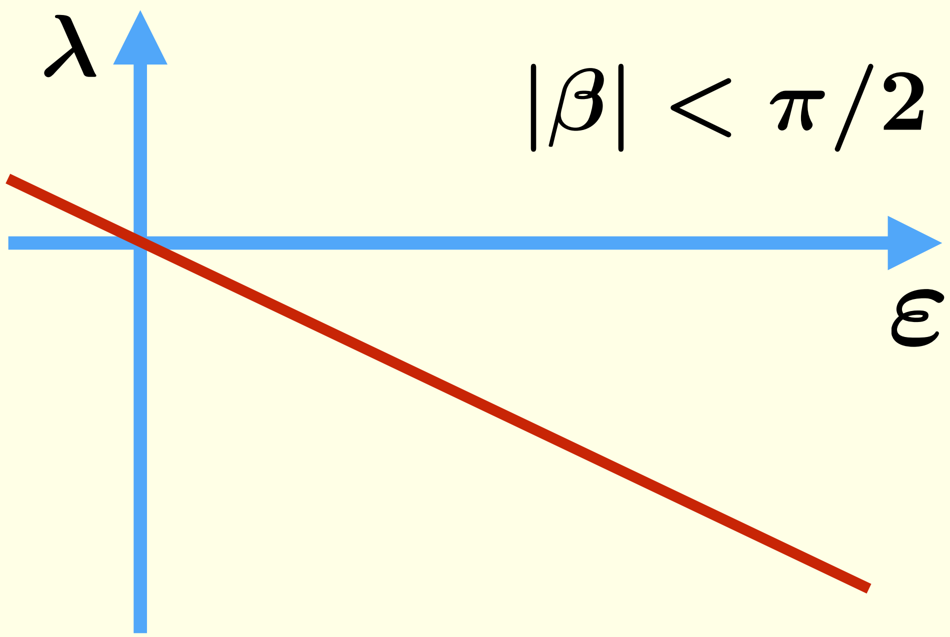
eigenvalue



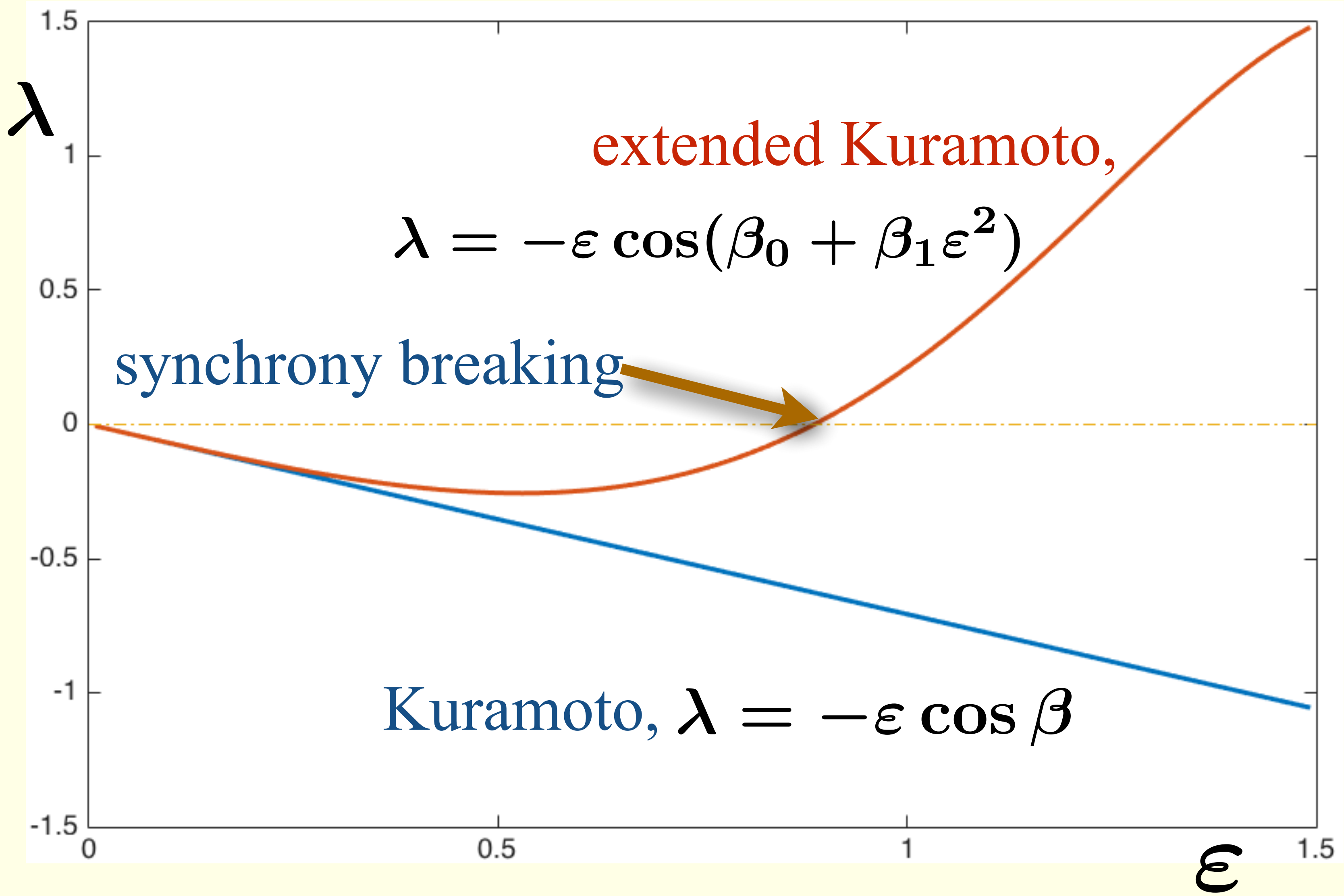
For this model: stability is proportional to coupling

==> tendency to synchrony increases with ε

Linear vs nonlinear coupling

Extended Kuramoto-Sakagichi model (particular case):

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta_0 + \beta_1 \varepsilon^2 R^2)$$



nonlinear coupling

linear coupling

Partial synchrony and quasiperiodic dynamics after synchrony breaking

A solvable model for Quasiperiodic Partial Synchrony

Nonlinearly coupled Stuart-Landau oscillators:

$$\dot{a}_k = (1 + i\omega_0)a_k - (1 + i\kappa)|a_k|^2 a_k$$
$$+(\varepsilon_1 + i\varepsilon_2)A - (\eta_1 + i\eta_2)|A|^2 A \,,$$

complex mean field:

$$A = N^{-1} \sum_j a_j$$

linear and nonlinear mean field coupling

The solvable model: phase approximation

Nonlinearly coupled Stuart-Landau oscillators:

$$\dot{a}_k = (1 + i\omega_0)a_k - (1 + i\kappa)|a_k|^2 a_k + (\varepsilon_1 + i\varepsilon_2)A - (\eta_1 + i\eta_2)|A|^2 A \,,$$

complex mean field:

$$A = N^{-1}\sum_j a_j$$

linear and nonlinear mean field coupling

Phase approximation: nonlinear Kuramoto-Sakagichi model

$$\boxed{\dot{\varphi}_k = \omega + \mathcal{E}(R;\varepsilon_{1,2},\eta_{1,2})R\sin[\Theta - \varphi_k + \beta(R;\varepsilon_{1,2},\eta_{1,2})]}$$

A solvable particular case:

$$\boxed{\dot{\varphi}_k = \omega + \varepsilon R\sin(\Theta - \varphi_k + \beta_0 + \beta_1\varepsilon^2 R^2)}$$

The solvable model: beyond phase approximation

Nonlinearly coupled Stuart-Landau oscillators:

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$$+(\varepsilon_1 + i\varepsilon_2)A - (\eta_1 + i\eta_2)|A|^2 A \,,$$

complex mean field:

$$A = N^{-1} \sum_j a_j$$

linear and nonlinear mean field coupling

Stability of the synchronous state

$$a_1 = a_2 = \ldots = a_N = re^{i\varphi} = A \quad \text{with}$$

$$r^2 = \frac{1 + \varepsilon_1}{1 + \eta_1} \quad \text{and} \quad \dot{\varphi} = \Omega = \omega_0 + \varepsilon_2 - \frac{(\kappa + \eta_2)(1 + \varepsilon_1)}{1 + \eta_1}$$

The solvable model: beyond phase approximation

Stability of the synchronous state

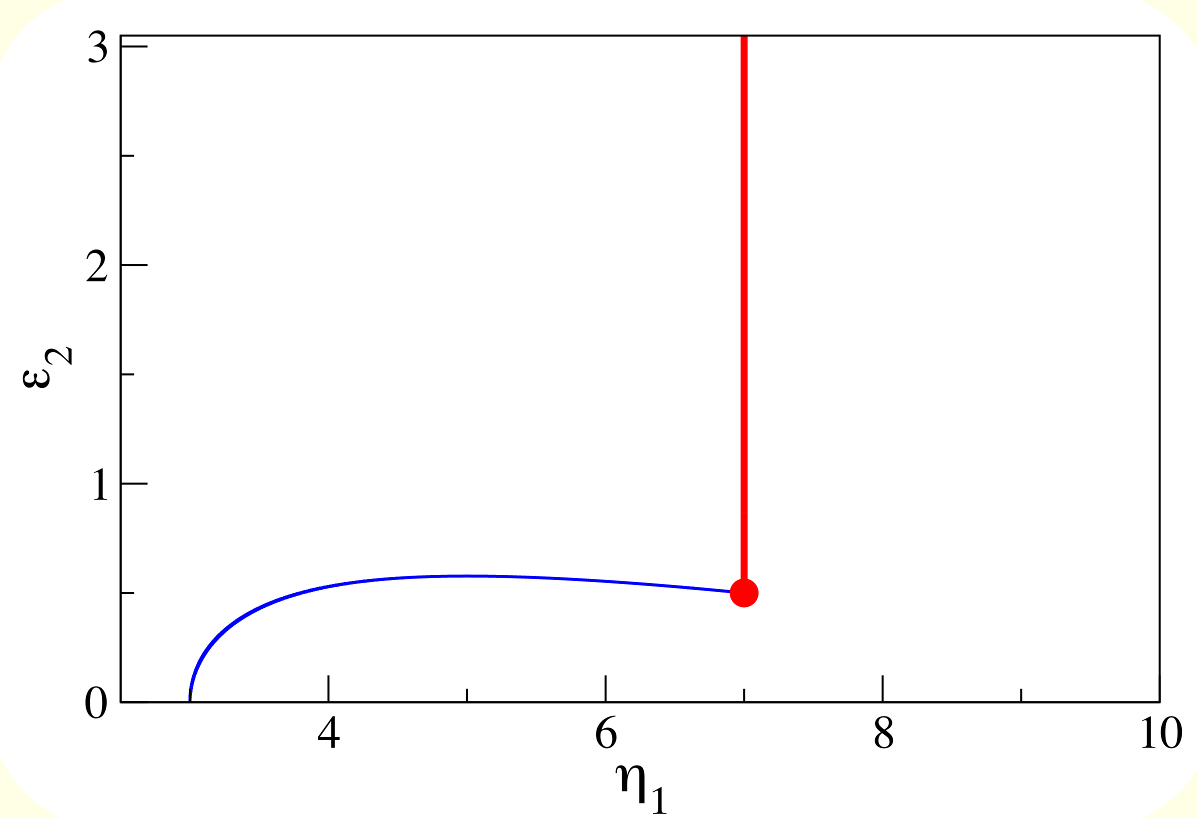
$$a_1 = a_2 = \ldots = a_N = re^{i\varphi} = A$$

Eigenvalues:

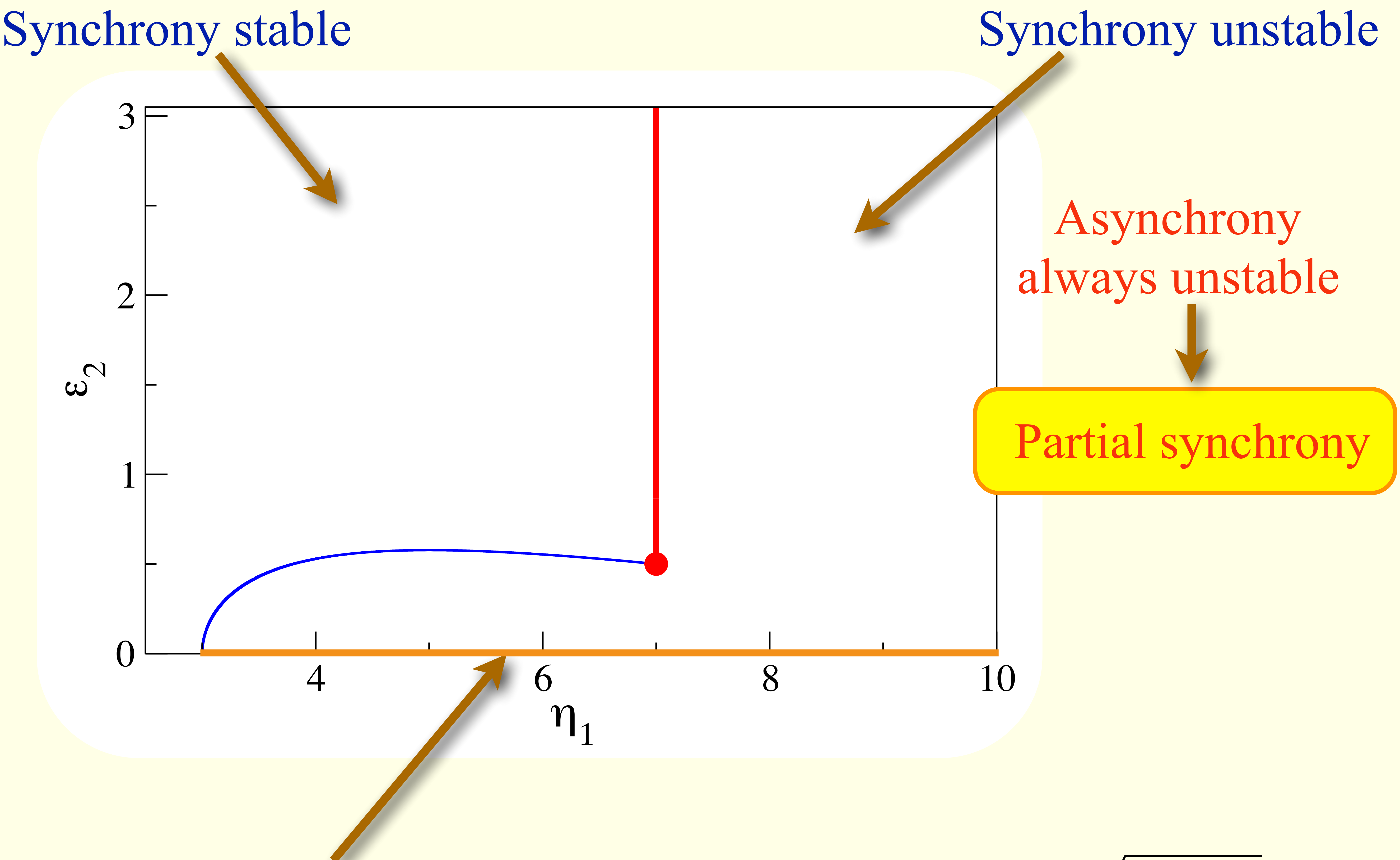
$$\lambda_{1,2} = (1 - 2r^2) \pm \sqrt{(1 - 3\kappa^2)r^4 + 4(\omega_0 - \Omega)\kappa r^2 - (\omega_0 - \Omega)^2}$$

A special case: $\kappa = 0, \eta_2 = 0, \varepsilon_1 = 3, \varepsilon_2 \geq 0$

$$\longrightarrow \quad \lambda_{1,2} = (1 - 2r^2) \pm \sqrt{r^4 - \varepsilon^2}$$



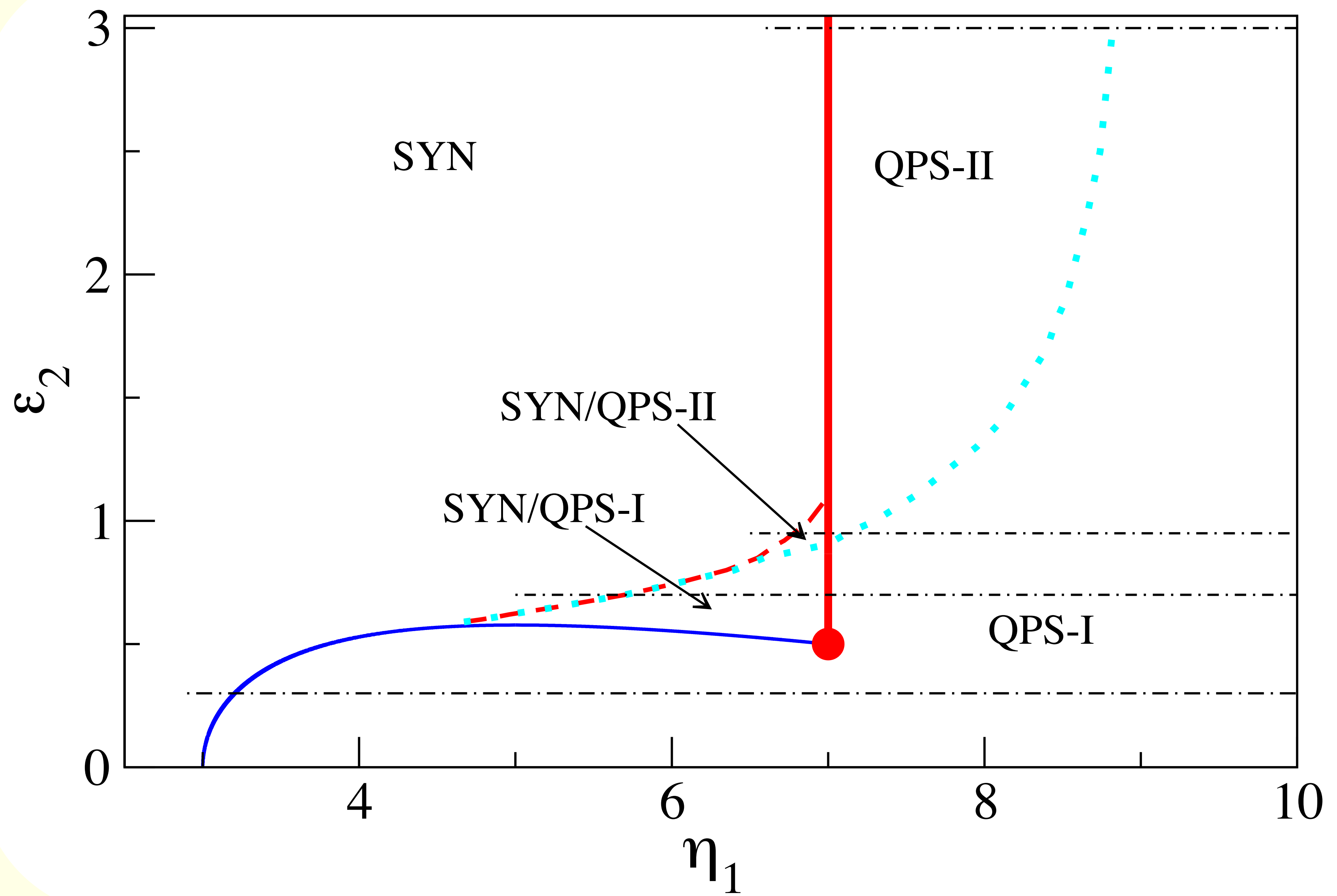
Stability diagram



Neutrally stable bunch state, $r = 1, \Omega = \omega_0, R = \sqrt{\varepsilon_1/\eta_1}$

Numerics

Mean field frequency ν, oscillators frequency $\Omega = \langle \dot{\varphi} \rangle$



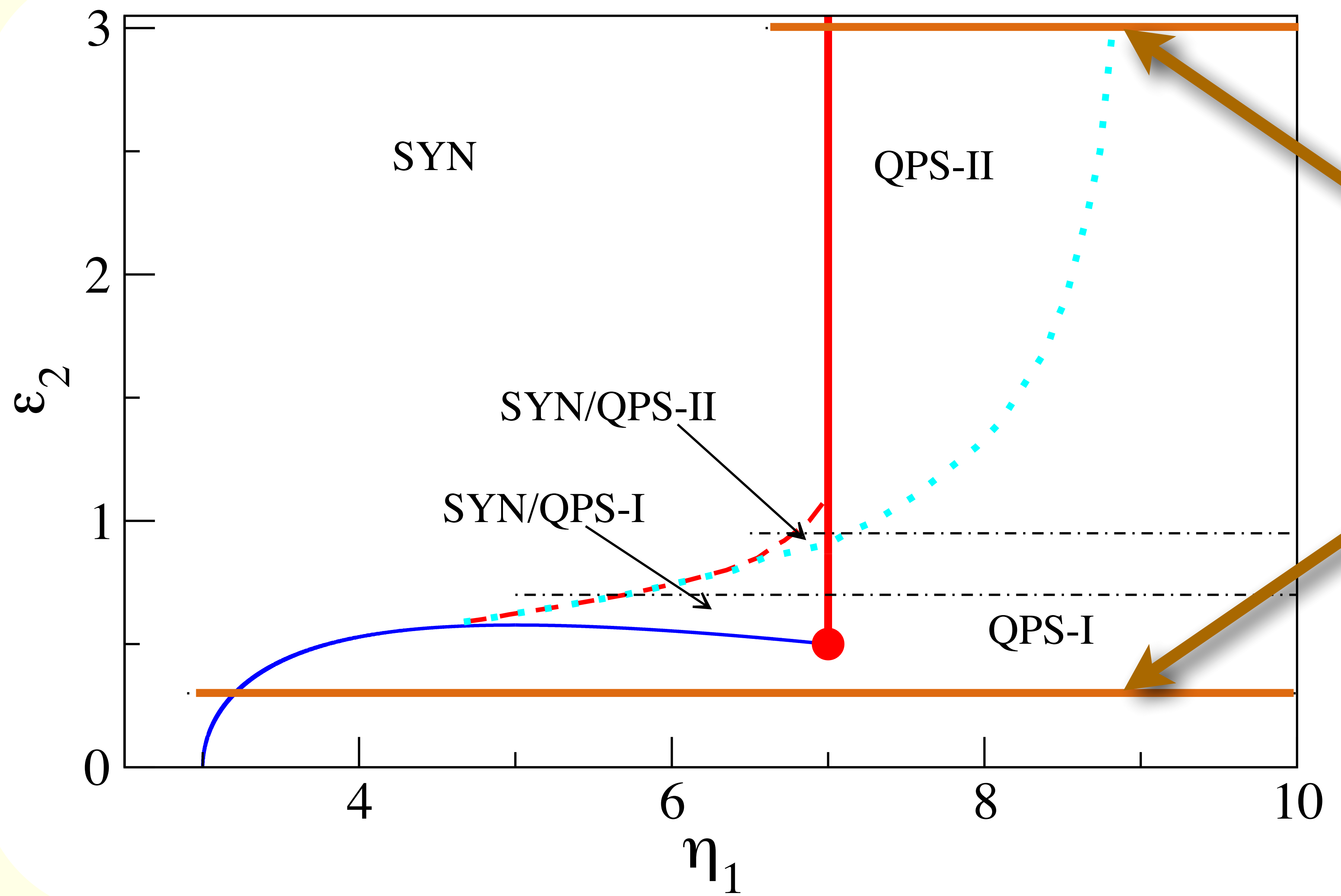
Quasiperiodic partial synchrony type I (QPS-I): $\nu \neq \Omega$

Quasiperiodic partial synchrony type II (QPS-II): $\nu = \Omega$,

quasiperiodicity due to **amplitude modulation**

Numerics

Mean field frequency ν, oscillators frequency $\Omega = \langle \dot{\varphi} \rangle$



$\varepsilon_2 = 3$

$\varepsilon_2 = 0.3$

Quasiperiodic partial synchrony type I (QPS-I): $\nu \neq \Omega$

Quasiperiodic partial synchrony type II (QPS-II): $\nu = \Omega$,

quasiperiodicity due to **amplitude modulation**

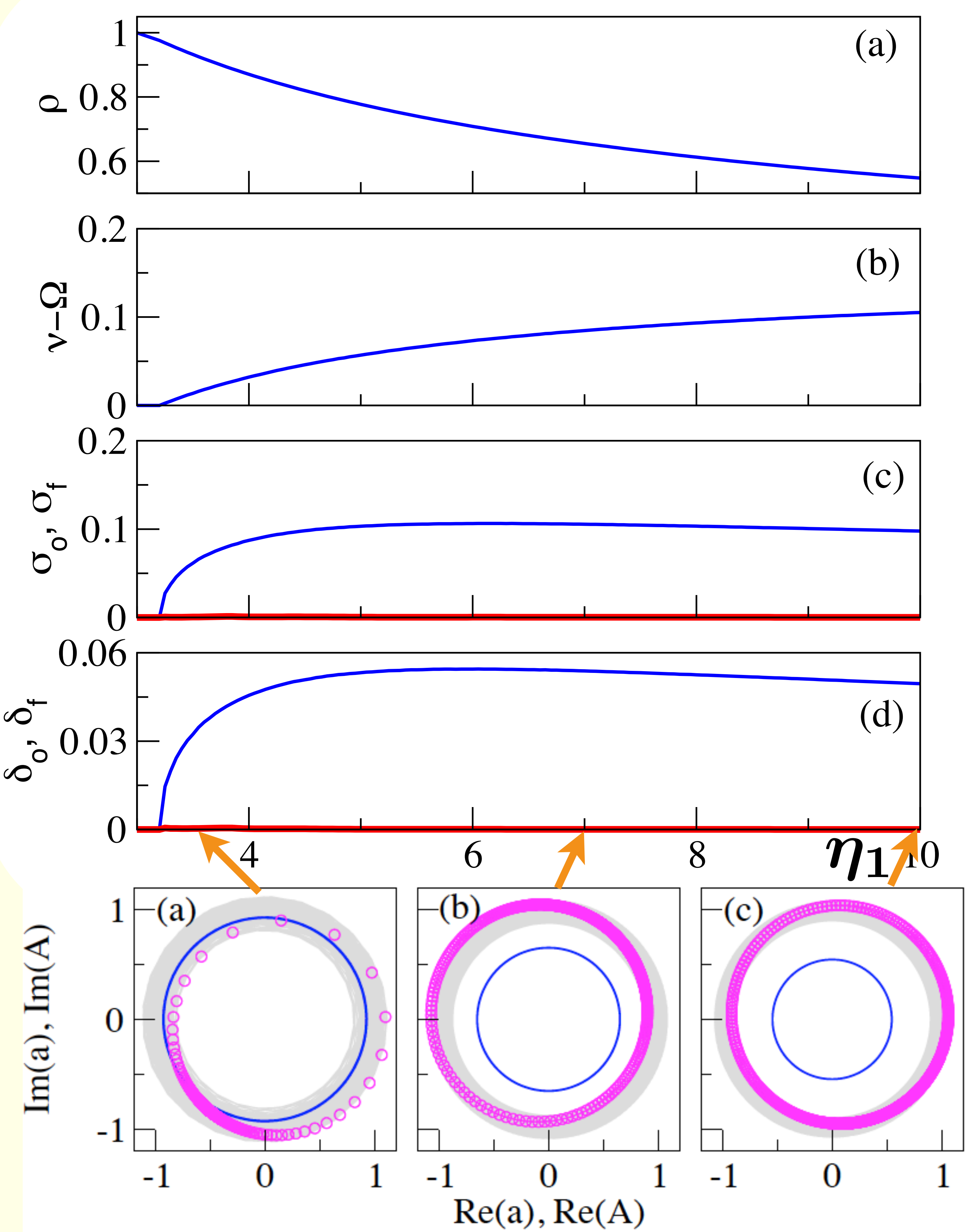
Transition for small $\varepsilon_2 = 0.3$

mean field amplitude

frequency difference

std of instantaneous frequency
(red: field, blue: oscillators)

std of instantaneous amplitude



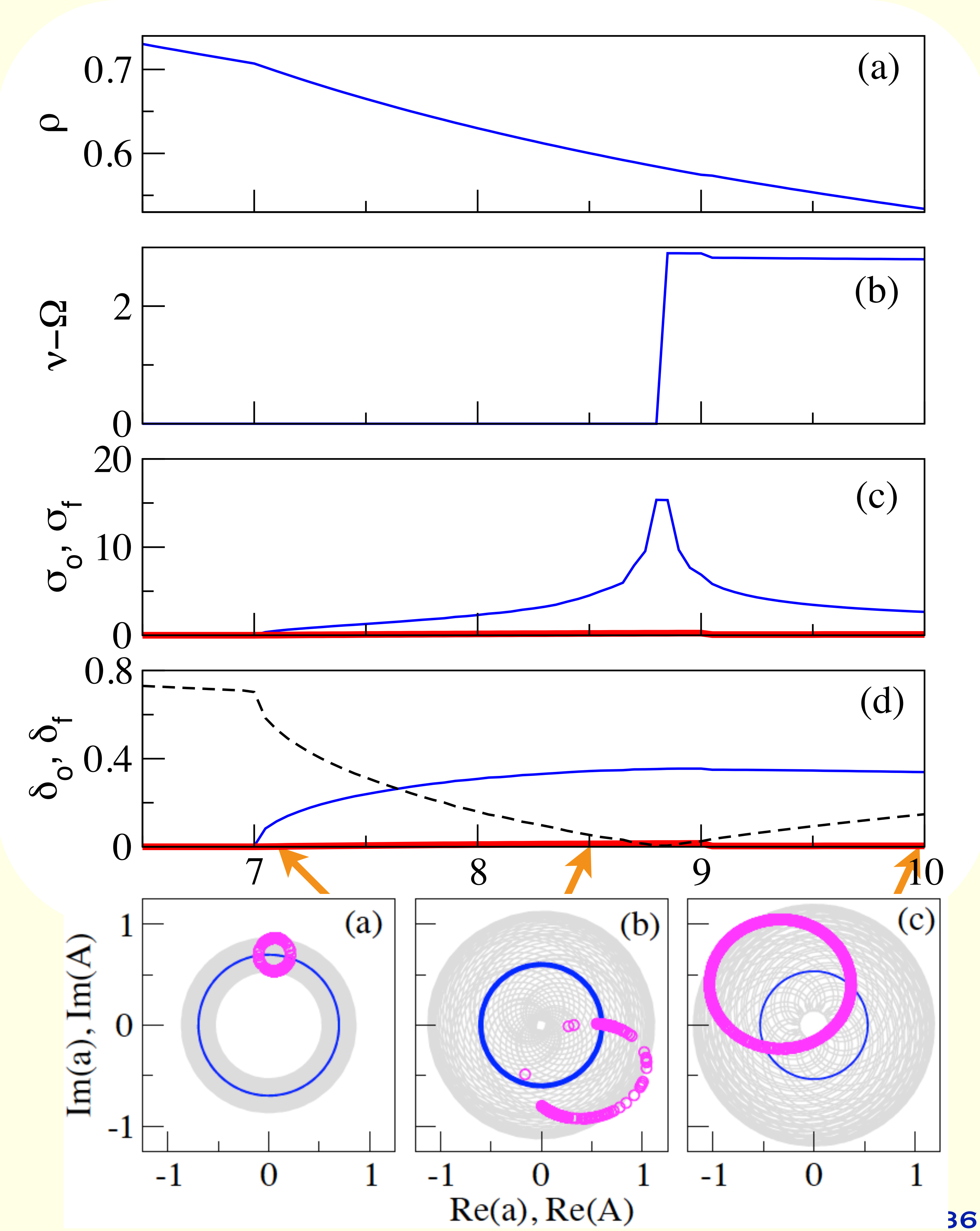
Transition for large $\varepsilon_2 = 3$

mean field amplitude

frequency difference

std of instantaneous frequency (red: field, blue: oscillators)

std of instantaneous amplitude



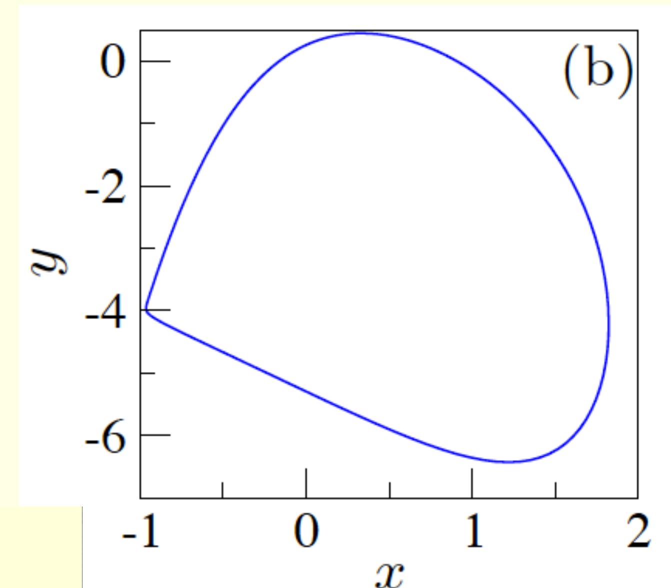
Globally coupled Hindmarsh-Rose neurons

$$\dot{x}_k = y_k - x_k^3 + 3x_k^2 - z_k + 5 + \varepsilon(X - x_k)$$

$$\dot{y}_k = 1 - 5x_k^2 - y_k$$

$$\dot{z}_k = 0.006\,[4(x_k + 1.56) - z_k]$$

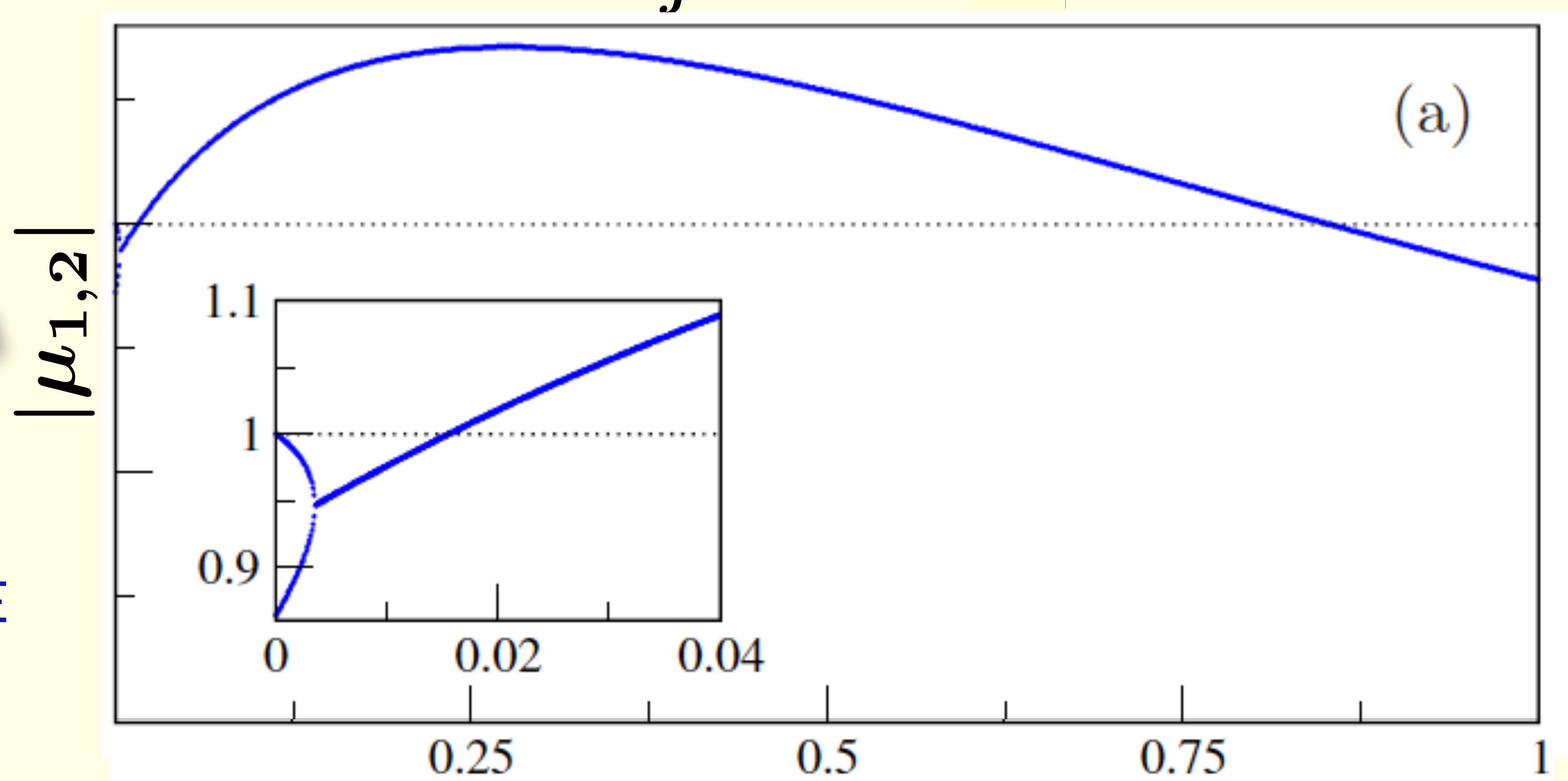
where mean field $X = N^{-1}\sum_{j}^{N} x_j$



multipliers

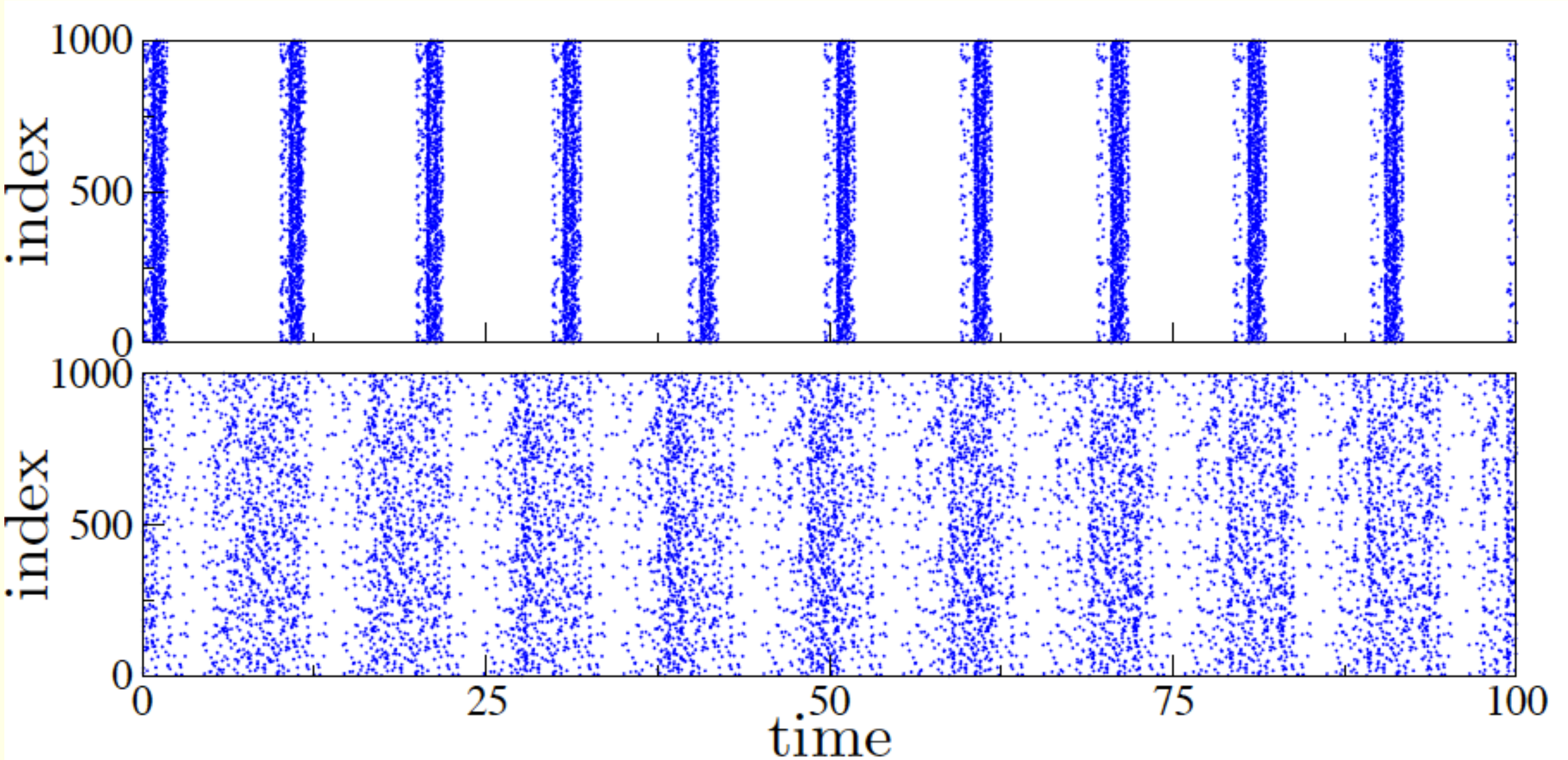
$$|\mu| = e^{\lambda T}$$

transversal LE



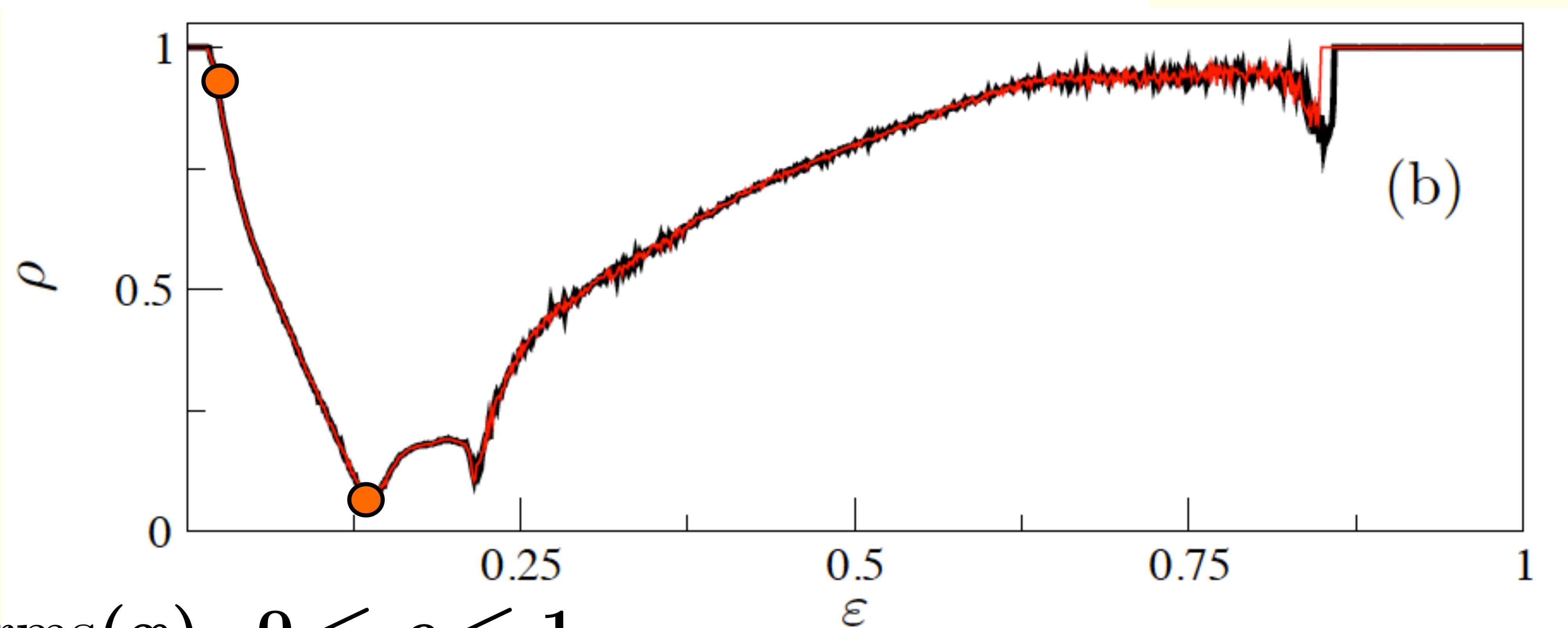
ε

Globally coupled Hindmarsh-Rose neurons: results II



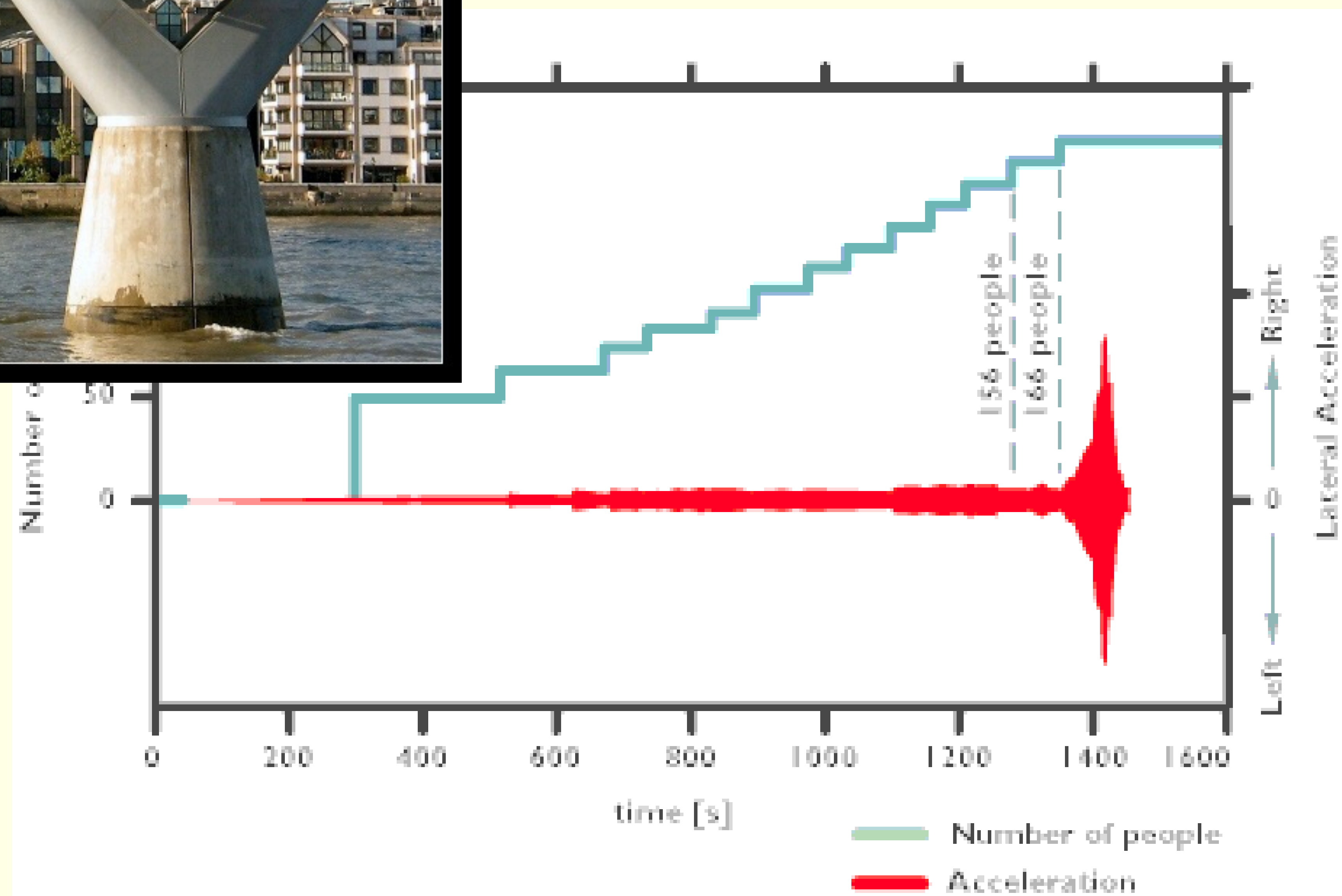
$\varepsilon = \mathbf{0.04}$

$\varepsilon = \mathbf{0.13}$



$$\boldsymbol{\rho} = \mathrm{rms}(\boldsymbol{X})/\mathrm{rms}(\boldsymbol{x}) \;, \mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1}$$

Illustrative example of collective synchrony: The Millennium Bridge



Bridge vibrations without synchrony

Tuesday, May 23

MS84

Oscillators and Networks with Switching Parameters

8:30 AM-10:30 AM

Room: Wasatch B

8:30-8:55 Bistable Gaits and Wobbling Induced by Pedestrian-Bridge Interactions

Igor Belykh and Russell Jeter, Georgia State University, USA; Vladimir Belykh, Lobachevsky State University of Nizhny Novgorod, Russia

CHAOS **26**, 116314 (2016)

Bistable gaits and wobbling induced by pedestrian-bridge interactions

Igor V. Belykh,[1] Russell Jeter,[1] and Vladimir N. Belykh[2,3]

Our results on the ability of a single pedestrian to initiate bridge wobbling when switching from one gait to another may give an additional insight into the initiation of wobbling without crowd synchrony as previously observed on the Singapore Airport's Changi Mezzanine Bridge[15] and the Clifton Suspension Bridge.[19] Both bridges wobbled during a crowd event; however, the averaged frequency of pedestrians' gaits was documented to be different from the bridge frequency, and the pedestrian walking showed no visible signs of synchrony.[29]

Bridge vibrations without synchrony

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[29] J. H. G. Macdonald, Proc. R. Soc. A **465**, 1055 (2009).

An example of quasiperiodic partial synchrony?

Qualitative discussion: frequency difference

In the partially synchronous state $R < 1$

Watanabe-Strogatz theory: cluster states are not possible

Hence, all phases are different

Hence, instantaneous frequencies

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta_0 + \beta_1 \varepsilon^2 R^2)$$

are all different as well

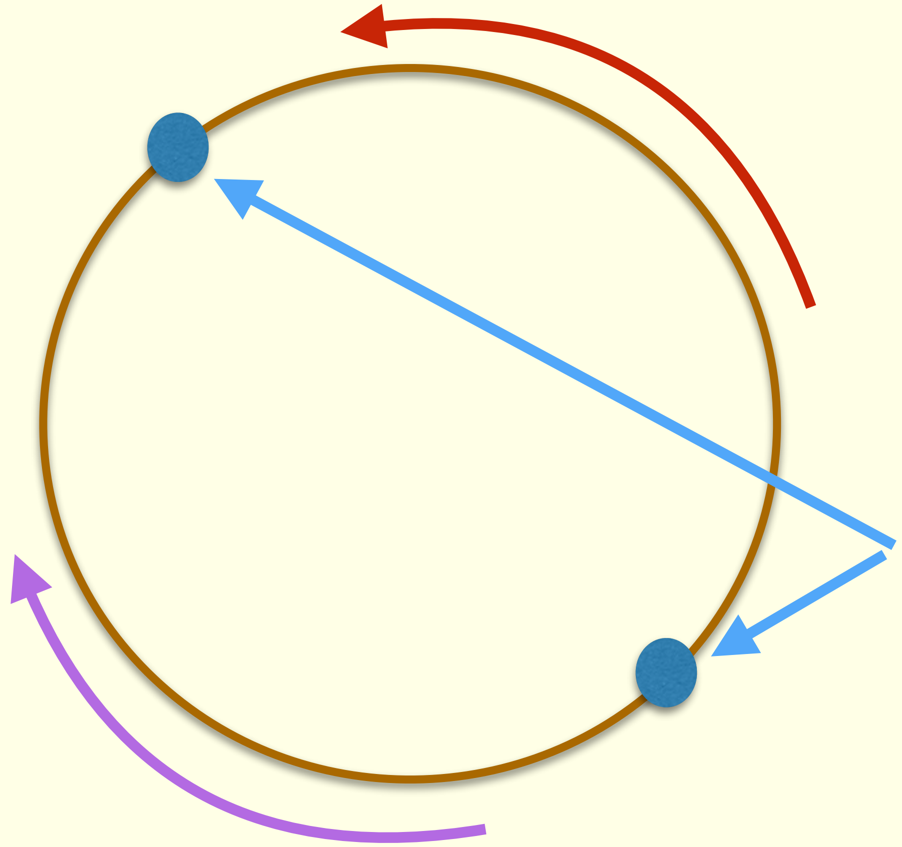
We denote: $\langle \dot{\phi} \rangle = \Omega$, $\langle \dot{\Theta} \rangle = \nu$

oscillator frequency mean field frequency

We argue that $\Omega \neq \nu$

Qualitative discussion: frequency difference II

Suppose the contrary, $\Omega = \nu$, and consider the motion in the frame, rotating with the mean field



Then, depending on their phase, some oscillators are faster than mean field, and some are slower

There must be points where relative velocity is zero $\longrightarrow$ the points should cluster

Clusters are not possible, hence oscillators are either always faster, or always slower than the mean field, thus $\boxed{\Omega \neq \nu}$

Theory: $\Omega = \omega + \frac{\varepsilon_{crit}^2}{\varepsilon}, \quad \nu = \omega + \frac{\varepsilon^2 + \varepsilon_{crit}^2}{2\varepsilon}$

Quasiperiodic dynamics

Conclusions

- Partially synchronous quasiperiodic dynamics appears at the border of stability of the synchronous state
- It appears in phase and full models, also for (weakly) inhomogeneous ensembles
- Further examples: van Vreeswijk model of coupled leaky integrate-and-fire neurons, electronic circuits (experiment)
- At least two non-trivial forms of quasiperiodicity
- Exact conditions for emergence of these states is not yet clear

References

1) P. Clusella, A. Politi, M. Rosenblum, *A minimal model of self-consistent partial synchrony*, New J. Physics 18 (2016) 093037
2) M. Rosenblum and A. Pikovsky, *Two types of quasiperiodic partial synchrony in oscillator ensembles*, PRE, **92,** 012919, 2015

Acknowledgements

Thanks to my co-authors: **P. Clusella, A. Pikovsky, A. Politi**

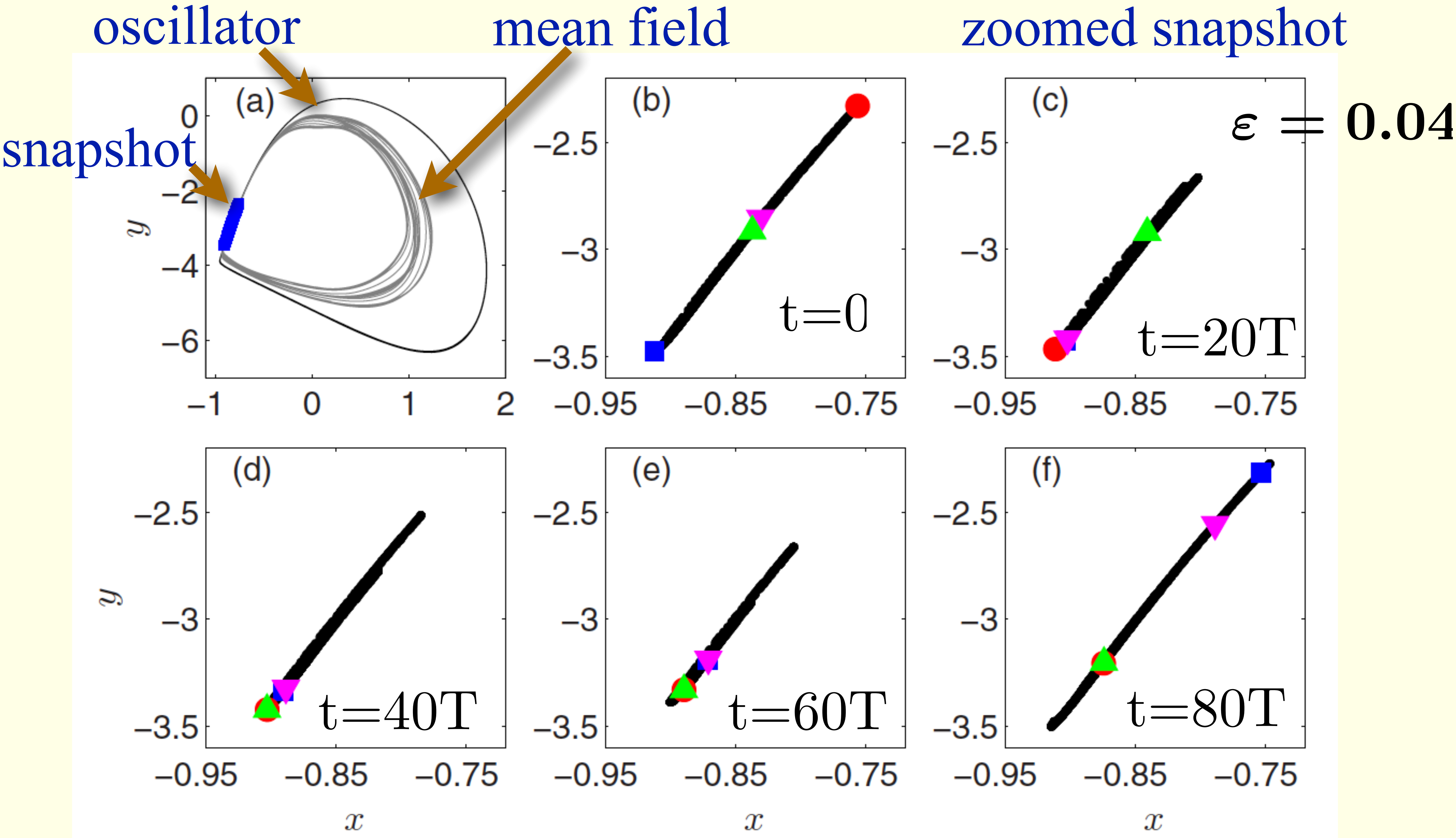
COSMOS

Complex Oscillatory Systems: Modeling and Analysis

Innovative Training Network

European Joint Doctorate

Thank you for your attention!



Different scenario of synchrony breaking (Hopf-like),
another type of quasiperiodic partial synchrony