INVASION FRONTS ON GRAPHS

Matt Holzer (George Mason University)

Collaborator:
Aaron Hoffman (Olin)

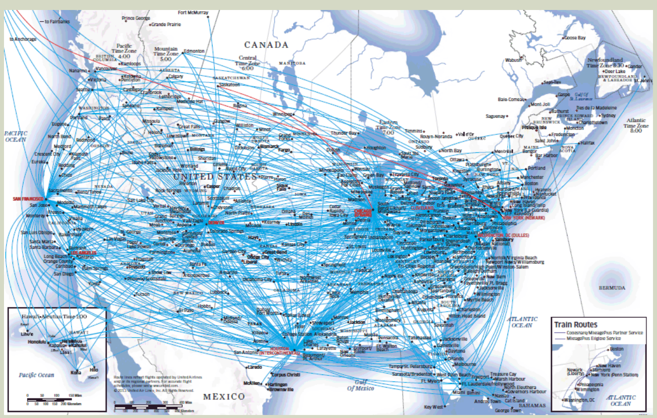
BIG PICTURE

Reaction-diffusion equations on networks

- Network given as a graph $G = (V, E)$
- Local reaction prescribed by an ordinary differential equation

Main Question: Given an unstable homogeneous state, if a local perturbation is applied how quickly does that perturbation spread through the network?

Motivating example: invasive species on a transportation network



Complications: heterogeneities,
network topology,
transport mechanism

SMALL PICTURE

We focus on the Fisher-KPP equation on a graph

$$u_t = \alpha \Delta_G u + u(1-u)$$

- $G = (V, E)$ is a undirected, unweighted graph

$$\Delta_G = A(G) - D(G)$$

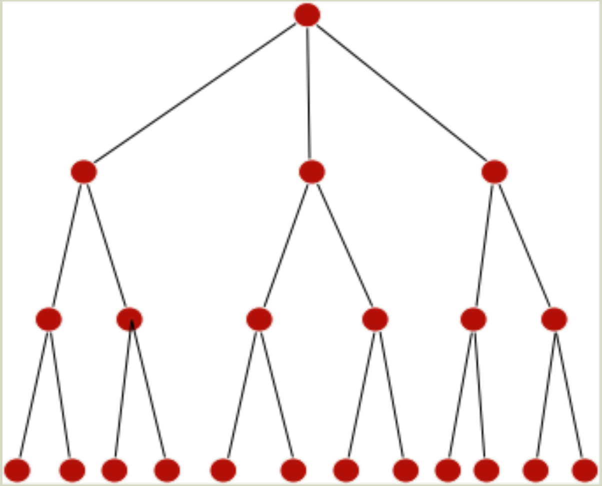
- Initial conditions – $u_1(0) = 1$, $u_j(0) = 0$ otherwise

We study two classes of graphs

- Homogeneous Trees
- Erdős Réyni random graphs

THE HOMOGENOUS TREE

Consider the infinite homogeneous tree of degree $k+1$



Due to symmetry we consider only a representative element $u_n(t)$, distance $n-1$ from the root

$$\frac{du_n}{dt} = \alpha\left(u_{n-1} - (k+1)u_n + ku_{n+1}\right) + u_n - u_n^2.$$

$$\frac{du_1}{dt} = \alpha(k+1)(-u_1 + u_2)) + u_1 - u_1^2.$$

KEY QUESTIONS

1. What is the behavior of $u_n(t)$, i.e. does $u_n(t)$ converge to one or zero?

2. How does the total population

$$P(t) = \sum_{n \in \mathbb{N}} k^{n-1} u_n(t)$$

evolve in time? The linear problem has exponential growth rate e^t, but what about the nonlinear equation?

EXISTENCE/NONEXISTENCE OF INVASION FRONTS

Factoring

$$\frac{du_n}{dt} = \alpha\left(u_{n-1} - 2u_n + u_{n+1}\right) + \alpha(k-1)(u_{n+1} - u_n) + u_n - u_n^2.$$

Two Observations

- If $\alpha \ll 1$, then reaction dominates and we expect traveling fronts
- If $\alpha \gg 1$, $\alpha = \frac{1}{(\Delta x)^2}$ the system can be viewed as a discretization of the PDE

  $$u_t = u_{xx} + \frac{k-1}{\Delta x} u_x + u(1-u),$$

  advection dominates and solution converges to zero

RELATED WORKS

- Fisher-KPP front propagation – many works

  Weinberger – linear determinacy

  Matano, Punzo, Tesei – propagation/extinction in hyperbolic space

- Reaction-Diffusion on Networks (meta-population models)

  Burioni, Chibbaro, Vergni, Vulpiani – population growth rates for Erdős Réyni graphs

    Observed sub-linear population growth rates

  Kouvaris, Kori, Mikhailov – pinned fronts for bistable reactions

  Brockmann, Helbing – SIR model

POINTWISE GROWTH AND LINEAR SPREADING SPEED

Linearize about zero state

$$\frac{du_n}{dt} = \alpha\left(u_{n-1} - (k+1)u_n + ku_{n+1}\right) + u_n,$$

Dispersion relation from ansatz $u_n(t) = e^{\lambda t}e^{-\gamma(n-st)}$

$$d_s(\lambda,\gamma) = \alpha\left(e^{\gamma} - k - 1 + ke^{-\gamma}\right) + s\gamma + 1 - \lambda$$

Defn: A *pinched double root* is a pair (λ^*, γ^*) for which

$$d_s(\lambda,\gamma) = 0, \quad \partial_\gamma d_s(\lambda,\gamma) = 0$$

λ^* gives the pointwise growth/decay rate in a frame moving with speed s_{lin}

See references related to absolute/convective instabilities – Bers, Briggs, etc.

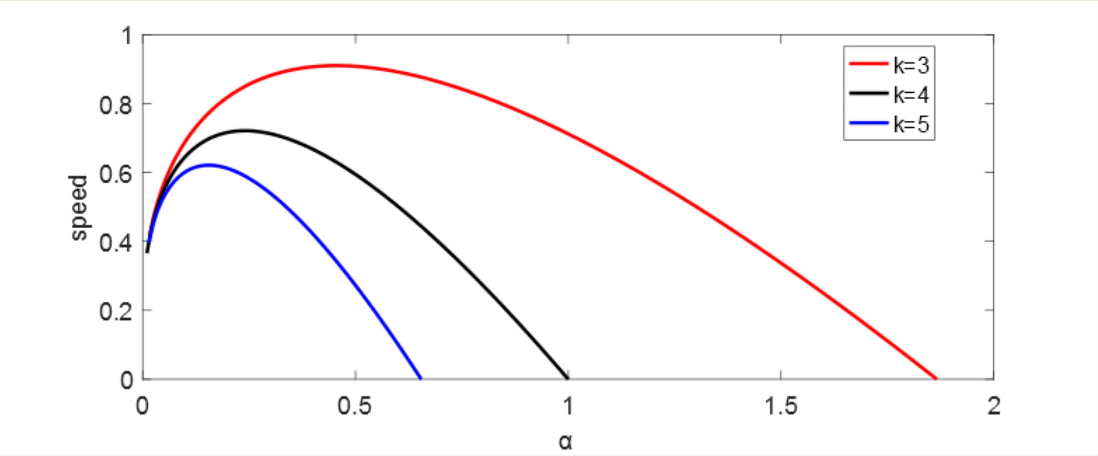
THE LINEAR SPREADING SPEED

Defn: The *linear spreading speed* is a solution of

$$d_{s_{lin}}(0,\gamma) = 0, \quad \partial_\gamma d_{s_{lin}}(0,\gamma) = 0,$$

Thus, we need to solve the system

$$F(s,\gamma) = \begin{pmatrix} \alpha\left(e^{\gamma} - k - 1 + ke^{-\gamma}\right) - s\gamma + 1 \\ \alpha\left(e^{\gamma} - ke^{-\gamma}\right) - s \end{pmatrix} = 0$$



CRITICAL DIFFUSION COEFFICIENTS

For $s_{lin} = 0$ we solve

$$\begin{pmatrix} \alpha\left(e^{\gamma} - k - 1 + ke^{-\gamma}\right) + 1 \\ \alpha\left(e^{\gamma} - ke^{-\gamma}\right) \end{pmatrix} = 0$$

We find solution for

$$\gamma_2 = \frac{\log(k)}{2}$$

$$\alpha_2 = \frac{1}{k + 1 - 2\sqrt{k}}$$

Note that γ_2 is l^2-critcal

$$||u||_2 = \left(\sum_{n \in \mathbb{N}} k^{n-1} u_n^2(t)\right)^{\frac{1}{2}}$$

For s_{lin} maximal we observe

$$\gamma_1 = \log(k)$$

$$\alpha_1 = \frac{1}{(k-1)\log(k)}$$

$$s_1 = \frac{1}{\log(k)}$$

solves $F(s_1, \gamma_1, \alpha_1) = 0$

Note that γ_1 is l^1-critcal

$$||u||_1 = \sum_{n \in \mathbb{N}} k^{n-1} u_n(t)$$

FURTHER PROPERTIES

- Spreading speed is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha_1 < \alpha < \alpha_2$
- Spreading speed for small α is independent of k to leading order

$$s_{lin} \sim \frac{1}{W\left(\frac{1}{\alpha}\right)} \text{ as } \alpha \to 0$$

- Properties generalize to periodic trees

  l^2-critical front stationary, l^1-critical front maximal

- **Theorem**: Selected spreading speed in the nonlinear system is the linear spreading speed.

  Proof by construction of sub and super solutions

TOTAL POPULATION DYNAMICS

Let $w_n(t) = k^{n-1}u_n(t)$. Linearization near zero is

$$\frac{dw_n}{dt} = \alpha\left(kw_{n-1} - (k+1)w_n + w_{n+1}\right) + w_n,$$

Dispersion relation

$$\tilde{d}_s(\lambda, \gamma) = \alpha\left(ke^{\gamma} - (k+1) + e^{-\gamma}\right) + s\gamma + 1 - \lambda$$

Calculation: If (λ^*, γ^*) is a pinched double root of $d_s(\lambda, \nu)$, then

$$(\lambda^* + s\log(k), \gamma^* - \log(k))$$

is a pinched root of $\tilde{d}_s$.

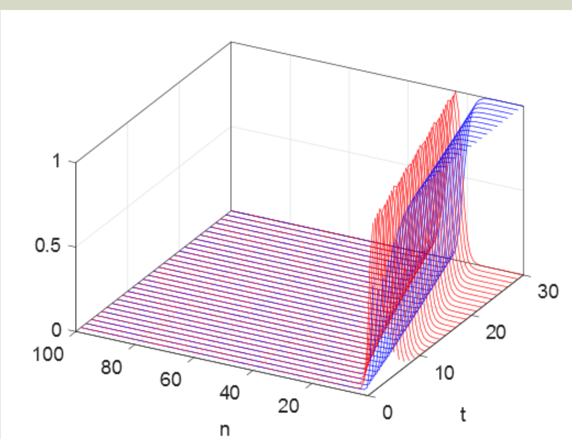
POINTWISE VERSUS POPULATION BEHAVIOR

Lemma The maximal linear growth rate is one and is achieved in a frame moving with

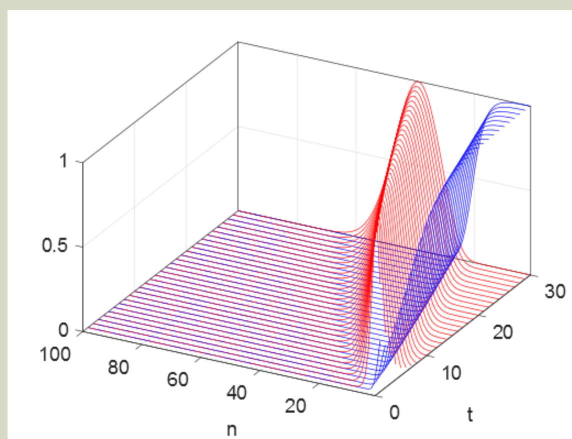
$$s^* = \alpha(k-1),$$

for which $s^* < s_1$ for $\alpha < \alpha_1$ and $s^* > s_1$ for $\alpha > \alpha_1$.

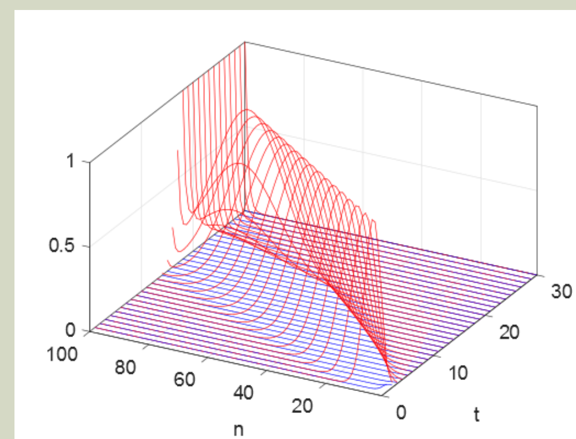
$\alpha = 0.2$



$\alpha = 0.8$



$\alpha = 2.2$



CONCLUSIONS: NON-MONOTONE SPEEDS

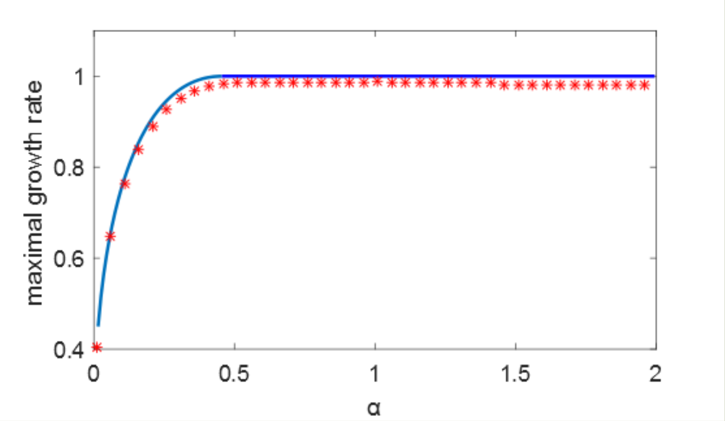
- For $\alpha < \alpha_1$, the maximal growth rate occurs in a frame moving slower than the linear spreading speed.

  Most of the population is added at the front interface

  Sublinear growth rates for the total population

- For $\alpha > \alpha_1$ the maximal growth rate is achieved ahead of the front interface

  Solution remains near zero and the system can achieve its maximum growth rate of one



FISHER-KPP ON RANDOM GRAPH

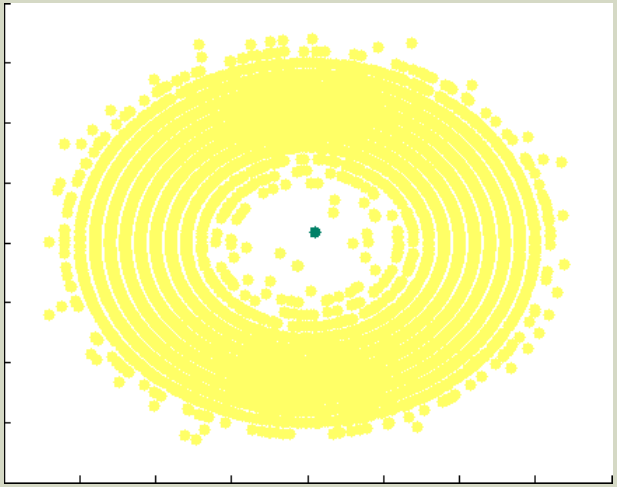
Consider Erdos-Reyni random graph $G = (V, E)$

$$u_t = \alpha \Delta_G u + u(1-u)$$

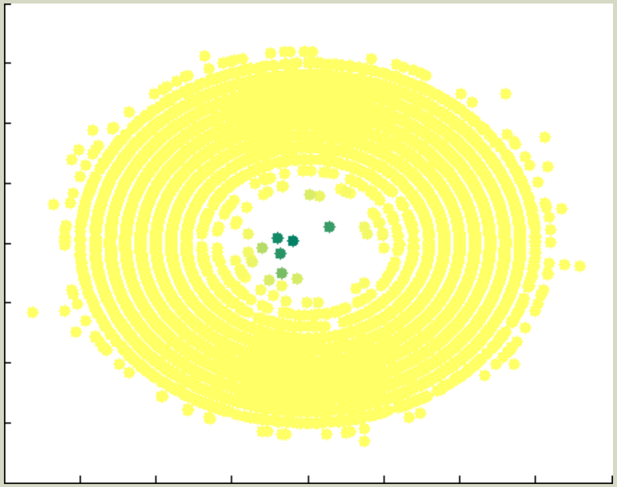
Nodes are assigned randomly with probability p

Example: $N = 600000$, $p = \frac{3}{60000}$

$\alpha = .01$



$\alpha = 2.0$



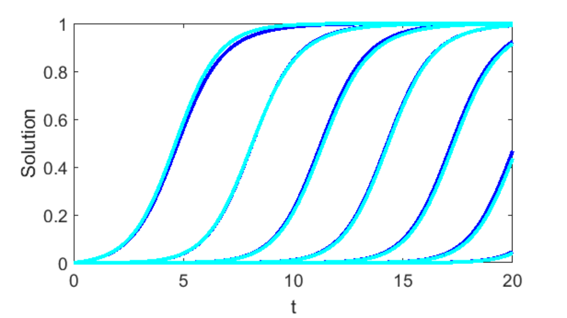
POINTWISE COMPARISON – NUMERICAL SIMULATIONS

$p = \frac{3}{60000}$

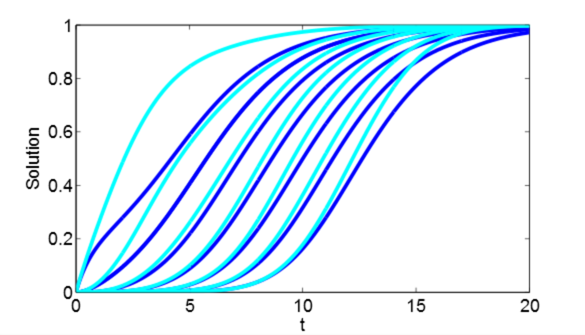
blue – tree with $k = 3$

cyan – mean over each level of random graph

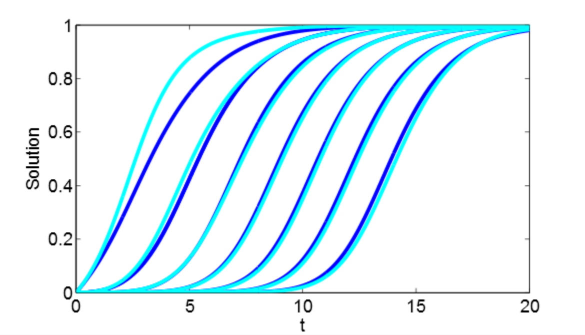
$\alpha = 0.01$



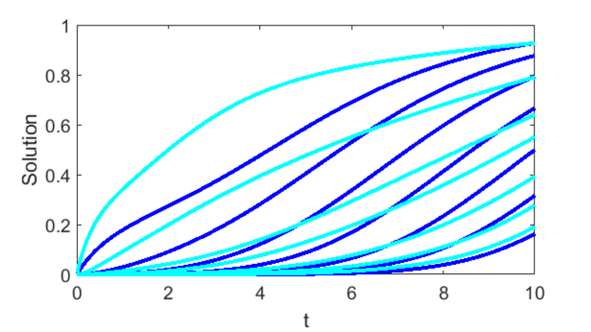
$\alpha = 0.3$



$\alpha = 0.1$



$\alpha = 0.8$

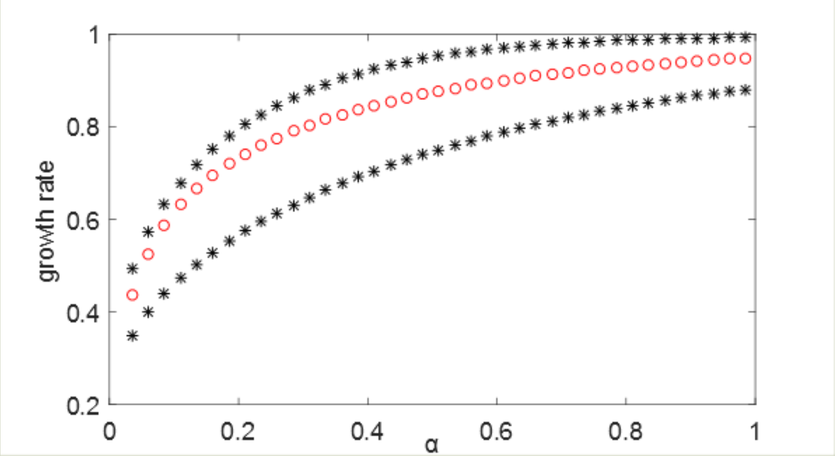


ERDOS-REYNI GRAPHS

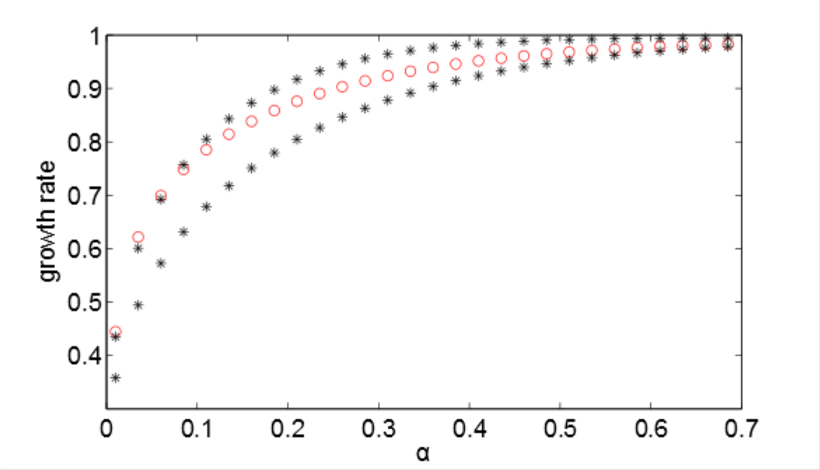
Exponential growth rates of $P(t)$ for Erdos-Reyni random graphs with N large and k small

These rates can be compared with those observed on the homogeneous tree

$N = 60000 \qquad p = \frac{3}{60000}$



$N = 500,000 \qquad p = \frac{4}{500,000}$



REFERENCES

References

A. Hoffman and M.H. *Invasion fronts on graphs.* under review (2016).

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