Delay Parameter Selection in Permutation Entropy Using Topological Data Analysis

Audun Myers and Firas Khasawneh

Department of Mechanical Engineering

May 22, 2019

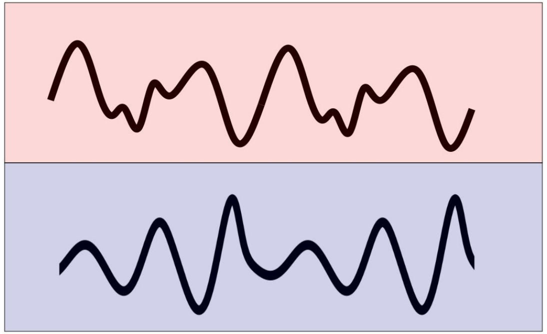
SIAM DS-19

MICHIGAN STATE UNIVERSITY

Permutation Entropy (PE) Introduction

Chaotic

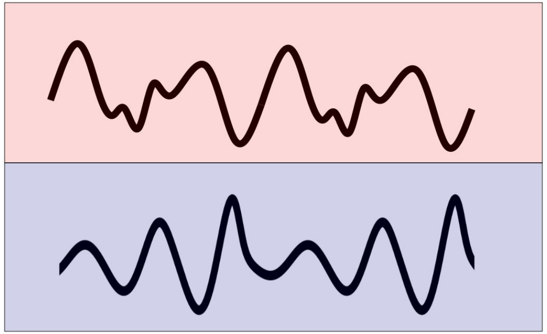
Periodic



1

1 **PE**
Example
2 **Motivation**
3 **Delay**
Freq. App.
SW1PerS
4 **Results**
5 **Summary**

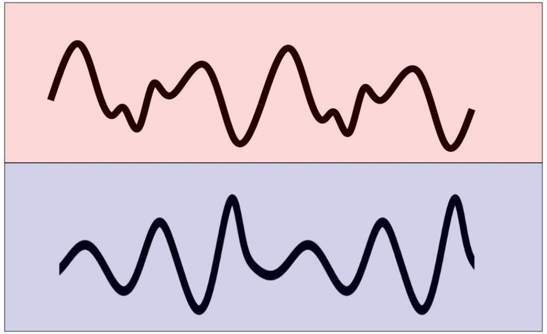
Permutation Entropy (PE) Introduction



Chaotic → Unpredictable

Periodic → Predictable

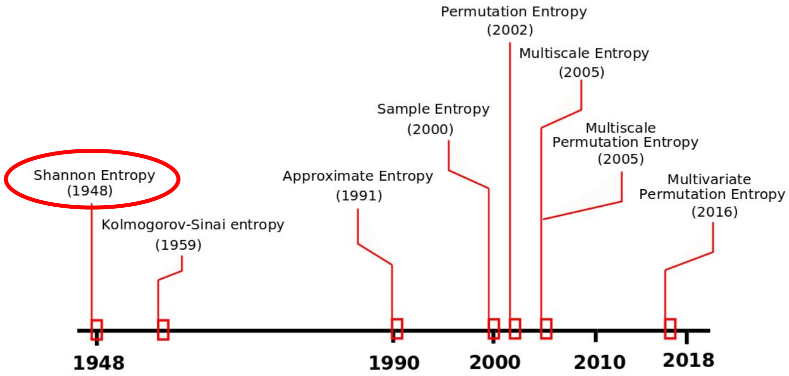
Permutation Entropy (PE) Introduction



Chaotic → Unpredictable → High Entropy

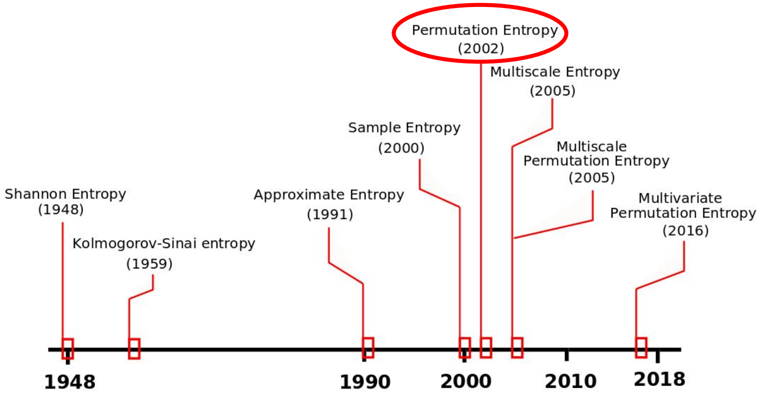
Periodic → Predictable → Low Entropy

Permutation Entropy (PE) Introduction



Claude E Shannon. *The mathematical theory of communication*. Bell System Technical Journal, **27** (3): 379–423, 1948.

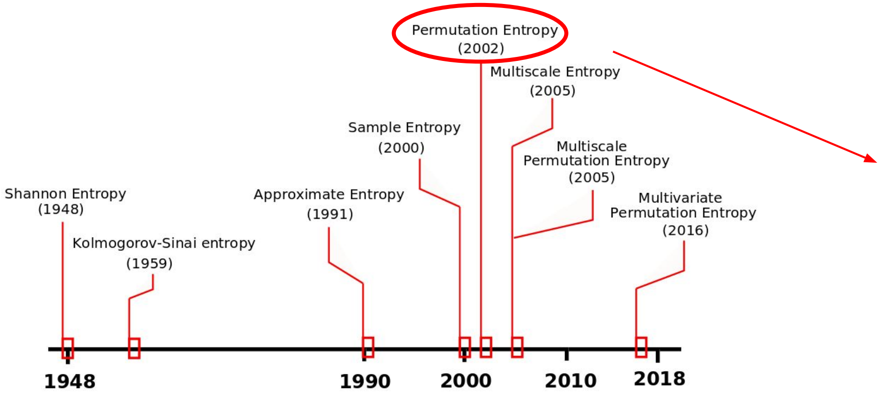
Permutation Entropy (PE) Introduction



Claude E Shannon. *The mathematical theory of communication*. Bell System Technical Journal, **27** (3): 379–423, 1948.

Christoph Bandt and Bernd Pompe. *Permutation entropy: a natural complexity measure for time series*. Physical review letters, 88(17): 174 102, 2002.

Permutation Entropy (PE) Introduction

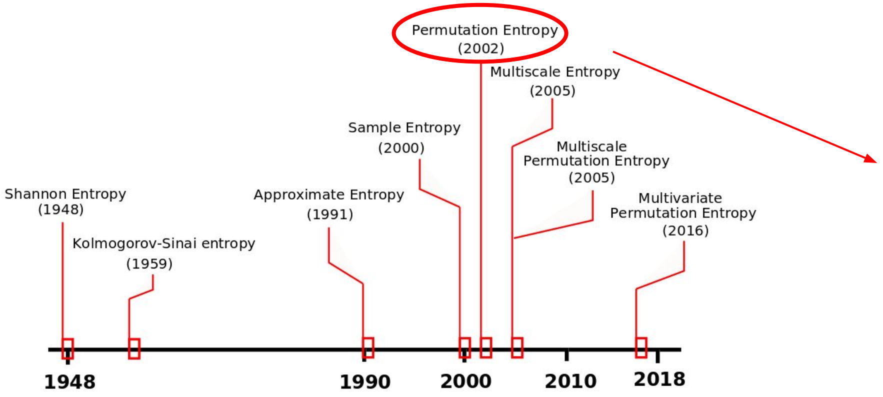


$$H(n) = -\sum p(\pi_i) \log p(\pi_i)$$

Claude E Shannon, Warren Weaver, and Arthur W Burks. The mathematical theory of communication. 1951.

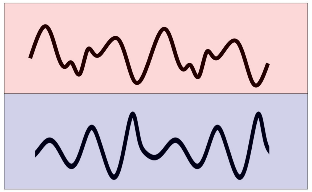
Christoph Bandt and Bernd Pompe. *Permutation entropy: a natural complexity measure for time series*. Physical review letters, 88(17): 174 102, 2002.

Permutation Entropy (PE) Introduction

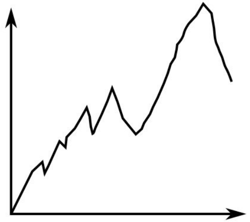


$$H(n) = -\sum p(\pi_i) \log p(\pi_i)$$

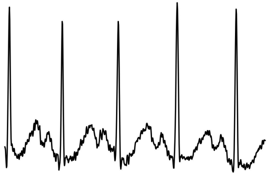
Dynamic State Detection



Financial Analysis $



Healthcare

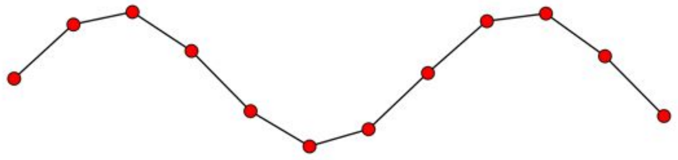


Claude E Shannon, Warren Weaver, and Arthur W Burks. The mathematical theory of communication. 1951.

Christoph Bandt and Bernd Pompe. *Permutation entropy: a natural complexity measure for time series*. Physical review letters, 88(17): 174 102, 2002.

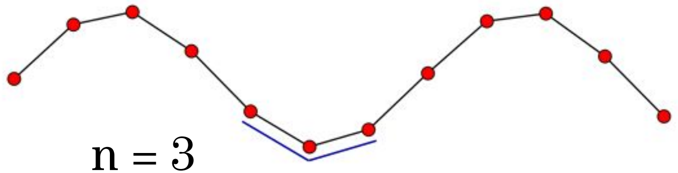
Permutation Entropy Parameters

Permutation Dimension: n



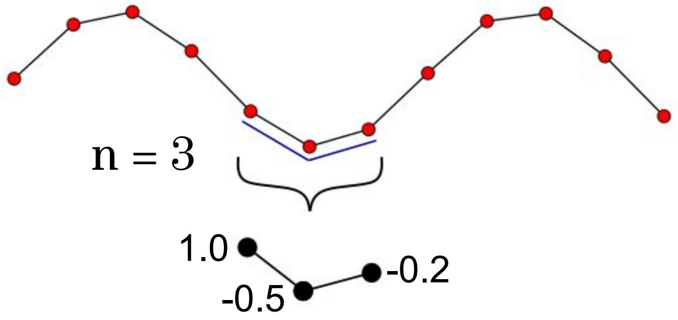
Permutation Entropy Parameters

Permutation Dimension: n



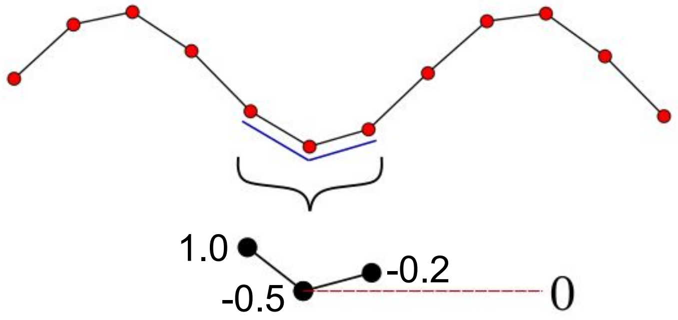
Permutation Entropy Parameters

Permutation Dimension: n



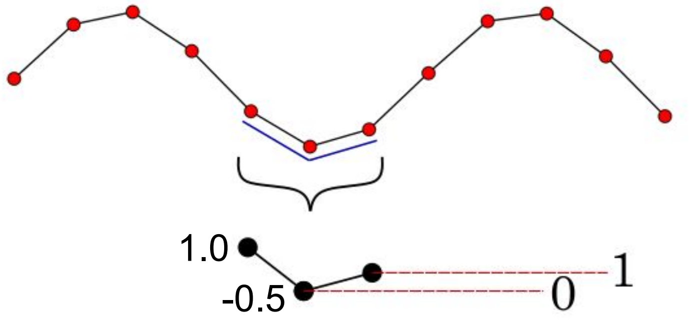
Permutation Entropy Parameters

Permutation Dimension: n



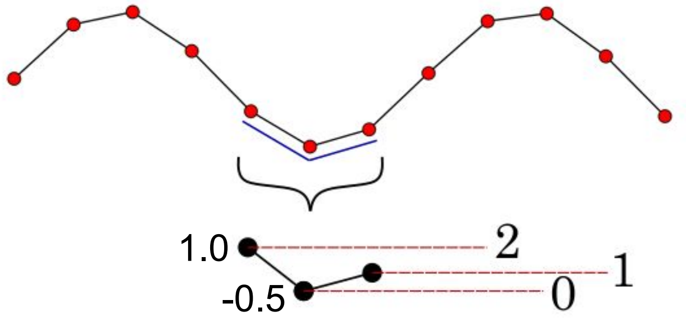
Permutation Entropy Parameters

Permutation Dimension: n



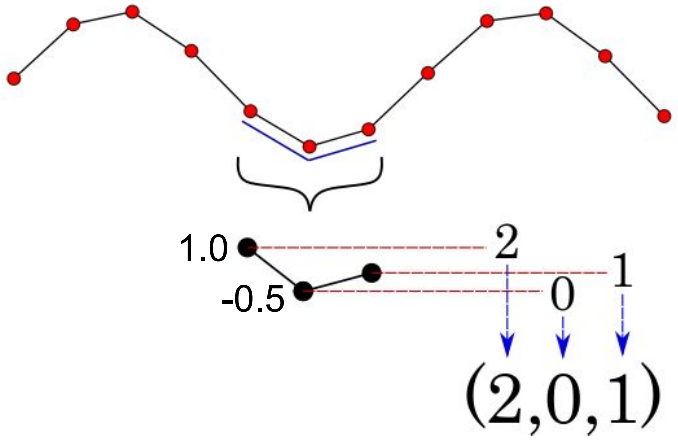
Permutation Entropy Parameters

Permutation Dimension: n



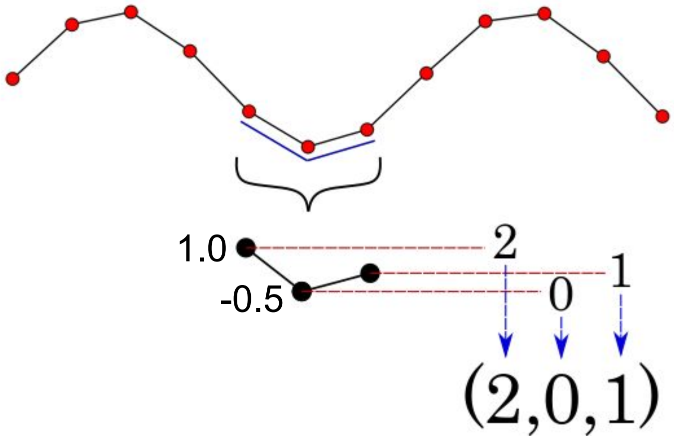
Permutation Entropy Parameters

Permutation Dimension: n



Permutation Entropy Parameters

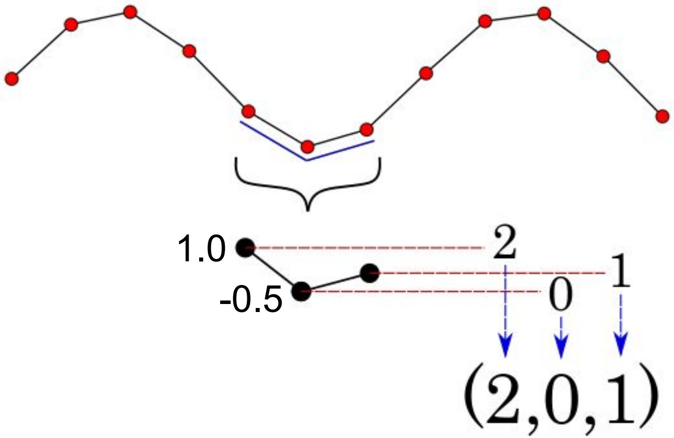
Permutation Dimension: n



Possible permutations: $n!$

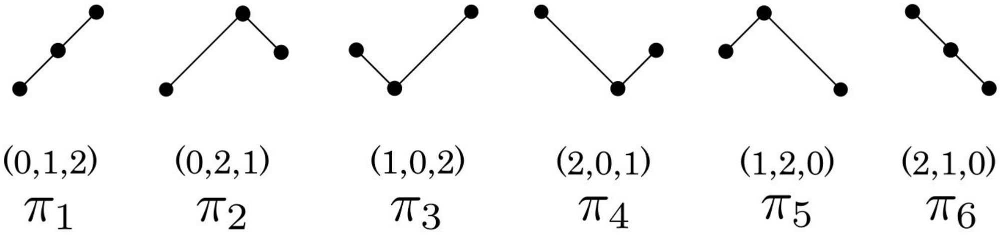
Permutation Entropy Parameters

Permutation Dimension: n



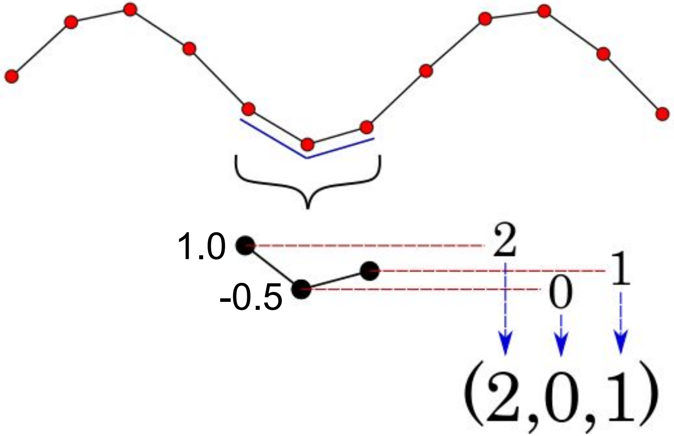
Possible permutations: $n!$

$n = 3$ →



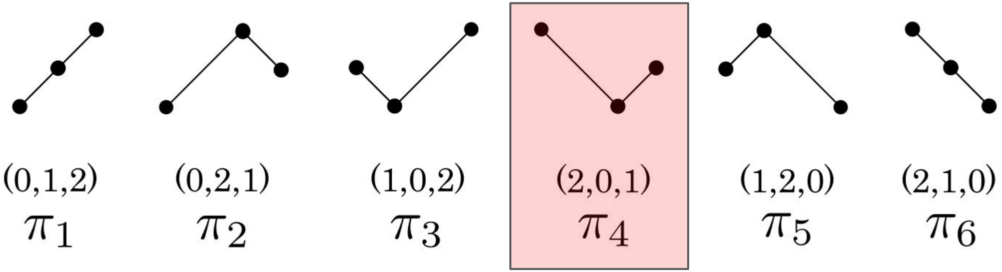
Permutation Entropy Parameters

Permutation Dimension: n



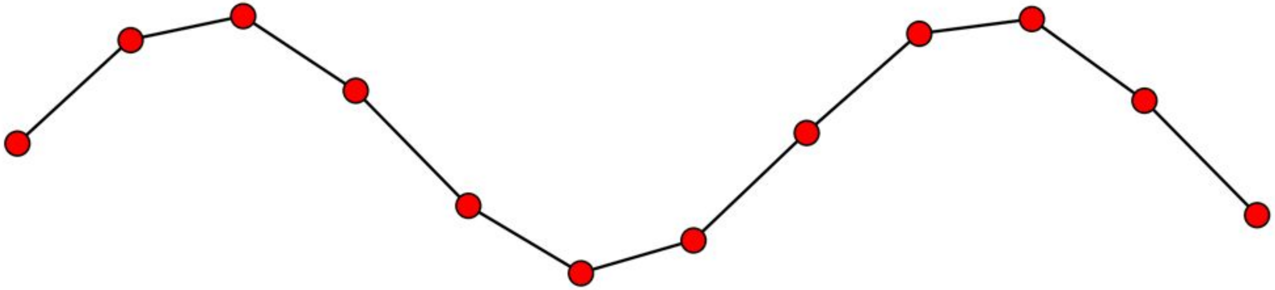
Possible permutations: $n!$

$n = 3 \longrightarrow$



Permutation Entropy Parameters

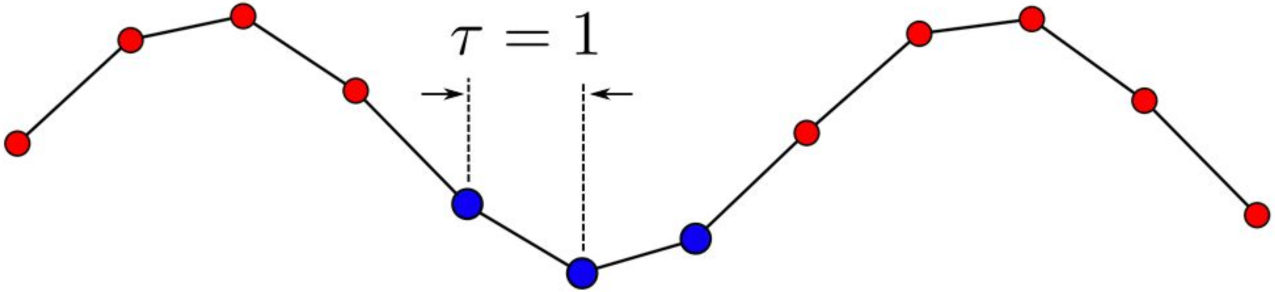
Embedding Delay: τ



Permutation Entropy Parameters

Embedding Delay: τ

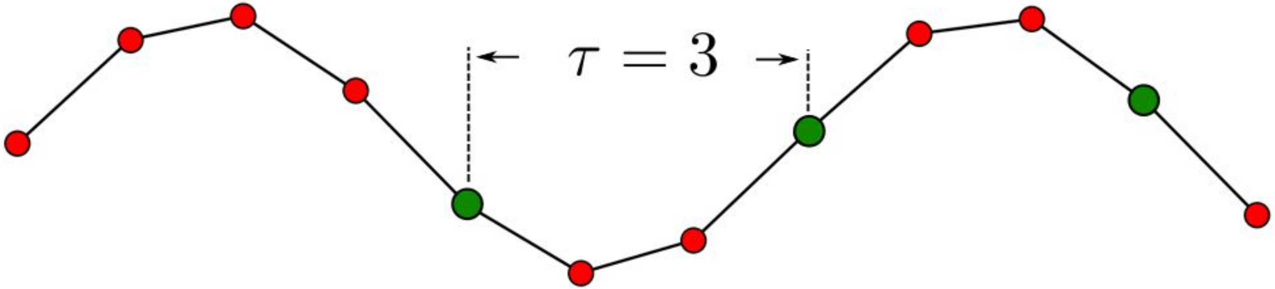
with n = 3



Permutation Entropy Parameters

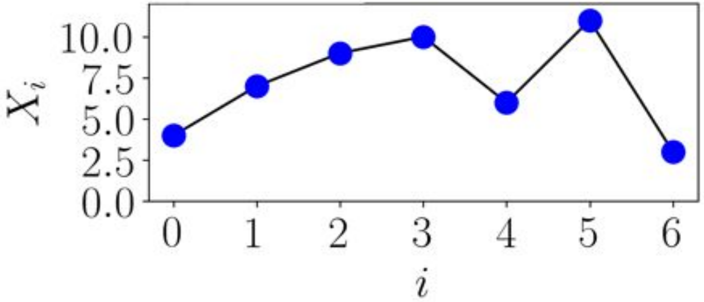
Embedding Delay: τ

with n = 3



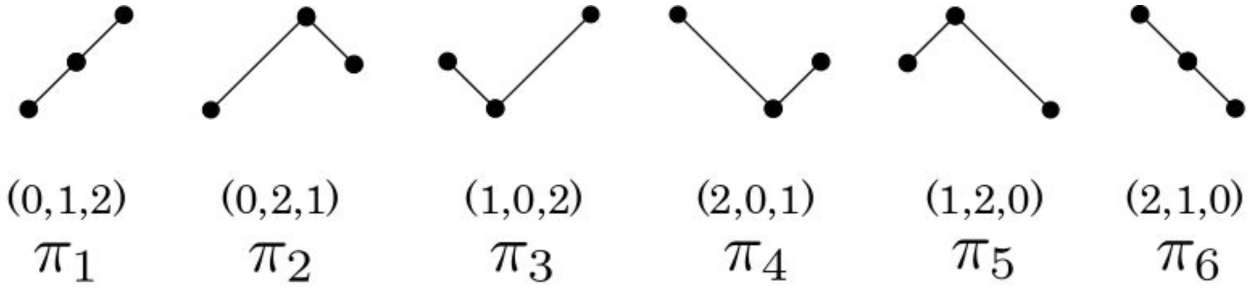
Permutation Entropy (PE) Example

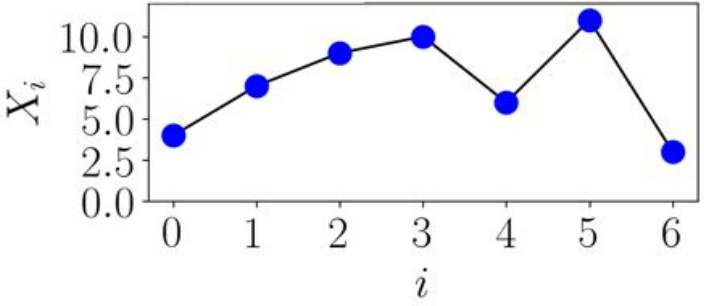
n = 3 and $\tau = 1$



Permutation Entropy (PE) Example

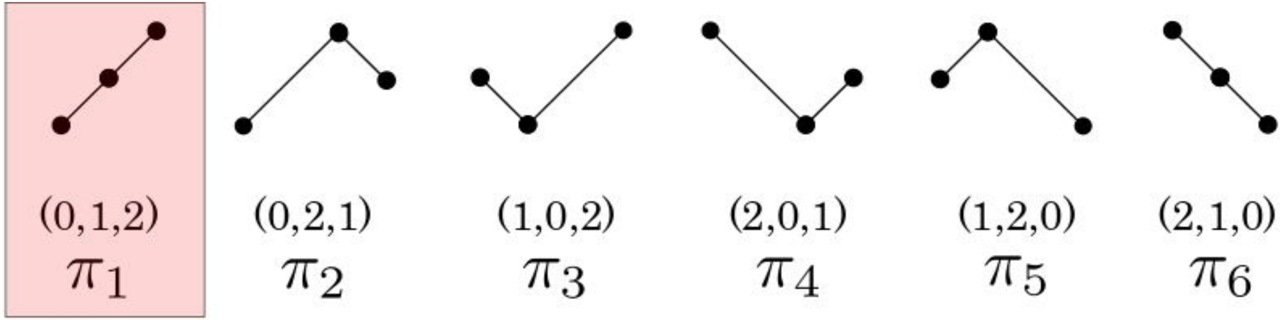
n = 3 and $\tau = 1$

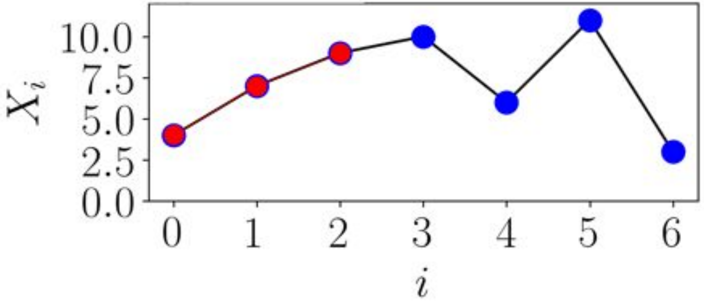




Permutation Entropy (PE) Example

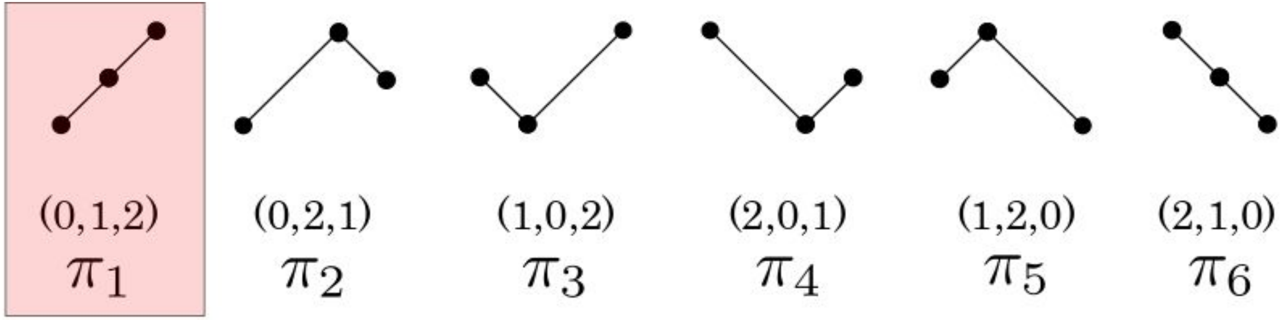
$n = 3$ and $\tau = 1$

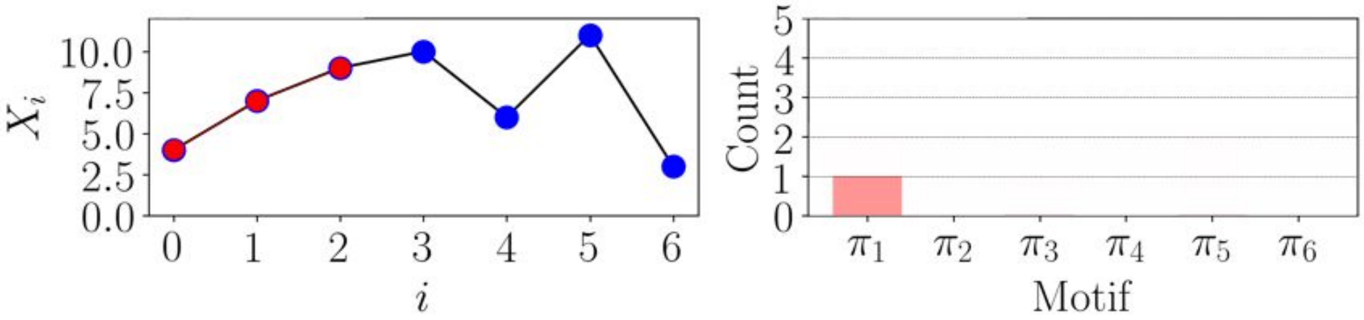




Permutation Entropy (PE) Example

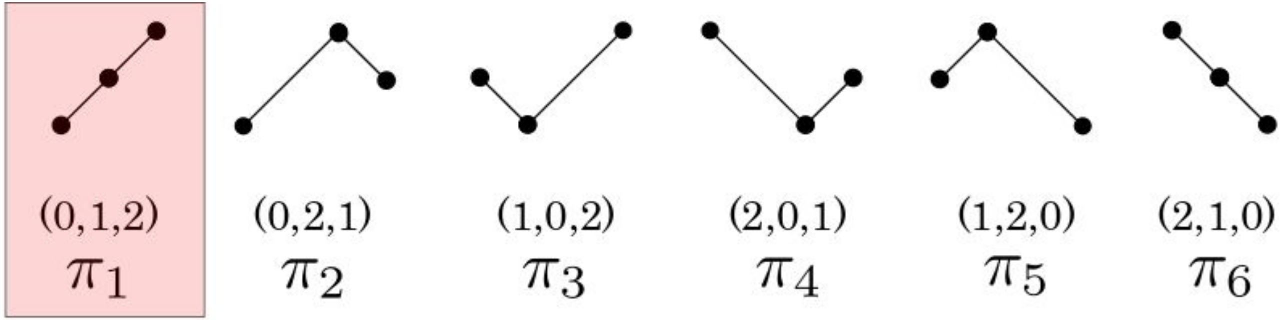
$n = 3$ and $\tau = 1$

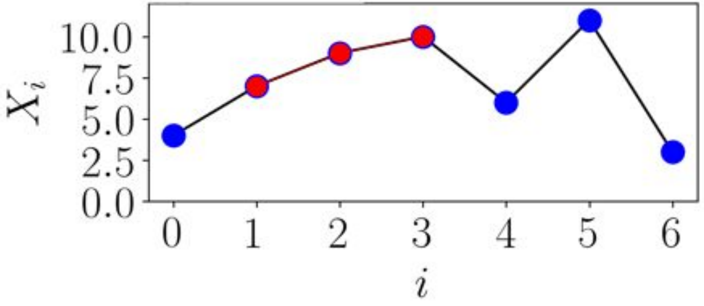


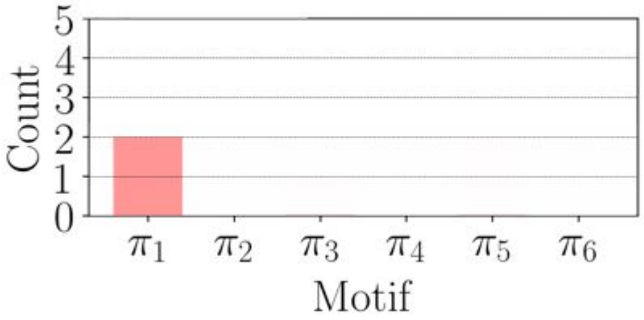


Permutation Entropy (PE) Example

$n = 3$ and $\tau = 1$

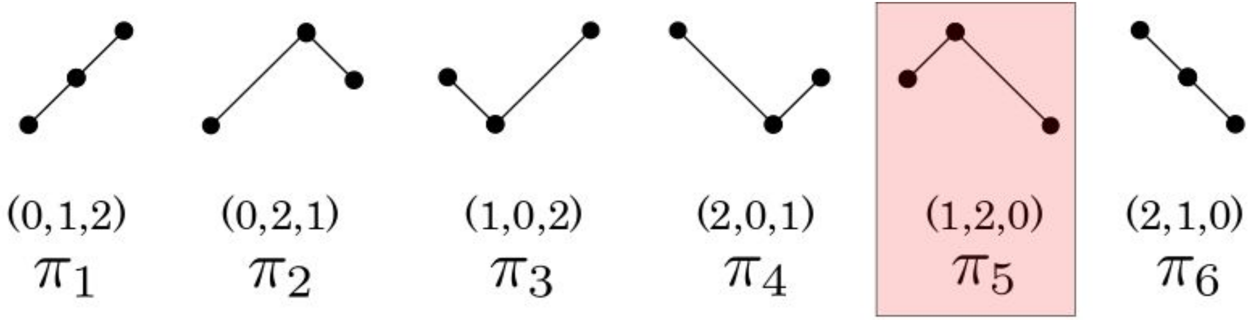


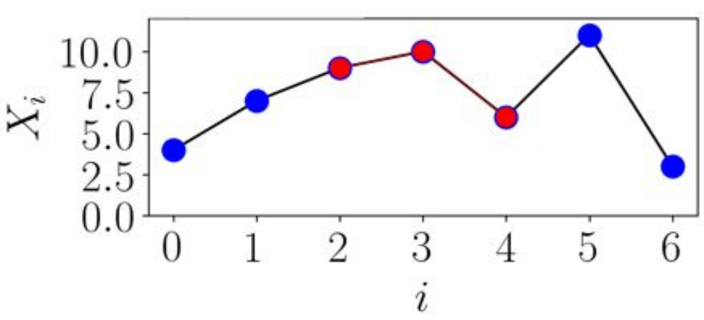


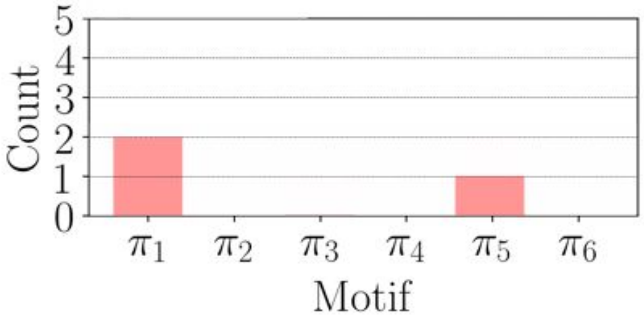


Permutation Entropy (PE) Example

$n = 3$ and $\tau = 1$

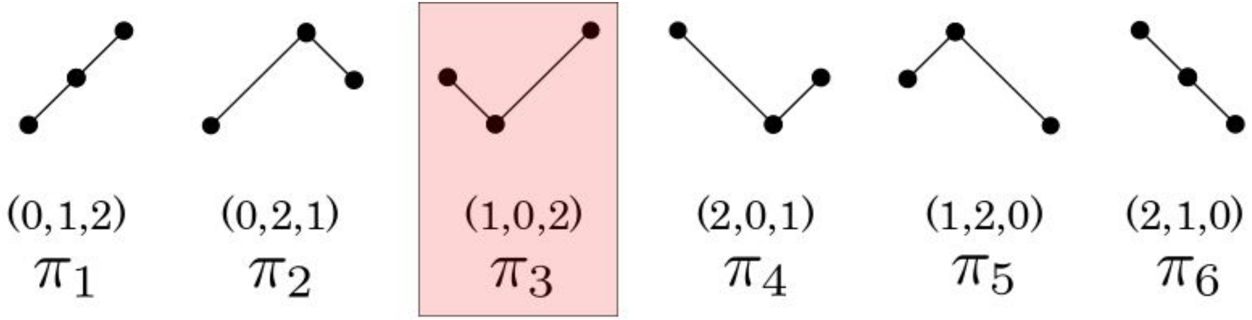


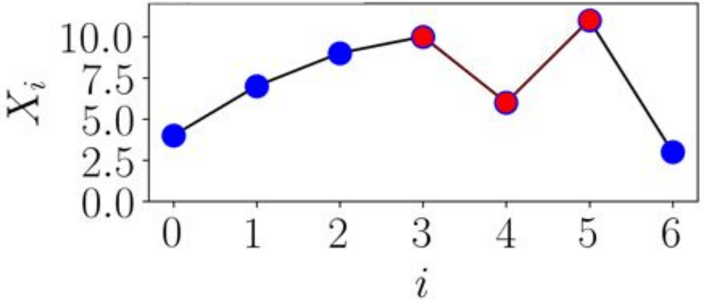


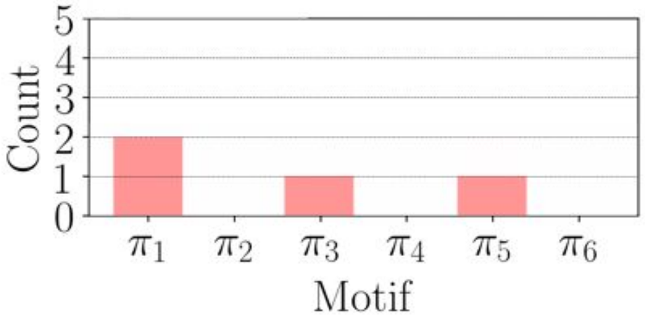


Permutation Entropy (PE) Example

n = 3 and $\tau = 1$

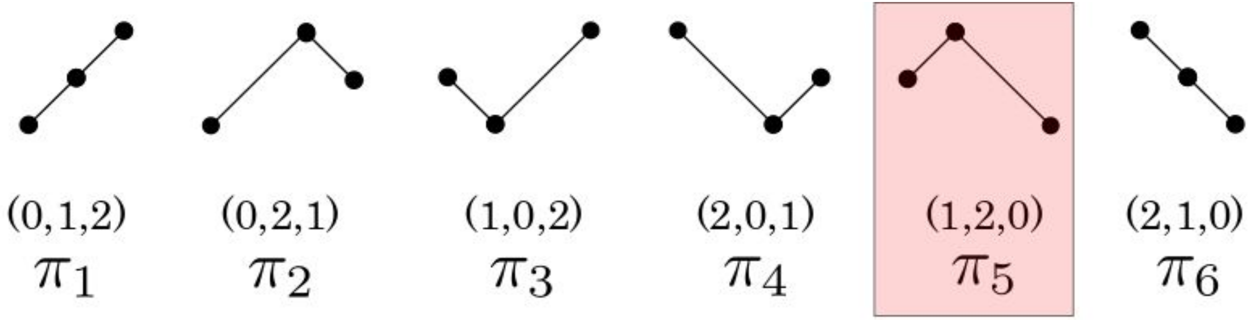


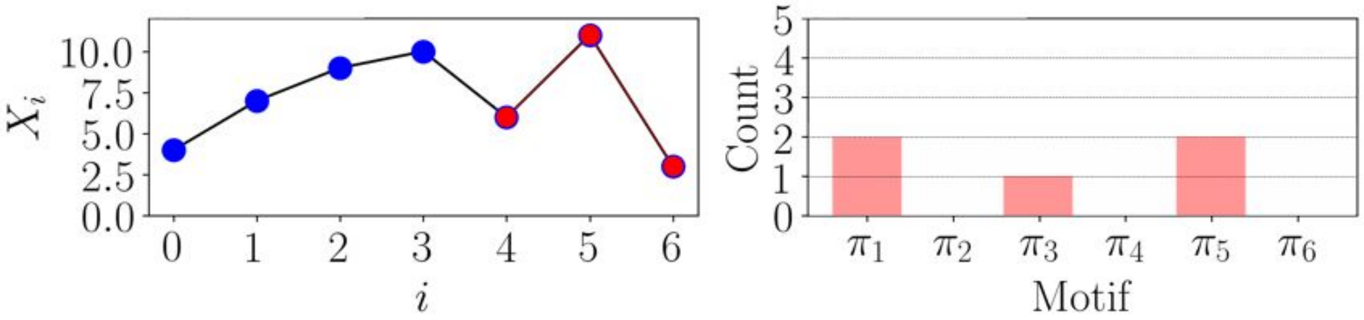




Permutation Entropy (PE) Example

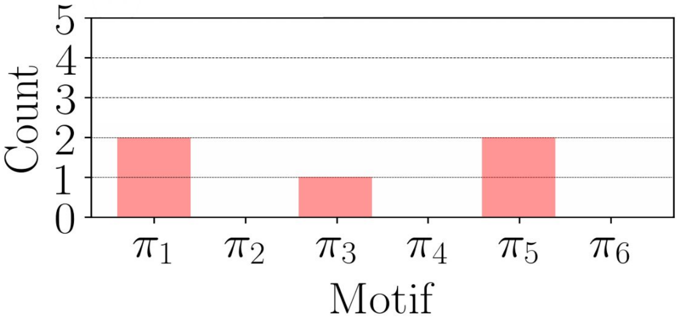
n = 3 and $\tau = 1$





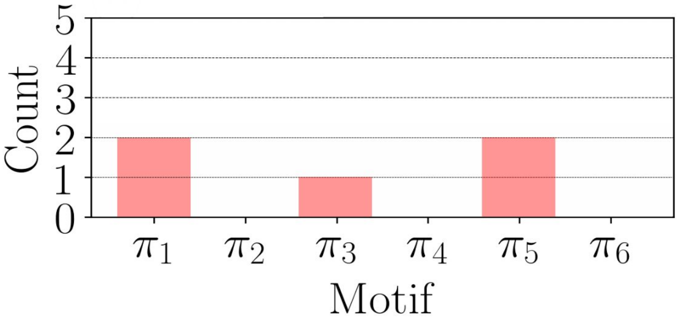
Permutation Entropy (PE) Example

n = 3 and $\tau = 1$



Permutation Entropy (PE) Example

n = 3 and $\tau = 1$

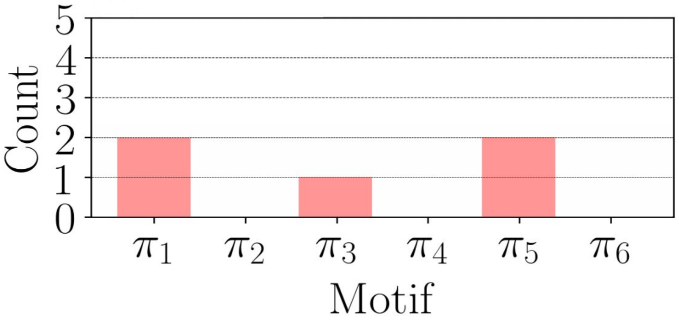


$\longrightarrow$

$$p(\pi_1) = 2/5 \qquad p(\pi_4) = 0$$
$$p(\pi_2) = 0 \qquad p(\pi_5) = 2/5$$
$$p(\pi_3) = 1/5 \qquad p(\pi_6) = 0$$

Permutation Entropy (PE) Example

$n = 3$ and $\tau = 1$



$$p(\pi_1) = 2/5 \qquad p(\pi_4) = 0$$

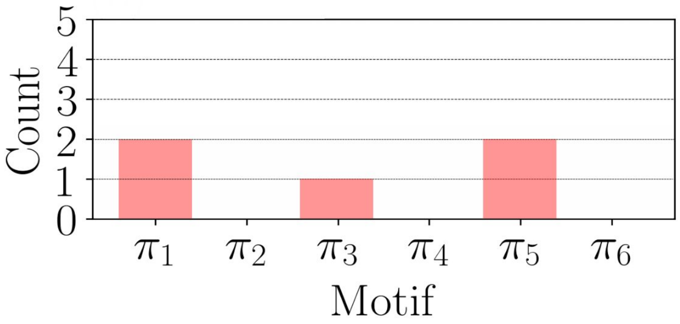
$$p(\pi_2) = 0 \qquad p(\pi_5) = 2/5$$

$$p(\pi_3) = 1/5 \qquad p(\pi_6) = 0$$

$$H(n) = -\sum p(\pi_i) \log p(\pi_i)$$

Permutation Entropy (PE) Example

n = 3 and $\tau = 1$



→

$$p(\pi_1) = 2/5 \qquad p(\pi_4) = 0$$

$$p(\pi_2) = 0 \qquad p(\pi_5) = 2/5$$

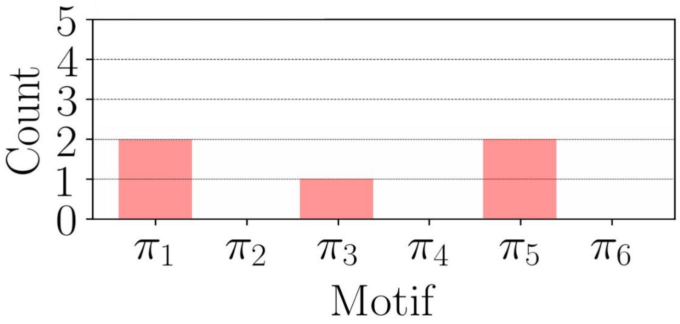
$$p(\pi_3) = 1/5 \qquad p(\pi_6) = 0$$

$$H(n) = -\sum p(\pi_i) \log p(\pi_i)$$

$$H(3) = -\frac{2}{5}\log\frac{2}{5} - \frac{1}{5}\log\frac{1}{5} - \frac{2}{5}\log\frac{2}{5} = 1.522 \text{ bits}$$

Permutation Entropy (PE) Example

n = 3 and $\tau = 1$



$$p(\pi_1) = 2/5 \qquad p(\pi_4) = 0$$
$$p(\pi_2) = 0 \qquad p(\pi_5) = 2/5$$
$$p(\pi_3) = 1/5 \qquad p(\pi_6) = 0$$

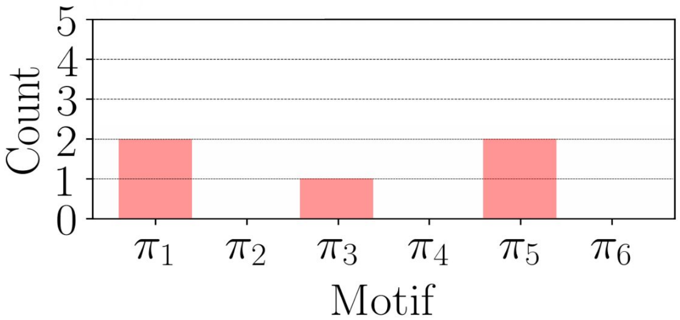
$$H(n) = -\sum p(\pi_i) \log p(\pi_i)$$

$$H(3) = -\frac{2}{5}\log\frac{2}{5} - \frac{1}{5}\log\frac{1}{5} - \frac{2}{5}\log\frac{2}{5} = 1.522 \text{ bits}$$

$$h_n = -\frac{1}{\log_2 n!}\sum p(\pi_i)\log_2 p(\pi_i)$$

Permutation Entropy (PE) Example

n = 3 and $\tau = 1$



$\longrightarrow$

$$p(\pi_1) = 2/5 \qquad p(\pi_4) = 0$$
$$p(\pi_2) = 0 \qquad p(\pi_5) = 2/5$$
$$p(\pi_3) = 1/5 \qquad p(\pi_6) = 0$$

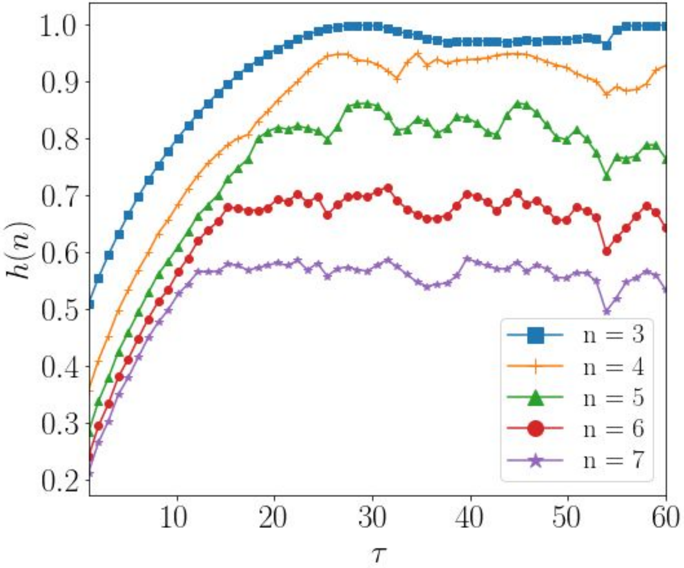
$$H(n) = -\sum p(\pi_i) \log p(\pi_i)$$

$$H(3) = -\frac{2}{5}\log\frac{2}{5} - \frac{1}{5}\log\frac{1}{5} - \frac{2}{5}\log\frac{2}{5} = 1.522 \text{ bits}$$

$$h_n = -\frac{1}{\log_2 n!}\sum p(\pi_i)\log_2 p(\pi_i) \longrightarrow h_3 \approx 0.5888$$

Problem

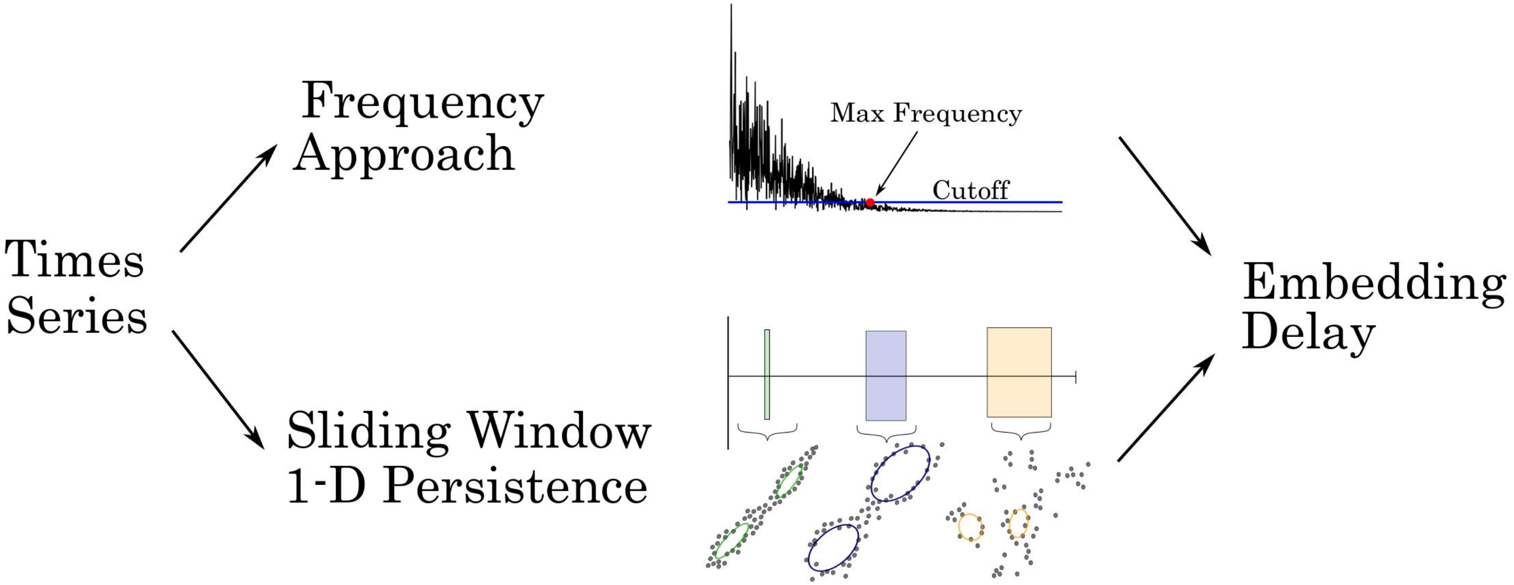
Problem: Permutation entropy is highly dependent on parameters: n and τ



Objective

Develop a method for automatically selecting of the delay parameter for permutation entropy using Topological Data Analysis (TDA).

Embedding Delay Overview



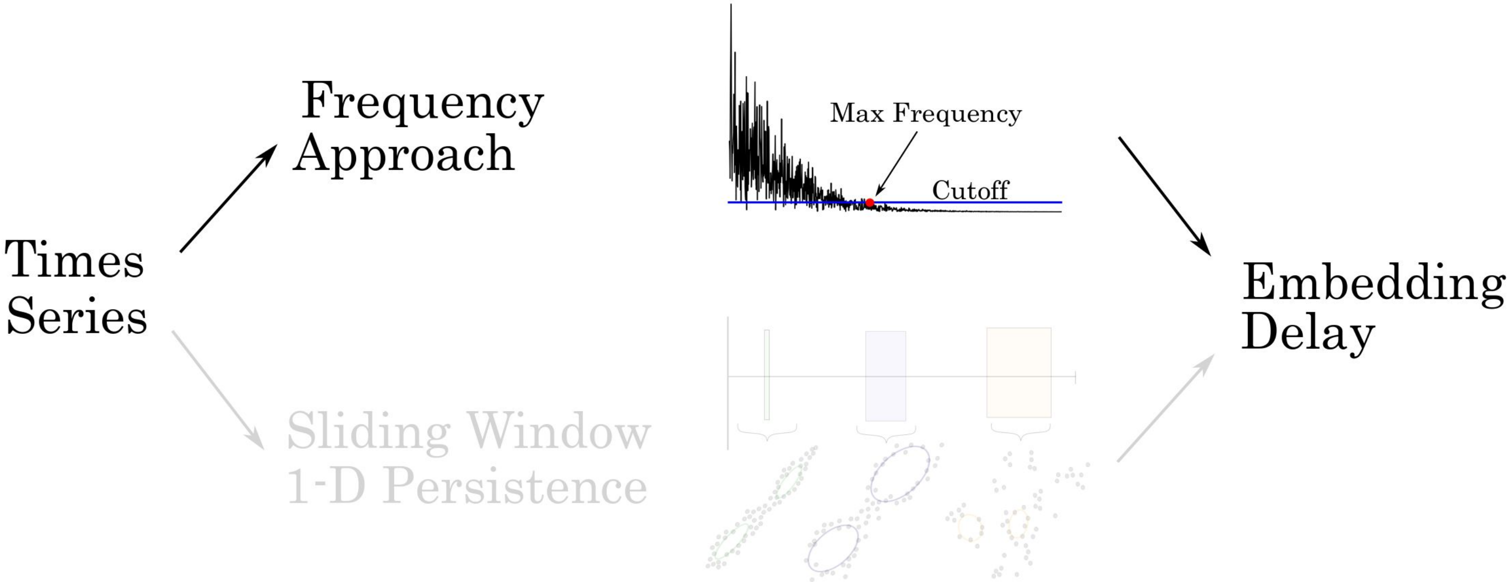
Times Series

Frequency Approach

Sliding Window 1-D Persistence

Embedding Delay

Embedding Delay Overview



Times Series

Frequency Approach

Sliding Window 1-D Persistence

Embedding Delay

Embedding Delay: Frequency Approach

$$2f_{\max} < f_s < 4f_{\max}$$

[1] Michał Melosik and W Marszalek. On the 0/1 test for chaos in continuous systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, 64(3):521 528, 2016.

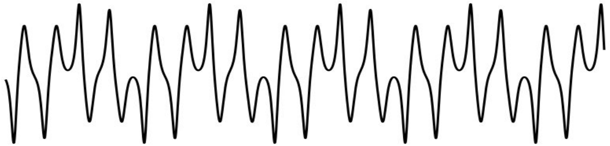
Embedding Delay: Frequency Approach

$$2f_{\max} < f_s < 4f_{\max} \longrightarrow \tau = \frac{f_s}{\alpha f_{\max}} \quad , \; 2 \leq \alpha \leq 4$$

[1] Michał Melosik and W Marszalek. On the 0/1 test for chaos in continuous systems. *Bulletin of the Polish Academy of Sciences Technical Sciences*, 64(3):521 528, 2016.

Finding Maximum Significant Frequency

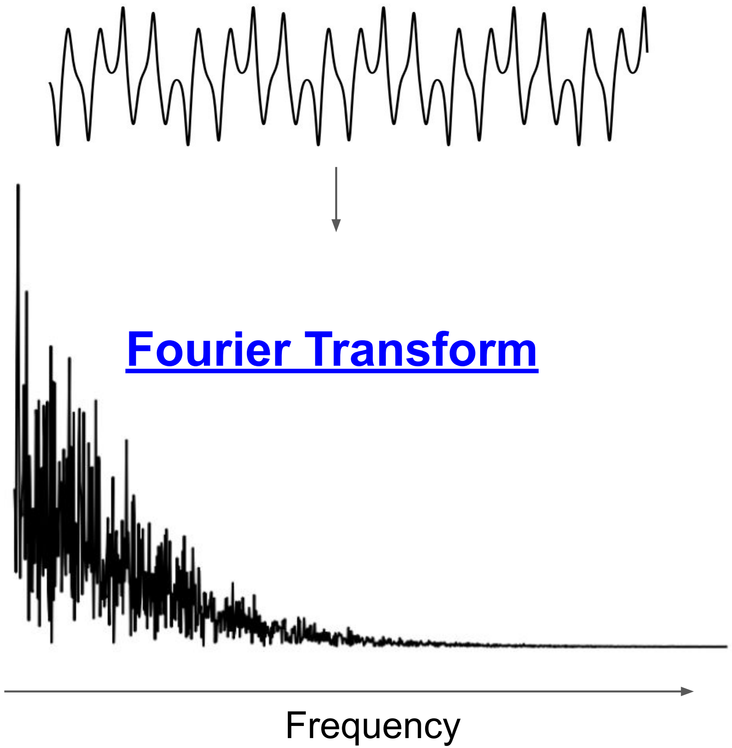
Time Series



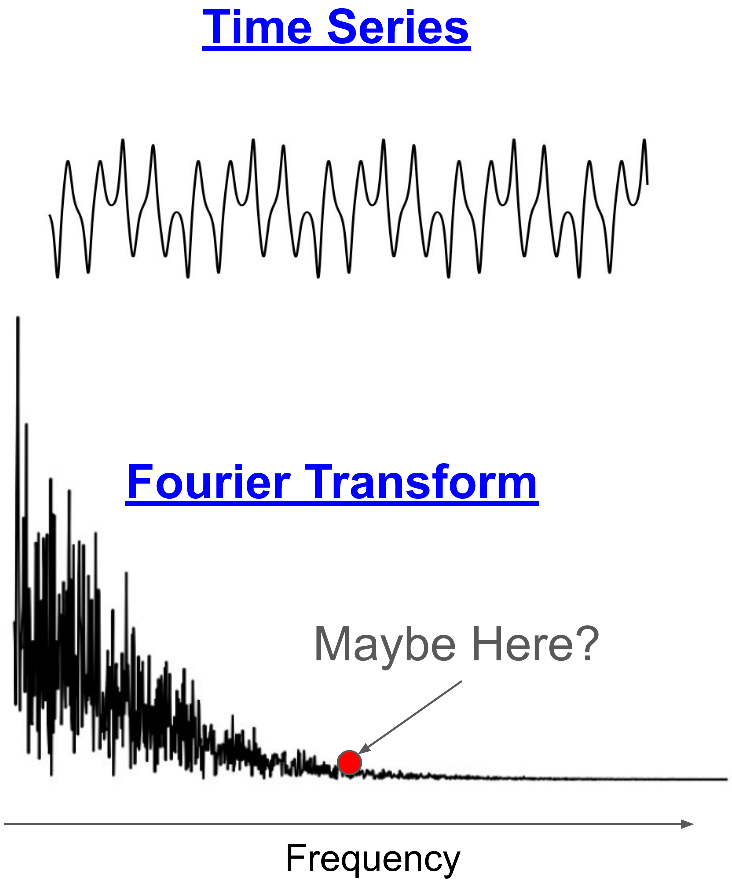
Finding Maximum Significant Frequency

1
PE
Example
2
Motivation
3
Delay
Freq. App.
SW1PerS
4
Results
5
Summary

[**Time Series**]

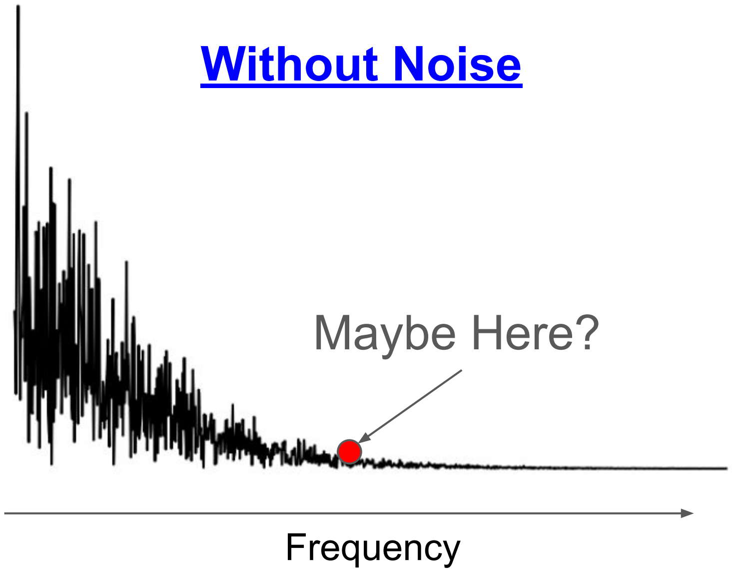


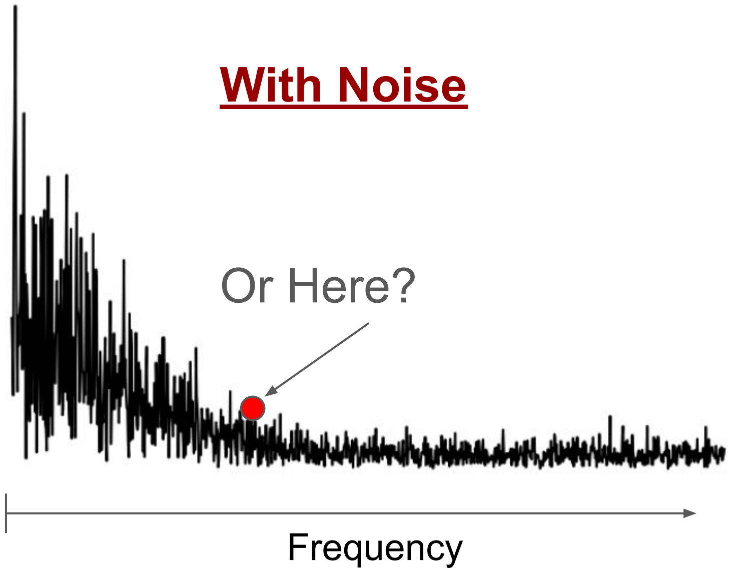
Finding Maximum Significant Frequency



Finding Maximum Significant Frequency

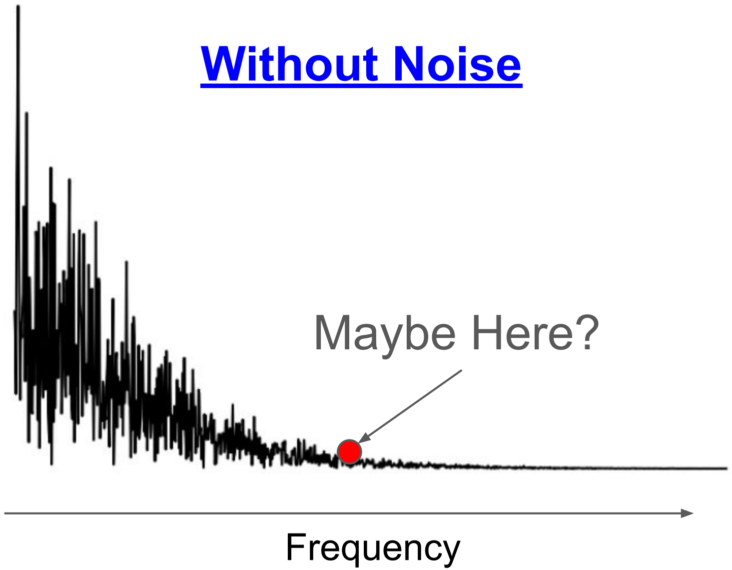
1
PE
Example
2
Motivation
3
Delay
Freq. App.
SW1PerS
4
Results
5
Summary



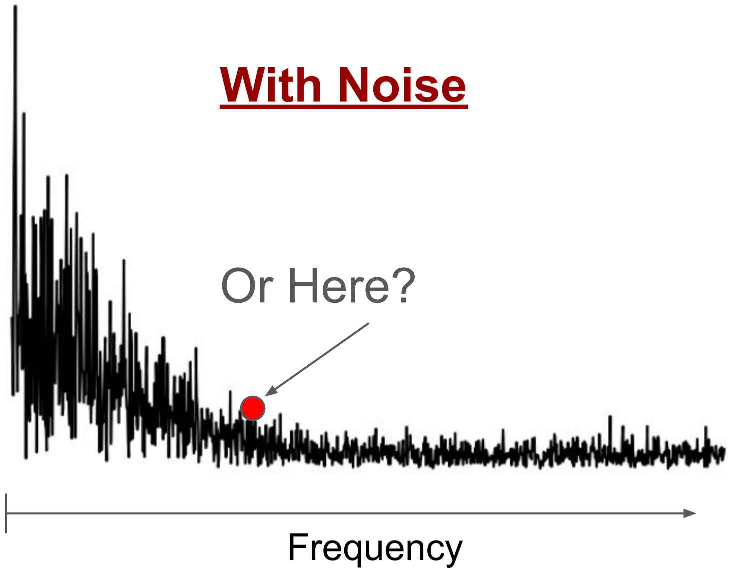


Finding Maximum Significant Frequency

0-D Sublevel Set Persistence

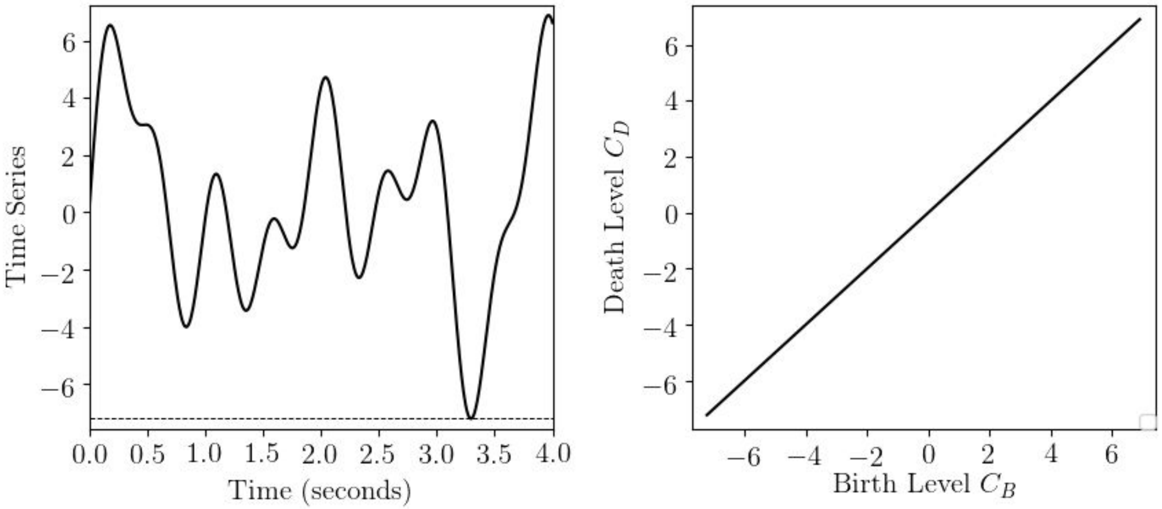

Without Noise

Maybe Here?


With Noise

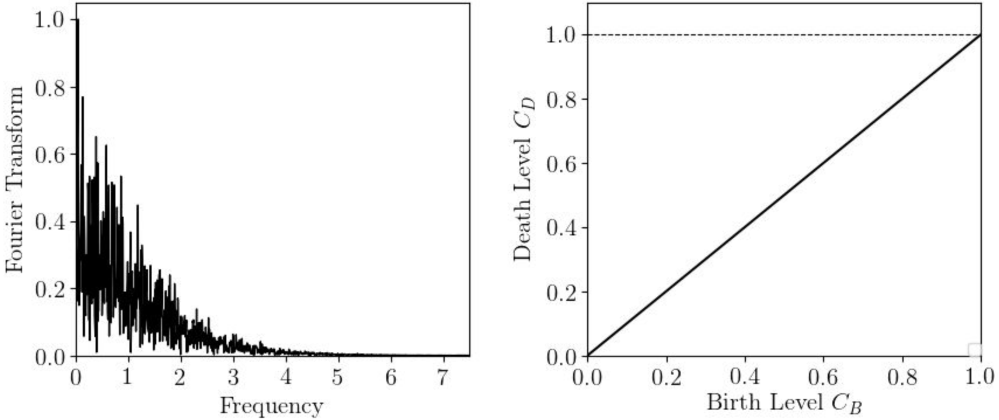
Or Here?

0-D Sublevel Set Persistence for 1-D function



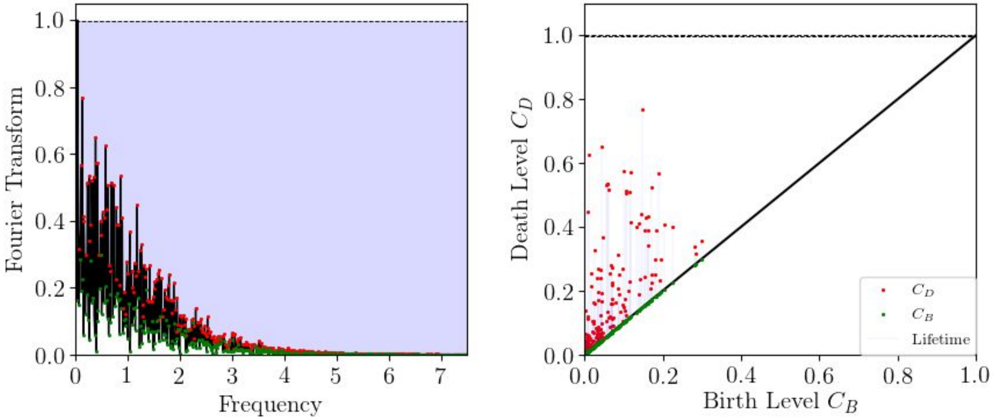
0-D Persistence of Fourier Transform

Without Noise →

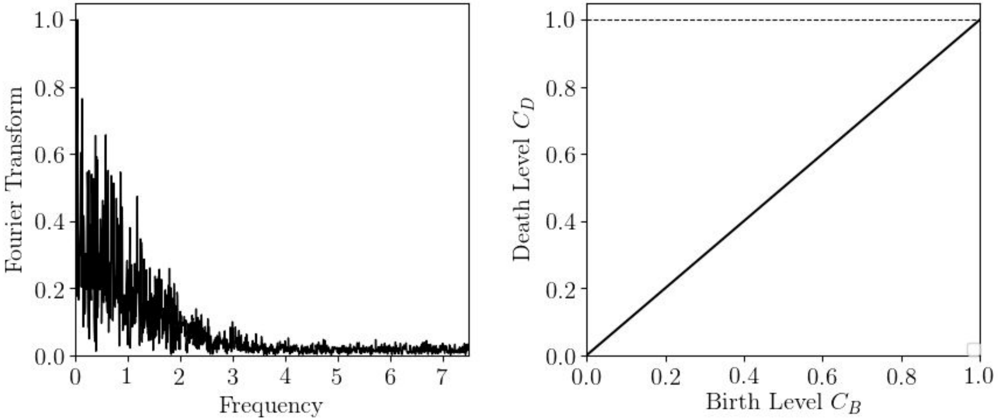


0-D Persistence of Fourier Transform

Without Noise →

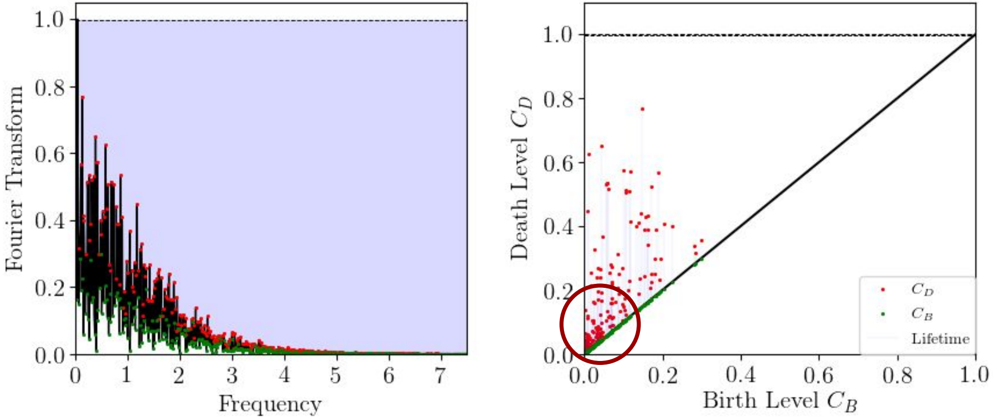


With Noise →

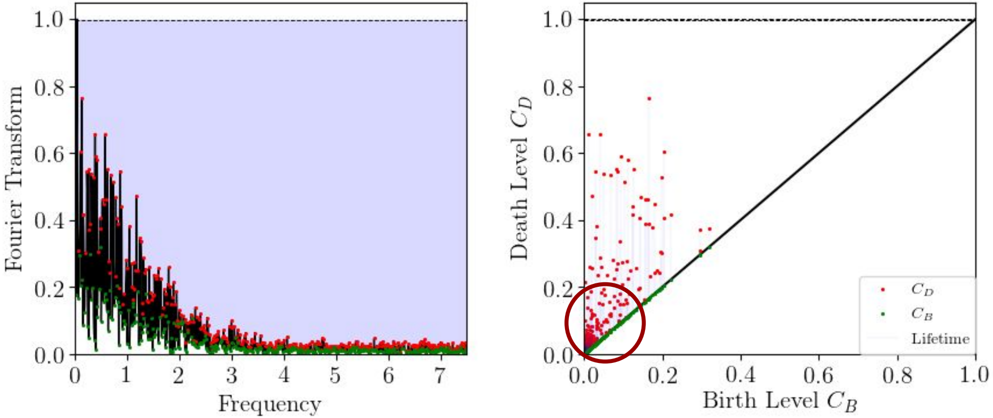


0-D Persistence of Fourier Transform

Without Noise →



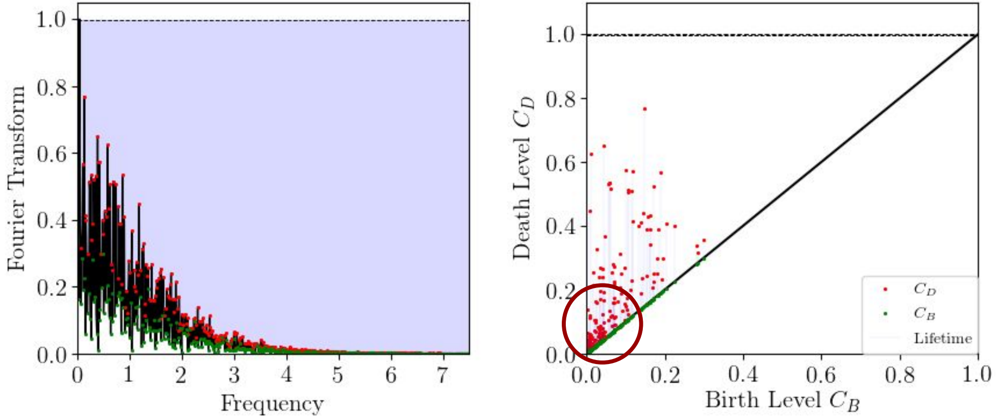
With Noise →



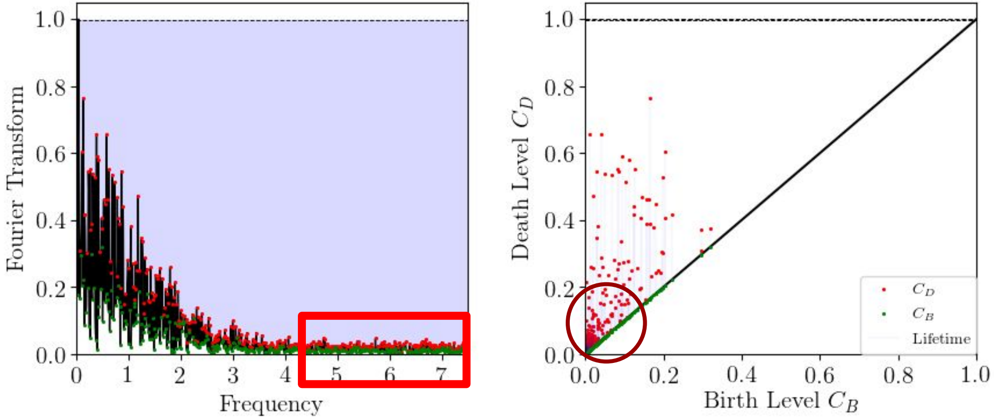
0-D Persistence of Fourier Transform

1 **PE** Example
2 **Motivation**
3 **Delay** Freq. App. SW1PerS
4 **Results**
5 **Summary**

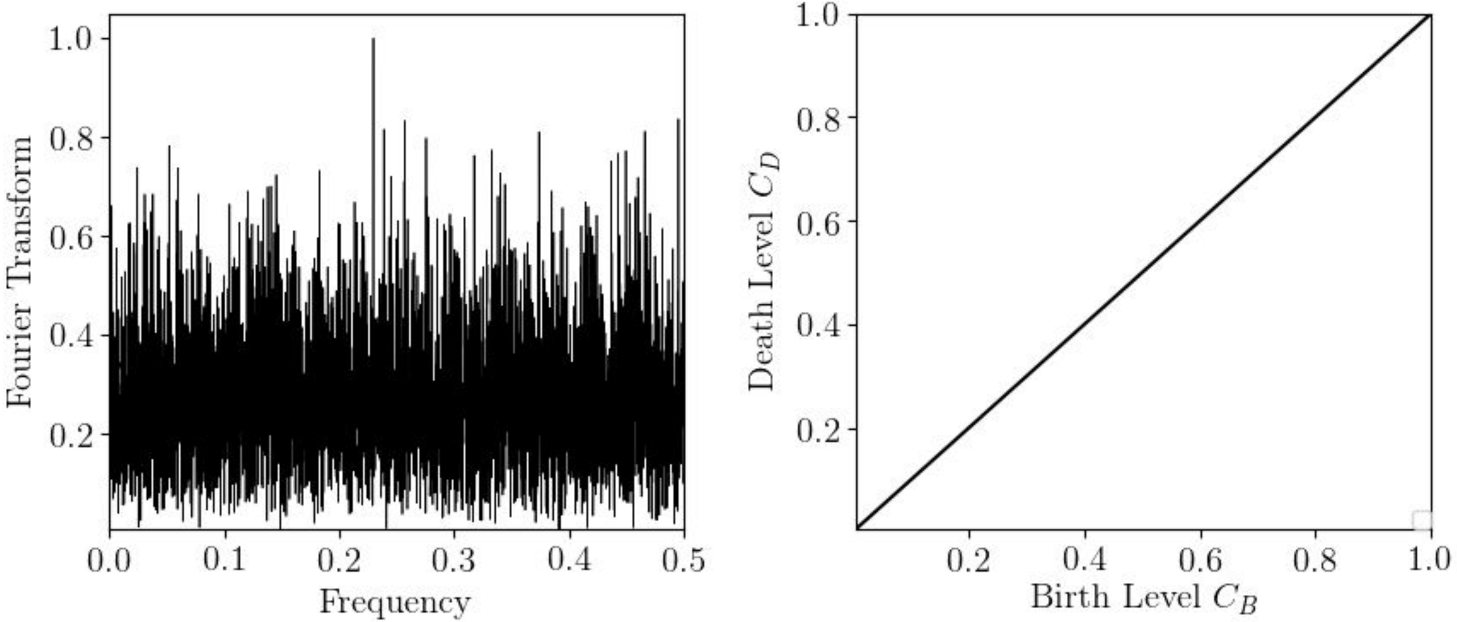
Without Noise →



With Noise →



0-D Persistence of Fourier Transform of Noise



Modified z-score for Noise Isolation

Standard z-score

$$z = \frac{x - \mu}{\sigma}$$

Modified z-score

$$z_m = 0.6745 \frac{x - \tilde{x}}{\mathrm{MAD}}$$

$$\mathrm{MAD} = \mathrm{median}(|x - \tilde{x}|)$$

Modified z-score for Noise Isolation

Standard z-score

$$z = \frac{x - \mu}{\sigma}$$

Modified z-score

$$z_m = 0.6745 \frac{x - \tilde{x}}{\text{MAD}}$$

$$\text{MAD} = \text{median}(|x - \tilde{x}|)$$

If z_m < Threshold → **Noise** 🔴

Otherwise → **Data** ●

Threshold = 5

Boris Iglewicz and David Hoaglin. Volume 16: how to detect and handle outliers, The ASQC basic references in quality control: statistical techniques, Edward F. Mykytka. PhD thesis, Ph.D., Editor, 1993.

Modified z-score for Noise Isolation

Standard z-score

$$z = \frac{x - \mu}{\sigma}$$

Modified z-score

$$z_m = 0.6745 \frac{x - \tilde{x}}{\text{MAD}}$$

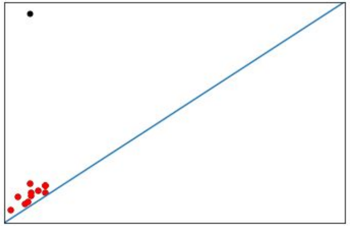
$$\text{MAD} = \text{median}(|x - \tilde{x}|)$$

If z_m < Threshold → **Noise** ●

Otherwise → **Data** ●

Threshold = 5

1 outlier out of 11



Boris Iglewicz and David Hoaglin. Volume 16: how to detect and handle outliers, The ASQC basic references in quality control: statistical techniques, Edward F. Mykytka. PhD thesis, Ph.D., Editor, 1993.

Modified z-score for Noise Isolation

Standard z-score

$$z = \frac{x - \mu}{\sigma}$$

Modified z-score

$$z_m = 0.6745\,\frac{x - \tilde{x}}{\mathrm{MAD}}$$

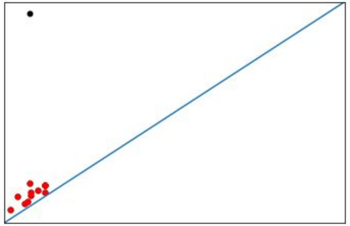
$$\mathrm{MAD} = \mathrm{median}(|x - \tilde{x}|)$$

If z_m < Threshold → **Noise** 🔴

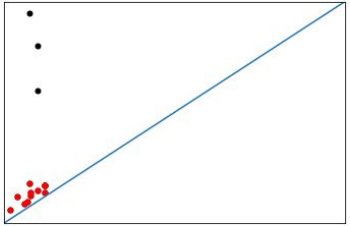
Otherwise → **Data** ●

Threshold = 5

1 outlier out of 11



3 outliers out of 14



Boris Iglewicz and David Hoaglin. Volume 16: how to detect and handle outliers, The ASQC basic references in quality control: statistical techniques, Edward F. Mykytka. PhD thesis, Ph.D., Editor, 1993.

Modified z-score for Noise Isolation

Standard z-score

$$z = \frac{x - \mu}{\sigma}$$

Modified z-score

$$z_m = 0.6745 \frac{x - \tilde{x}}{\mathrm{MAD}}$$

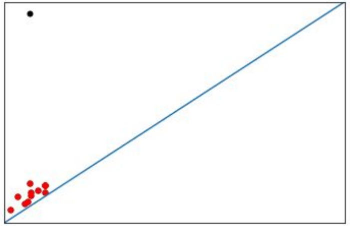
$$\mathrm{MAD} = \mathrm{median}(|x - \tilde{x}|)$$

If z_m < Threshold → Noise 🔴

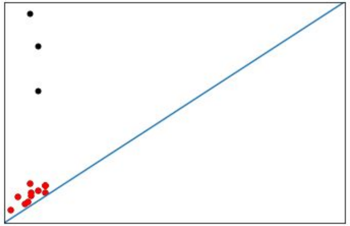
Otherwise → Data ●

Threshold = 5

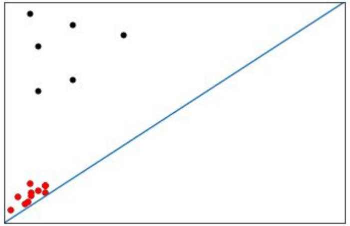
1 outlier out of 11



3 outliers out of 14



6 outliers out of 17



Boris Iglewicz and David Hoaglin. Volume 16: how to detect and handle outliers, The ASQC basic references in quality control: statistical techniques, Edward F. Mykytka. PhD thesis, Ph.D., Editor, 1993.

Modified z-score for Noise Isolation

Standard z-score

$$z = \frac{x - \mu}{\sigma}$$

Modified z-score

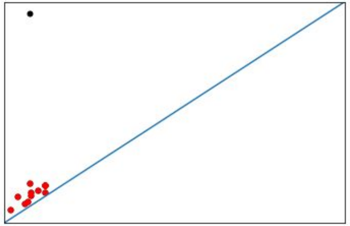
$$z_m = 0.6745 \frac{x - \tilde{x}}{\text{MAD}}$$

$$\text{MAD} = \text{median}(|x - \tilde{x}|)$$

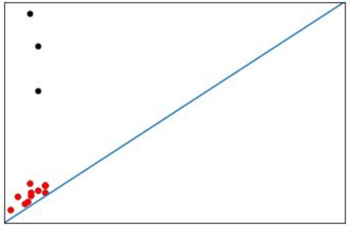
If z_m < Threshold → Noise 🔴

Otherwise → Data ●

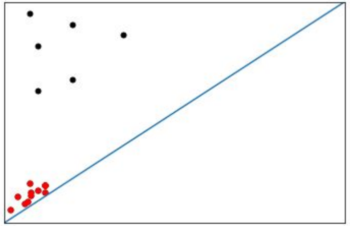
Threshold = 5



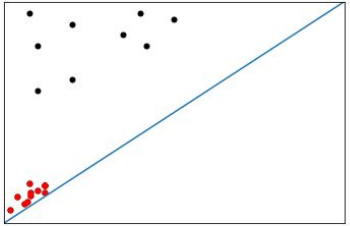
1 outlier out of 11



3 outliers out of 14



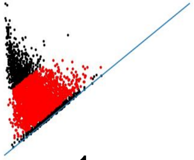
6 outliers out of 17



9 outliers out of 20

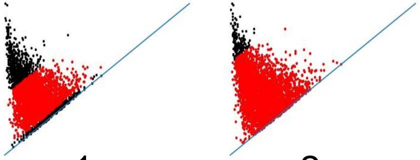
Boris Iglewicz and David Hoaglin. Volume 16: how to detect and handle outliers, The ASQC basic references in quality control: statistical techniques, Edward F. Mykytka. PhD thesis, Ph.D., Editor, 1993.

Modified z-score Threshold



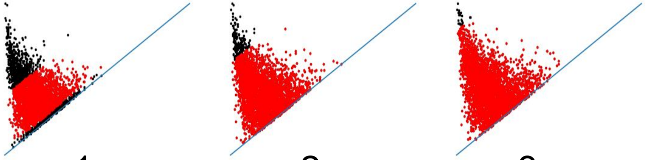
Threshold: 1

Modified z-score Threshold



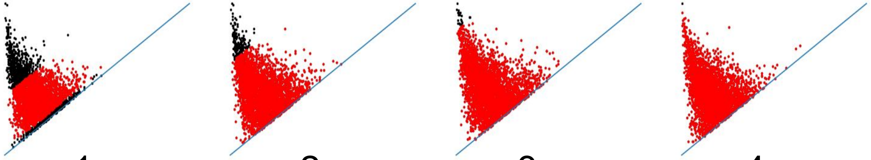
Threshold: 1 2

Modified z-score Threshold



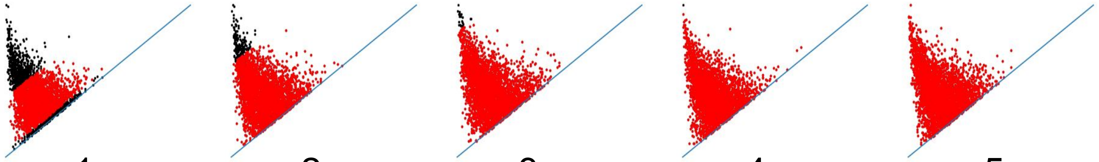
Threshold: 1 2 3

Modified z-score Threshold



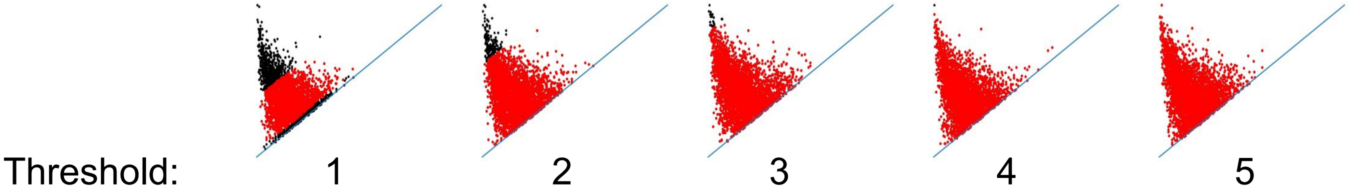
Threshold: 1 2 3 4

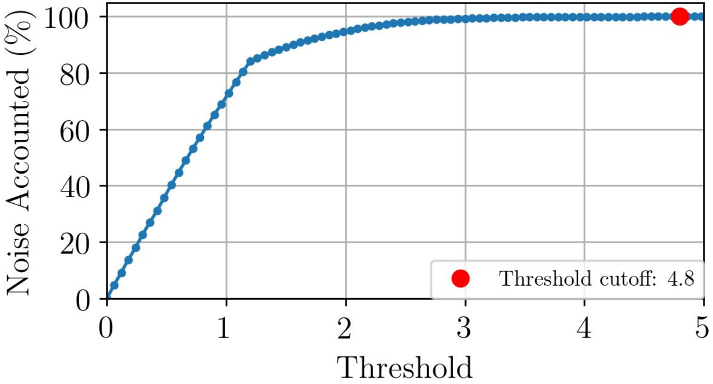
Modified z-score Threshold



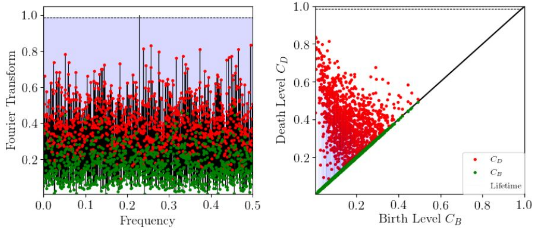
Threshold: 1 2 3 4 5

Modified z-score Threshold

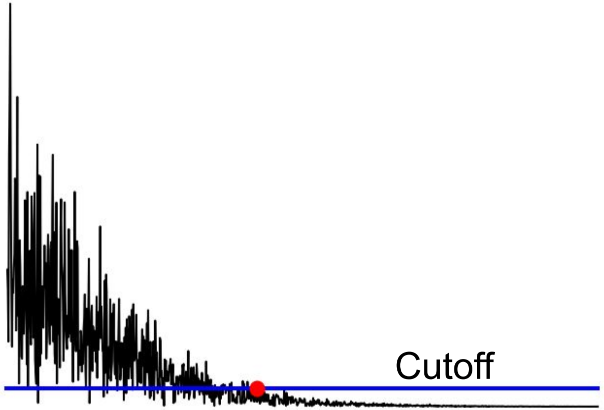




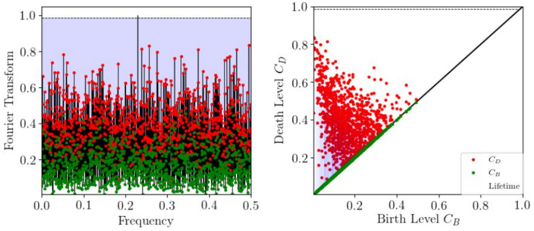
0-D Persistence for Delay



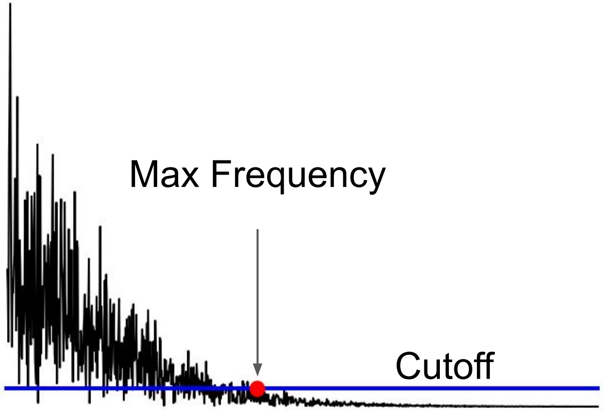
$$\rightarrow \quad \text{Cutoff}_{\text{noise}} = \max\left(\text{lifetime}_{\text{noise}}\right) + \text{median}(\text{lifetime}_{\text{noise}})$$



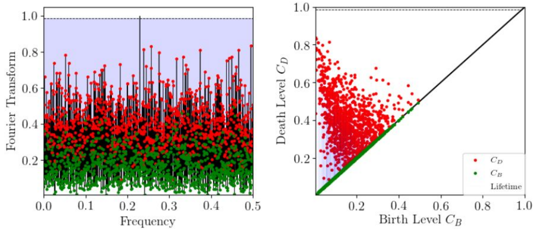
0-D Persistence for Delay



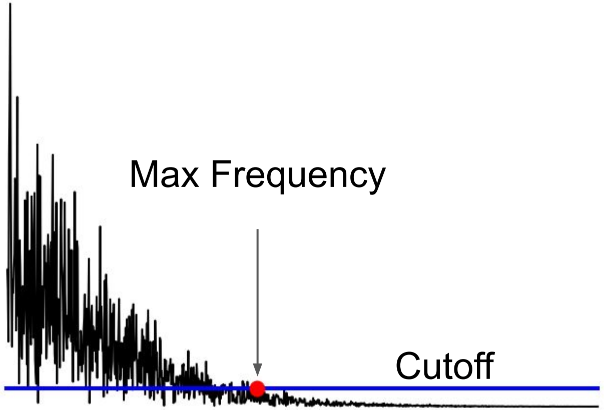
$$\longrightarrow \quad \text{Cutoff}_{\text{noise}} = \max\left(\text{lifetime}_{\text{noise}}\right) + \text{median}(\text{lifetime}_{\text{noise}})$$



0-D Persistence for Delay

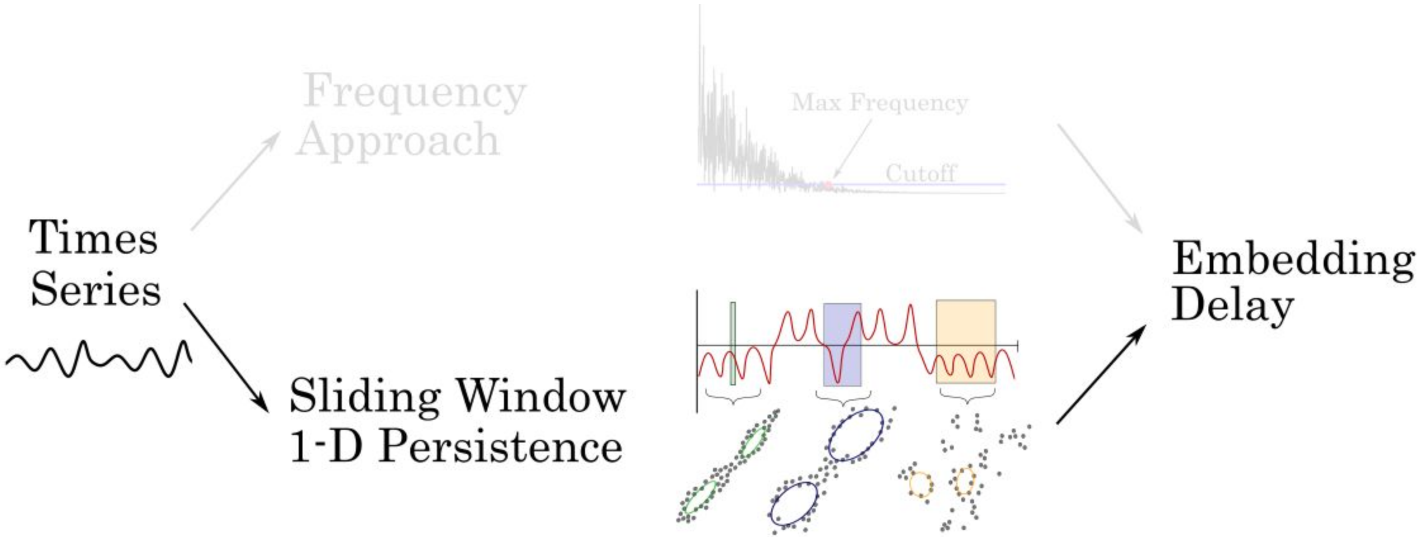


$$\longrightarrow \text{Cutoff}_{\text{noise}} = \max\left(\text{lifetime}_{\text{noise}}\right) + \text{median}(\text{lifetime}_{\text{noise}})$$



$$\longrightarrow \quad \tau = \frac{f_s}{\alpha f_{\max}}$$

SW1PerS



Times Series

Frequency Approach

Sliding Window 1-D Persistence

Embedding Delay

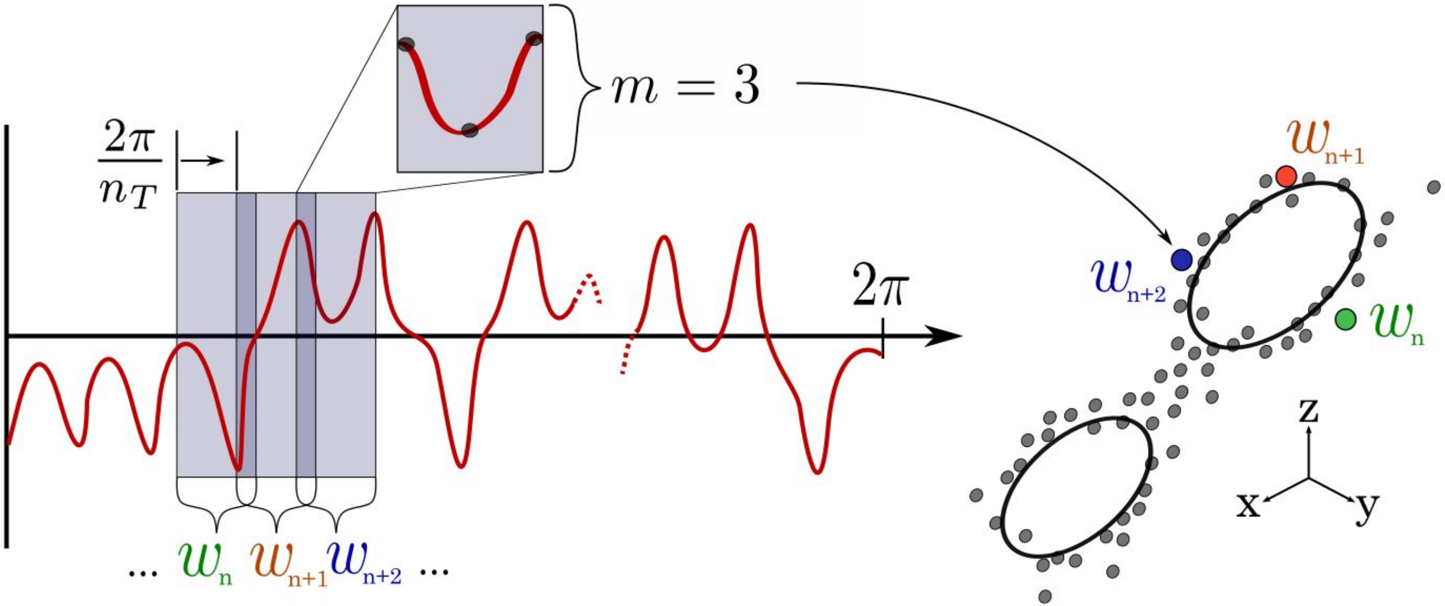
Jose A Perea and John Harer. Sliding windows and persistence: An application of topological methods to signal analysis. Foundations of Computational Mathematics, 15(3):799–838, 2015.

Sliding Window Embedding

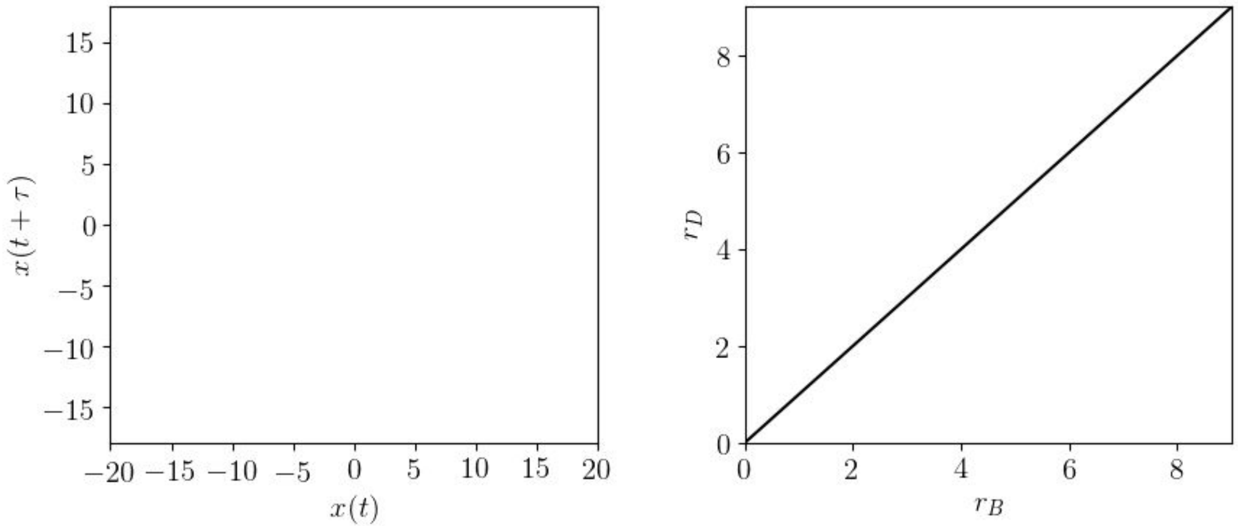
$$SW_{m,\tau_s} f(t) = [f(t_0), f(t_0 + \tau_s), \ldots, f(t_0 + m\tau_s)]$$

Sliding Window Embedding

$$SW_{m,\tau_s} f(t) = [f(t_0), f(t_0 + \tau_s), \ldots, f(t_0 + m\tau_s)]$$

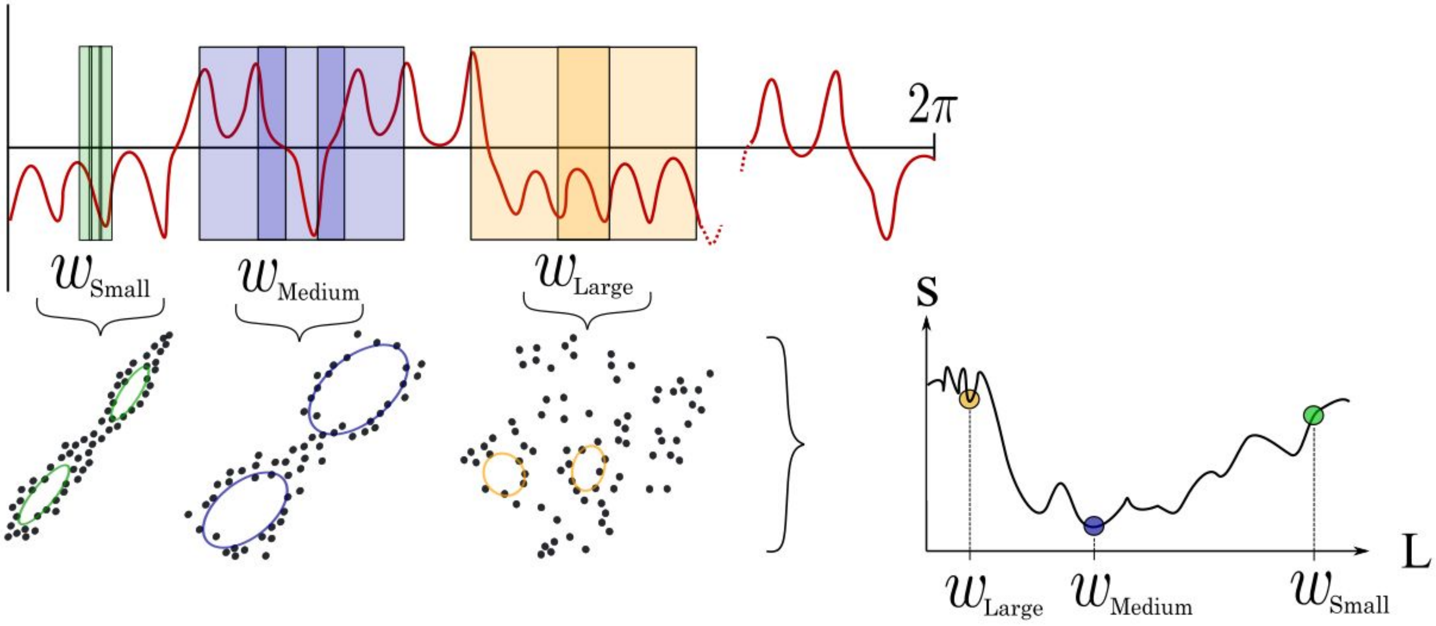


1-D Persistent Homology



Window Size Variation

$$w = \frac{2\pi m}{L(m+1)}$$

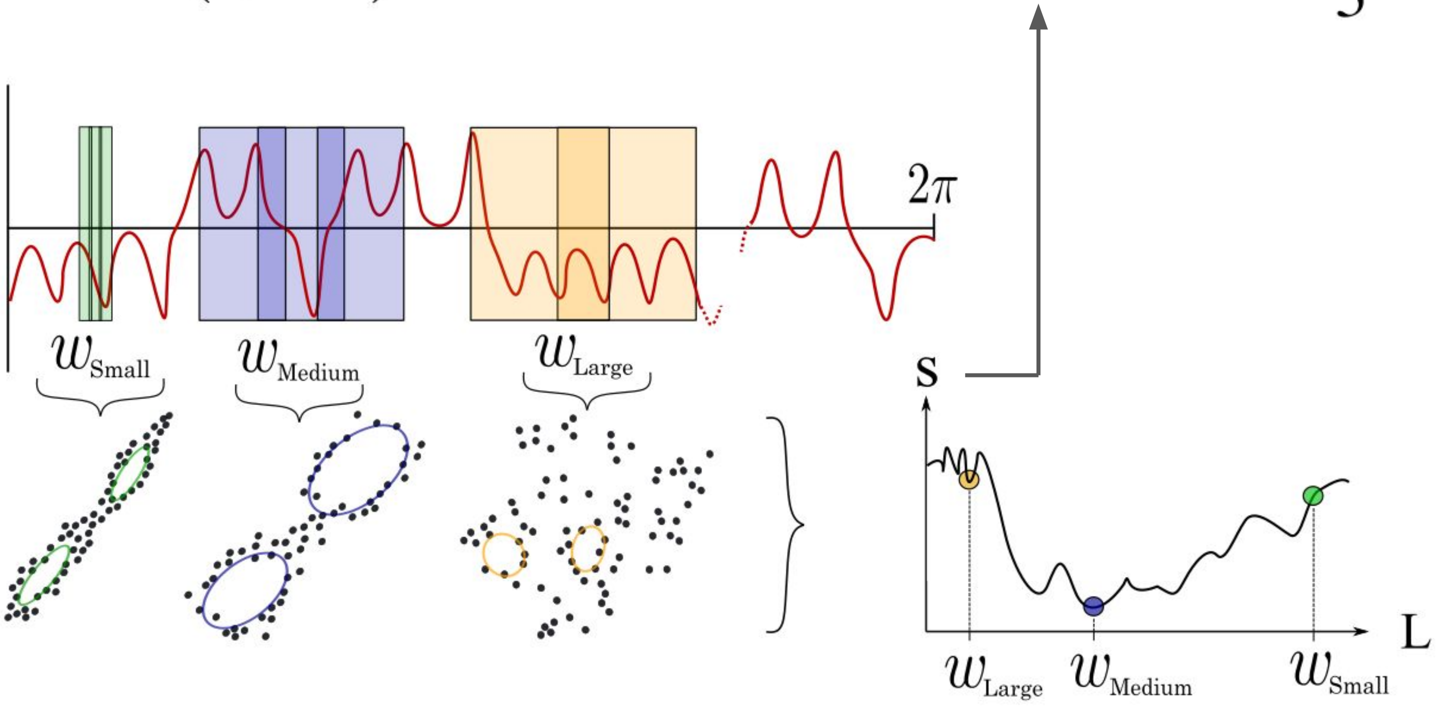


Window Size Variation

1 **PE**
Example
2 **Motivation**
3 **Delay**
Freq. App.
SW1PerS
4 Results
5 Summary

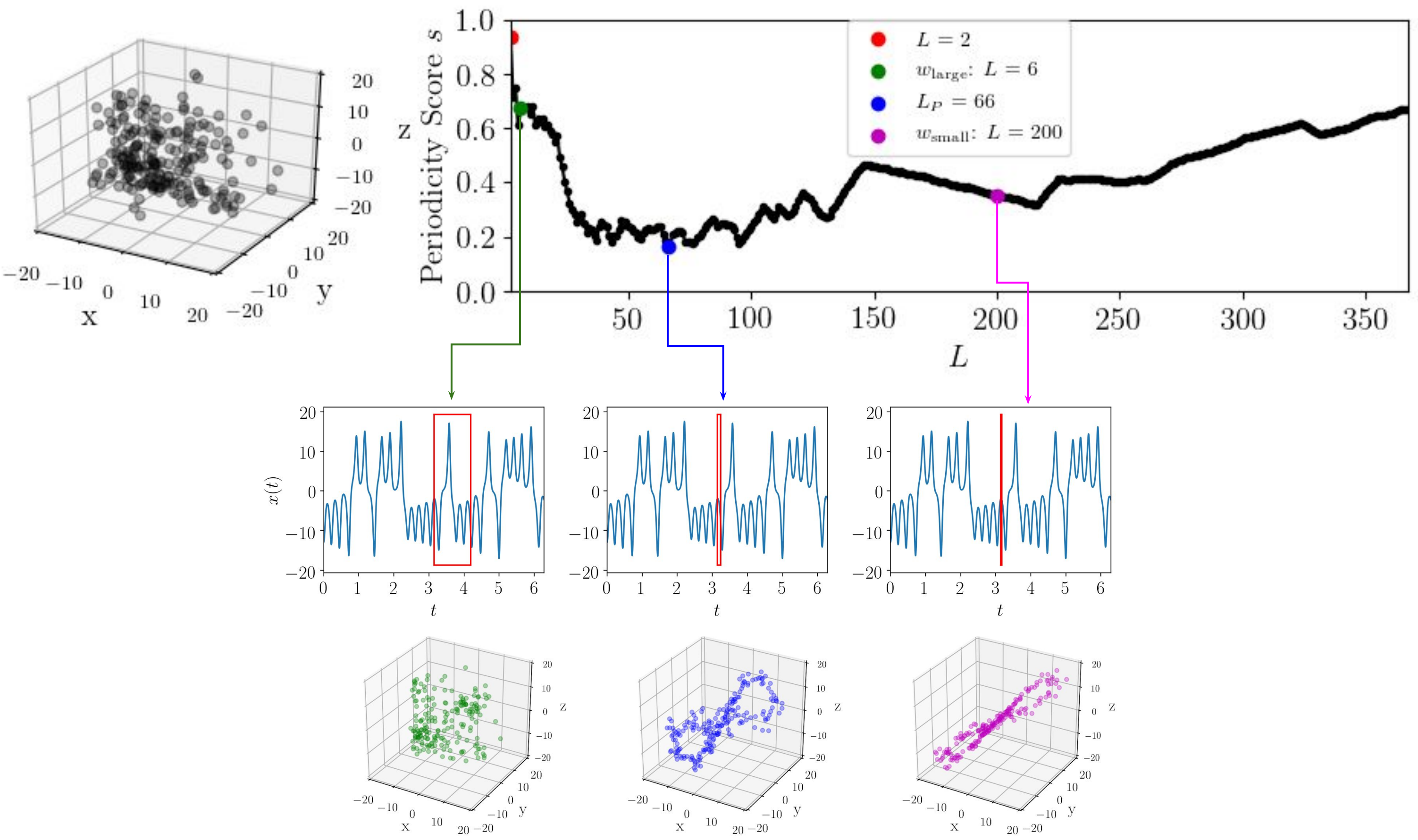
$$w = \frac{2\pi m}{L(m+1)}$$

$$s = 1 - \frac{r_B^2 - r_D^2}{3}$$

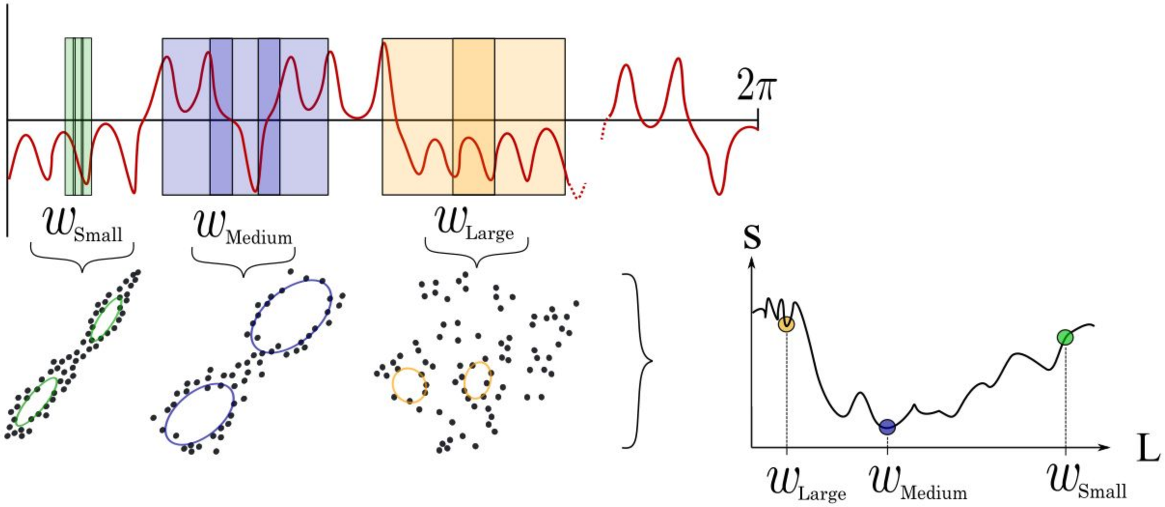


Window Size Variation

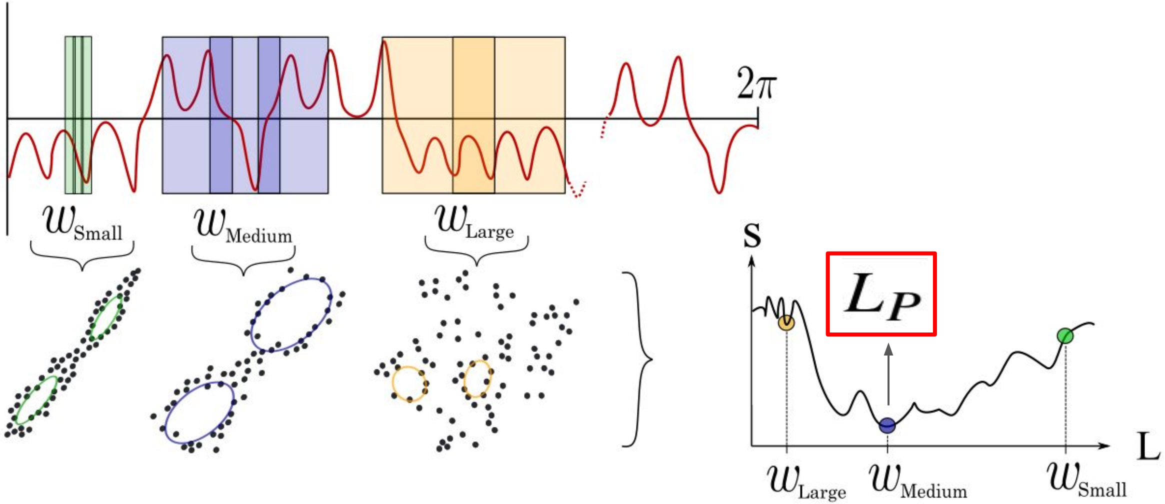
1 **PE** Example
2 **Motivation**
3 **Delay** Freq. App. SW1PerS
4 **Results**
5 **Summary**



Finding the Delay

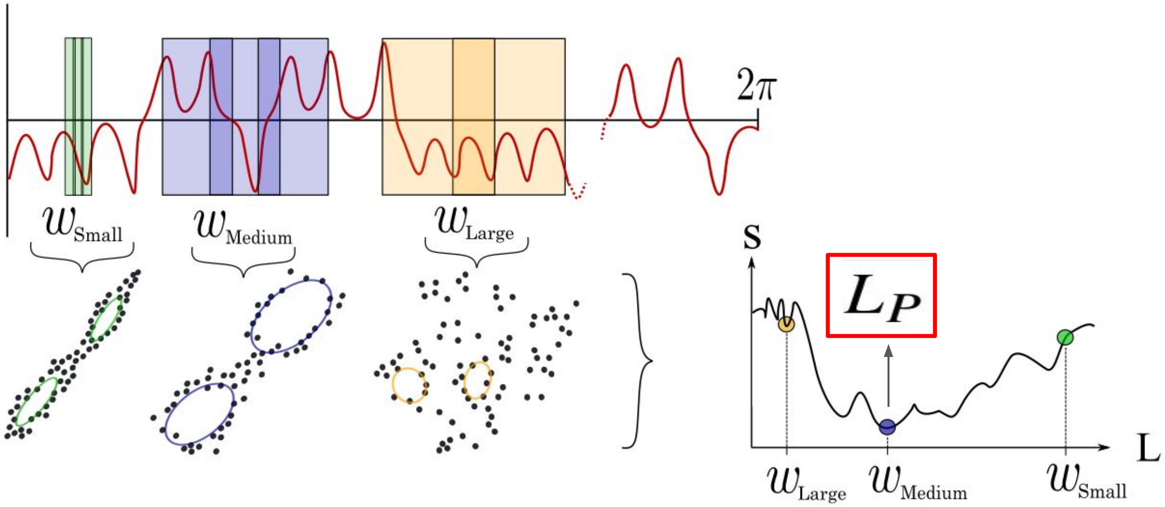


Finding the Delay



1
PE
Example
2
Motivation
3
Delay
Freq. App.
SW1PerS
4
Results
5
Summary

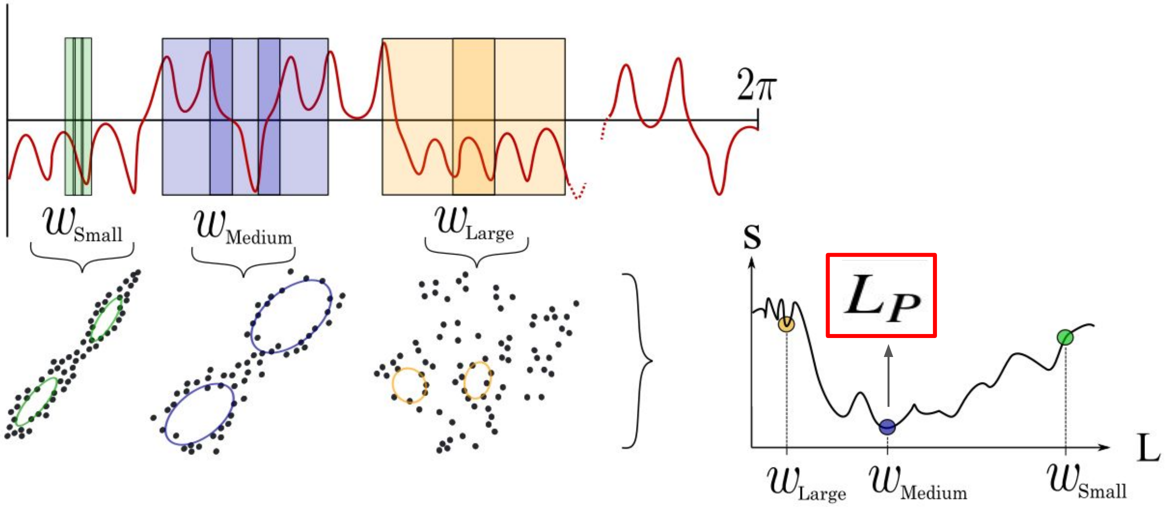
Finding the Delay



$$L_P \longrightarrow \quad P = \frac{mT}{(m+1)L_P}$$

1
PE
Example
2
Motivation
3
Delay
Freq. App.
SW1PerS
4
Results
5
Summary

Finding the Delay



$$L_P \longrightarrow P = \frac{mT}{(m+1)L_P} \longrightarrow \tau = \frac{f_s P}{\alpha}$$

Results

System	Method		Suggested Delay
	SW1PerS	Frequency Approach	
Lorenz	6-12	5-11	10
Rössler	15-30	4-9	9
Bi-directional Rössler	5-9	8-16	15
Mackey-glass	7-13	2-4	10
Sine Wave	15-30	14-29	20
EEG	2-3	3-7	3
ECG	7-15	8-16	4

Müller Riedl, A Müller, and N Wessel. Practical considerations of permutation entropy. The European Physical Journal Special Topics, 222 (2): 249–262, 2013.

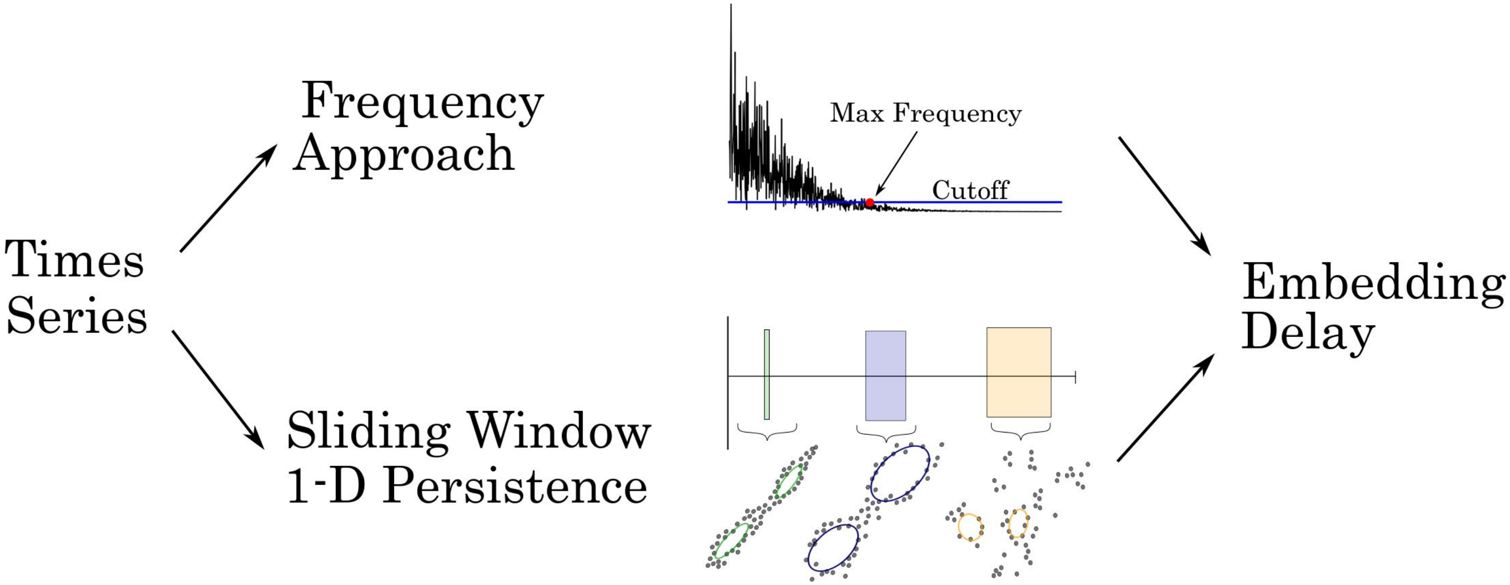
Results

System	Method		Suggested Delay
	SW1PerS	Frequency Approach	
Lorenz	6-12	5-11	10
Rössler	15-30	4-9	9
Bi-directional Rössler	5-9	8-16	15
Mackey-glass	7-13	2-4	10
Sine Wave	15-30	14-29	20
EEG	2-3	3-7	3
ECG	7-15	8-16	4

Müller Riedl, A Müller, and N Wessel. Practical considerations of permutation entropy. The European Physical Journal Special Topics, 222 (2): 249–262, 2013.

Summary

- What we did:



Times Series

Frequency Approach

Sliding Window 1-D Persistence

Embedding Delay

Thank you!

✉ myersau3@msu.edu

- Myers, Khasawneh. Delay Parameter Selection in Permutation Entropy Using Topological Data Analysis, arXiv: 1905.04329, 2019.
- Myers, Khasawneh. On the Automatic Parameter Selection for Permutation Entropy, arXiv: 1905.06443, 2019.
- Myers, Munch, Khasawneh. *Persistent Homology of Complex Networks for Dynamic State Detection*, arXiv:1904.07403, 2019.

Firas Khasawneh

Elizabeth Munch

MICHIGAN STATE UNIVERSITY

CDS&E-MSS # 1759824

Dynamics, Control and System # 1759823