SIAM Analysis of Partial Differential Equations

Invariant measures for passive scalars in the small noise inviscid limit

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2D turbulence

Navier-Stokes equations:

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = f, \qquad \mathrm{div}\boldsymbol{u} = 0$$

Euler equations:

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = f, \qquad \mathrm{div}\boldsymbol{u} = 0$$

Problem: understand the limit as $\nu \to 0$.

Inviscid limit

There is a huge literature on the problem, and many points of view. Here:

- No boundaries: we will consider the periodic domain $\mathbb{T}^d = [0, 2\pi]^d$ and mean free solutions.
- Longtime behavior: qualitative description with "infinite time" objects, such as attractors (in weak topologies) and invariant measures.
- Stochastic forcing: encodes a sufficiently generic behavior, extremely helpful from the invariant measure viewpoint.

Invariant measures

In vorticity formulation (d=2) we have: $\omega = \partial_x u_2 - \partial_y u_1$.
Navier-Stokes equations:

$$\omega_t + \boldsymbol{u} \cdot \nabla\omega - \nu\Delta\omega = g$$

Euler equations:

$$\omega_t + \boldsymbol{u} \cdot \nabla\omega = g$$

GOAL: construction of "meaningful" invariant measures for inviscid equations (having in mind Euler).
EXAMPLE: free Euler ($g = 0$) defines a weak-$*$ continuous semigroup $S(t) : B_{L^\infty}(R) \to B_{L^\infty}(R)$. Does there exist an invariant measure whose support is the whole weak-$*$ attractor?

Kuksin approach

Stochastic forcing balanced with viscosity (Freidlin-Wentzell scaling). Navier-Stokes equations:

$$\mathrm{d}\omega + [\boldsymbol{u}\cdot\nabla\omega - \nu\Delta\omega]\mathrm{d}t = \sqrt{\nu}\,\Psi\,\mathrm{d}W_t = \sqrt{\nu}\sum_{k\in\mathbb{N}}\psi_k e_k\,\mathrm{d}W_t^k$$

with

$$\|\Psi\|^2 = \sum_{k\in\mathbb{N}}|\psi_k|^2 < \infty.$$

For each $\nu > 0$, there exists a (unique) invariant measure μ_ν on L^2 (Flandoli, Maslowski, Eckmann, Hairer, Mattingly, Ferrario...).

Properties of invariant measures (I)

- Compactness: $\{\mu_\nu\}$ is tight, hence $\mu_\nu \to \mu_0$ up to subsequences. The possibly (non-unique) measure μ_0 is invariant for free Euler (Kuksin measure).
- μ_0 satisfies (Kuksin, Shirikyan)

$$\int_{L^2} \|\nabla\zeta\|^2_{L^2} \mathrm{d}\mu_0(\zeta) < \infty$$

and (Glatt-Holtz, Sverak, Vicol)

$$\int_{L^2} \|\zeta\|^2_{L^\infty} \mathrm{d}\mu_0(\zeta) < \infty$$

Properties of invariant measures (II)

Morever

- μ_0 canNOT be concentrated on points. This is a consequence of balance relations (ω_S^ν is a statistically stationary solution with distribution μ_ν)

$$\mathbb{E}\|\omega_S^\nu\|_{L^2}^2 = \int_{L^2} \|\zeta\|_{L^2}^2 \mathrm{d}\mu_\nu(\zeta) = \text{ constant}$$

and

$$\mathbb{E}\|\nabla\omega_S^\nu\|_{L^2}^2 = \int_{L^2} \|\nabla\zeta\|_{L^2}^2 \mathrm{d}\mu_\nu(\zeta) = \text{ constant.}$$

Is it possible to say more on the support of μ_0, in equations with similar structure, for example linear transport equations?

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Setting

For $x \in \mathbb{T}^d$ and $t \geq 0$, we study

$$\partial_t f + \boldsymbol{u} \cdot \nabla f = 0, \qquad f(0) = f_0.$$

Here $\boldsymbol{u} = \boldsymbol{u}(x) : \mathbb{T}^d \to \mathbb{R}^d$ is a *given* Lipschitz, divergence-free, time-independent velocity vector field. We call $\{S(t)\}_{t \in \mathbb{R}}$ the family of linear solution operators $S(t) : L^2 \to L^2$ acting as

$$f_0 \mapsto S(t) f_0 = f(t),$$

fulfilling the group properties

$$S(0) = \mathrm{Id}_{L^2}, \qquad S(t+\tau) = S(t)S(\tau), \qquad S(t)^* = S(-t), \qquad \forall t, \tau \in \mathbb{R}.$$

Spectral properties

We define the closed subspace

$$E = \overline{\mathrm{span}\left\{\varphi \in H^1 : \boldsymbol{u} \cdot \nabla\varphi = i\lambda\varphi,\ \lambda \in \mathbb{R}\right\}}^{L^2},$$

generated by H^1-eigenfunctions of $\boldsymbol{u} \cdot \nabla$. We can then write $L^2 = E \oplus E^\perp$ and denote by

$$\Pi_e : L^2 \to E \quad \text{and} \quad \Pi_e^\perp : L^2 \to E^\perp.$$

These subspaces are invariant under $S(t)$.

Rigidity

Theorem (BCZGH '15)

Let $\mathcal{K} \subset H^1 \cap E^\perp$ be a nonempty compact set in L^2 such that $0 \notin \mathcal{K}$. Then the solution operator $S(t) : L^2 \to L^2$ satisfies

$$\lim_{T\to\infty} \inf_{f_0\in\mathcal{K}} \frac{1}{T}\int_0^T \|S(t)f_0\|_{H^1}^2 \mathrm{d}t = \infty.$$

Some ingredients of proof: spectral theory on $E^\perp$, RAGE theorem to control growth of continuous spectrum of $\boldsymbol{u} \cdot \nabla$, estimates on linear evolution on the point spectrum from Constantin, Kiselev, Ryzhik, Zlatos (CKRZ, 2008).

Consequences for invariant measures

Corollary (BCZGH '15)

Let μ_0 be an invariant measure for $S(t)$ such that

$$\int_{L^2} \|\zeta\|_{H^1}^2 \mathrm{d}\mu_0(\zeta) < \infty.$$

Then $\mu_0(H^1 \cap E) = 1$. In particular, $\mathrm{spt}(\mu_0) \subset E$.

Short proof: if not, there is a compact set $\mathcal{K} \subset E^\perp$ with positive measure. From $\mathcal{K}$, solutions grow in H^1 and eventually violate the above condition.

A few remarks

- Same results hold for dynamical systems posed on (finite dimensional) Riemannian manifolds without boundaries.
- Nothing special about H^1. Everything works by replacing with H^s, $s > 0$.
- If H^s-growth can be proved from a compact set, then that compact set has measure 0 (nonlinear problems too).

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Stochastically forced viscous problems

The (Kuksin) measures μ_0 can be constructed as limits of

$$\mathrm{d}f + (\boldsymbol{u} \cdot \nabla f - \nu \Delta f)\,\mathrm{d}t = \sqrt{\nu}\,\Psi\,\mathrm{d}W_t = \sqrt{\nu}\sum_{k\in\mathbb{N}} \psi_k e_k\,\mathrm{d}W_t^k.$$

- we have $\mu_\nu = \mathcal{N}(0, Q_\nu)$, a Gaussian centered at 0 with covariance operator given by ($S_\nu(t)$ is the linear viscous semigroup)

$$Q_\nu = \nu \int_0^\infty S_\nu(t)\Psi\Psi^* S_\nu(t)^* \mathrm{d}t.$$

- any statistically stationary solution obeys

$$\mathbb{E}\|f_S^\nu(t)\|_{H^1}^2 = \int_{L^2} \|\zeta\|_{H^1}^2 \mathrm{d}\mu_\nu(\zeta) = \frac{1}{2}\|\Psi\|^2, \qquad \forall t \geq 0.$$

- Letting $\nu \to 0$ we have

$$\int_{L^2} \|\zeta\|_{H^1}^2 \mathrm{d}\mu_0(\zeta) \leq \frac{1}{2}\|\Psi\|^2.$$

Main result

Theorem (BCZGH '15)

Let μ_0 be a Kuksin measure for the linear inviscid problem. Then

- $\mu_0(L^\infty \cap H^1 \cap E) = 1.$
- $\mu_0 = \mathcal{N}(0, Q_0)$, *where Q_0 is a limit point of $\{Q_\nu\}_{\nu \in (0,1]}$ in the weak operator topology.*

Recall $E = \overline{\operatorname{span}\{\varphi \in H^1 : \boldsymbol{u} \cdot \nabla \varphi = i\lambda\varphi,\ \lambda \in \mathbb{R}\}}^{L^2}$.

Consequences

- What can we say about E, given a flow $\boldsymbol{u}$? Example: if $\boldsymbol{u}$ is weakly mixing (only continuous spectrum), then $\mu_0 = \delta_0$. Very different from Euler (no H^{-1} conservation)
- Kuksin measures should mostly retain information about the long-time dynamics of the large scales in the solutions, rather than information about the "enstrophy" in the small scales.
- Decay properties of $S_\nu(t)$ affect the covariance Q_ν and hence its limit $\nu \to 0$.
- The rate at which L^2 density is dissipated by $S_\nu(t)$ is crucial: if dissipation happens at scales faster than $O(\nu^{-1})$, then small scales are created and then rapidly annihilated by the dissipation.

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Relaxation enhancing flows

$\boldsymbol{u} : \mathbb{T}^d \to \mathbb{R}^d$ is called relaxation enhancing if for every $\tau > 0$ and $\delta > 0$, there exists $\nu_0 = \nu_0(\tau, \delta)$ such that for any $\nu < \nu_0$ and any $f_0 \in L^2$ we have

$$\|S_\nu(\tau/\nu) f_0\|_{L^2} < \delta \|f_0\|_{L^2}.$$

From (CKRZ, 2008),

$$\boldsymbol{u} \text{ relaxation enhancing} \;\Leftrightarrow\; \text{no eigenfunctions in } H^1$$

In particular, weakly mixing flows are relaxation enhancing.

Theorem (Bedrossian, CZ, Glatt-Holtz '15)

Let $\boldsymbol{u}$ be a relaxation enhancing flow. Then δ_0, the Dirac mass centered at zero, is the unique Kuksin measure for the linear inviscid evolution $S(t)$.

Shear flows

Take $\boldsymbol{u} = (u(y), 0)$, we assume that $u \in C^{n_0+2}(\mathbb{T})$ has a *finite* number of critical points, where $n_0 \in \mathbb{N}$ denotes the maximal order of vanishing of u' at the critical points.

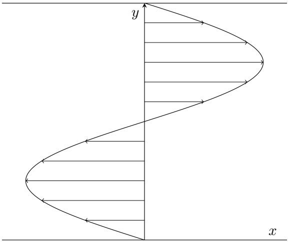

Figure: $u(y) = \sin y$

Shear flows

Consider more generally

$$\mathrm{d}f + (u(y)\partial_x f - \nu\Delta f)\,\mathrm{d}t = \nu^{a/2}\,\Psi\,\mathrm{d}W_t, \qquad a \in [0,1]$$

with

$$\Psi \mathrm{d}W_t = \sum_{(k,j)\in\mathbb{Z}^2} \psi_{k,j} e_{k,j} \mathrm{d}W_t^{k,j}, \quad e_{k,j} = \frac{1}{4\pi^2}\mathrm{e}^{-ikx-ijy}.$$

Theorem (Bedrossian, CZ, Glatt-Holtz '15, $a = 1$)

The resulting Kuksin measure is given uniquely by a Gaussian $\mathcal{N}(0, Q_0)$ with

$$Q_0\varphi = \sum_{j\neq 0} \frac{|\psi_{0,j}|^2}{2|j|^2} \langle e_{0,j}, \varphi\rangle e_{0,j}, \qquad \varphi \in L^2.$$

Shear flows

Theorem (Bedrossian, CZ '15, $a < 1$)

Suppose that $\psi_{0,j} = 0$ for all $j \in \mathbb{Z}$, and let the parameter a satisfy

$$a \in \left(\frac{n_0 + 1}{n_0 + 3}, 1 \right].$$

Then, as $\nu \to 0$, we have that $\mu_\nu \to \delta_0$, a Dirac mass centered at 0, in the sense of measures.

Rather surprising, any statistically stationary (with respect to the invariant measure μ_ν) solution f_S^ν satisfies

$$\mathbb{E}\|f_S^\nu(t)\|_{H^1}^2 = \frac{\nu^{a-1}}{2}\|\Psi\|^2, \qquad \forall t \geq 0.$$

Shear flows

Based on

Theorem (Bedrossian, CZ '15)

We have

$$\|S_\nu(t)P_k\|_{L^2\to L^2} \leq C\mathrm{e}^{-\varepsilon\lambda_{\nu,k}t}, \qquad \forall t \geq 0,$$

where P_k denotes the projection to the k-th Fourier mode in x and

$$\lambda_{\nu,k} = \frac{\nu^{\frac{n_0+1}{n_0+3}}|k|^{\frac{2}{n_0+3}}}{(1+\log|k|+\log\nu^{-1})^2}$$

is the decay rate.

THANK YOU