- We want to investigate:
  - the effect of peer/group pressure
  - heterogenity in voters behaviour
  - stubborn voters (zealots)
- These have been explored individually in the following:
  - Castellano, C., Muñoz, M. A. & Pastor-Satorras, R. **Nonlinear q-voter model**. Physical Review E 80, 041129 (2009).
  - Mobilia, M., Petersen, A. & Redner, S. **On the role of zealotry in the voter model**. Journal of Statistical Mechanics: Theory and Experiment 2007, P08029 (2007).
  - Mobilia, M. **Nonlinear q-voter model with inflexible zealots**. Physical Review E 92, 012803 (2015).

We find that considering heterogeneous populations leds to non-trivial behaviour - in particular a *non-equilibrium steady state (NESS)*.

Looking for Higher (and Higher) Meaning
The draw of the crowd is devilishly strong.
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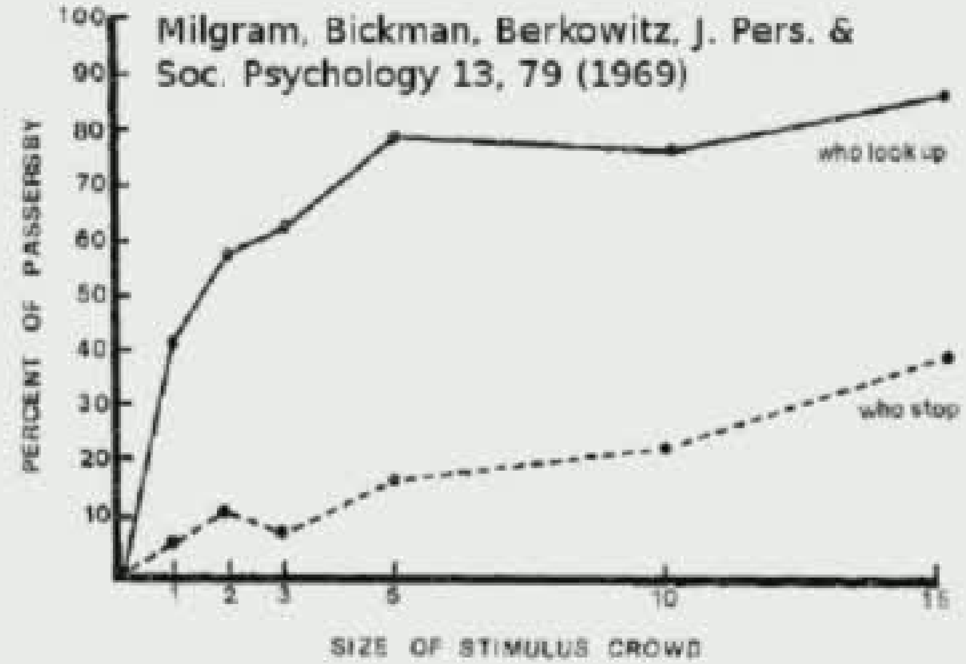
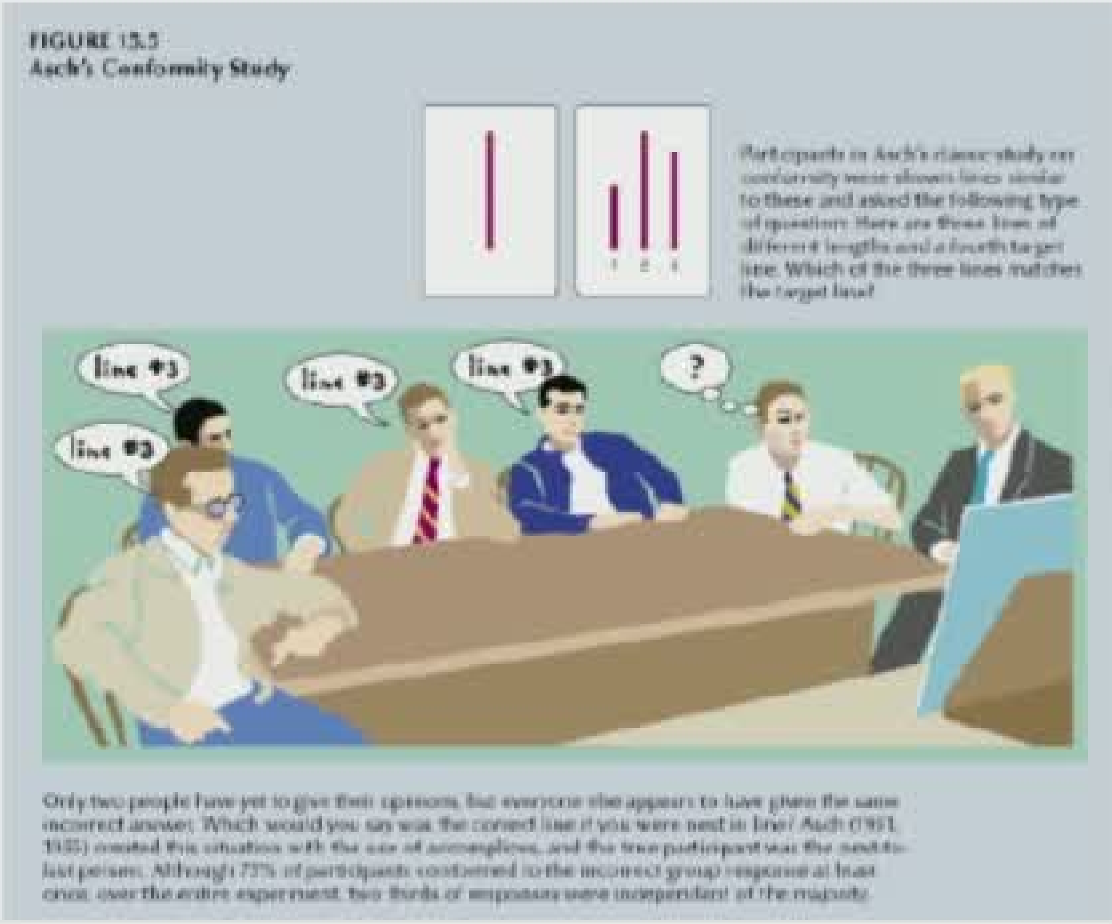


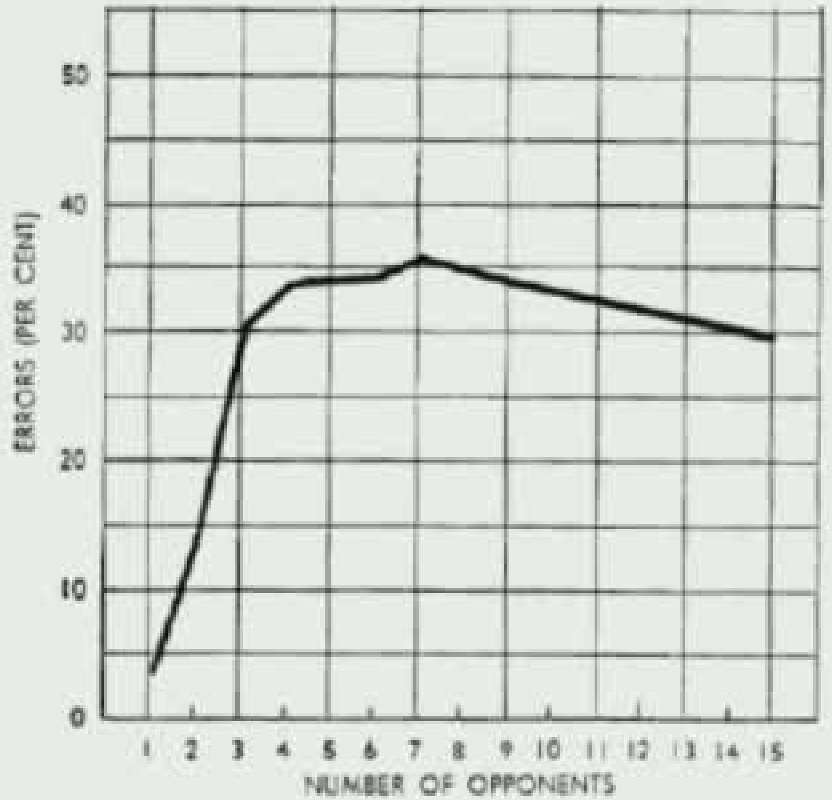
Fig. 1. Mean percentage of passersby who look up and who stop, as a function of the size of the stimulus crowd.

Figure: Milgram's Crowd Experiment. Cartoon (left) and results (right).

Experiment to understand conformity.

- A group of N people on a street, looking up.
- An individual has a large effect ($N = 1$).
- Conformity increases greatly from $N = 1$ to $N = 5$.





SIZE OF MAJORITY which opposed them had an effect on the subjects. With a single opponent the subject erred only 3.6 per cent of the time; with two opponents he erred 13.6 per cent; three, 31.8 per cent; four, 35.1 per cent; six, 35.2 per cent; seven, 37.1 per cent; nine, 35.1 per cent; 15, 31.2 per cent.

Figure: Asch's Conformity Study. Cartoon (left) and results (right).

- Experiment to study how many people it takes to conform - even if the majority is wrong.
- Dependent on group size, up to a point.
- Our pressure to conform is individually different and depends on how many people we observe with that opinion.

- Population of N voters who support one of two opinions (denoted by ± 1)
- Some voters are inflexible zealots, they never change their opinion, the number of which is denoted by $Z_{\pm}$.
- The rest are swing voters of two types, labelled as q_1- and q_2-susceptibles, with numbers S_i $(i = 1, 2)$.

The number of voters is conserved, i.e,

$$S_1 + S_2 + Z_+ + Z_- = N.$$

We track the number of $+1$ voters in each population, $\boldsymbol{n} = (n_1, n_2)$.

Consider a well-mixed system (complete graph). The system can be described exactly by the master equation

$$P(\boldsymbol{n}, T+1) = \sum_{\boldsymbol{n}'} \mathcal{G}(\boldsymbol{n}', \boldsymbol{n})\, P(\boldsymbol{n}', T)$$

where $\boldsymbol{n}' \in \{\boldsymbol{n}, \boldsymbol{n} \pm \boldsymbol{e}_1, \boldsymbol{n} \pm \boldsymbol{e}_2\}$ and $\mathcal{G}$ the transition matrix consisting of the rates $W_i^{\pm}(\boldsymbol{n})$ and with $\boldsymbol{e}_1 := (1,0)$ and $\boldsymbol{e}_2 := (0,1)$.

We can specify the system state with (n_1, n_2) so the configuration space is a discrete set of $S_1 \times S_2$ points arranged in a square lattice.

Probability Currents:

$$\boldsymbol{K}(\boldsymbol{n}, T) = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} W_1^+(\boldsymbol{n})P(\boldsymbol{n},T) - W_1^-(\boldsymbol{n})P(\boldsymbol{n},T) \\ W_2^+(\boldsymbol{n})P(\boldsymbol{n},T) - W_2^-(\boldsymbol{n})P(\boldsymbol{n},T) \end{pmatrix}$$

MEAN-FIELD APPROXIMATION

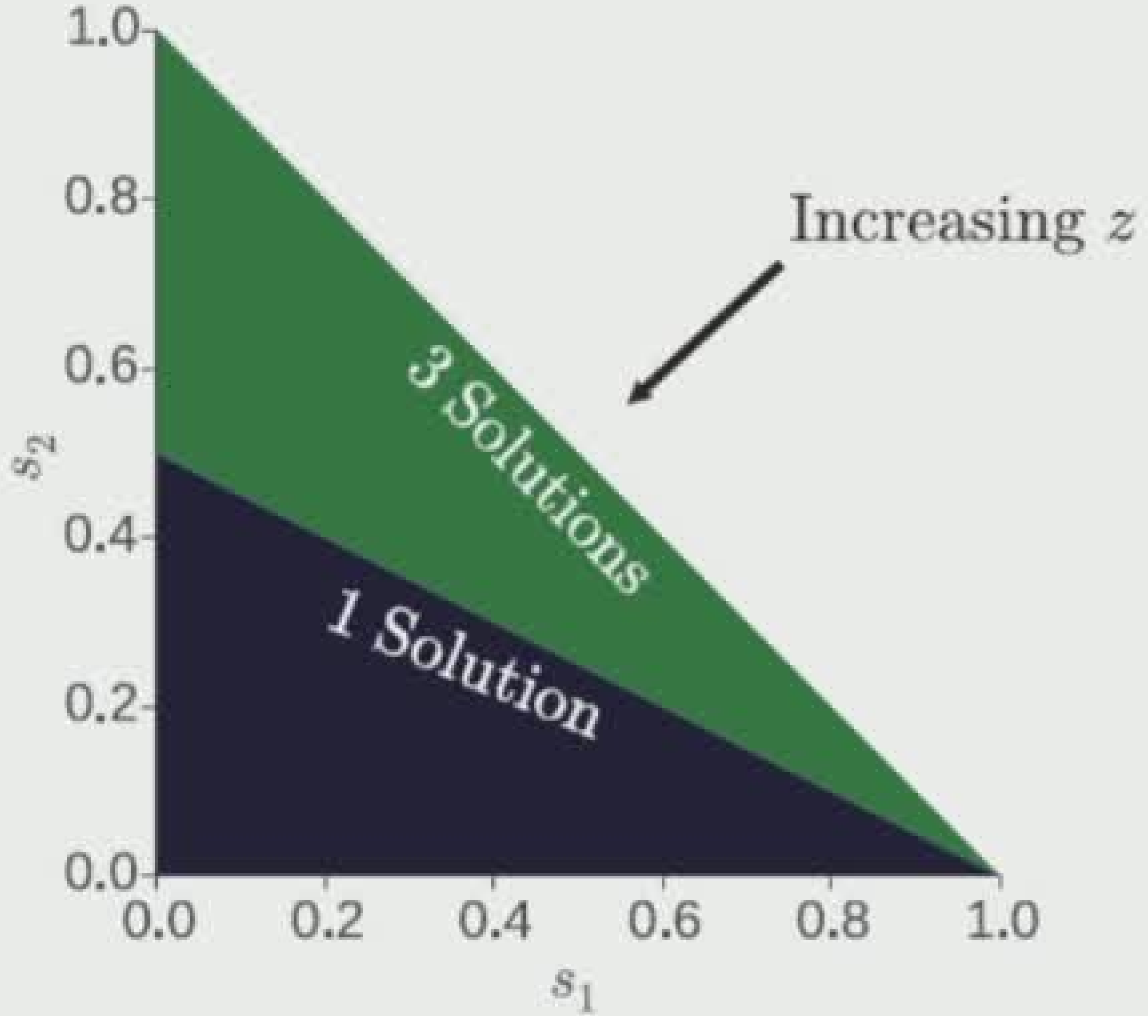
Let $N \to \infty$ and $x_i = n_i/N$.

The mean field equations can be written as

$$\begin{aligned} \dot{x}_1 &= (s_1 - x_1)\mu^{q_1} - x_1(1-\mu)^{q_1} \\ \dot{x}_2 &= (s_2 - x_2)\mu^{q_2} - x_2(1-\mu)^{q_2}, \end{aligned}$$

where $\mu = M/N = z_+ + x_1 + x_2$ is the total fraction of holding the $+1$ opinion.

- Linear stability analysis gives either one (stable) or three (stable/unstable/stable) fixed points.
- Pitchfork bifurcation as the number of zealots is decreased.



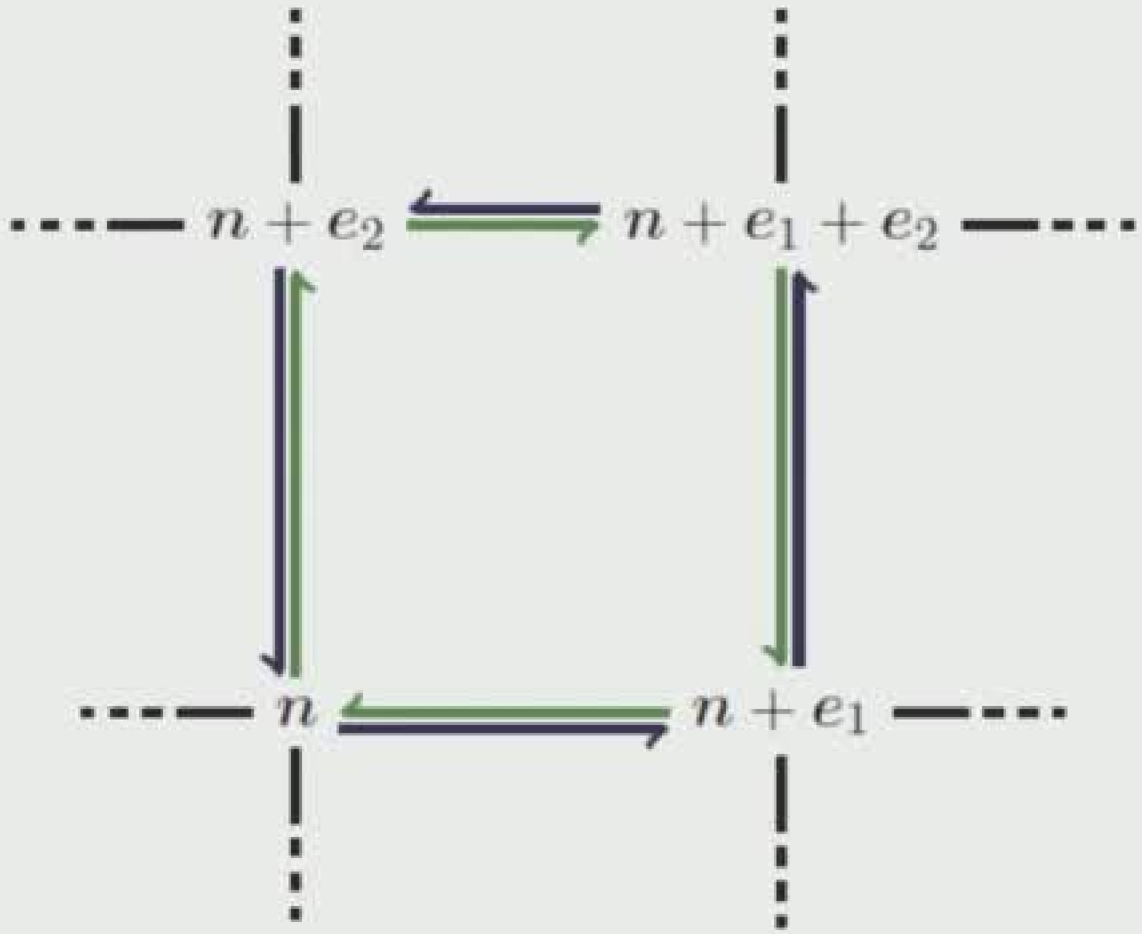
A Markov chain is stochastically identical to its time-reversed version if and only if its transition probabilities satisfy

$$\mathcal{G}(j_1, j_2) \cdots \mathcal{G}(j_n, j_1) = \mathcal{G}(j_1, j_n) \cdots \mathcal{G}(j_2, j_1)$$

for all finite sequences of states $j_1, j_2, \ldots, j_n \in S$.

What does this mean?

- The system settles into an NESS with a non-trivial $P^*(\boldsymbol{n})$ and stationary probability current $\boldsymbol{K}^*(\boldsymbol{n})$
- The dynamics are *time-irreversible*



We can show that the $2q$VM violates detailed balance if $q_1 \neq q_2$.

LINEAR GAUSSIAN APPROXIMATION (LGA)

The LGA of the FPE consists of linearising the drift term and evaluating the diffusion term at the fixed point.

To linearise, we consider small deviations from the FP

$$\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{x}^*$$

This results in the LGA FPE for the stationary distribution:

$$\nabla \cdot \left[\mathbb{F}\boldsymbol{\xi} + \mathbb{D}\nabla\right] P^*(\boldsymbol{\xi}) = 0.$$

where $\mathbb{F}$ and $\mathbb{D}$ represent the linear drift, and scaled Gaussian white noise (diffusion) at the fixed point respectively.

By means of taking a Fourier transform and using a Gaussian ansatz we see that the LGA FPE has solution

$$P^*(\boldsymbol{\xi}) = \frac{1}{2\pi\sqrt{\mathbb{C}}} \exp\left[-\frac{1}{2}\boldsymbol{\xi}^T \mathbb{C}^{-1} \boldsymbol{\xi}\right],$$

where $\mathbb{C}$ is the covariance matrix, with elements

$$C_{ij} = \langle \xi_i \xi_j \rangle^*$$

- Introduced a new modification of the voter model which allows nodes to have different 'resolves' (number of nodes in consensus required to switch).
- No detailed balance!
- Two distinct regimes of behaviour - switching dynamics and oscillations around a central FP.
- Characterised the fixed points and the non-trivial non-equilibrium steady state.
- Applied the linear Gaussian approximation to find NESS properties.

Questions?

References:

[1] Mellor, A., Mobilia, M. & Zia, R. K. P. *Characterization of the nonequilibrium steady state of a heterogeneous nonlinear q-voter model with zealotry.* **EPL (Europhysics Letters)** 113, 48001 (2016).

[2] Mellor, A., Mobilia, M. & Zia, R. K. P. *Heterogeneous out-of-equilibrium nonlinear q-voter model with zealotry.* **Physical Review E** 95, 012104 (2017).

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Workshop on Higher-order interaction Networks

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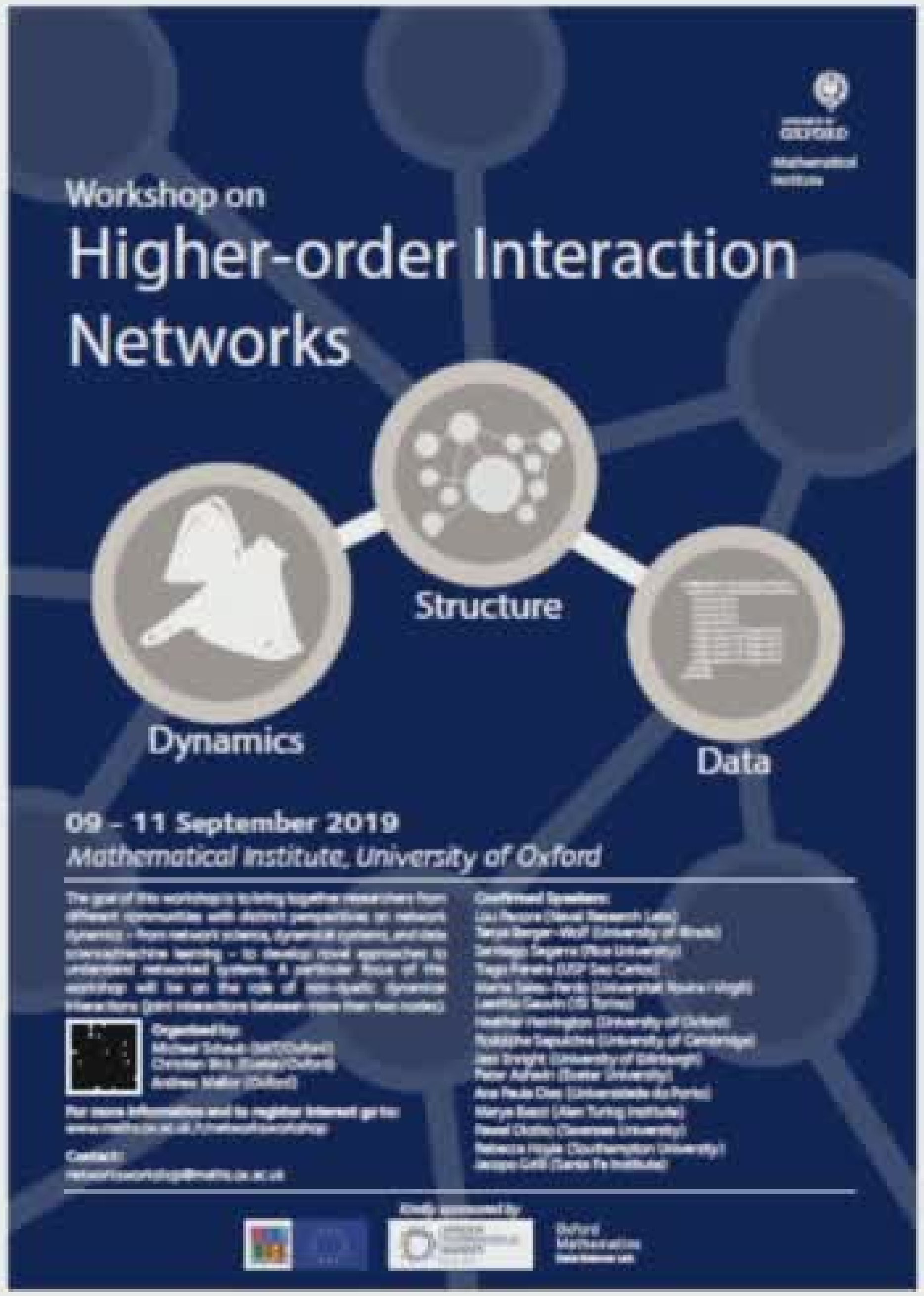

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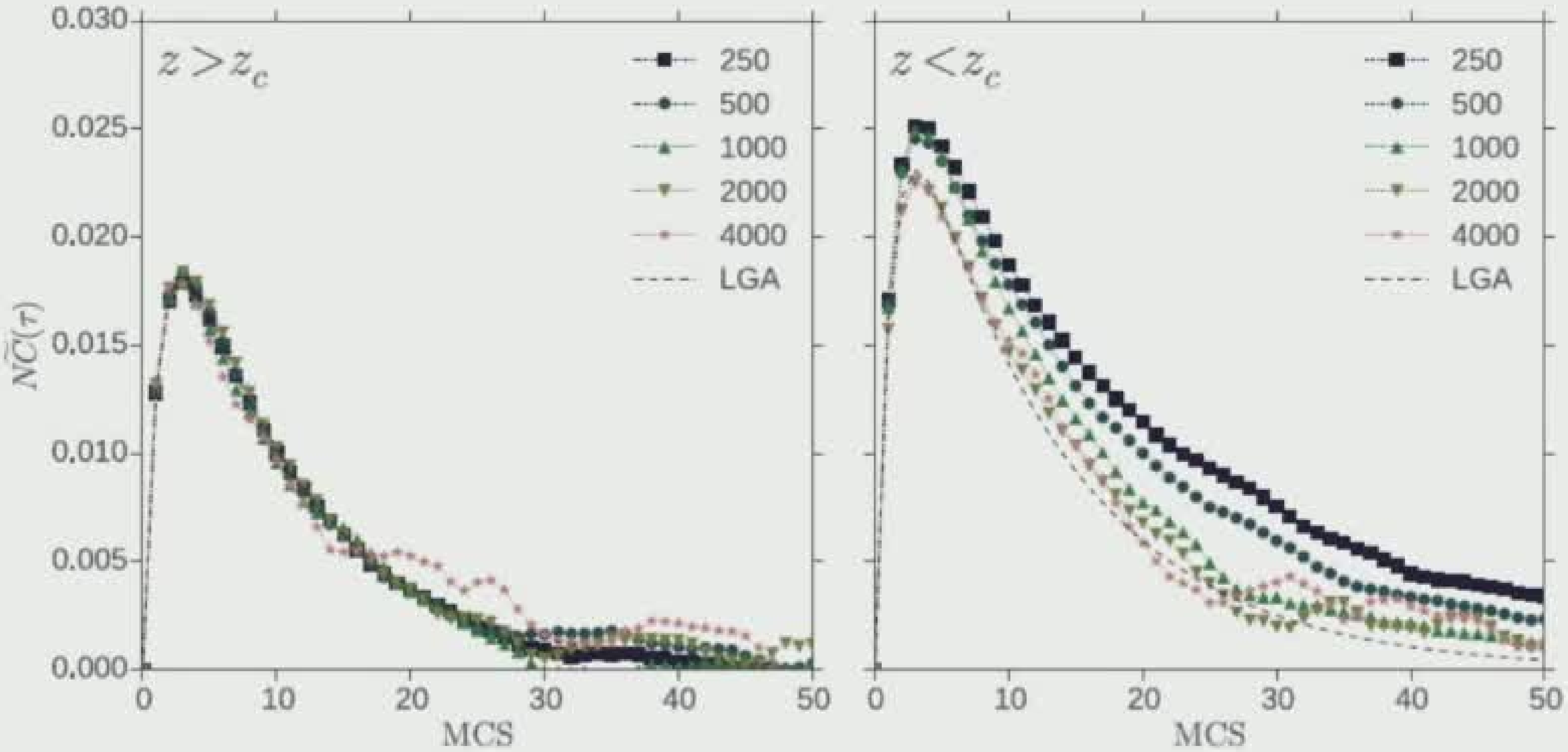
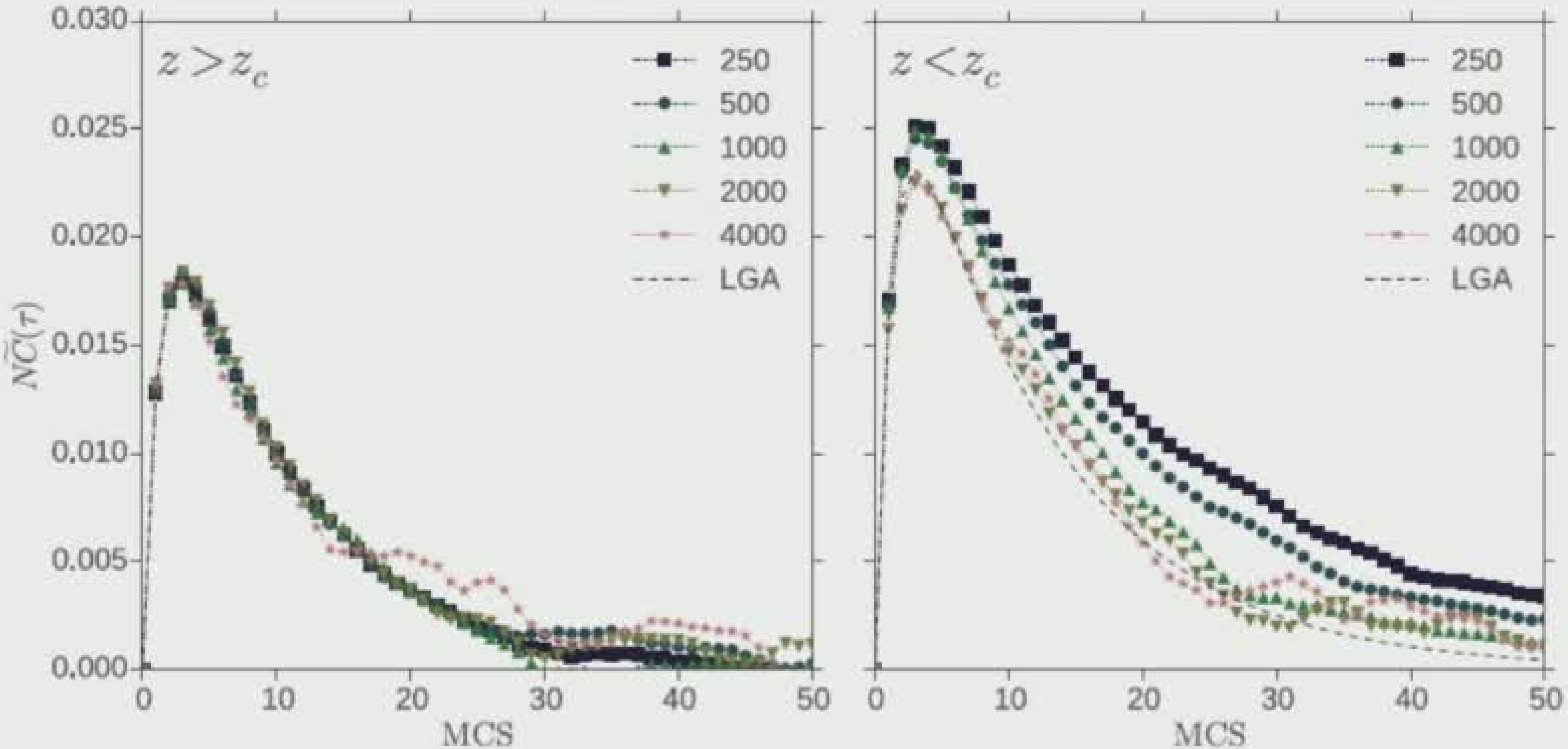


Plots of the lagged correlation function $C_{12} = \langle \xi_1 \xi_2 \rangle_\tau$ for the two regimes.

In the LGA, $\mathbb{C}_{ij}(\tau) = \langle \xi_i \xi_j \rangle_\tau$ is given explicitly by

$$\mathbb{C}(\tau) = \mathbb{C} \exp\left[-\mathbb{F}^T \tau\right].$$

LGA CORRELATIONS



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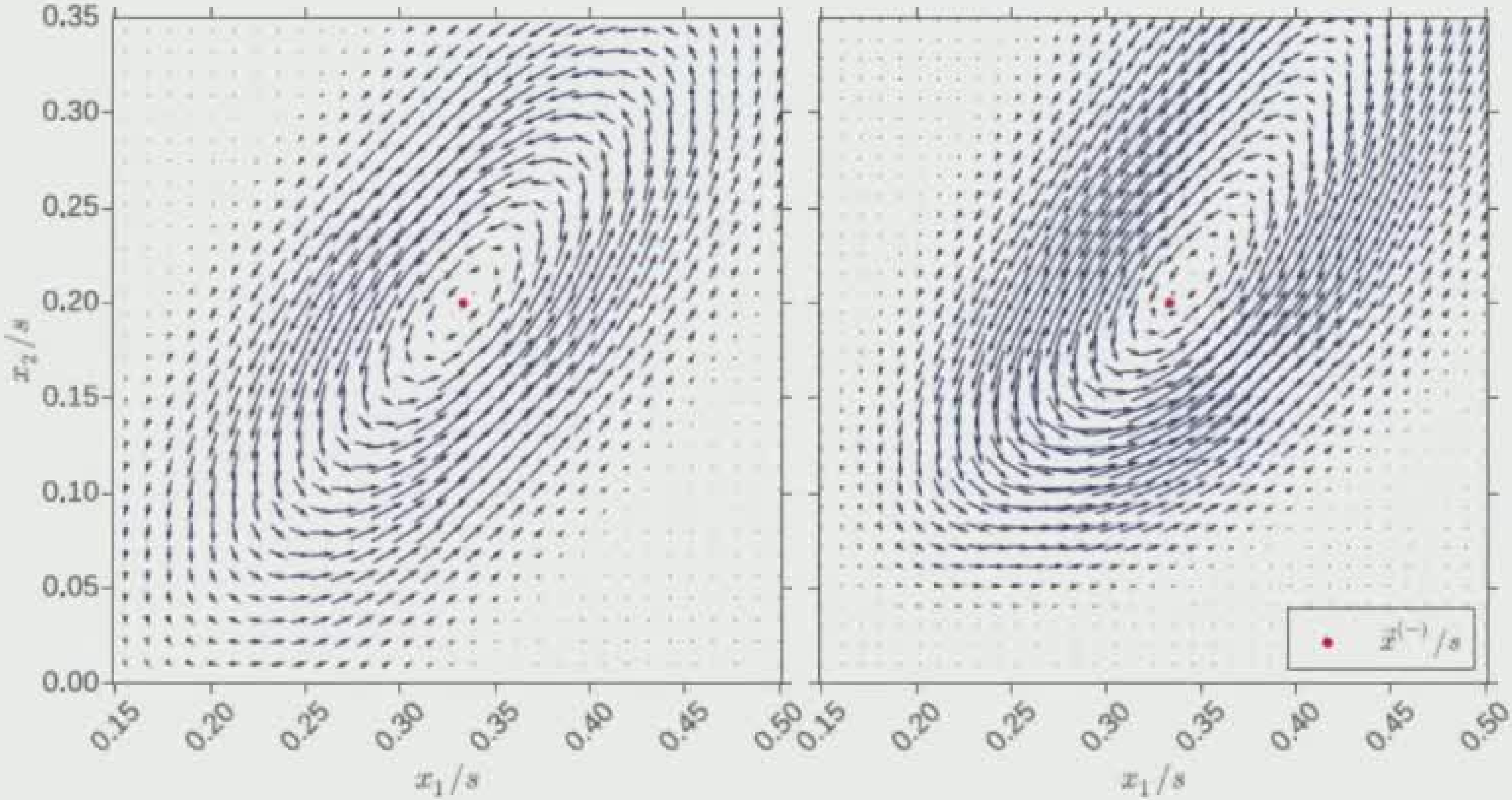

LGA predictions of the stationary flows around a critical point in the low zealotry regime.

In the LGA the probability flow is given explicitly by

$$K^*(\xi) = \{\mathbb{D}\mathbb{C}^{-1} - \mathbb{F}\}\,\xi P^*(\xi).$$

In finite populations, demographic fluctuations are important.

For large (but finite) N, the fluctuations are accounted for in the probability density $P(x, t)$, where $t = T/N$ also becomes continuous.

A Markov chain is stochastically identical to its time-reversed version if and only if its transition probabilities satisfy

$$\mathcal{G}(j_1, j_2) \cdots \mathcal{G}(j_n, j_1) = \mathcal{G}(j_1, j_n) \cdots \mathcal{G}(j_2, j_1)$$

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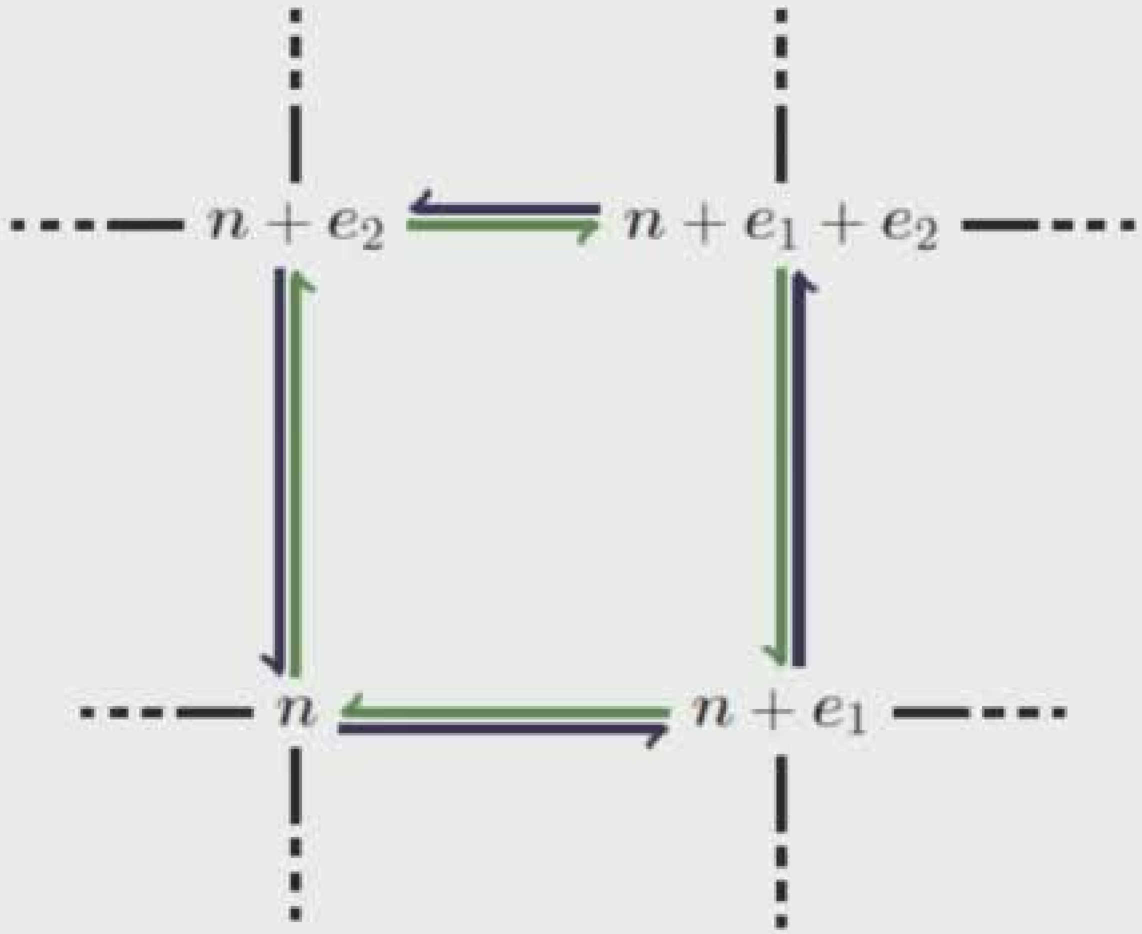

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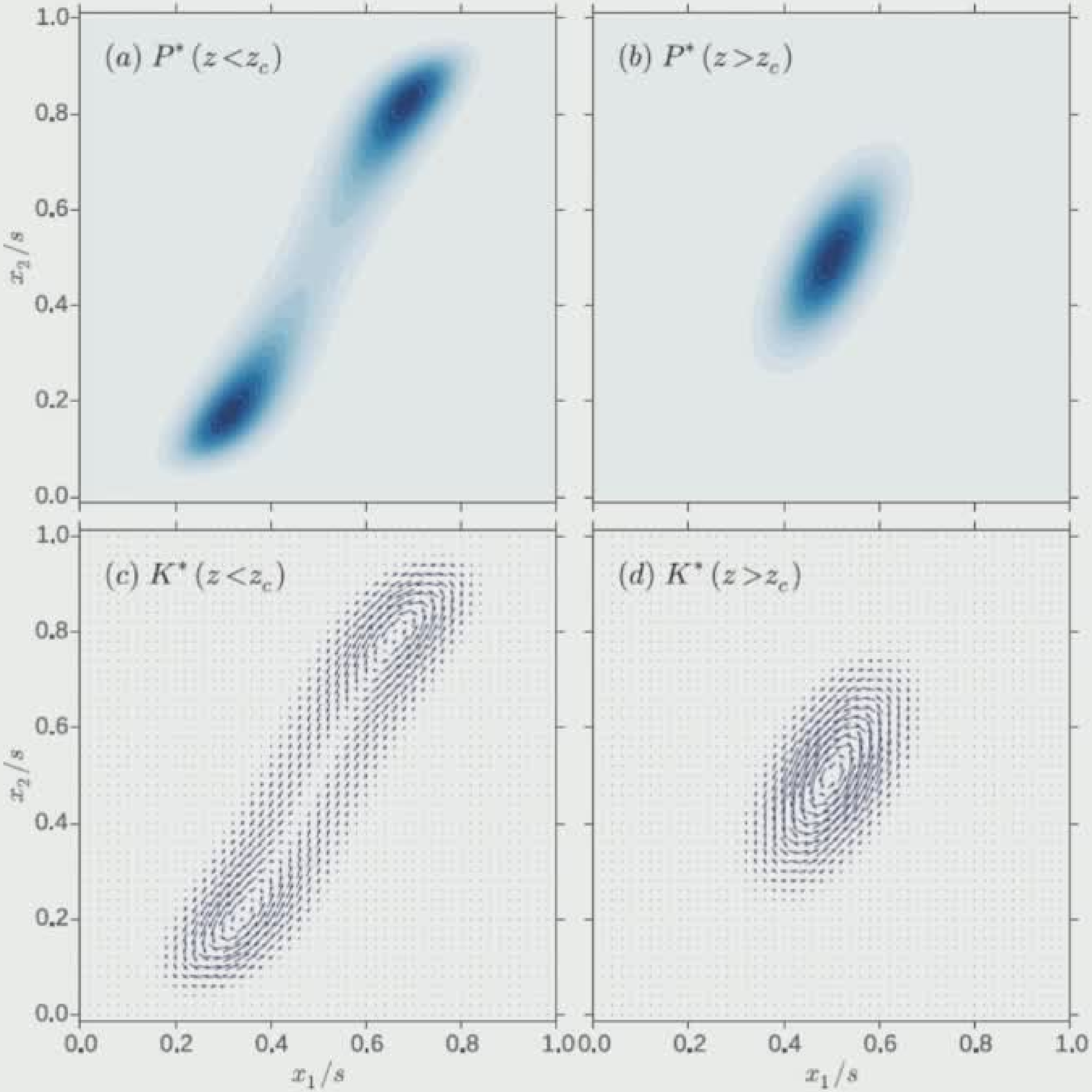
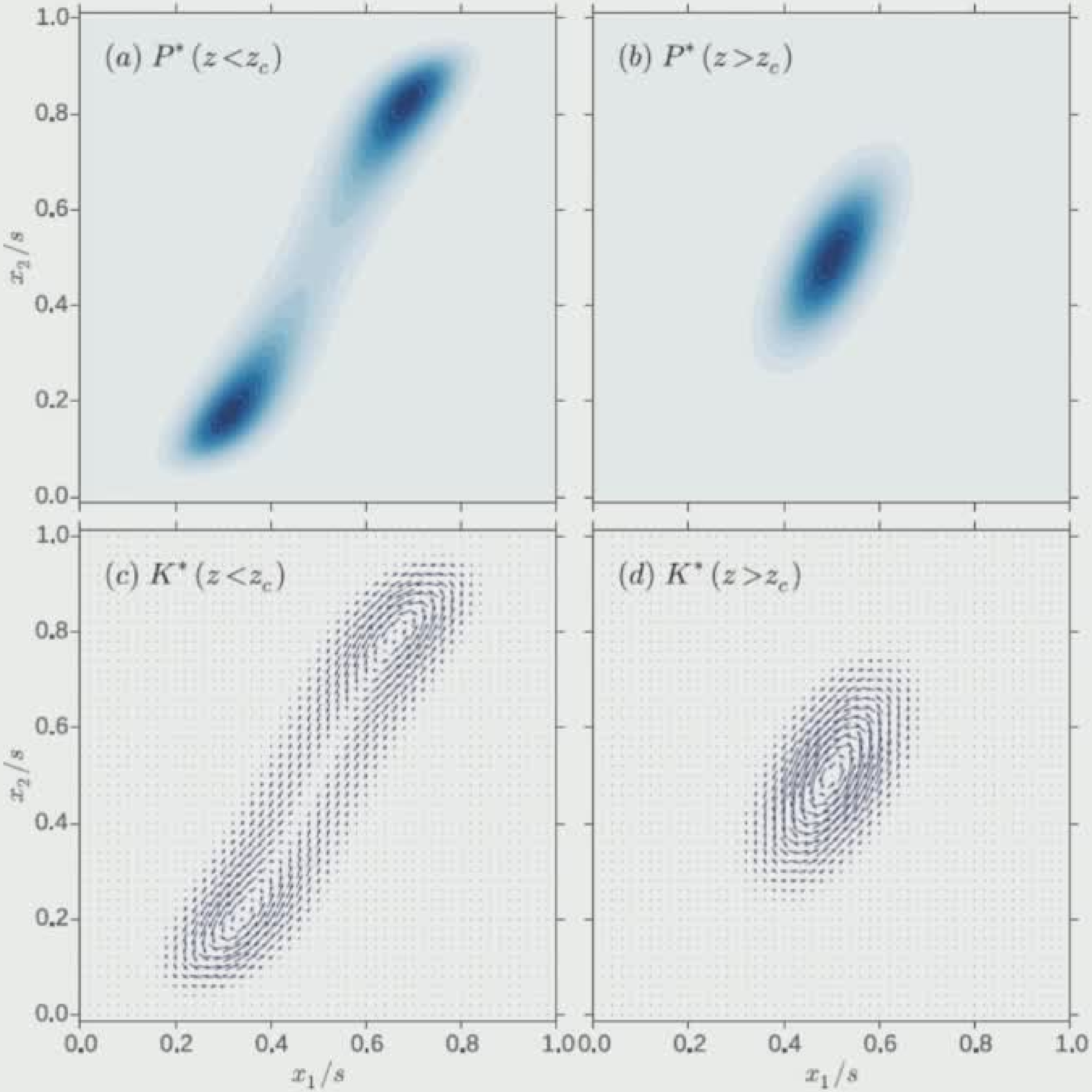

(a) P^* $(z < z_c)$
(b) P^* $(z > z_c)$
(c) K^* $(z < z_c)$
(d) K^* $(z > z_c)$
x_2/s
x_1/s
0.0
0.2
0.4
0.6
0.8
1.0


(a) P* (z<z_c)
(b) P* (z>z_c)
(c) K* (z<z_c)
(d) K* (z>z_c)
x_2/s
x_1/s
0.0
0.2
0.4
0.6
0.8
1.0