VU UNIVERSITY AMSTERDAM

Validated continuation and Hopf bifurcations in ODEs

Elena Queirolo

Joint work with
J.P. Lessard and J.B van den Berg

Outline

Goal:
validation of Hopf bifurcation and validated continuation of periodic orbits

Steps:

- validation of a periodic orbit
- validation of a branch of periodic orbits
- reformulation of Hopf into that framework

First problem

Periodic orbit validation

$$\dot{u} = f(\lambda, u)$$

$\lambda \in \mathbb{R}^M$ scalar variables, $u : [0, 2\pi] \to \mathbb{R}^N$ periodic,
$f : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^N$ polynomial

$$u(t) = \sum_{k \in \mathbb{Z}} \hat{u}_k e^{\mathbf{i}kt}, \qquad \hat{u}_k \in \mathbb{C}^N$$

First problem

Periodic orbit validation

$$\{\hat{u}_k^i\}_{k\in\mathbb{Z},i=1,\ldots,N}$$

with the analytic norm

$$\|\hat{u}^i\|_\nu = \sum_{k\in\mathbb{Z}} |\hat{u}_k^i|\nu^{|k|}, \qquad \nu > 1$$

First problem

Periodic orbit validation

Some interesting properties of this space

$$(u^i \cdot u^j)(t) \longleftrightarrow (\hat{u}^i * \hat{u}^j)_k$$

$$\dot{u}(t) \longleftrightarrow \mathbf{i} k \hat{u}_k$$

$$\|\hat{u}^i * \hat{u}^j\|_\nu \leq \|\hat{u}^i\|_\nu \|\hat{u}^j\|_\nu$$

Thus it is a Banach algebra.

First problem

Periodic orbit validation

We can reformulate f in the sequence space as a sum of convolutions.

Then

$$\dot{u} = f(\lambda, u) \longleftrightarrow \underbrace{\mathbf{i}k\hat{u}_k - \mathcal{F}(f(\lambda, u))_k}_{F_k(\lambda, \hat{u})} = 0$$

for all $k \in \mathbb{Z}$

$$F(\lambda, u) = \begin{pmatrix} \vdots \\ F_{-1}(\lambda, u) \\ F_0(\lambda, u) \\ F_1(\lambda, u) \\ \vdots \end{pmatrix}$$

First problem

Periodic orbit validation

We can reformulate f in the sequence space as a sum of convolutions.

Then

$$\dot{u} = f(\lambda, u) \longleftrightarrow \underbrace{\mathbf{i}k\hat{u}_k - \mathcal{F}(f(\lambda, u))_k}_{F_k(\lambda,\hat{u})} = 0$$

for all $k = -N, \ldots, N$

$$F^N(\lambda, \hat{u}^N) = \begin{bmatrix} F_{-N}(\lambda, \hat{u}^N) \\ \vdots \\ F_0(\lambda, \hat{u}^N) \\ \vdots \\ F_N(\lambda, \hat{u}^N) \end{bmatrix}$$

First problem

Periodic orbit validation

Underdetermined system:

$$F^N(\lambda, u^N) = 0$$

We need M extra scalar equations.
We assume that the scalar equations G are polynomials depending on λ and u^N (to avoid truncations).

The full problem

$$H(\lambda, u) = \begin{pmatrix} G(\lambda, u) \\ F(\lambda, u) \end{pmatrix}$$

First problem

Periodic orbit validation

Assume we have a numerical solution

$$\tilde{x} = (\tilde{\lambda}, \tilde{u}^N)$$

of the truncated system

$$H^N(\lambda, u^N) = \begin{pmatrix} G(\lambda, u^N) \\ F^N(\lambda, u^N) \end{pmatrix}$$

we want to validate it as solution of the full problem.

First problem

Periodic orbit validation

$$T(x) = x - DH^{-1}(\tilde{x})H(x)$$

$$T(x) = x - AH(x)$$

and we prove that T is a contraction with the radii polynomials:

$$\|T(\tilde{x}) - \tilde{x}\| \leq Y$$

$$\|DT(\tilde{x} + rz)\| \leq Z(r)$$

If it exists r

$$Y + Z(r) \leq r$$

the solution is validated.

First problem

Periodic orbit validation

Remarks:

- the bounds are little affected by the dimension of the problem
- they are not very affected by the nonlinearity of f
- they are affected by the norm of the solution
- they are affected by the number of nodes and the conditioning of DH

Second problem

Periodic orbit continuation

By adding a parameter, $\lambda \in \mathbb{R}^{M+1}$

$$H(\lambda, u) = \begin{pmatrix} G(\lambda, u) \\ F(\lambda, u) \end{pmatrix}$$

H has, generically, a 1D zero-curve.

Second problem

Periodic orbit continuation

Assume we have a numerical solution

$$x_0 = (\lambda_0, u_0)$$

next proposed solution step:

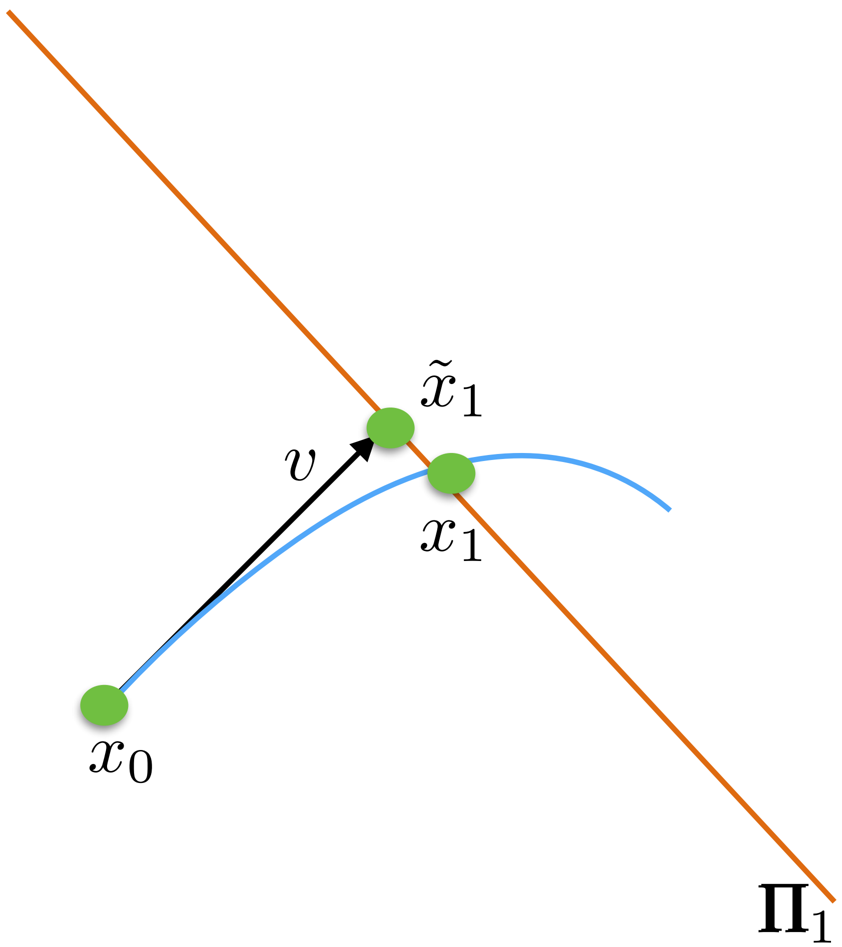
$$\tilde{x}_1 = x_0 + hv$$

with h step size and

$$v : DH(x_0)v \approx 0$$

then we search for x_1 such that

$$H(x_1) = 0,$$
$$(x_1 - \tilde{x}_1, v) = 0$$



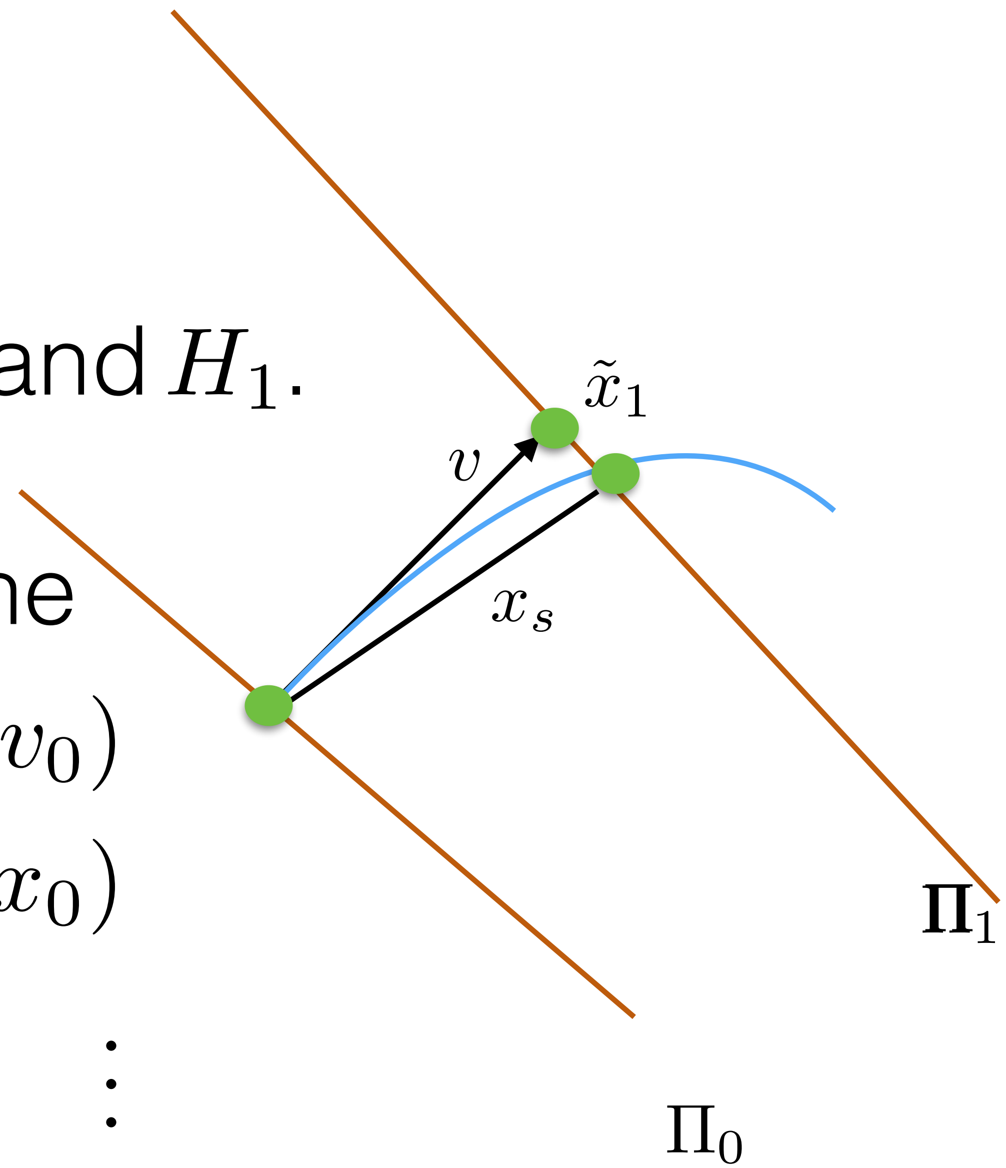
Second problem

Periodic orbit continuation

x_0 satisfies a similar problem.

We call the extended problems H_0 and H_1.

Introducing a parameter s, we define



$$v_s = v_0 + s(v_1 - v_0)$$

$$x_s = x_0 + s(x_1 - x_0)$$

$$\vdots$$

$$H_s(x_s) = \begin{pmatrix} (x_s - \tilde{x}_s, v_s) \\ H(x_s) \end{pmatrix}$$

Second problem

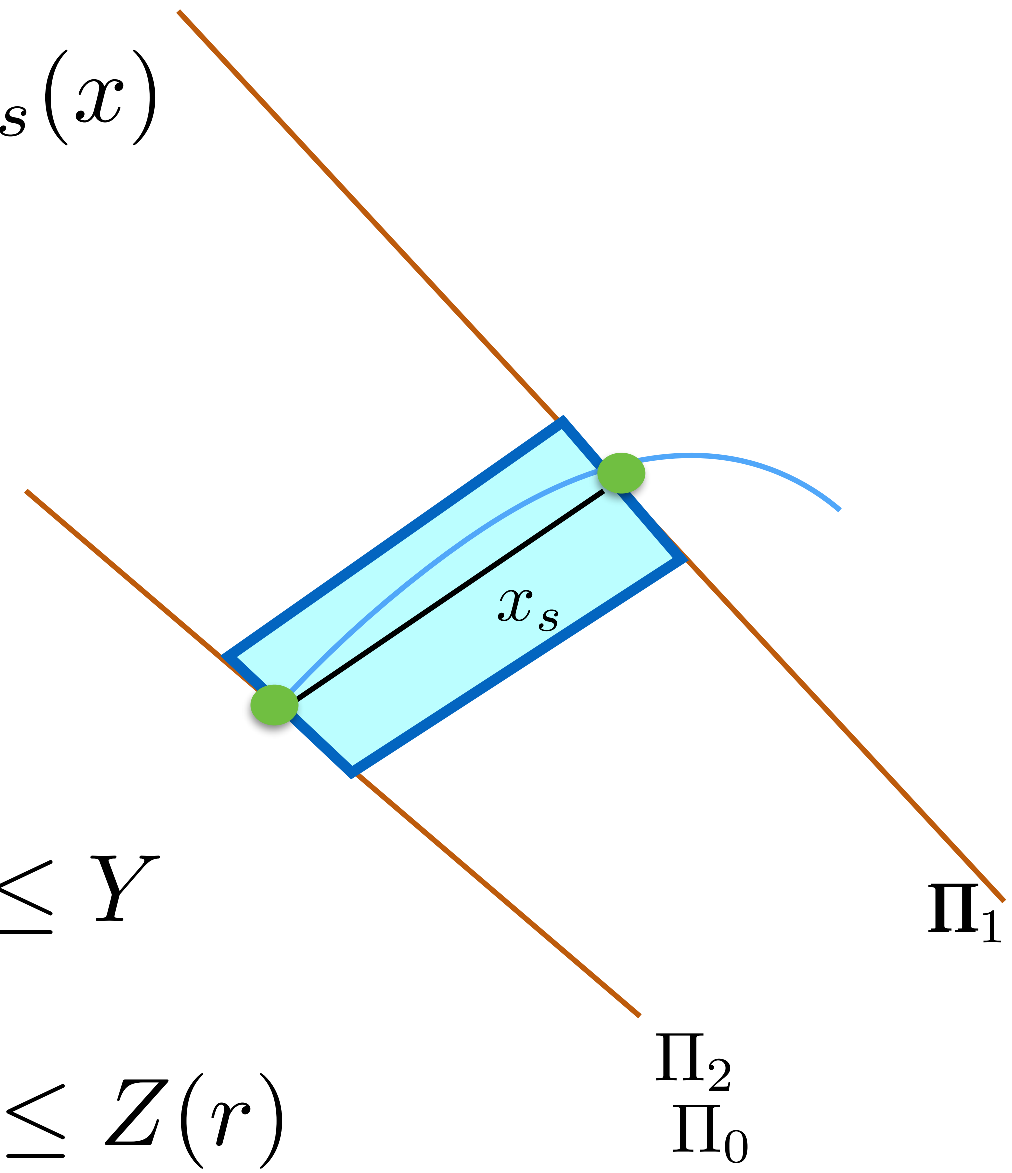
Periodic orbit continuation

The Newton-like operator gets to be

$$T_s(x) = x - DH_s^{-1}(x_s)H_s(x)$$

$$A_s = A_0 + s(A_1 - A_0)$$

$$T_s(x) = x - A_s H_s(x)$$



and we solve for

$$\max_s \|T_s(\tilde{x}_s) - \tilde{x}_s\| \leq Y$$

$$\max_s \|DT_s(\tilde{x}_s + rz)\| \leq Z(r)$$

$$Y + Z(r) \leq r$$

Second problem

Periodic orbit continuation

A useful theorem

$f : [0,1] \to \mathcal{B}, \quad \mathcal{B}$ Banach,

$$\max_{s\in[0,1]} |f(s)| \leq \max\{|f(0)|, |f(1)|\} + \frac{1}{8} \max_{h\in[0,1]} |f''(h)|$$

Second problem

Periodic orbit continuation

Example: Lorenz system

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) + y$$
$$\dot{z} = xy + \beta z$$

$$\sigma = 10, \beta = \frac{8}{3}, \rho = 28$$

Second problem

Periodic orbit continuation

Example: 19 coupled Lorenz systems

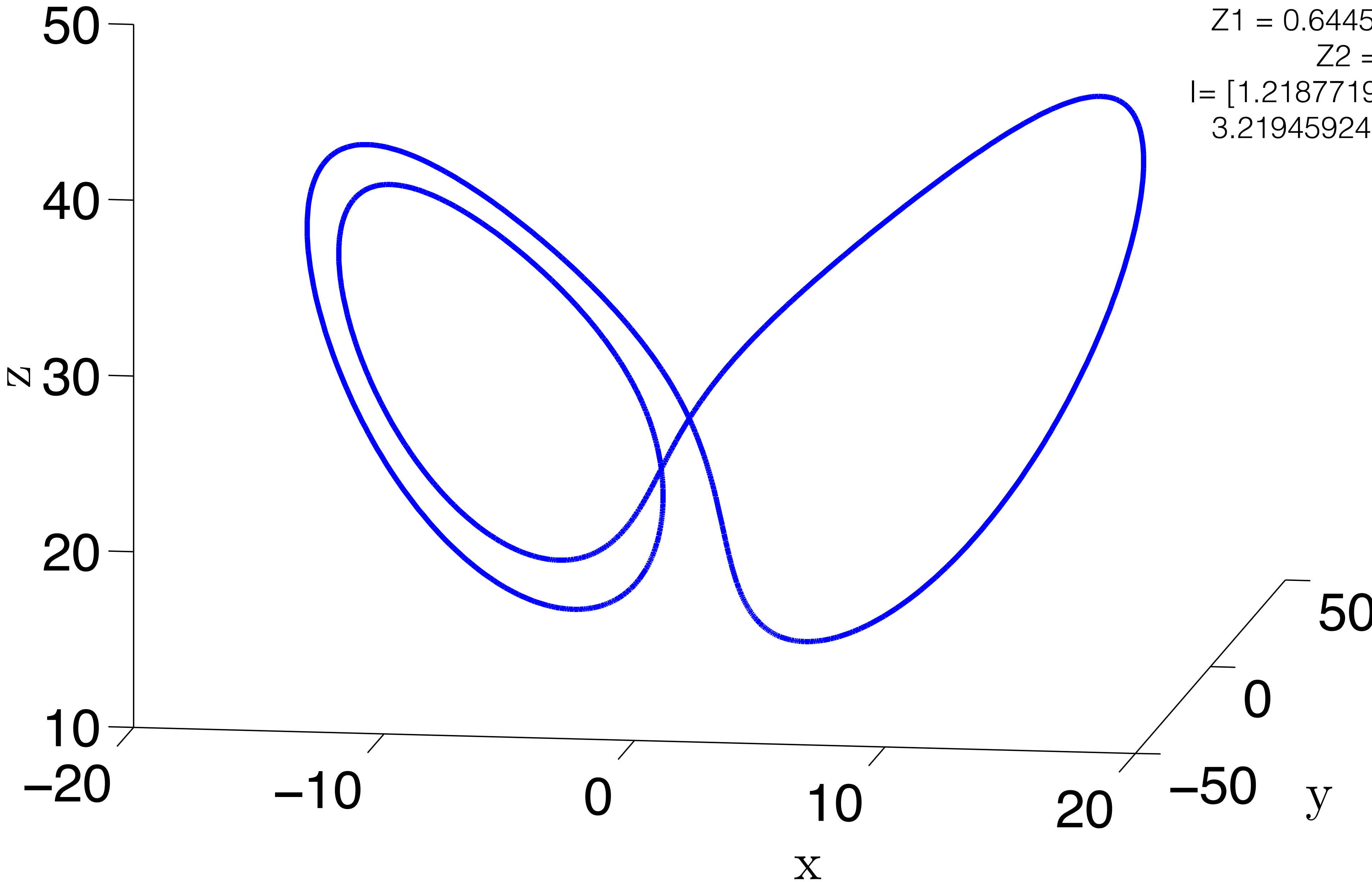
$$\dot{x}_i = \sigma(y - x) + \epsilon x_{i-1}$$
$$\dot{y}_i = x(\rho - z) + y \qquad i = 1, \ldots, 19$$
$$\dot{z}_i = xy + \beta z$$

$$\sigma = 10, \beta = \frac{8}{3}, \rho = 28$$

Second problem

Periodic orbit continuation

Example: 19 coupled Lorenz systems

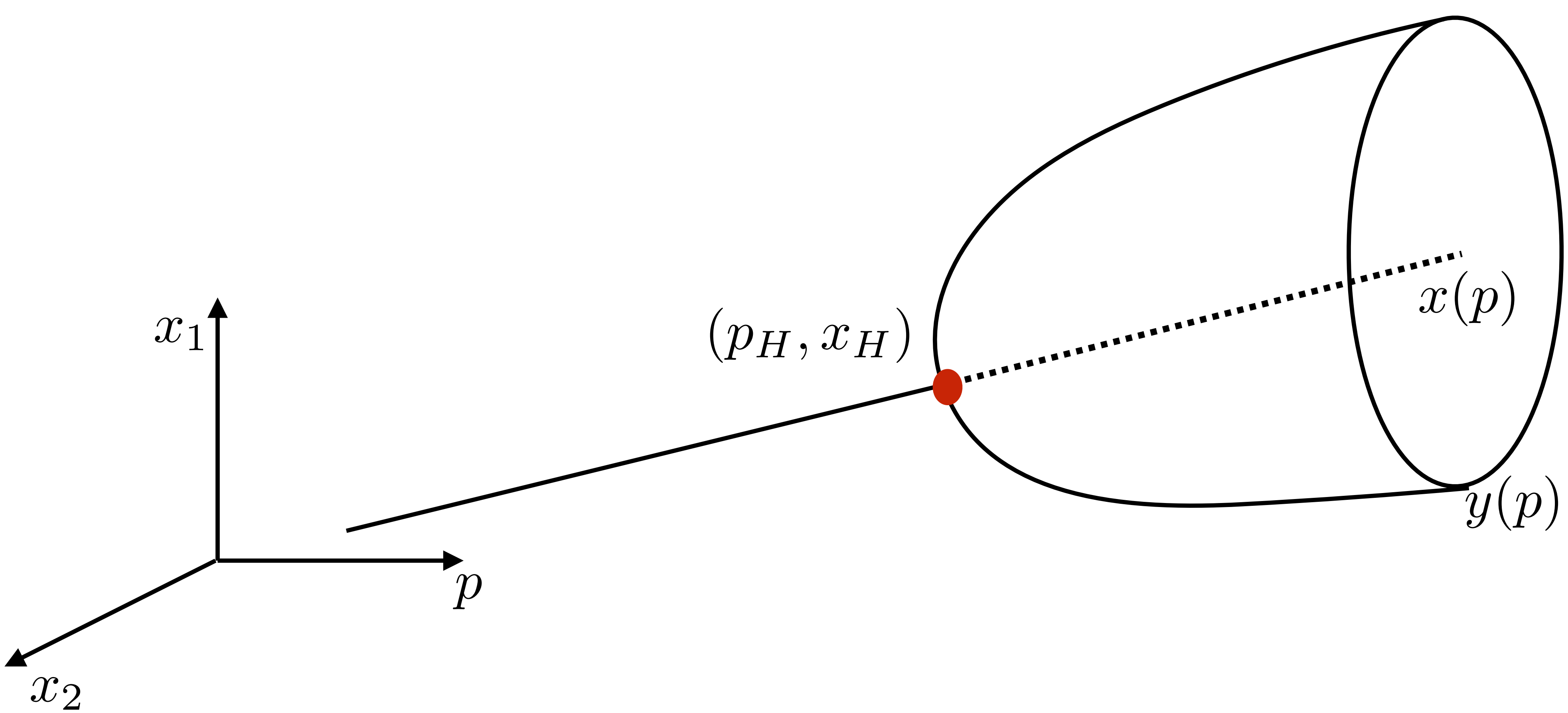


Hopf bifurcation

$$\dot{x} = f(p, x)$$

2 steps:

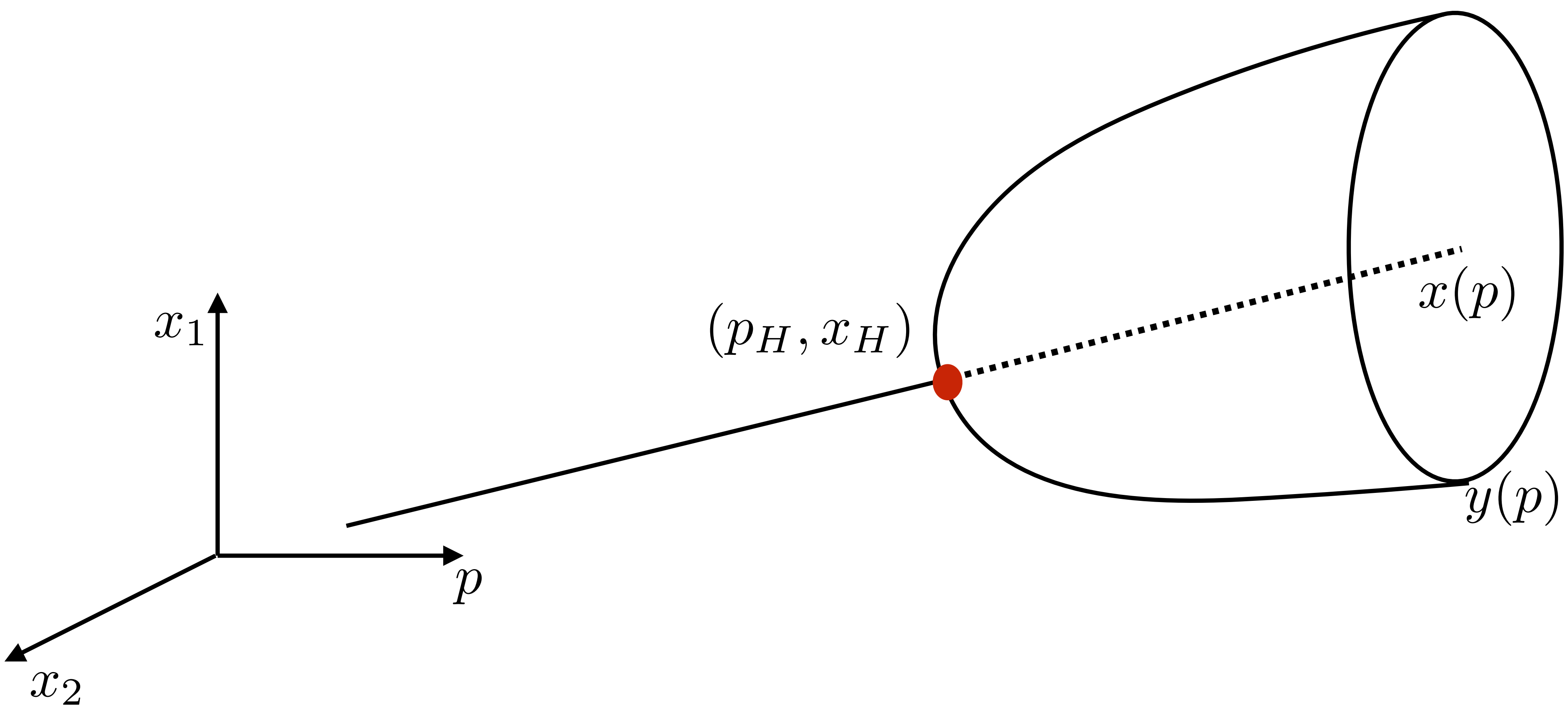
- find the Hopf bifurcation
- validated continuation of the periodic orbits



Hopf bifurcation

$$\dot{x} = f(p, x)$$

At (p_H, x_H), a pair of eigenvalues of Df passes the imaginary axes with non-zero velocity



Hopf bifurcation

$$\dot{x} = f(p, x)$$

$$\Downarrow$$

$$f(p, x) = 0$$
$$Df(p, x)v = \mathbf{i}\beta v$$

$$\beta \in \mathbb{R}, v \in \mathbb{C}^N$$

$$\Downarrow$$

$$f(p, x) = 0$$
$$Df(p, x)v_1 = -\beta v_2$$
$$Df(p, x)v_2 = \beta v_1$$

$$\beta \in \mathbb{R}, v_1, v_2 \in \mathbb{R}^N$$

+ 2 phase conditions

Hopf bifurcation

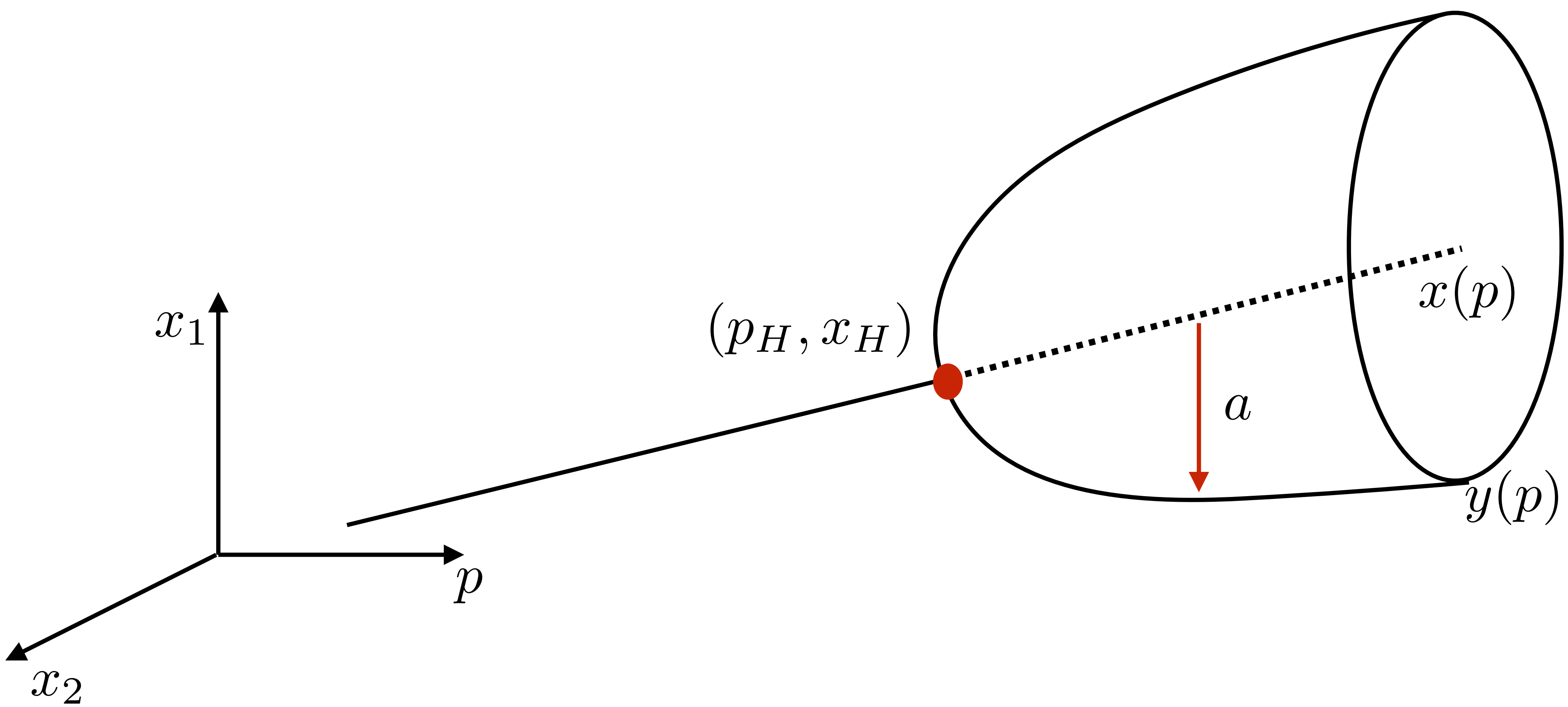
Finite dimensional system, solved numerically and then validated.

During validation, a posteriori check of the first Lyapunov coefficient.

A posteriori check for no other eigenvalues of Df along the imaginary axes.

Hopf bifurcation

New variable: a
the amplitude of $y(p,t) - x(p)$

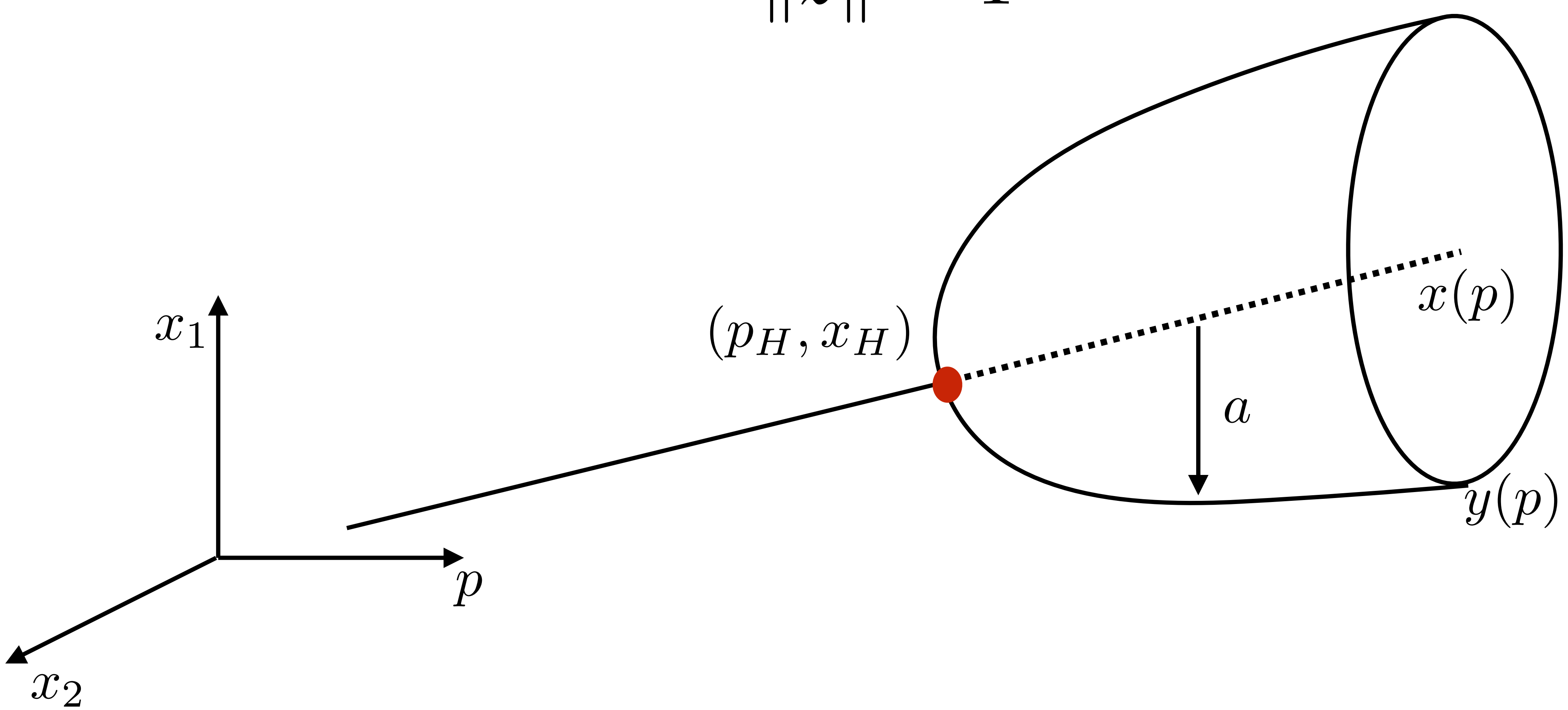


Hopf bifurcation

Rescaling

$$z = \frac{y - x}{a}$$

$$\|z\| = 1$$



Hopf bifurcation

$$\begin{aligned} x, p: & \qquad f(p, x) = 0 \\ z: & \qquad \|z\| = 1 \\ & \qquad \dot{z} = \frac{\dot{y}}{a} = \frac{f(p, az + x)}{a} \end{aligned}$$

Hopf bifurcation

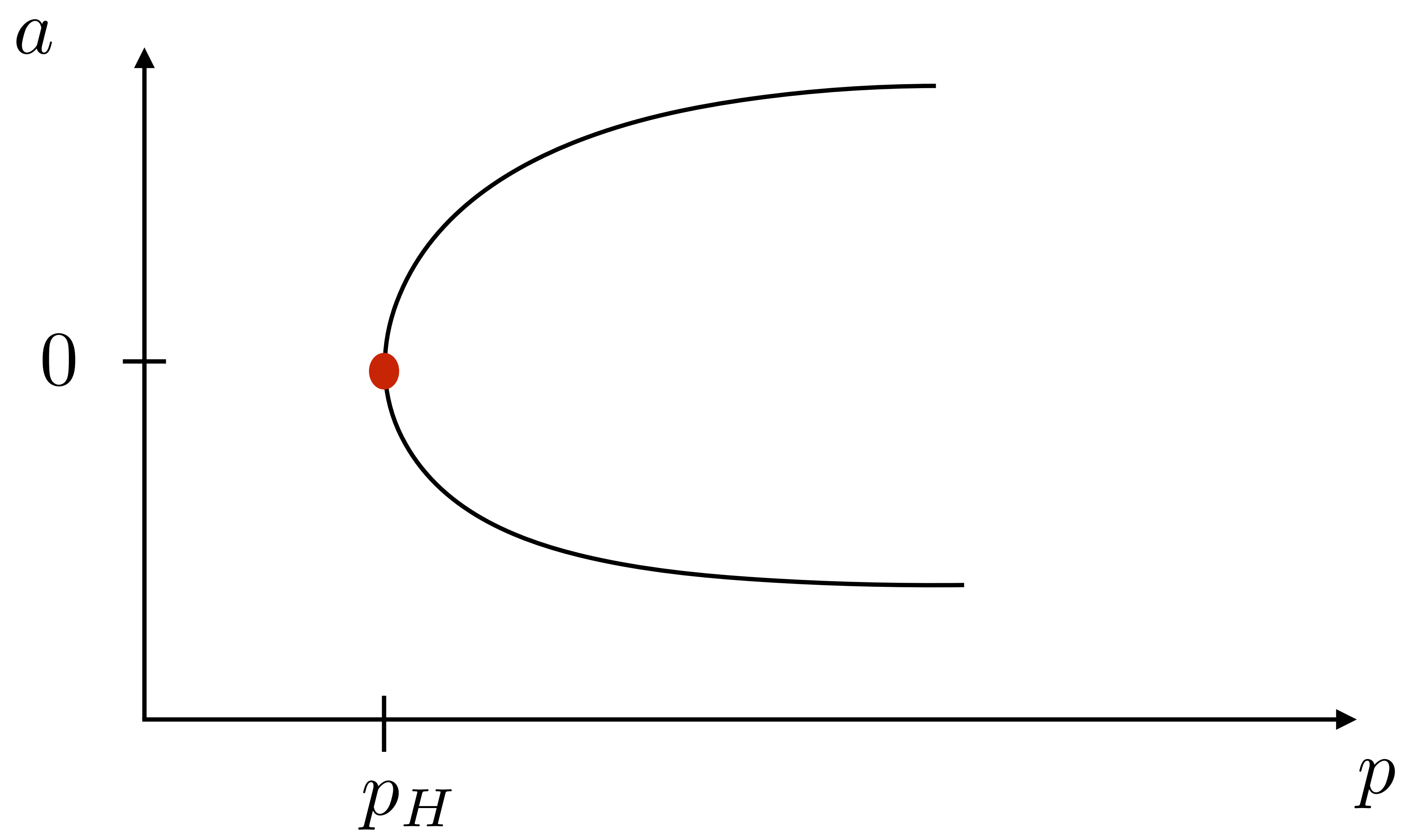
$$\begin{aligned} x, p: &\qquad f(p, x) = 0 \\ z: &\qquad \|z\| = 1 \\ &\qquad \dot{z} = \frac{f(p, x)}{a} + \sum_{k \in \mathbb{N}^N} \frac{1}{k!} \frac{d^k f}{dx^k}(x, p)(az)^k \end{aligned}$$

Hopf bifurcation

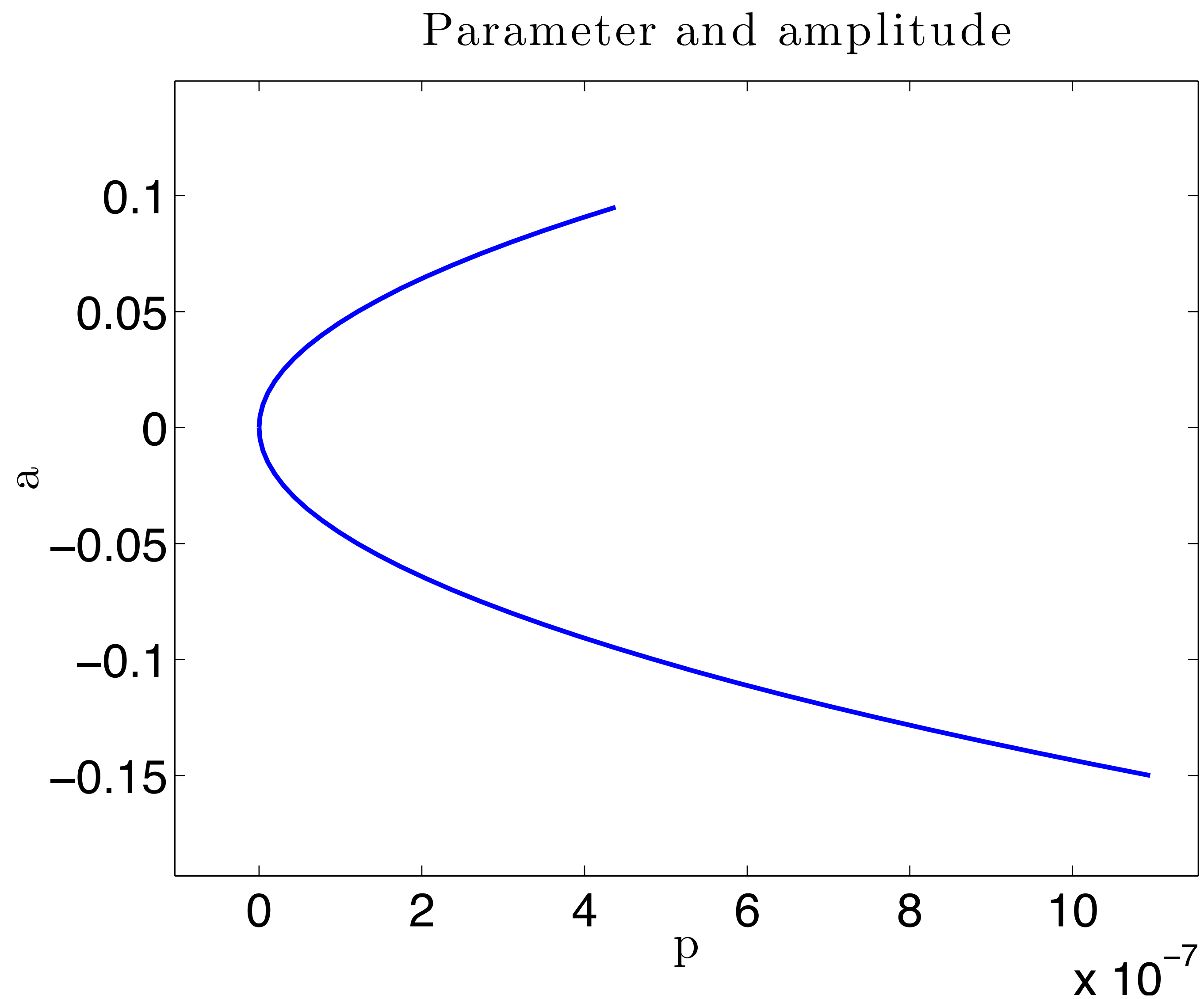
$$
\begin{aligned}
x, p: \qquad & f(p, x) = 0 \\
z: \qquad & \|z\| = 1 \\
& \dot{z} = \sum_{k \in \mathbb{N}^N} \frac{1}{k!} \frac{d^k f}{dx^k}(x, p) a^{|k|-1} z^k
\end{aligned}
$$

Back to the continuation formulation!

Hopf bifurcation



Hopf bifurcation



$$\dot{x} = p(x - y) - yz + w$$
$$\dot{y} = -by + xz$$
$$\dot{z} = -cz + dx + xy$$
$$\dot{w} = -e(x + y)$$

$$b = c = 4$$
$$d = 0.04$$
$$e = 1.4$$

Thanks for your attention!