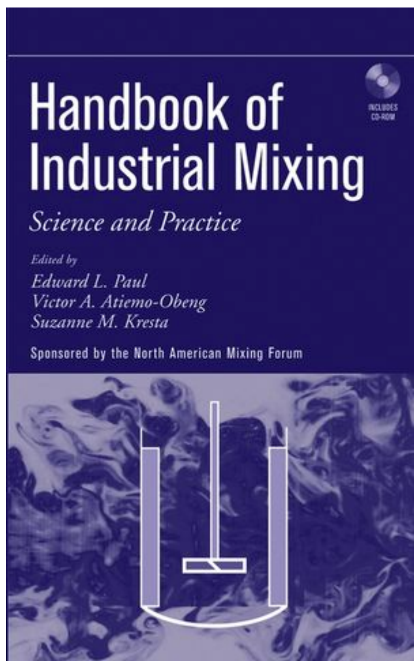
Diffusion-limited mixing by incompressible flows

*Emmanuelle Gouillart, et al. PRL 2007

Christopher J. Miles
Charles R. Doering
University of Michigan


INCLUDES CD-ROM
Handbook of Industrial Mixing
Science and Practice
Edited by
Edward L. Paul
Victor A. Atiemo-Obeng
Suzanne M. Kresta
Sponsored by the North American Mixing Forum

What do we want to know about mixing?

How do we know we are doing well at mixing?

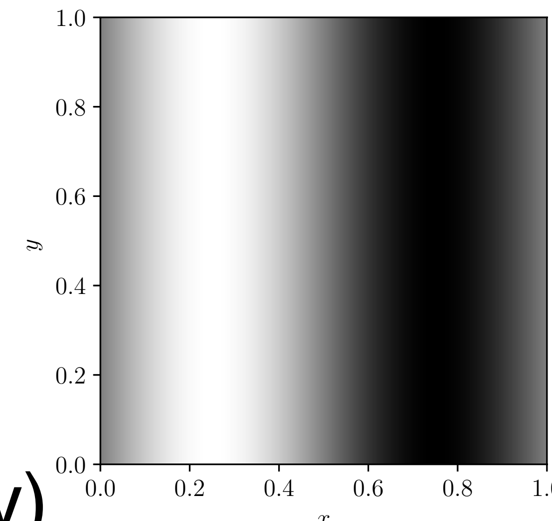
How do we improve our mixing performance through stirring?

What is the effect of diffusion on mixing?

Setup of the problem:

Advection-diffusion equation:

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta \qquad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x})$$



Question: What velocity field mixes well?

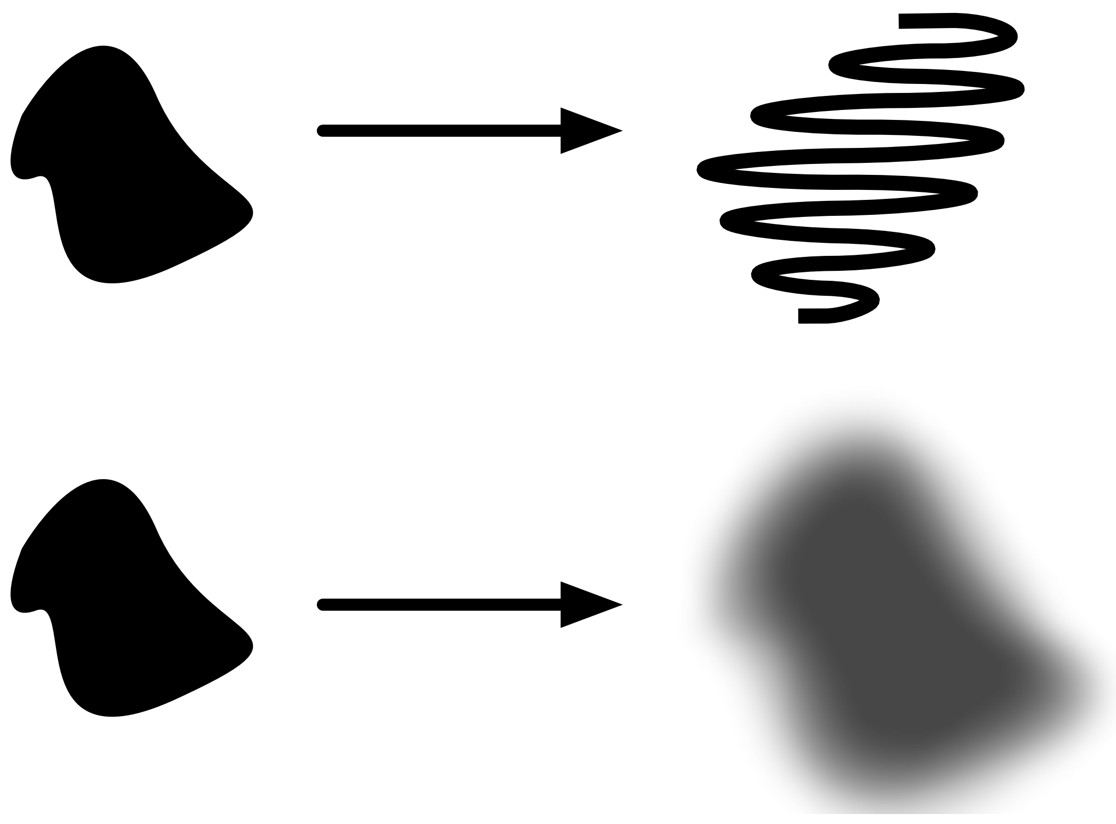
Incompressibility: $\nabla \cdot \mathbf{u} = 0$

Stirring budget: $\int_D d^d\mathbf{x}\, |\mathbf{u}|^2 = U^2 L^d$ (energy)

or

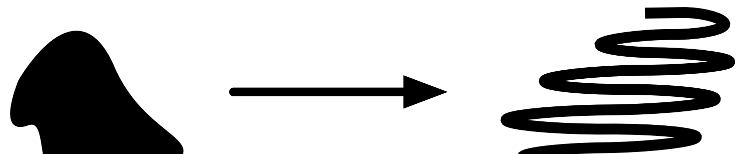
$\int_D d^d\mathbf{x}\, |\nabla \mathbf{u}|^2 = \Gamma^2 L^d$ (enstrophy)

How do we measure mixing?



How do we measure mixing?

$$\|\theta\|_{H^{-1}} = \sqrt{\sum_{\mathbf{k}\neq\mathbf{0}} L^d \frac{|\hat{\theta}_{\mathbf{k}}(t)|^2}{|\mathbf{k}|^2}}$$



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Physica D 211 (2005) 23–46

PHYSICA D

www.elsevier.com/locate/physd

A multiscale measure for mixing

George Mathew*, Igor Mezić, Linda Petzold

Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, CA 93106, USA

Received 23 March 2005; received in revised form 26 July 2005; accepted 28 July 2005

Available online 19 August 2005

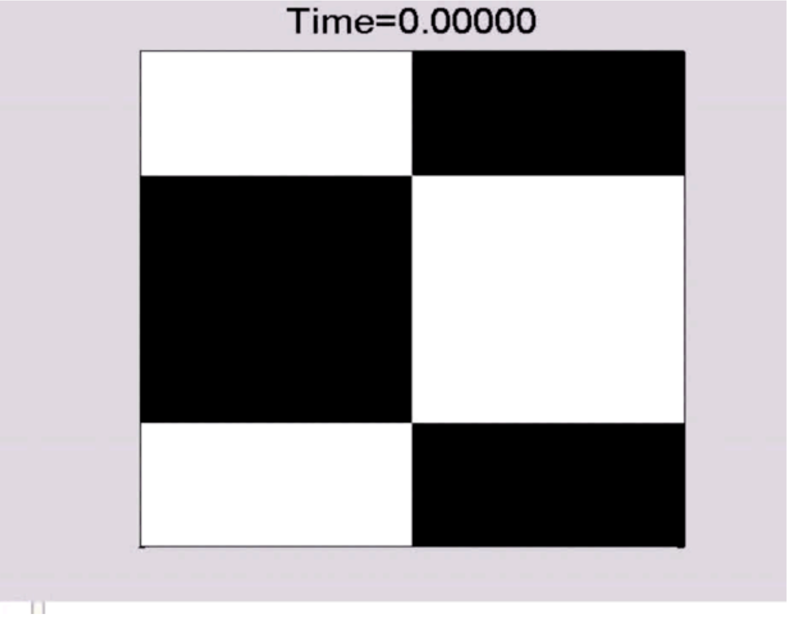
Communicated by C.K.R.T. Jones

For energy constraint without diffusion, perfect mixing in finite time is possible.

JOURNAL OF MATHEMATICAL PHYSICS **53**, 115611 (2012)

Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows

Evelyn Lunasin,[1] Zhi Lin,[2,a)] Alexei Novikov,[3] Anna Mazzucato,[3] and Charles R. Doering[4]



For *enstrophy* constraint without diffusion, perfect mixing in finite time is *not* possible.

IOP Publishing | Nonlinearity
Nonlinearity **26** (2013) 3279–3289 | doi:10.1088/0951-7715/26/12/3279

Maximal mixing by incompressible fluid flows

Christian Seis

Department of Mathematics, University of Toronto, 40 St. George Street, M5S 2E4, Toronto, Ontario, Canada

IOP Publishing | London Mathematical Society | Nonlinearity
Nonlinearity **27** (2014) 973–985 | doi:10.1088/0951-7715/27/5/973

Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows

Gautam Iyer[1], Alexander Kiselev[2] and Xiaoqian Xu[2,3]

[1] Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213, USA
[2] Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

Local-in-time (LIT) optimization:

$$\min_{\mathbf{u}} \frac{d}{dt} \|\theta(\,\cdot\,, t)\|^2_{H^{-1}}$$

Instantaneous flow intensity budget constraints:

(energy) $$\int_D d^d\mathbf{x}\, |\mathbf{u}|^2 = U^2 L^d$$

or

(enstrophy) $$\int_D d^d\mathbf{x}\, |\nabla\mathbf{u}|^2 = \Gamma^2 L^d$$

I. A Shell Model for Mixing

J Nonlinear Sci
DOI 10.1007/s00332-017-9400-7

Journal of Nonlinear Science

CrossMark

A Shell Model for Optimal Mixing

Christopher J. Miles[1,2,3] · Charles R. Doering[1,2,3]

Received: 1 November 2016 / Accepted: 2 June 2017

Abstract What is the maximum mixing efficiency of an incompressible flow? To

PDE

$$\partial_t \hat{\theta}(\mathbf{k}, t) + i \sum_i \sum_{\mathbf{k}' \in K} \hat{u}_i(\mathbf{k} - \mathbf{k}', t)\, k_i' \,\hat{\theta}(\mathbf{k}', t) + \kappa\, \mathbf{k}^2 \hat{\theta}(\mathbf{k}, t) = 0.$$

Shell Model

$$\frac{d}{dt}\theta_n - k_{n-1} u_{n-1} \theta_{n-1} + k_n u_n \theta_{n+1} + \kappa\, k_n^2 \theta_n = 0, \quad n = 1, 2, \ldots,$$

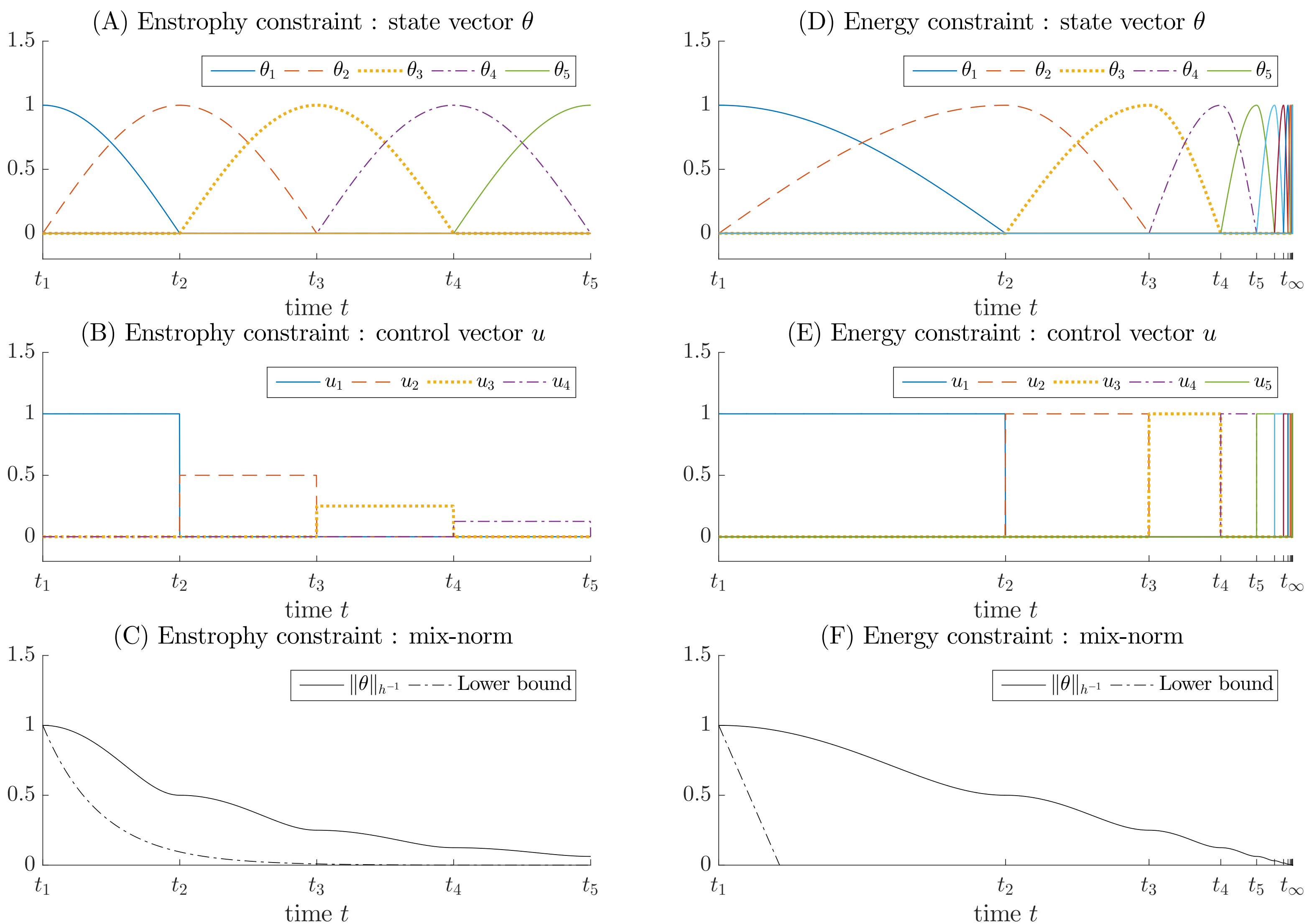
Local-in-time optimization

$$\min_u \frac{d}{dt} \sum_n \frac{\theta_n^2}{k_n^2}$$

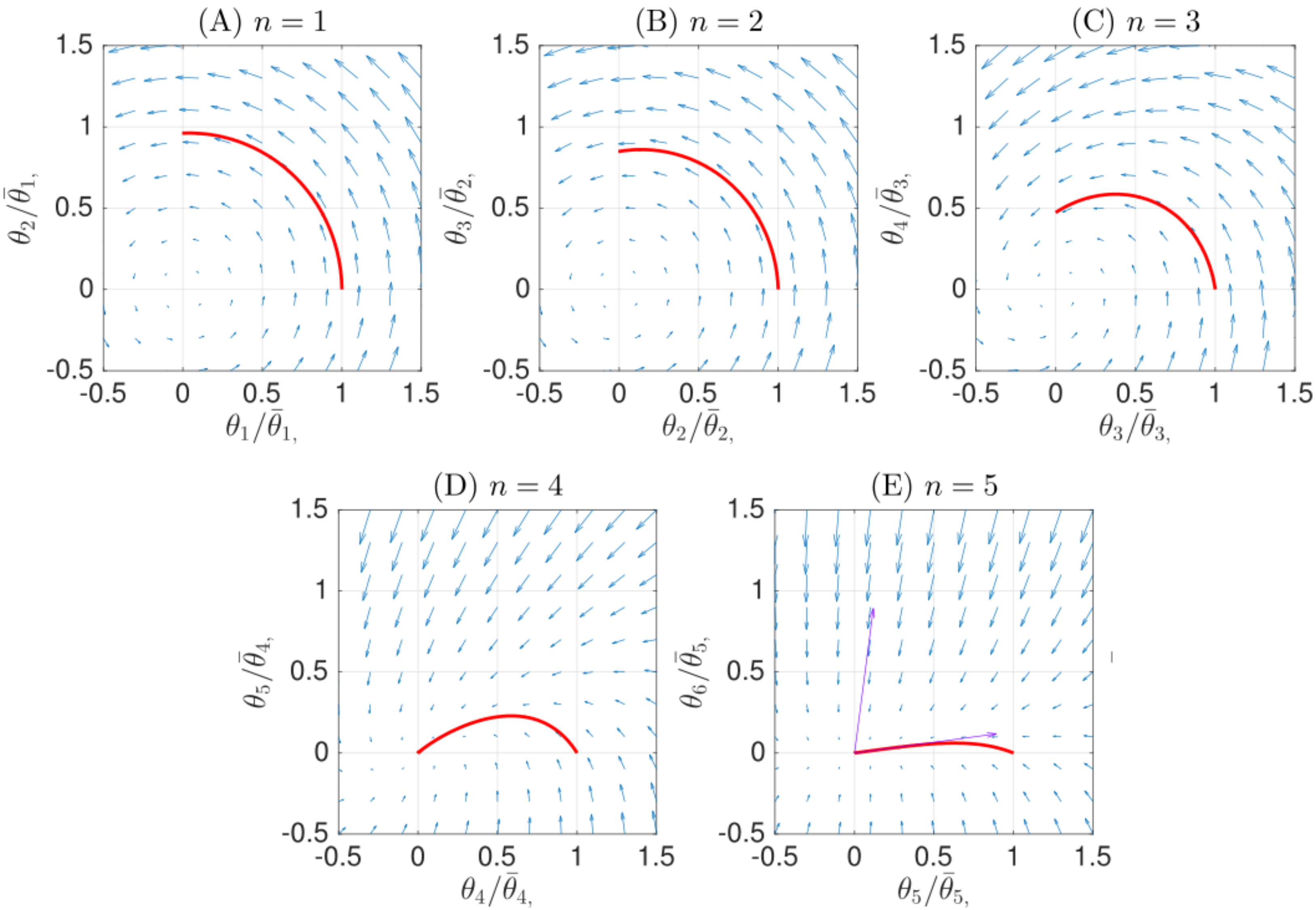
$$\text{Energy:} \qquad \sum_n u_n^2 = U^2$$

$$\text{Enstrophy:} \qquad \sum_n k_n^2 u_n^2 = \Gamma^2$$

Local-in-time optimization without diffusion

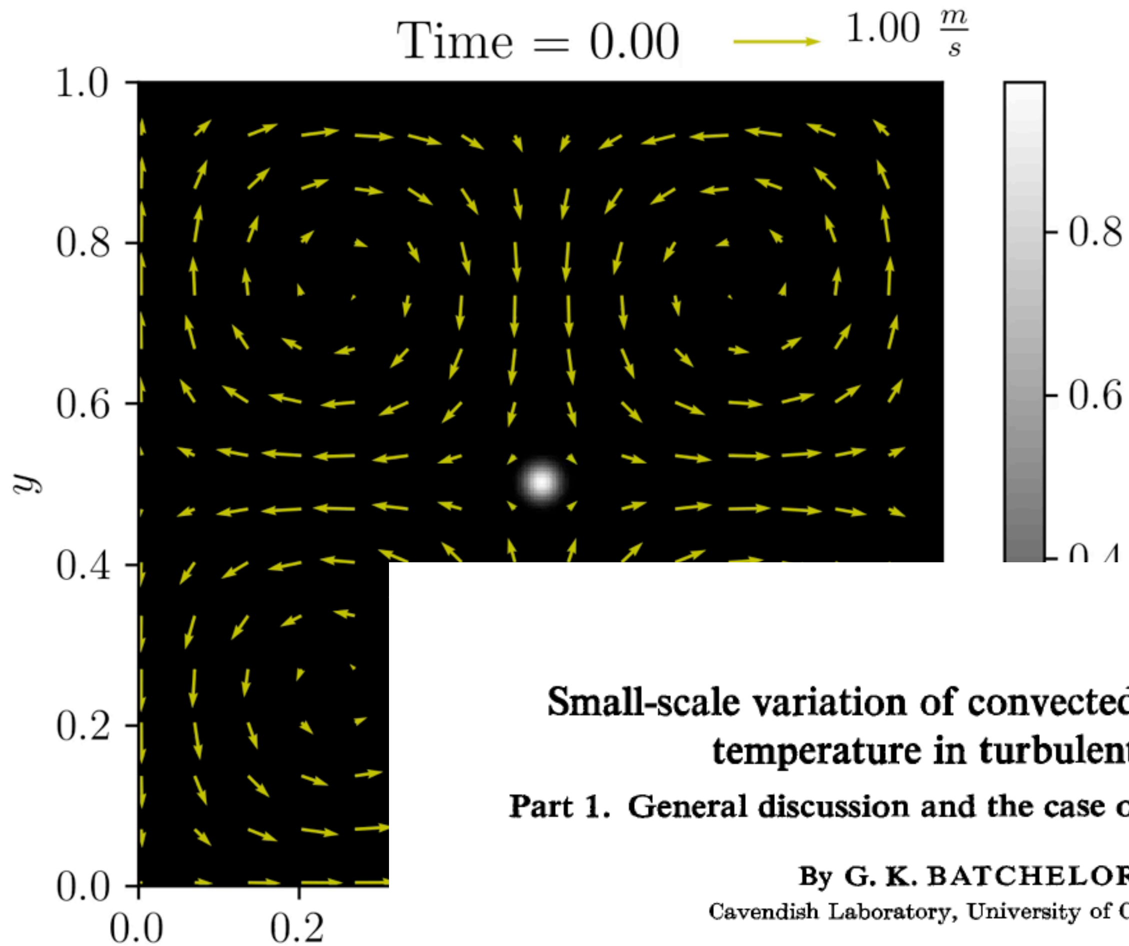


Local-in-time optimization with diffusion



What is the Batchelor scale?

$$\lambda_\Gamma = \sqrt{\frac{\kappa}{\Gamma}} \qquad \lambda_U = \frac{\kappa}{U}$$



113

Small-scale variation of convected quantities like temperature in turbulent fluid

Part 1. General discussion and the case of small conductivity

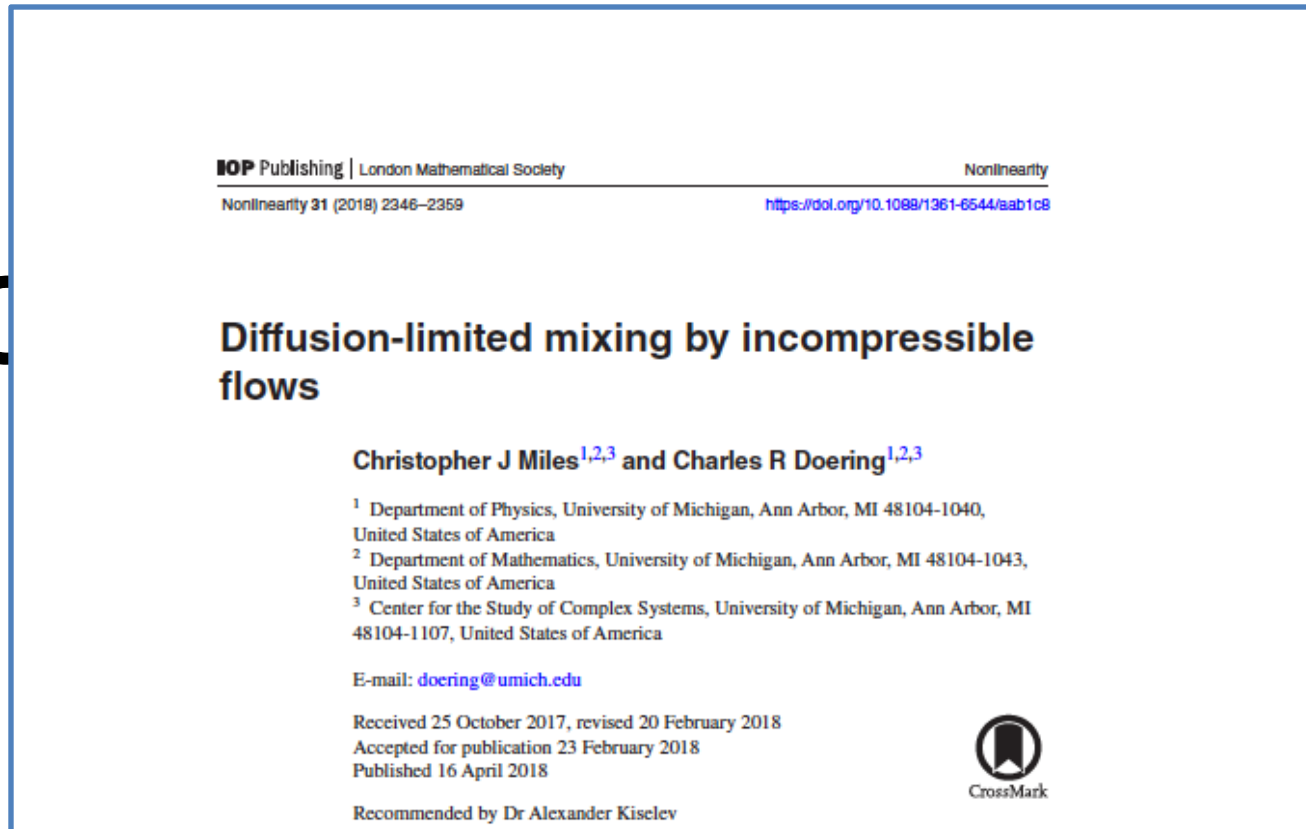
By G. K. BATCHELOR

Cavendish Laboratory, University of Cambridge

(Received 1 June 1958)

When some external agency imposes on a fluid large-scale variations of some

II.
Local-in-time Optimization of Advect Equatio



IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **31** (2018) 2346–2359 https://doi.org/10.1088/1361-6544/aab1c8

Diffusion-limited mixing by incompressible flows

Christopher J Miles[1,2,3] and Charles R Doering[1,2,3]

[1] Department of Physics, University of Michigan, Ann Arbor, MI 48104-1040, United States of America
[2] Department of Mathematics, University of Michigan, Ann Arbor, MI 48104-1043, United States of America
[3] Center for the Study of Complex Systems, University of Michigan, Ann Arbor, MI 48104-1107, United States of America

E-mail: doering@umich.edu

Received 25 October 2017, revised 20 February 2018
Accepted for publication 23 February 2018
Published 16 April 2018

Recommended by Dr Alexander Kiselev

We are interested in the following optimization problem:

$$\min_{\mathbf{u}} \frac{d}{dt} \|\theta(\,\cdot\,, t)\|^2_{H^{-1}}$$

subject to the constraints

$$\partial_t \theta + \mathbf{u} \cdot \nabla\theta = \kappa\Delta\theta$$

with

$$\nabla \cdot \mathbf{u} = 0$$

and the flow intensity is constrained by a fixed enstrophy

$$\int d^d x dt |\nabla \mathbf{u}|^2 = \Gamma^2 L^d.$$

or energy

$$\int d^d x dt |\mathbf{u}|^2 = U^2 L^d.$$

In addition, we are provided with initial data

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}).$$

For the enstrophy-bounded flow problem, we choose:

- the length scale L
- the velocity scale $L\Gamma$, and
- the time scale $1/\Gamma$.

For the energy-bounded flow problem, we choose

- the length scale L
- the velocity scale U, and
- the time scale L/U.

Both scalings produce the following form of the advection-diffusion equation,

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \frac{1}{Pe} \Delta \theta,$$

where $Pe = \frac{\Gamma L^2}{\kappa}$ for the enstrophy-constrained case and $Pe = \frac{UL}{\kappa}$ for the energy-constrained case. The non-dimensional flow constraints become $\|\nabla \mathbf{u}\|_{L^2} = 1$ or $\|\mathbf{u}\|_{L^2} = 1$.

The optimal velocity fields are given instantaneously for the enstrophy case by (in non-dimensional form)

$$\mathbf{u} = \frac{-\Delta^{-1}\mathbb{P}(\theta\nabla\Delta^{-1}\theta)}{\langle|\nabla^{-1}\mathbb{P}(\theta\nabla\Delta^{-1}\theta)|^2\rangle^{1/2}}$$

and for the energy case by

$$\mathbf{u} = \frac{\mathbb{P}(\theta\nabla\Delta^{-1}\theta)}{\langle|\mathbb{P}(\theta\nabla\Delta^{-1}\theta)|^2\rangle^{1/2}}$$

where

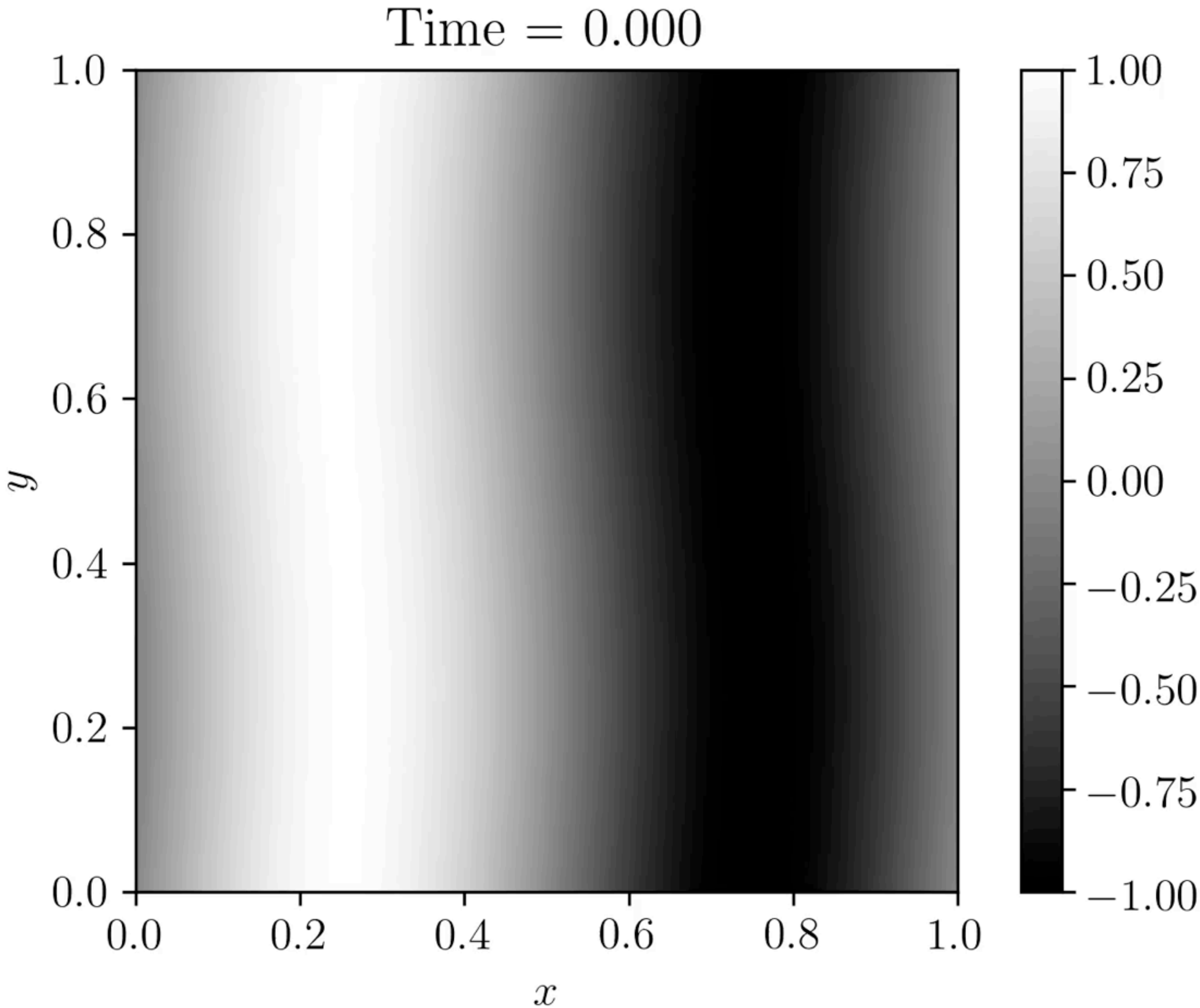
- The operator Δ^{-1} acting on ρ returns the solution ϕ of $\Delta\phi = \rho$.
- $\mathbb{P}$ is the di
- $\langle\cdot\rangle$ is a spa

J. Fluid Mech. (2011), *vol.* 675, *pp.* 465–476.
doi:10.1017/S0022112011000292

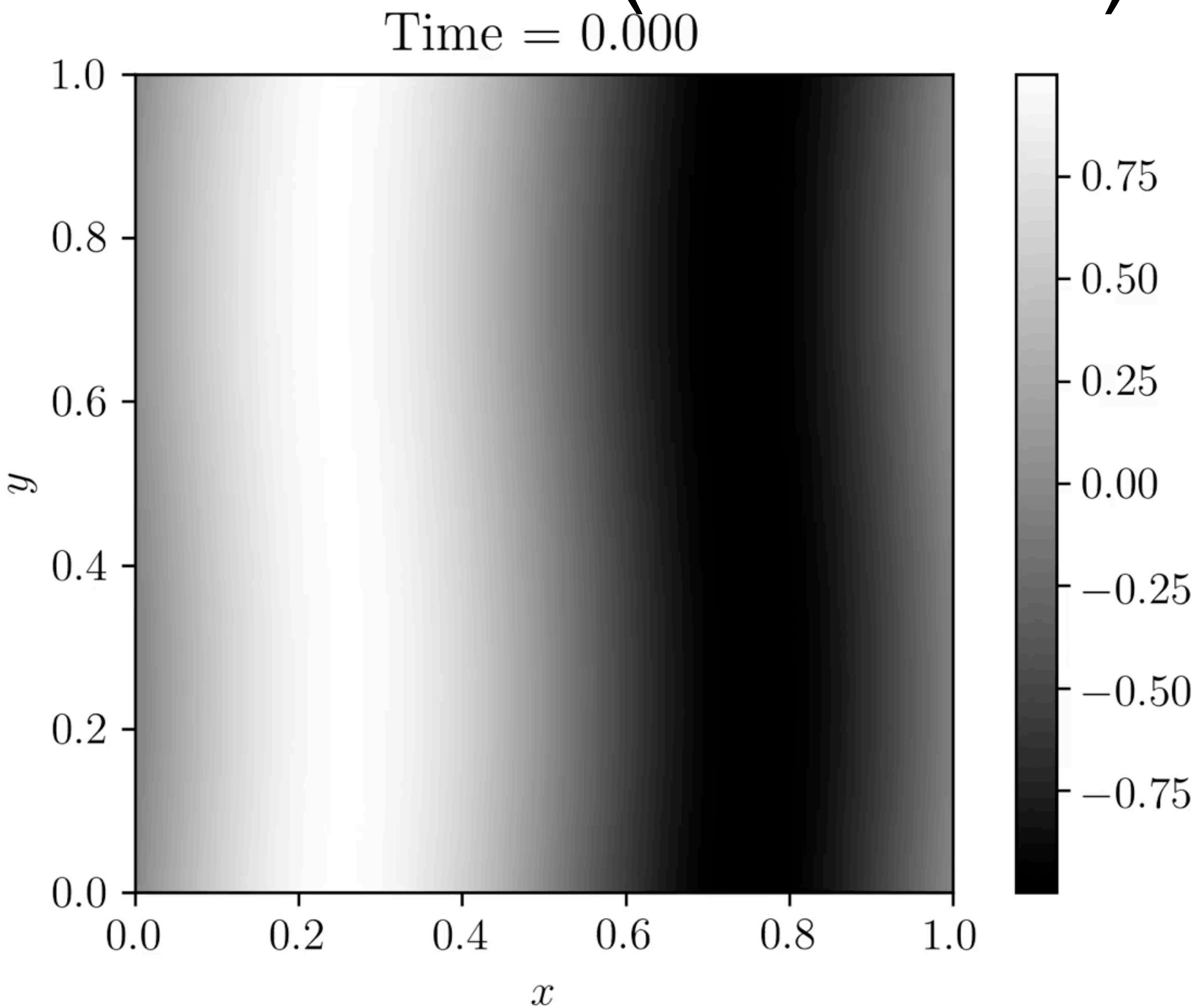
Optimal stirring strategies for passive scalar mixing

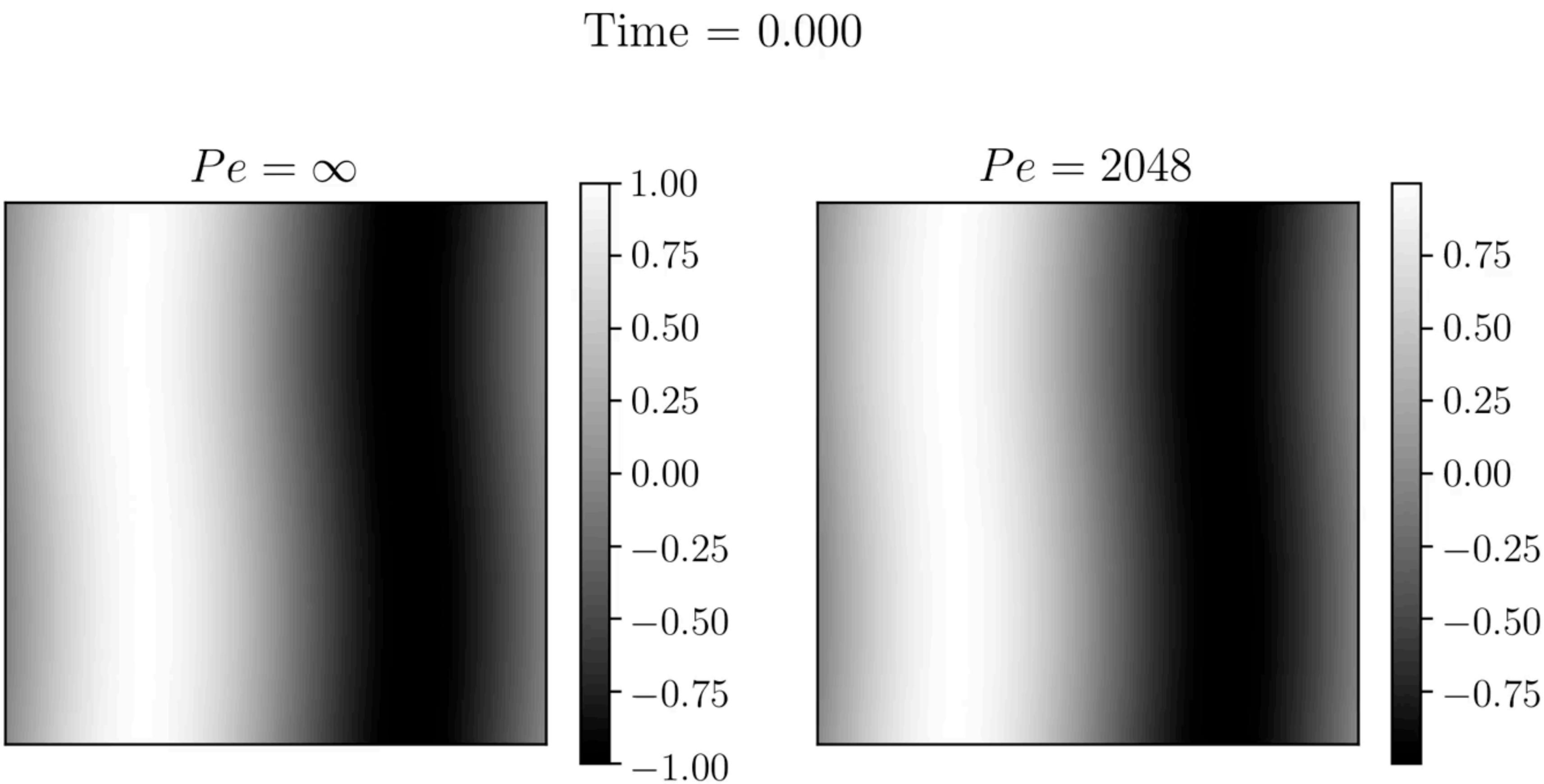
ZHI LIN[1], JEAN-LUC THIFFEAULT[2]
AND CHARLES R. DOERING[3]†

Without diffusion



With Diffusion (Pe = 2048)

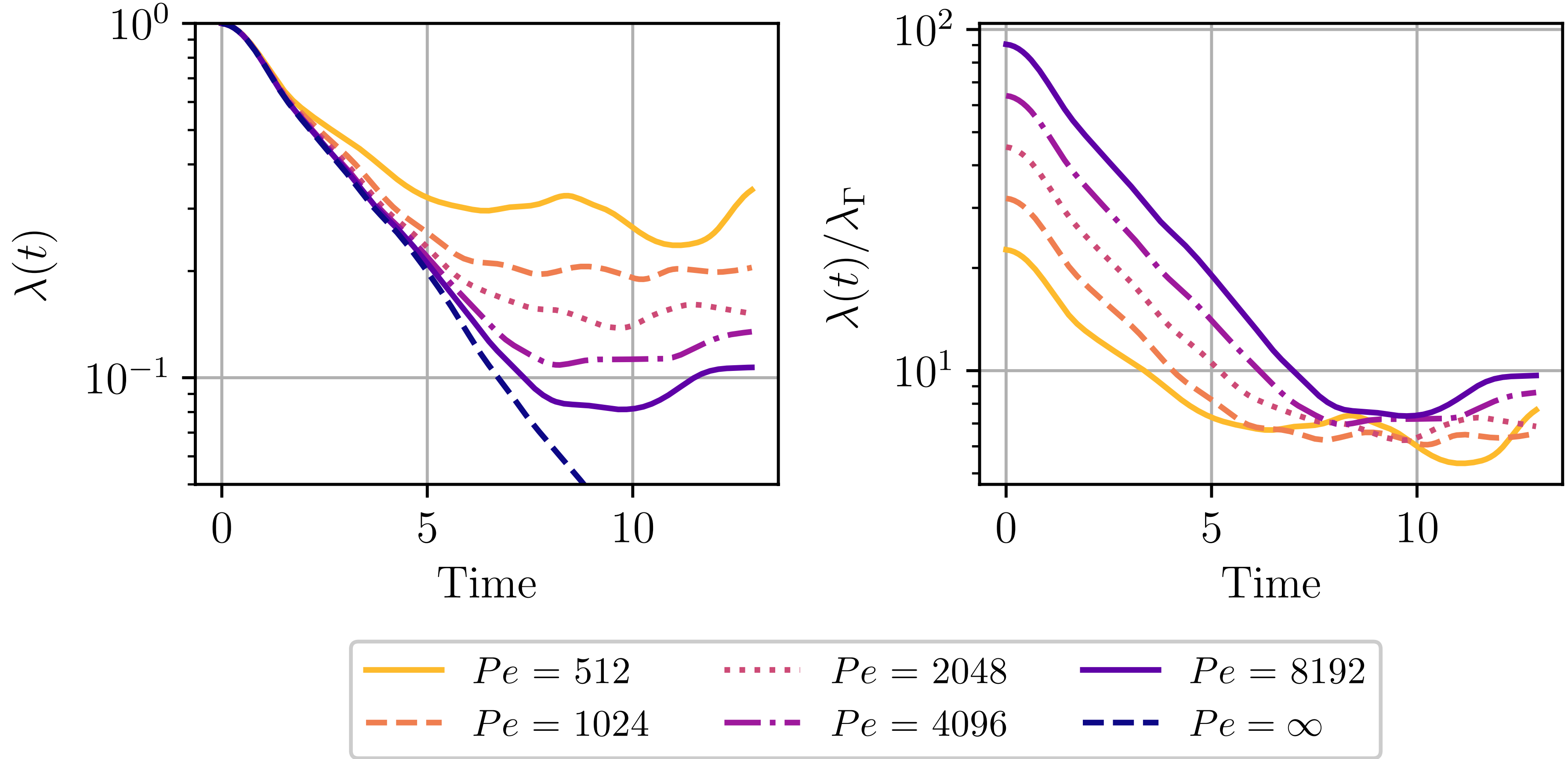


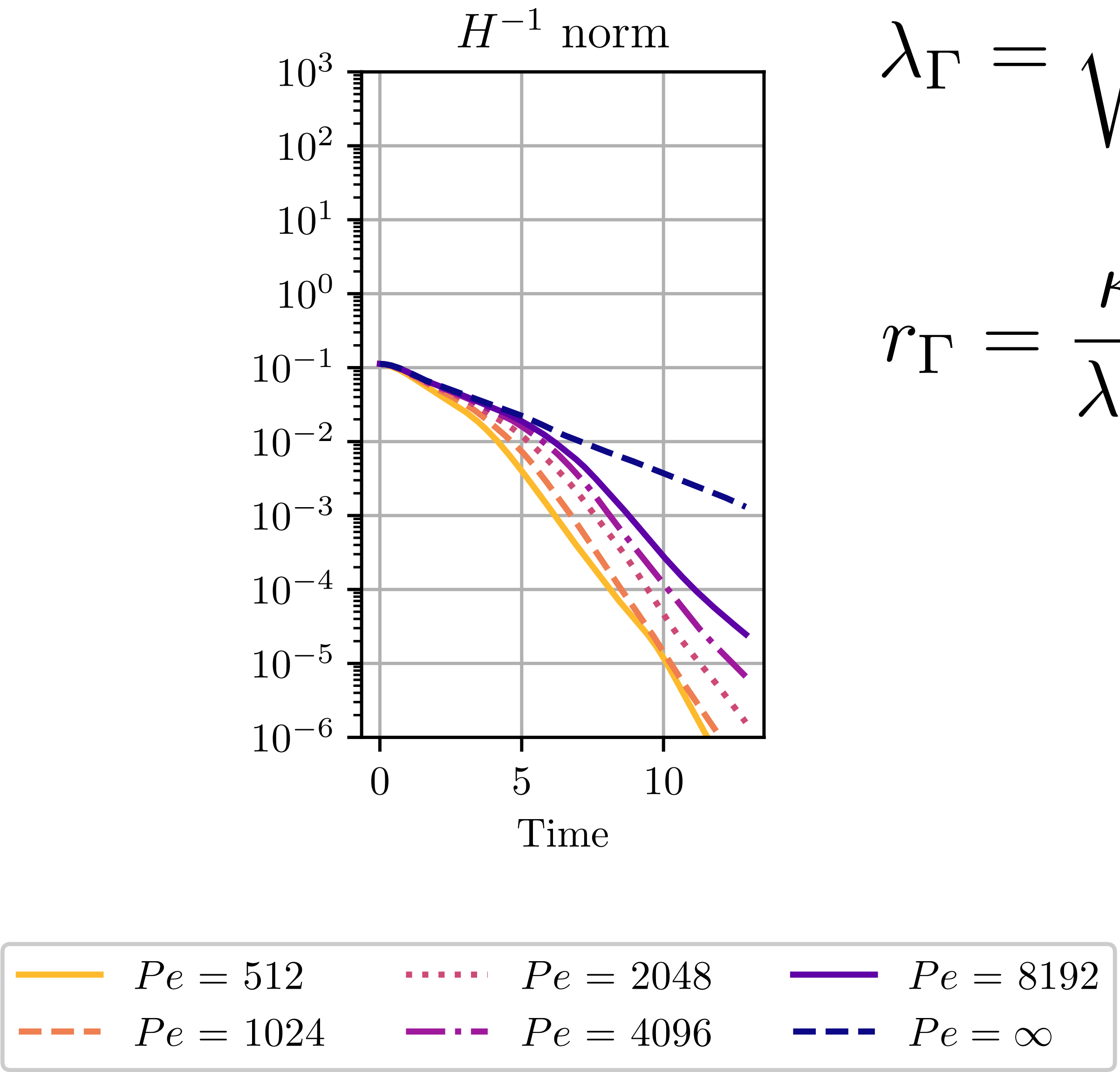


Time = 0.000
Pe = ∞
Pe = 2048
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75

$$\lambda(t) = \frac{\|\theta(\,\cdot\,,t)\|_{H^{-1}}}{\|\theta(\,\cdot\,,t)\|_{L^2}}$$

$$\lambda_\Gamma = \sqrt{\frac{\kappa}{\Gamma}}$$

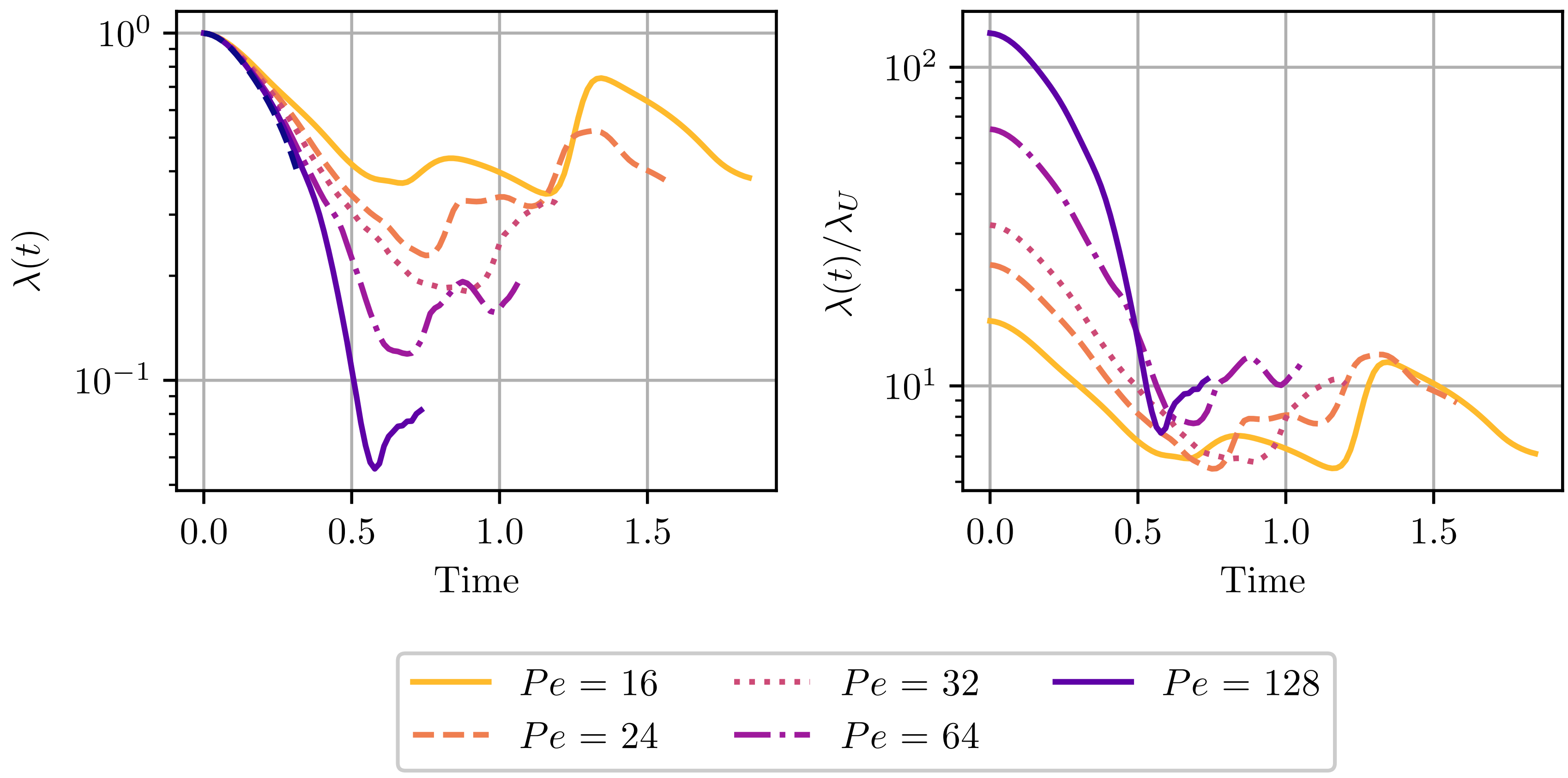


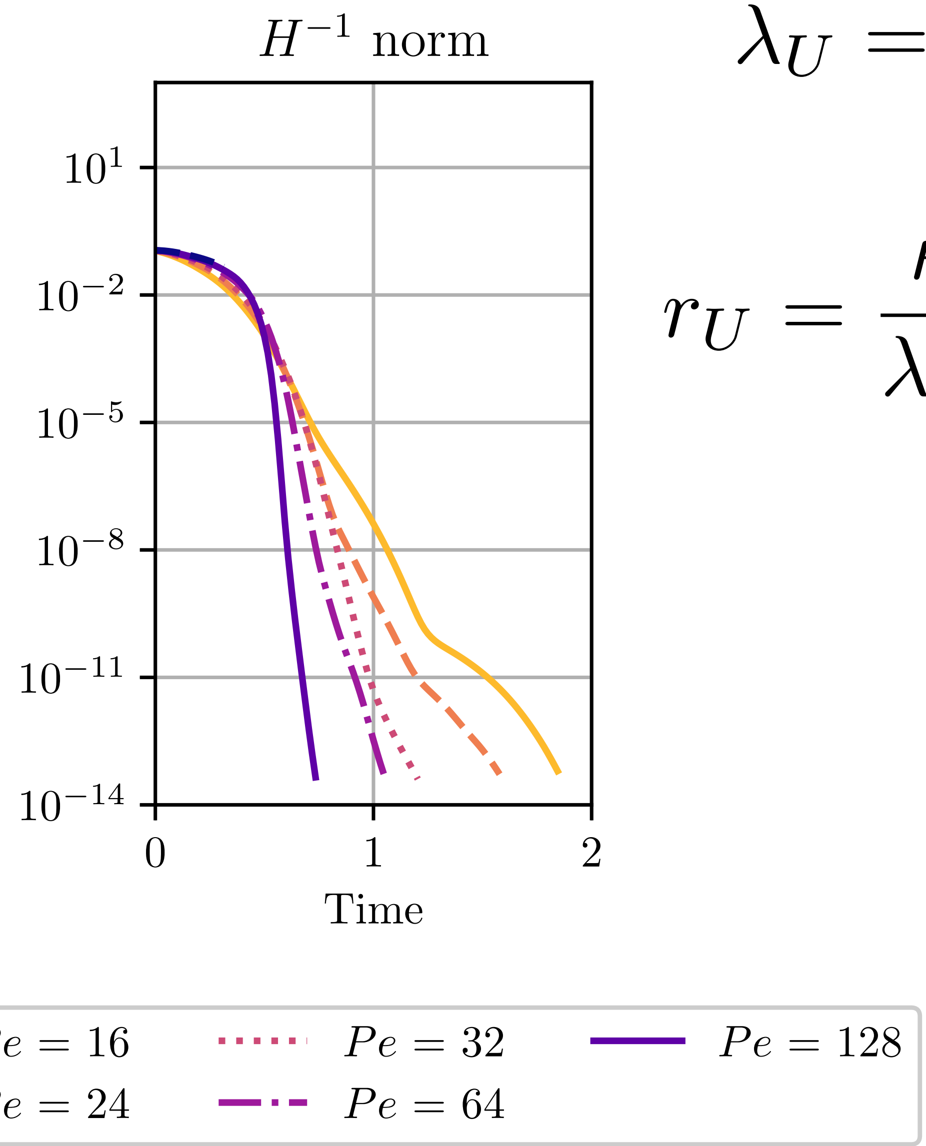


$$\lambda_\Gamma = \sqrt{\frac{\kappa}{\Gamma}}$$

$$r_\Gamma = \frac{\kappa}{\lambda_\Gamma^2} = \Gamma$$

$$\lambda(t) = \frac{\|\theta(\cdot\,,t)\|_{H^{-1}}}{\|\theta(\cdot\,,t)\|_{L^2}} \qquad \lambda_U = \frac{\kappa}{U}$$





$$\lambda_U = \frac{\kappa}{U}$$

$$r_U = \frac{\kappa}{\lambda_U^2} = \frac{U^2}{\kappa}$$

Lower bounds on Mix-norm for L^∞ bounded flows

Bounded rate-of-strain $\|\nabla \mathbf{u}\|_{L^\infty} = 1$

$$\|\nabla^{-1}\theta\|_{L^2} \geq \|\nabla^{-1}\theta_0\|_{L^2} \exp\left[-t - \frac{1}{2Pe}\frac{\|\nabla\theta_0\|_{L^2}^2}{\|\theta_0\|_{L^2}^2}\left(e^{2t}-1\right)\right]$$

Bounded speed $\|\mathbf{u}\|_{L^\infty} = 1$

$$\|\nabla^{-1}\theta\|_{L^2} \geq \frac{\|\theta_0\|_{L^2}^2}{\|\nabla\theta_0\|_{L^2}} \exp\left[-\frac{Pe}{2}\ t - \frac{1}{Pe^2}\frac{\|\nabla\theta_0\|_{L^2}^2}{\|\theta_0\|_{L^2}^2}\left(e^{Pe\ t}-1\right)\right]$$

Conclusion

- Shell model shows similarities to PDE.

- LIT optimization of PDE demonstrated the impact of the Batchelor scale on the mixing rate. Diffusion can negatively affect the mixing rate in some cases!