Applied Harmonic Analysis Methods in Imaging Science Part I

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Outline

1. Continuous Wavelet Transform
2. Continuous Shearlet Transform
   - Shearlet analysis of singularities
3. Applications
   - Edge analysis and detection
   - Soma detection in neuronal images
   - Classification with scattering transform

The Continuous Wavelet Transform

The classical continuous wavelet transform on $\mathbb{R}$ is associated with the **affine systems** of functions

$$\{\psi_{a,t}(x) = a^{-\frac{1}{2}}\psi(a^{-1}(x-t)) :\ a > 0, t \in \mathbb{R}\},$$

where $\psi \in L^2(\mathbb{R})$.

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where $\psi \in L^2(\mathbb{R})$.

Provided that ψ satisfies the **admissibility condition** [Calderón, 1964]

$$\int_{a>0} |\psi(a\xi)|^2 \frac{da}{a} = 1, \qquad \text{for a.e. } \xi \in \mathbb{R},$$

the **continuous wavelet transform** of f

$$\mathcal{W}_\psi : \, f \to \mathcal{W}_\psi f(a,t) = \langle f, \psi_{a,t} \rangle\,, \qquad \text{for } a > 0, t \in \mathbb{R}^d,$$

is a linear isometry (from $L^2(\mathbb{R})$ to $L^2(\mathbb{A})$).

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That is,

$$\|f\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \int_{a>0} |\mathcal{W}_\psi f(a,t)|^2 \, \frac{da}{a} dt,$$

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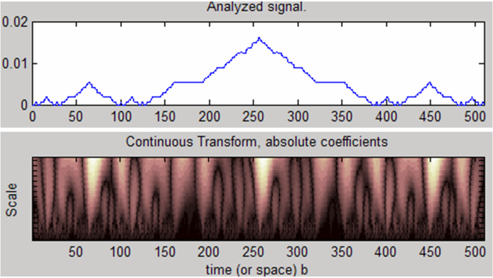
- $\mathcal{W}_\psi f(a,t)$ measures the content of f at **scale** a and **location** t.



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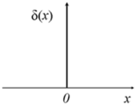
• This property is a manifestation of the *sparsity and locality* of the wavelet representation and it is critical in multiple signal/image processing applications.

Example: Dirac Delta

Let ψ be a well-localized wavelet (e.g., Schwartz class) on $\mathbb{R}$, and δ be the Dirac delta.



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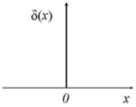

We have:

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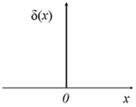

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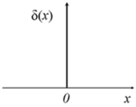

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- If $t \neq 0$, then, for each $k \in \mathbb{N}$, there is a constant C_k such that

$$|\mathcal{W}_\psi \delta(a,t)| = |\psi_{a,t}(0)| \le C_k\, a^k, \quad a \to 0.$$

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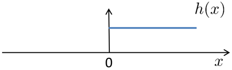

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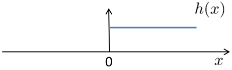

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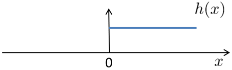

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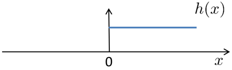

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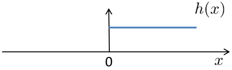

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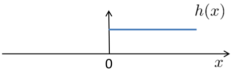

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- If $t = 0$, provided $\int \hat{\gamma}(\eta) d\eta \neq 0$, then

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- If $t \neq 0$, for any $k \in \mathbb{N}$,

$$|\mathcal{W}_\psi h(a,0)| \leq C_k\, a^k, \quad a \to 0.$$

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The continuous wavelet transform resolves the singular support

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The simplest way to extend the continuous wavelet transform to $\mathbb{R}^d$ is by considering the affine systems

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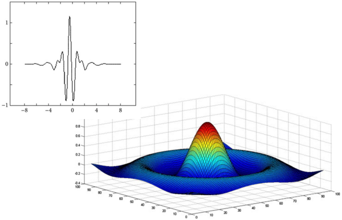

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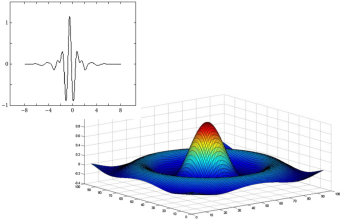

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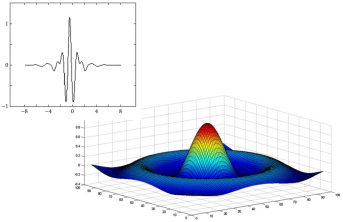

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However, it provides very limited information about the geometry of singularities of multivariate functions and distributions.

Continuous Shearlet Transform (D=2)

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We consider subgroups $\mathbb{A}_G$ of $\mathbb{A}$ of the form

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where G is referred to as the **dilation subgroup**.
The **affine system** generated by $\psi \in L^2(\mathbb{R}^2)$ and $\mathbb{A}_G$ is

$$\left\{\psi_{M,t}(x) = |\det M|^{-1/2}\psi(M^{-1}(x-t)) : \; (M,t) \in \mathbb{A}_G\right\}.$$

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For all $f \in L^2(\mathbb{R}^2)$

$$f(x) = \int_{\mathbb{R}^2} \int_G \langle f, \psi_{M,t} \rangle \, \psi_{M,t}(x) \, d\lambda(M) \, dt,$$

where λ is the left Haar measure on G.

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Admissibility is given by the classical Calderón condition.
This group is associated with the conventional continuous wavelet systems

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A system associated with this group is a **continuous shearlet system**

$$\left\{ \psi_{a,s,t}(x) = a^{-3/4}\, \psi(M_{as}^{-1}(x-t)) :\ a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2 \right\}$$

Construction of Continuous Shearlets

There are many admissible shearlets.

Band-limited shearlets [Guo,Kutyniok,L, 2006]. We choose:

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\, \hat{\psi}_2(\frac{\xi_2}{\xi_1}),$$

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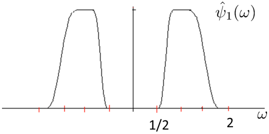
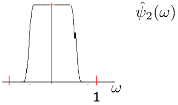


- ψ_2 satisfies $\hat{\psi}_2 \in C^\infty(\mathbb{R})$, supp $\hat{\psi}_2 \subset [-1, 1]$ and $\|\psi_2\| = 1$.



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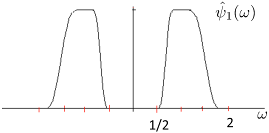
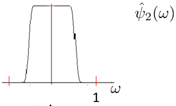
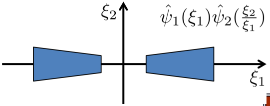


- ψ_2 satisfies $\hat{\psi}_2 \in C^\infty(\mathbb{R})$, supp $\hat{\psi}_2 \subset [-1, 1]$ and $\|\psi_2\| = 1$.



Hence ψ is a smooth bandlimited function.



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Compactly supported shearlets
[Lim,Kutyniok,2011] [Kutyniok,Petersen,2015]. We choose:

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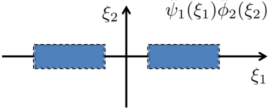

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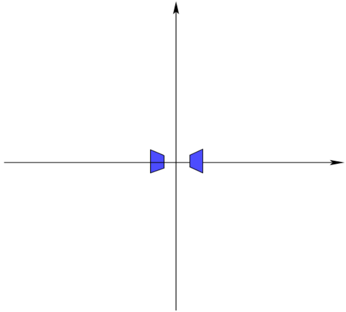
The elements of a shearlet system $\{\psi_{a,s,t}\}$ are a well localized waveforms, with **orientation** controlled by the shear parameter s, and increasingly **elongated** at fine scales ($a \to 0$).



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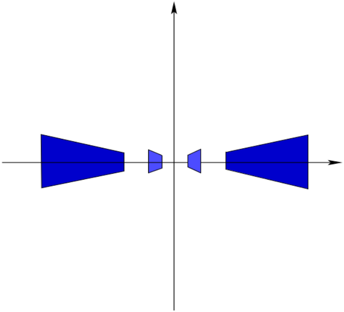

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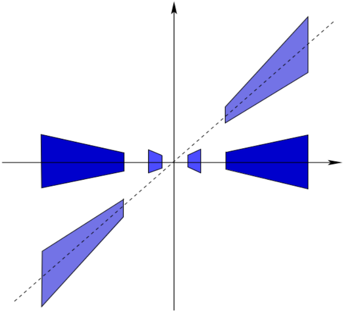

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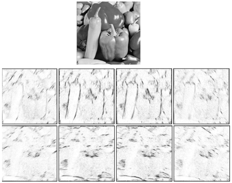

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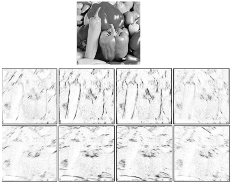
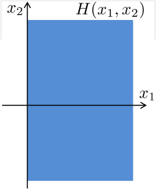


It is able to resolve both the location and orientation of singularities.

Example: Heaviside function (2D)



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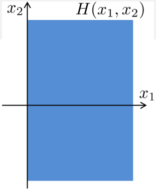

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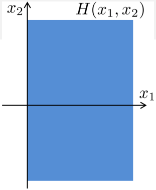

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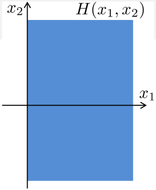

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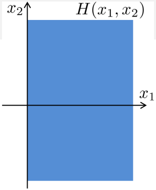

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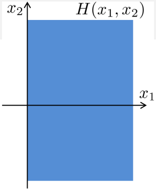

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- If $t_1 = 0$ and $s = 0$, provided $\hat{\psi}_2(0) \neq 0$ and $\int_{\mathbb{R}} \hat{\gamma}(\eta)\,d\eta \neq 0$, we have

$$\mathcal{SH}_\psi H(a,0,(0,t_2)) = O(a^{\frac{3}{4}}).$$

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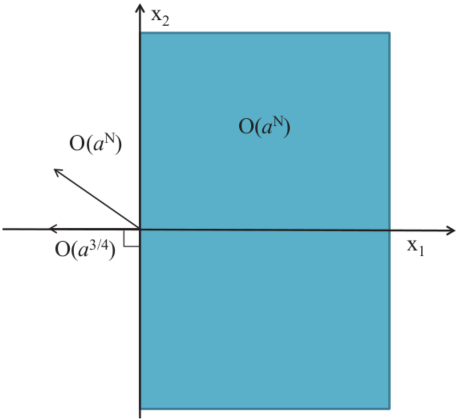

$\mathcal{SH}_\psi H(a, s, t)$ decays rapidly for all values of s and $t = (t_1, t_2)$, except for $s = 0$ and $t_2 = 0$

Resolution of the Wavefront Set

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- The *continuous curvelet transform* has similar properties [Candès,Donoho,2005].
- $\mathcal{SH}_\psi f$ provides a precise description of the **geometry** of **piecewise-smooth edges** of f through its asymptotic decay at fine scales [Guo,L,2008-2015]. This holds also in 3D.

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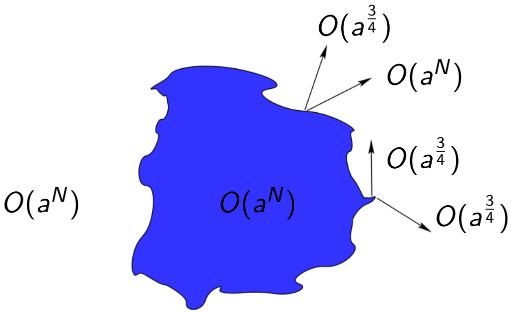
$$0 < \lim_{a \to 0^+} a^{-\frac{3}{4}} |\mathcal{SH}_\psi B(a, s_0, t)| < \infty.$$

That is, $\mathcal{SH}_\psi B$ has slow asymptotic decay only at the edge points for normal orientations, where

$$\mathcal{SH}_\psi B(a, s_0, t) = O(a^{\frac{3}{4}}) \quad \text{as } a \to 0$$

Resolution of Edges (D=2)



At the **regular points** t on an edge, for normal orientation, the shearlet transform decays as $O(a^{\frac{3}{4}})$. For all other values of s, the decay is as fast as $O(a^N)$, for any $N \in \mathbb{N}$.

At the **corner points**, the shearlet transform decays as $O(a^{\frac{3}{4}})$ for both normal orientations.

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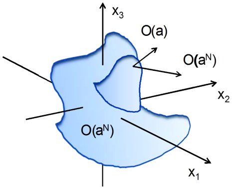

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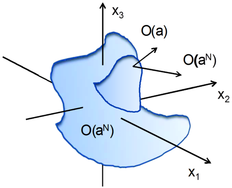

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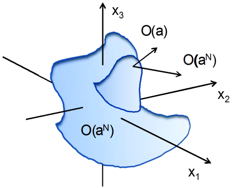
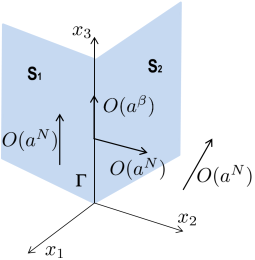


- Analysis of 3D edges and corners [Kutyniok,Petersen,2015].
- Analysis of one-dimensional manifolds, such as the curve of intersection of 2 surfaces. [Houska,L,2015] [Guo,L,2015]



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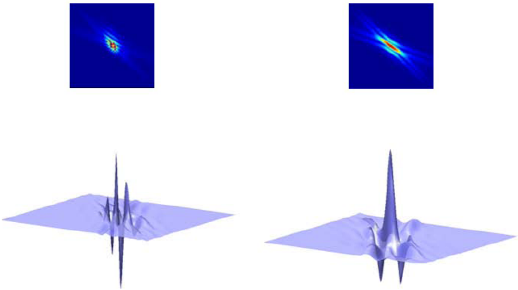
It is possible to separate, in a precise sense, point and curvilinear singularities in 2D [Donoho,Kutyniok, 2013] or points and piecewise linear singularities (polyhedral singularities) in 3D [Guo & L, 2014].

Applications



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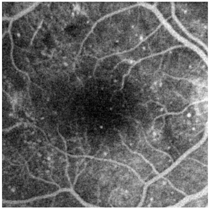
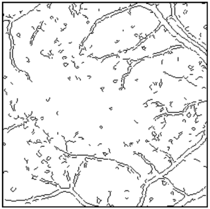


Original retina image



Wavelet/Canny result
FOM=0.27

Shearlet result
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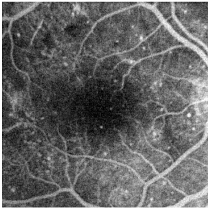

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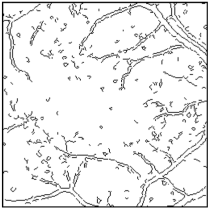

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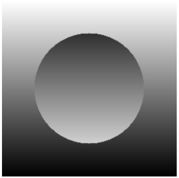
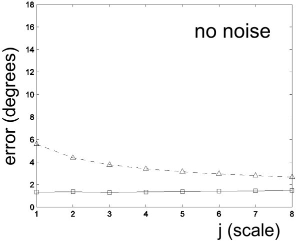
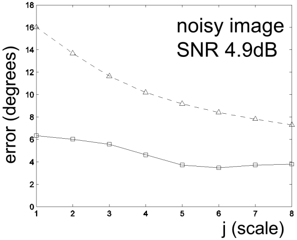
Shearlet-based methods yield extremely competitive results.

Edge Orientation

With respect to conventional multiscale methods, shearlets enable more accurate and robust estimation of **edge orientation**.







Average error (degrees) in estimating edge orientation using a <u>wavelet method</u> (dashed line) versus a <u>shearlet method</u> (solid line), as a function of the scale 2^{-j}.

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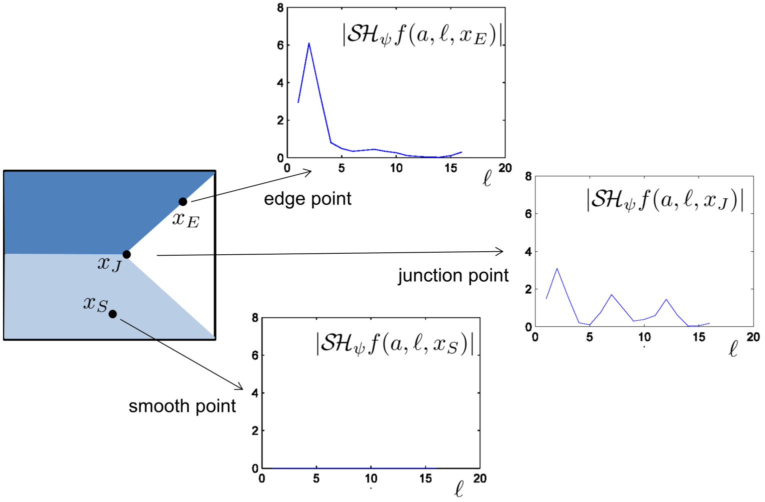
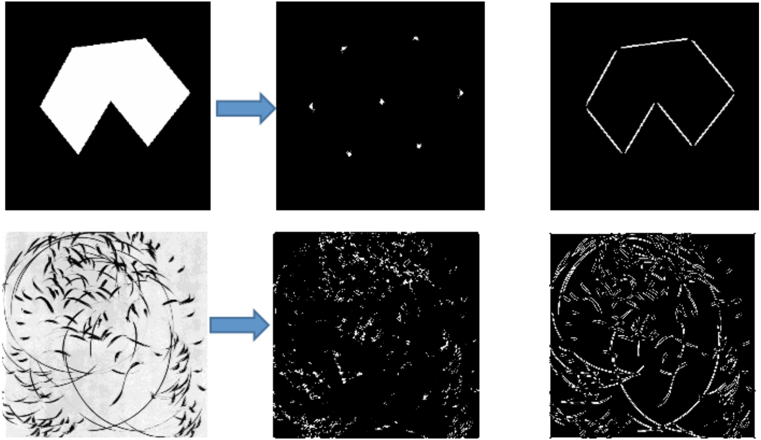
Feature Extraction

Single-scale shearlet analysis of **corners and junctions** [Easley,Labate,Yi,2008]



Feature Extraction

This idea can be used to **classify smooth regions, edges, corner points** [Easley,Labate,Yi,2008].



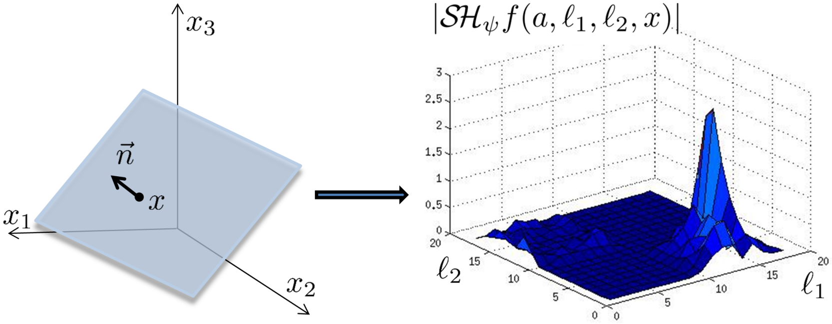
Feature Extraction

A multiscale variant of this idea can be used to define a **corner detector** that is stable to viewpoint and illumination change, and robust to blur and noise [Duval,Odone,De Vito,2015].

Shearlet multiscale corner detection: $j = 0$ (Blue); $j = 1$ (Green); $j = 2$ (Red); $j = 3$ (Magenta).

Surface Orientation

Same idea extends to 3D. The 3D shearlet transform can be used to estimate the **local surface orientation** [L,Negi,2013].



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Surface Orientation

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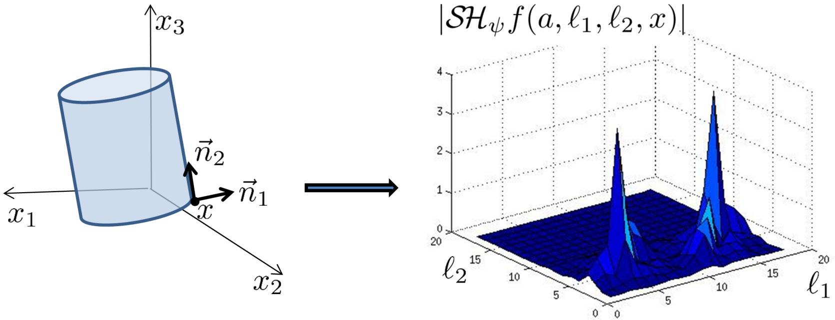

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Classification

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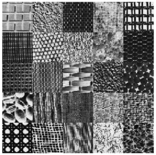

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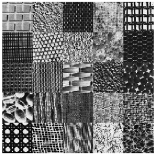

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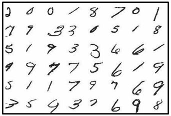

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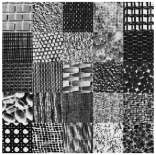

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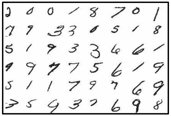

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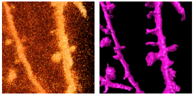

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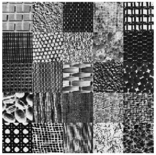

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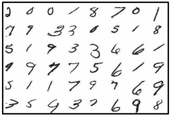

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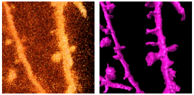

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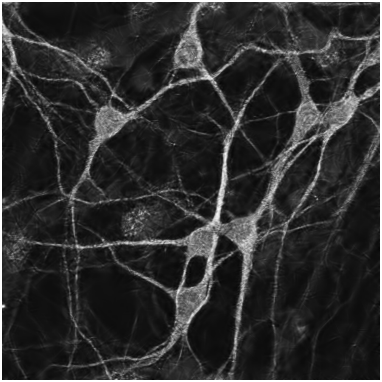
Problem: Soma Extraction

In neuroscience imaging, it is useful to automatically separate somas from dendrites in fluorescent images of neurons.



It may be challenging to accurately **detect** and **extract** somas due to large variations in size and shape and irregularities of fluorescence intensity.

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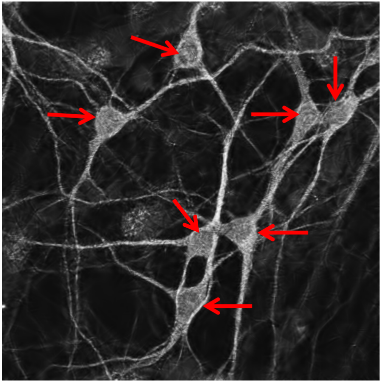

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Due to its *directional sensitivity*, the shearlet transform will exhibit a very different behavior at points of local isotropy (inside soma) vs. points of local anisotropy (inside dendrites)

Directionality Ratio

We define the **directionality ratio** of an image $f \in L^2(\mathbb{R}^2)$ at scale $a > 0$ and location $t \in \mathbb{R}^2$ as the quantity

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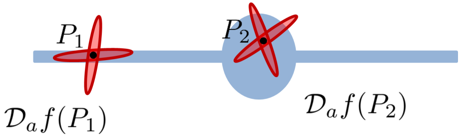
The directionality ratio $\mathcal{D}_a f(t)$ will be very different depending on t being a point of local isotropy of f or not.



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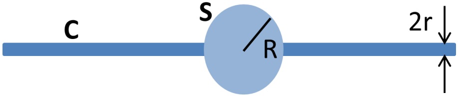
Theorem [Labate,Negi,Ozcan,Papadakis,2014]: Let $f = \chi_N$, where N is the union of two subsets: a ball S with radius $R > 0$ and a cylinder C of size $2r \times L$, where $r > 0$, $L \gg R$.



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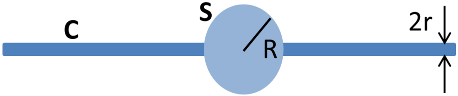

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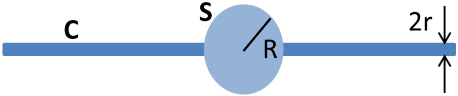

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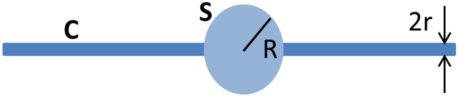

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On the other hand, the directionality ratio of f is **large (close to 1) inside the ball** S.

Soma Extraction. Segmentation

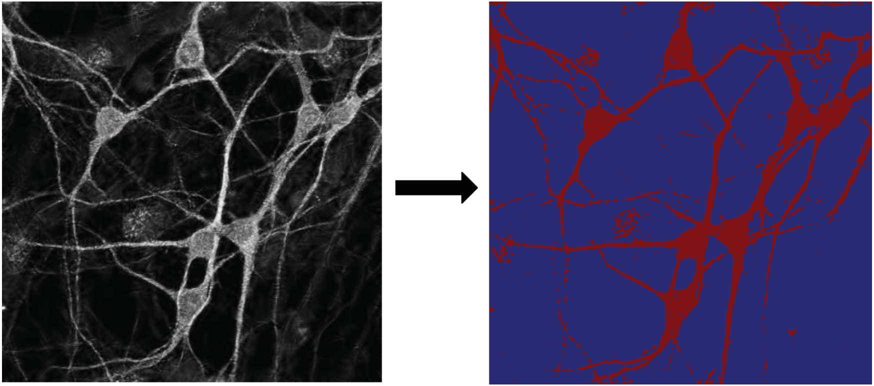
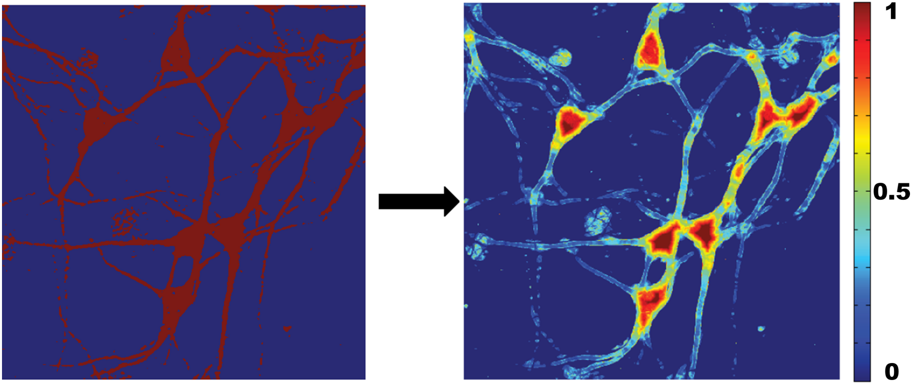


Image segmentation (SVM based)

Soma Extraction. Directionality ratio



Computation of directionality ratio

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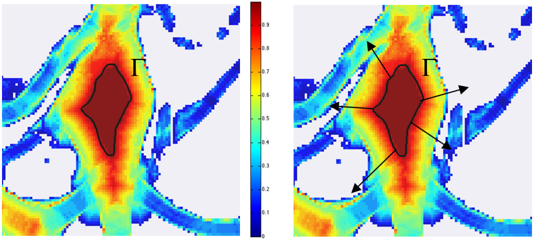

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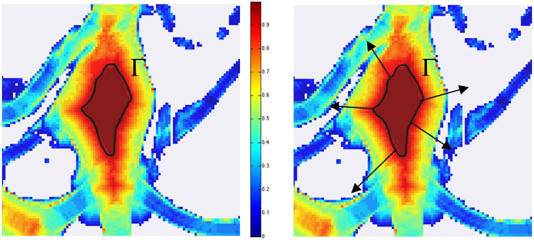
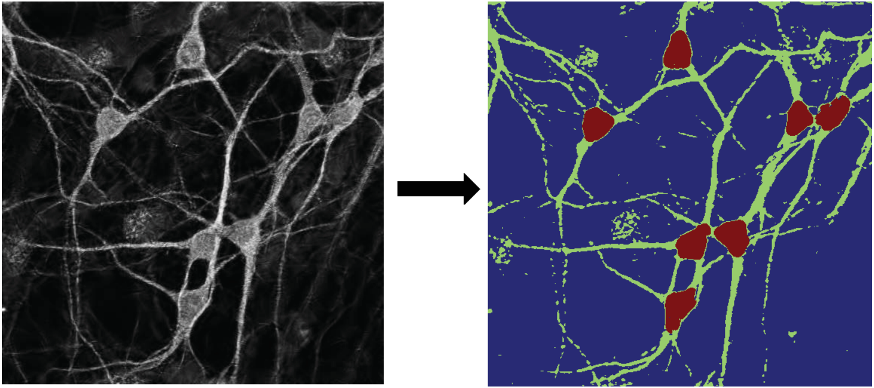


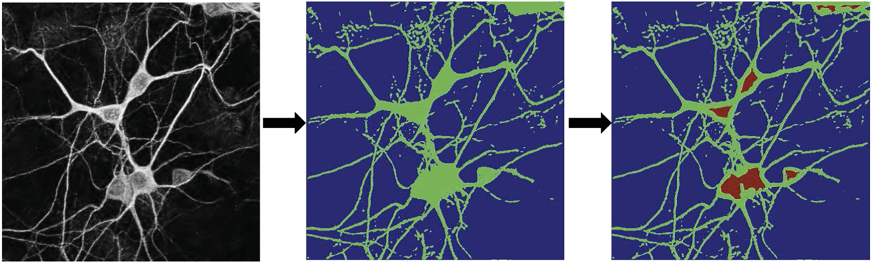
We also use this method to separate clustered somas.

Soma Extraction



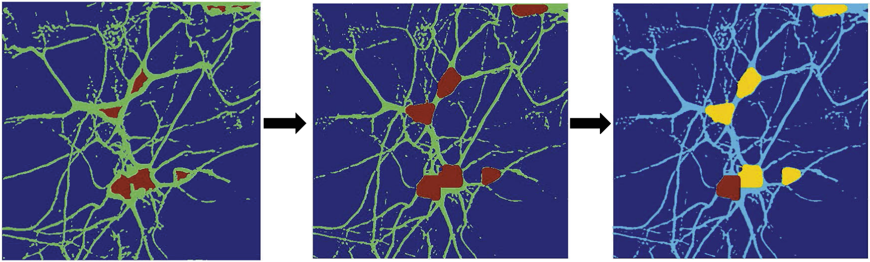
Directionality ratio + level set: soma detection

Soma Extraction. Another example



Identification of somas

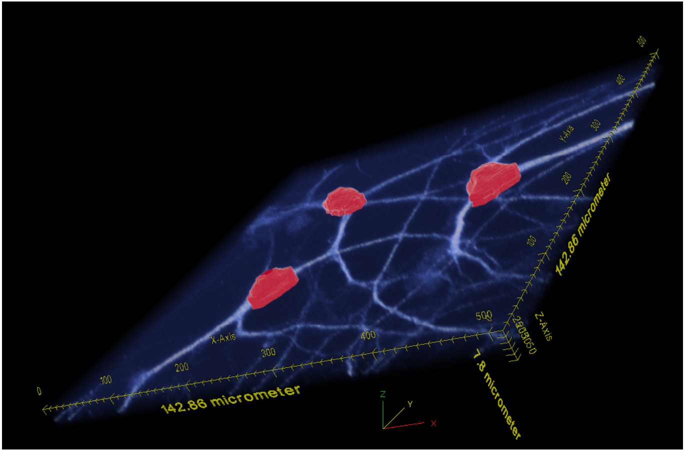
Soma Extraction. Another example



Identification of somas and separation of clustered ones

Soma Extraction (3D)

Method extends to 3D where soma detection can be combined with the extraction of soma morphology [Bozcan,L,Laezza,Negi,Papadakis,2014]



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Key features:

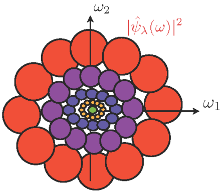
- It extracts locally invariant, stable, highly informative features.
- It is implemented through a cascade of **wavelet filters** and **modulus operators** over multiple layers (deep convolution network).

Dilated wavelets are also rotated with elements $r \in G$:

$$\psi_\lambda(x) = a^{-1}\psi(a^{-1}rx)$$

with $\lambda = (a, r)$, $a > 0, r \in G$.

$$\mathcal{W}_\psi : f \mapsto \mathcal{W}_\psi f(a, t) = f * \psi_\lambda(t)$$



Scattering Transform

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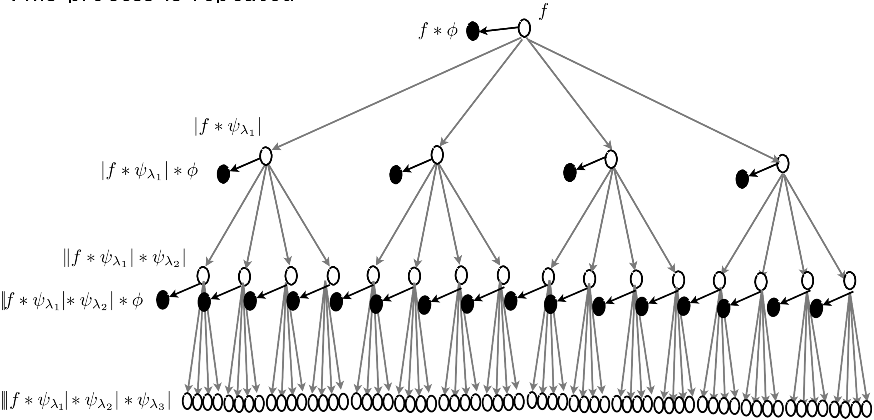
This process is repeated



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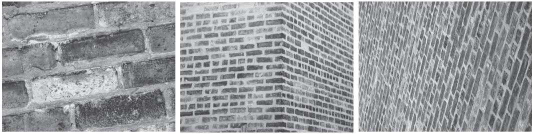

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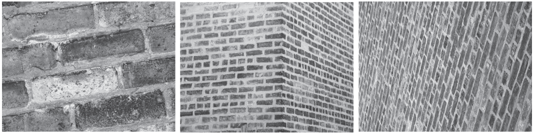
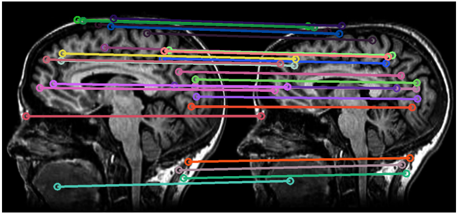
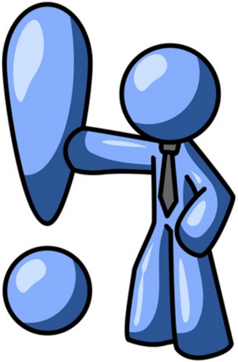


- image registration [Easley,Mc-Innis,L,2015]



Conclusion



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References + codes at:

`www.math.uh.edu\~dlabate`

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