MODELS FOR THIN PRESTRAINED STRUCTURES

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Focus

1. HIERARCHY OF MODELS FOR THIN ELASTIC STRUCTURES
2. MIX POINT 1 WITH A "PRESTRAINED" ASSUMPTION

1. HIERARCHY OF MODELS FOR THIN ELASTIC STRUCTURES

Membrane, plate, von Kármán,...

Sort the models in a hierarchy. In terms of?

- either the external world action (load magnitude, boundary conditions),
- or, equivalently, the internal energy of the structure.

Tool? thickness $h \to 0$, identify limit models.

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“PRESTRAINED” ASSUMPTION

A body deforms in $\mathbb{R}^3$, $\Phi : \Omega \mapsto \mathbb{R}^3$,

Usually:

$$I(\Phi) = \int_\Omega W(\nabla\Phi(x))\,\mathrm{d}x, \quad W : \mathbb{M}_3^> \mapsto \mathbb{R}^+ \text{ stored energy density,}$$

$$\mathbb{M}_3^> := \{F \in \mathbb{M}_3; \det F > 0\}.$$

$W(\mathrm{Id}) = 0$ and $W(F) = 0$ on $SO(3)$, $T_R(\mathrm{Id}) = DW(\mathrm{Id}) = 0$, **$\Omega$ natural state**.

T_R: First PK stress tensor.

Heterogeneity can be added still with $W(x, \mathrm{Id}) = 0, T_R(x, \mathrm{Id}) = 0$.

Prestrain (cont'd): We defined

$$I(\Phi) = \int_\Omega W(\nabla\Phi(x)A^{-1}(x))\,\mathrm{d}x, \quad W \geq 0,\ W(\cdot) = 0 \text{ on } \mathrm{SO}(3).$$

In other words,

$I(\Phi) = \int_\Omega Z(x, \nabla\Phi(x))\,\mathrm{d}x$ where the space-dependent stored energy density

$$Z(x, F) := W(FA^{-1}(x)),\ \det F > 0, \text{ satisfies}$$

$$Z(x, F) = 0 \text{ for } FA^{-1}(x) \in \mathrm{SO}(3), \text{or equivalently, } F^T F = G(x).$$

Why such energy densities? **Allow to model situations where**

for any $x \in \Omega$, the material aims at **reaching a prescribed metric G(x),**

$$(\nabla\Phi(x))^T \nabla\Phi(x) = G(x).$$

IF realized, then the changes of lengths between material points along a deformation Φ follow G.

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First PK stress tensor at x: $T_R(x, F) = D_F Z(x, F) = DW(FA^{-1}(x))A^{-1}(x)$,

$$T_R(x, F) = 0 \text{ for } F^T F = G(x).$$

Find a stress-free configuration? Means $\Phi : \Omega \mapsto \mathbb{R}^3$ such that

$$(\nabla\Phi(x))^T \nabla\Phi(x) = G(x) \quad \forall x \in \Omega, \text{ or } a.e. \quad \textbf{Exists?}$$

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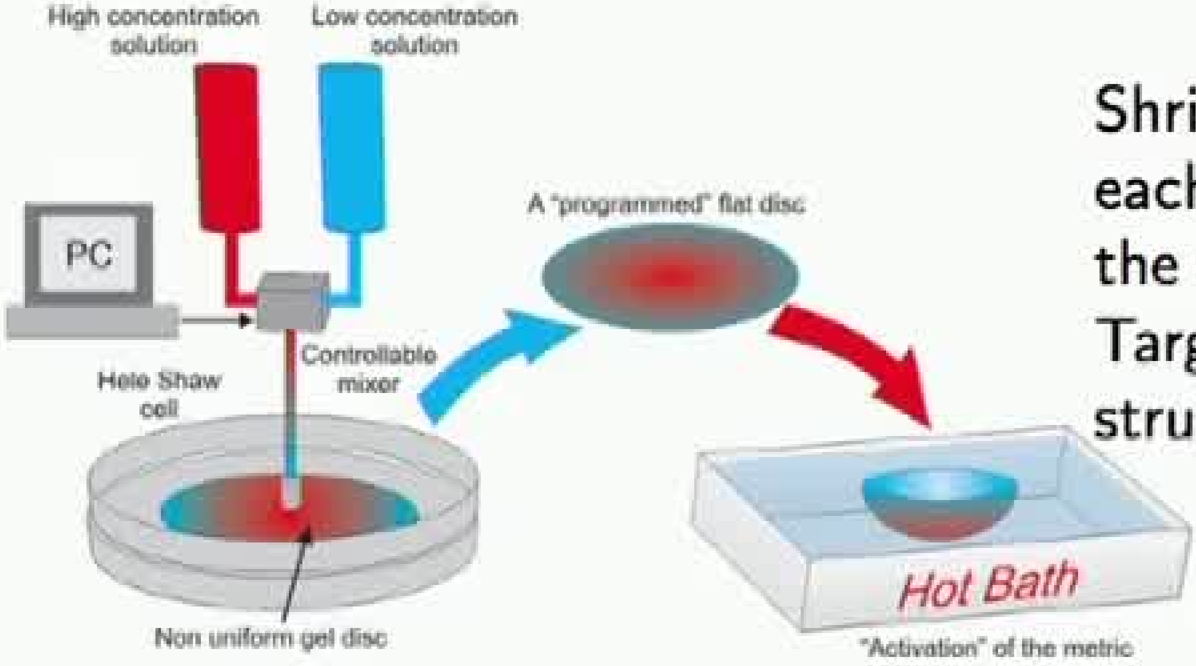
IF realized, then the changes of lengths between material points along a deformation Φ follow G.

Let $A = G^{1/2} \ldots$, the formalism is no longer mysterious.

See: Lewicka & Pakzad (2011), Bhattacharya, Lewicka & Schaffner (2016), Efrati, Sharon, Klein, Kupferman and coauthors (2007, ...).

In mind: growth-induced changes of target lengths, differential shrinking or swelling of materials (responsive gels).

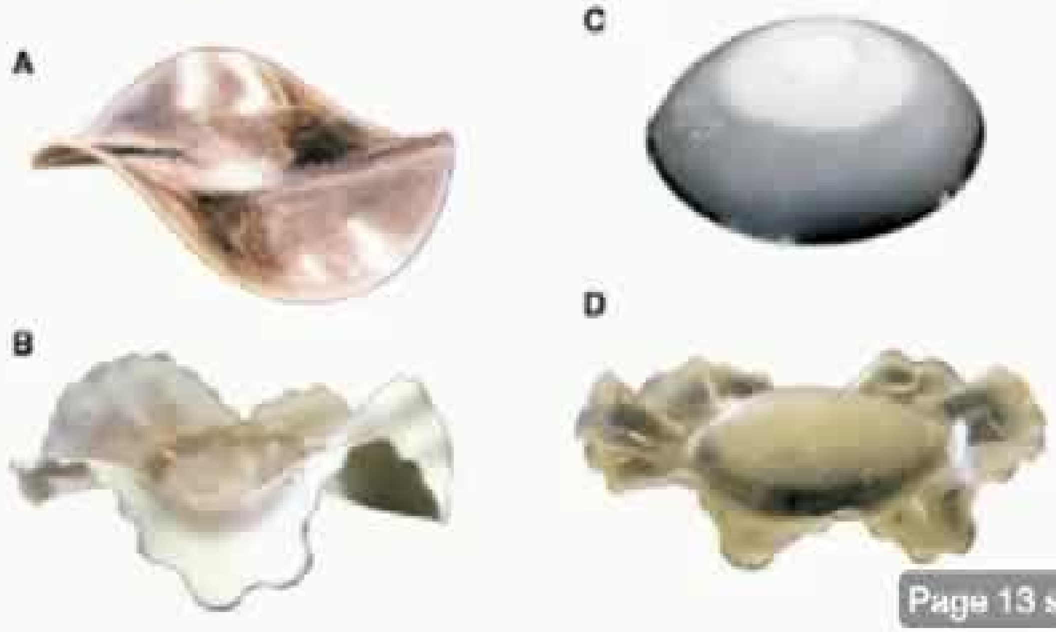
Klein, Efrati, Sharon experiment, Science (2007)



Shrinking by a different ratio $\eta(r)$ at each radius r both in the radial and the azimuthal directions.
Target metric of this initially planar structure:

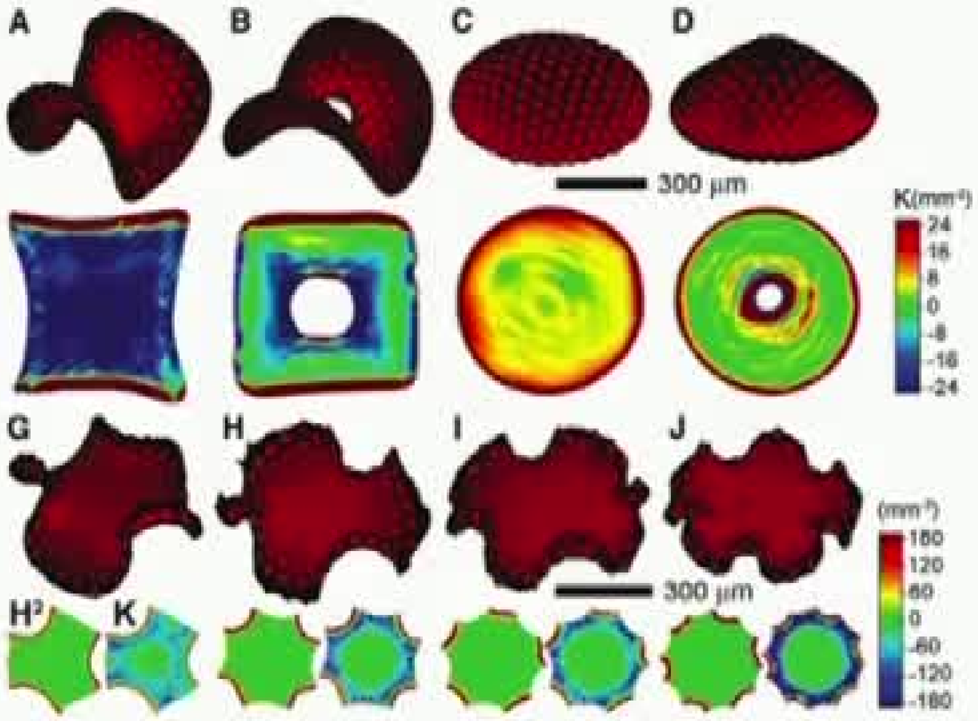
$$g(r) = \begin{bmatrix} \eta^2(r) & 0 \\ 0 & r^2\eta^2(r) \end{bmatrix}$$

The initially planar sheet aims at deforming in a surface in $\mathbb{R}^3$ whose curvature is encoded in $g(r)$ (Gauss Egregium theorem). A little more complicated because the sheet has a thickness. See, R. Kohn's talk.



The structure deforms in space not because of loads, or boundary conditions, but because it has to accommodate lengths (and thickness).

Kim, Hanna, Byun, Santangelo, Hayward experiment, Science (2012)



Photopatterning of polymer films

Remark: In both examples, the structures are thin. Of importance also for living tissues (leaves, skin).

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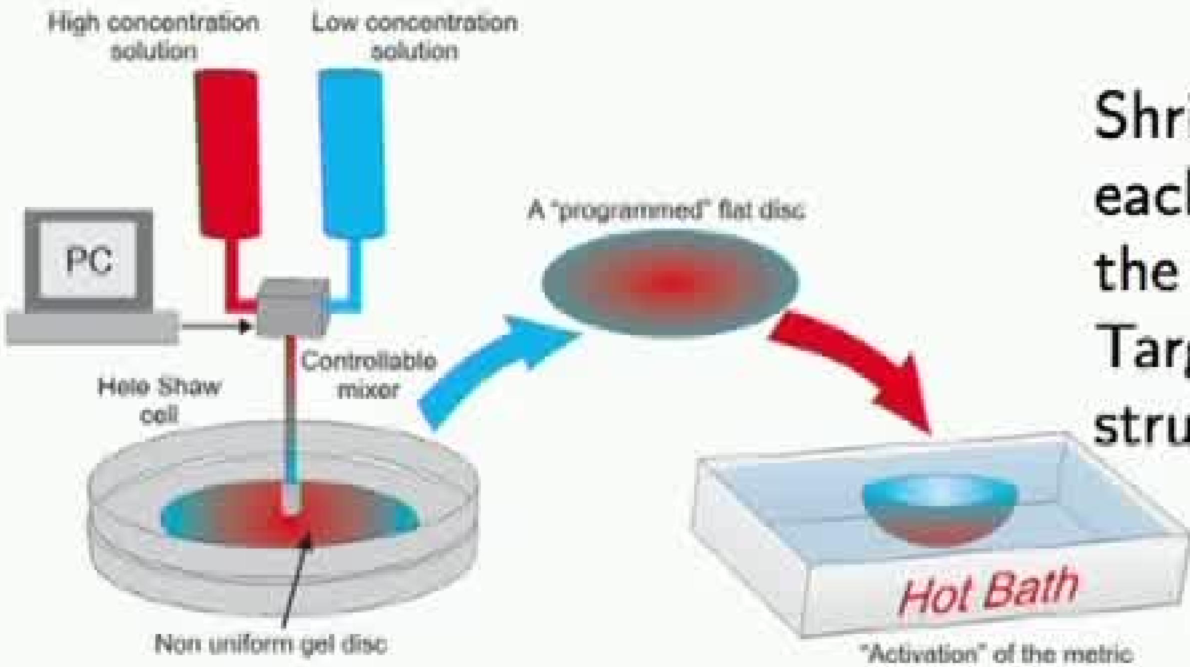

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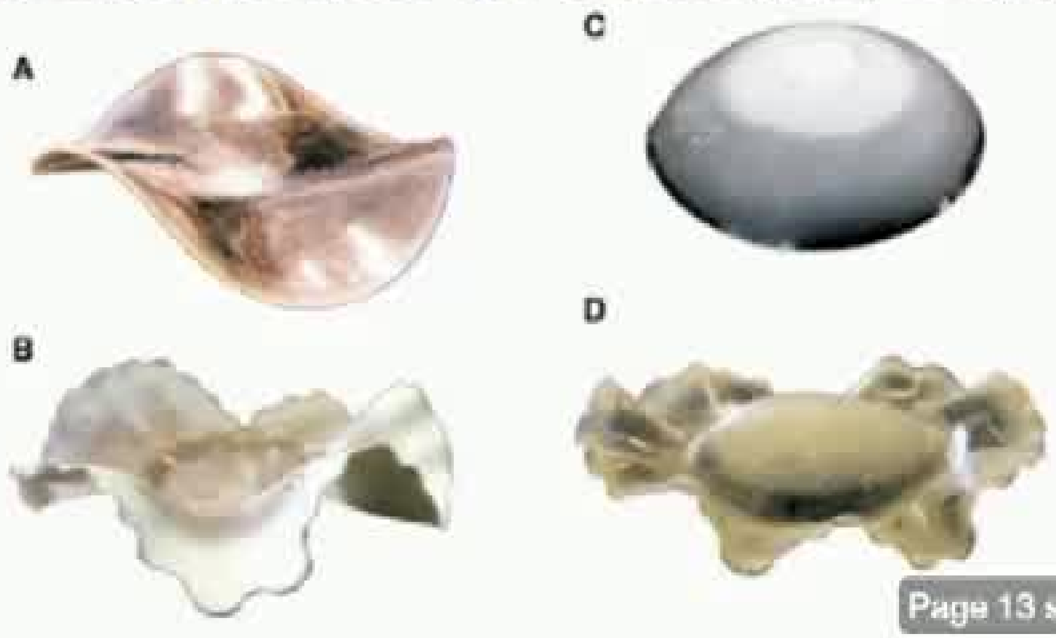

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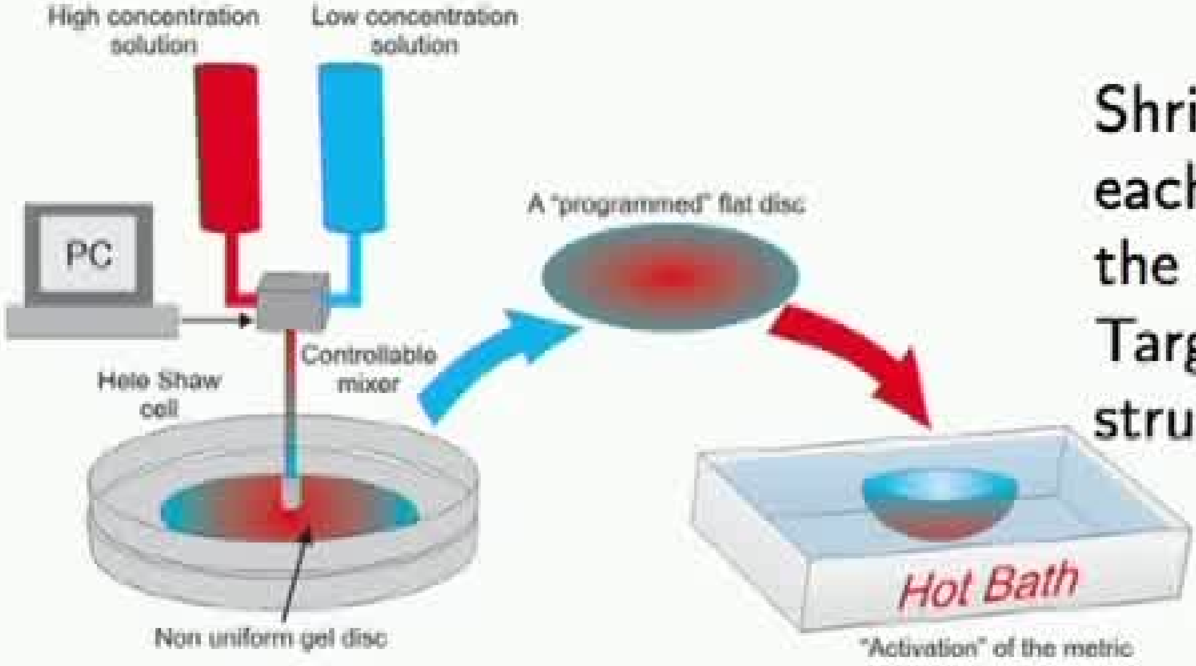

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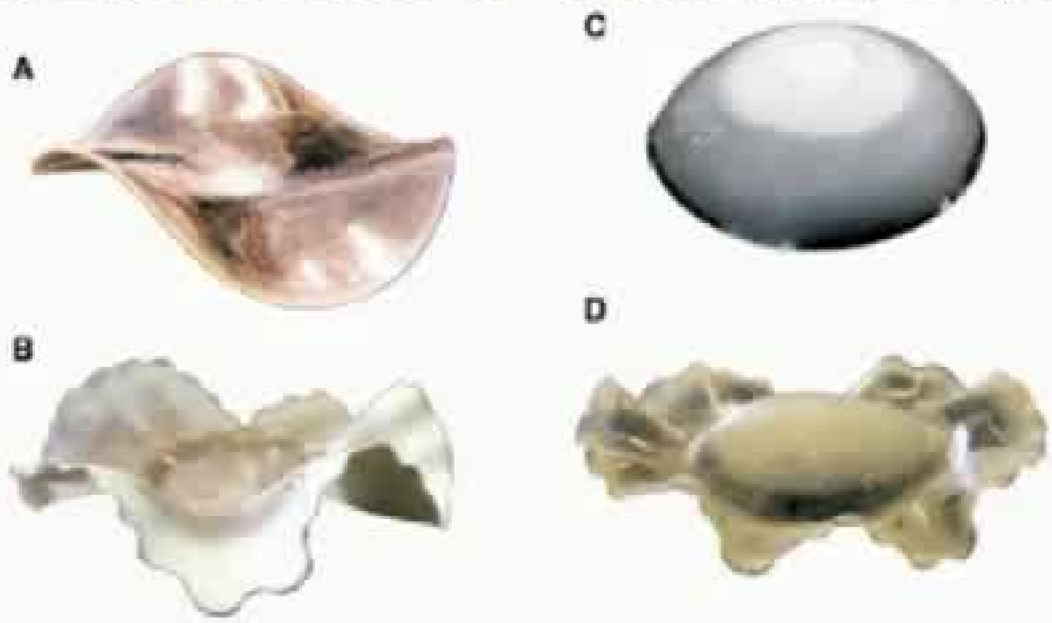

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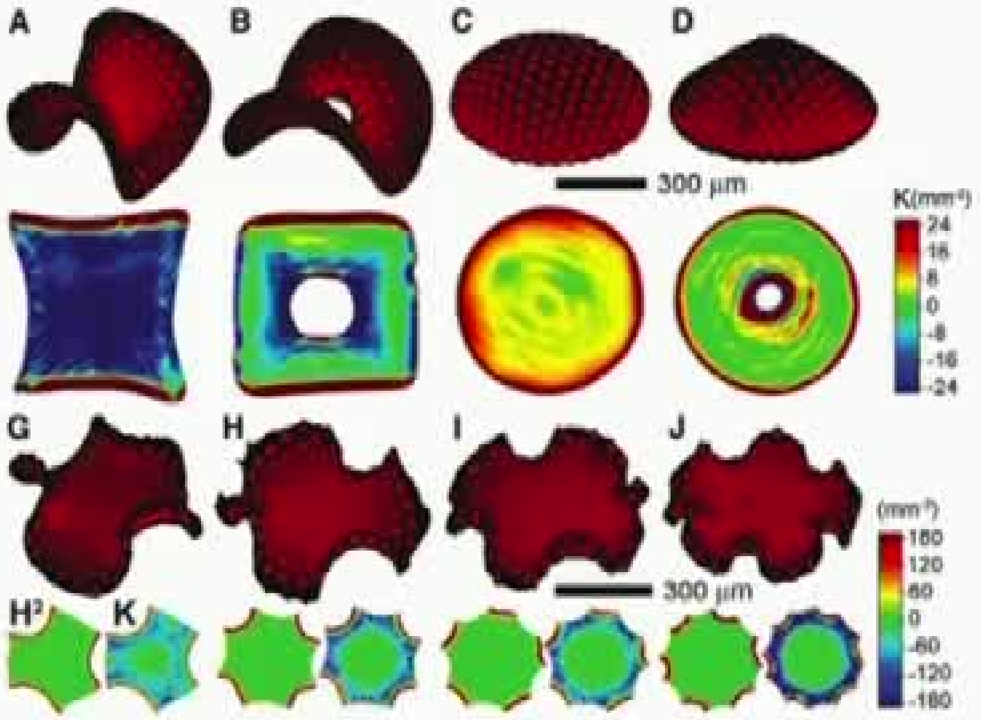

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NATURAL QUESTION: Rigorous derivation of models for prestrained thin structures from prestrained 3d models

Back to 3d: basic problem on a 3d-domain Ω. Let $G(x) \in \mathbb{S}_3^>$ be given (smooth). Can we find

$$\Phi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3, \ (\nabla\Phi(x))^T \nabla\Phi(x) = G(x), \ \det\nabla\Phi(x) > 0?$$

- if $G(x) = \mathrm{Id}$, then $\Phi(x) = Qx$ with $Q \in SO(3)$ (Liouville),
- arbitrary G: yes iff $\mathscr{R} = 0$, G said flat, where

$$\mathscr{R}_{qijk} = \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}, \ \text{“six” entries,}$$

$$2\Gamma_{ijq} = \partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}, \ \Gamma_{ij}^p = g^{pq}\Gamma_{ijq}, \ (g^{pq}) = G^{-1}.$$

$$I^h(\Phi) = \int_\Omega W(\nabla_h \Phi(x) A^{-1}(\bar{x}))\, \mathrm{d}x, \quad \nabla_h \Phi = (\partial_1 \Phi, \partial_2 \Phi, \frac{1}{h}\partial_3 \Phi).$$

Order 0 model: Generalized membrane model

Expected that "Φ^h converges to some Φ with some lim. behavior for $\frac{1}{h}\partial_3 \Phi^h$". Natural to define

$$W_0(\bar{x}, \bar{F}) := \min\{W\left([\bar{F}|b]A^{-1}(\bar{x})\right); b \in \mathbb{R}^3\} \text{ for } \bar{F} \in \mathbb{M}_{3,2}.$$

Then,

$$I^h \xrightarrow{\Gamma - L^p(\Omega)} I_0 \qquad \text{"effectively" defined on } W^{1,p}(\omega; \mathbb{R}^3),$$

$$\forall \Phi = \varphi \in W^{1,p}(\omega; \mathbb{R}^3),\ I_0(\varphi) = \int_\omega QW_0(\bar{x}, \bar{\nabla}\varphi(\bar{x}))\, \mathrm{d}\bar{x}.$$

Question: $\min I_0 = 0$? First, when does $W_0(\bar{x}, \bar{F}) = 0$?

When does $W_0(\bar{x},\bar{F})=0$? Recall $W(FA^{-1}(\bar{x}))=0 \Leftrightarrow F\in SO(3)A(\bar{x})$.

$$W_0(\bar{x},\bar{F}) := \min_b W([\bar{F}|b]A^{-1}(\bar{x})) = 0$$

when

$$\exists b\in\mathbb{R}^3,\ [\bar{F}|b]A^{-1}(\bar{x})\in SO(3), i.e., [\bar{F}|b]^T[\bar{F}|b]=G(\bar{x}),$$

$$i.e., \begin{bmatrix}\bar{F}^T\bar{F} & \bar{F}^Tb\\ b^T\bar{F} & |b|^2\end{bmatrix} = G(\bar{x}),\ i.e.,\ \bar{F}^T\bar{F}=G_{2\times 2}(\bar{x}).$$

Indeed, complete $\bar{F}$ with b s.t.
$b\cdot f_1 = g_{13}(\bar{x}), b\cdot f_2 = g_{23}(\bar{x}), |b|^2 = g_{33}(\bar{x}), \det[\bar{F}|b]>0.$

Second, consequence on QW_0?

Pipkin's results and extensions: write $W_0(F)=\tilde{W}_0(F^TF)$,

$$QW_0(\bar{x},\bar{F}) \le \inf\{\tilde{W}_0(\bar{x},\bar{F}^T\bar{F}+S); S\in\mathbb{S}_2^+\}.$$

Consequence: $QW_0(\bar{x},\bar{F})=0$ for any $\bar{F}$ s.t. $\bar{F}^T\bar{F}\le G_{2\times 2}(\bar{x})$,

Third, consequence on the mappings?

$$I_0(\varphi) = 0 \text{ for } \varphi \in W^{1,p}(\omega, \mathbb{R}^3),\ (\bar{\nabla}\varphi)^T \bar{\nabla}\varphi \leq G_{2\times 2},$$

that are the **short maps.**

Remark: one of the rare instances when a result on quasiconvex envelopes is obtained algebraically.

Is the obtained zero-order model sound?

- with loads (of adequate magnitude) and boundary conditions, then "yes" (contains some information).
- we decided: no loads, no B.C. All short maps make I_0 equal to 0.

How many short maps?

- arbitrary $G_{2\times 2}$,

  $$\bar{\nabla}\varphi^T \bar{\nabla}\varphi = G_{2\times 2} \quad \textbf{is possible! (isometric immersion)}$$

  Nash-Kuiper *circa* 1954, with C^1-regularity, not C^2,
- and the "really short" maps.

Comments:

- totally different from the $3d \mapsto 3d$ framework,
- Conti, Delellis & Szekelyhidi (2010) proved $C^{1,\alpha}$-regularity $\alpha < \frac{1}{7}$, Delellis, Inauen & Szekelyhidi (2015), $\alpha < \frac{1}{5}$,
- Nirenberg (1953): smooth iso. immersion for $G_{2\times 2}$ with $\mathscr{K} > 0$, Poznyak & Shikin (1995): $\mathscr{K} < 0$.
- Conti & Maggi, Pakzad, Hornung & Velčić, Olbermann, comments in R. Kohn's talk ...

Footnote: Isometric immersion of the flat torus into $\mathbb{R}^3$, $\mathscr{K} = 0$, Hevea project.

Order 2 model: Generalized bending model

From now on, $W(\cdot) \geq C\,\mathrm{dist}^2(\cdot, \mathrm{SO}(3))$.

The energy magnitude is smaller than h^0. Can it be of order 2 "as usual"?

For $\dfrac{\inf I^h}{h^2}$ to converge to a finite value (and conversely),
there must exist a $H^2(\omega)$-regular isometric immersion of $G_{2\times 2}$.

Where does it come from?

$$\text{From} \quad I^h(\Phi^h) \leq Ch^2, \quad \text{we have} \quad \|\mathrm{dist}(\nabla_h \Phi^h A^{-1}, \mathrm{SO}(3))\|_{L^2(\Omega)} \leq Ch.$$

By a generalized version (LP, BLS) of the quantitative rigidity estimate,

$\|\nabla_h \Phi^h - Q^h\|_{L^2(\Omega)} \leq Ch$, $\|\bar{\nabla} Q^h\|_{L^2(\omega)} \leq C$, where $Q^h \in H^1(\omega; \mathbb{M}^3)$ (not rigid).

From stage 0,

$$\nabla_h \Phi^h \rightharpoonup [\bar{\nabla}\varphi | b] \text{ in } L^2.$$

Then, Q^h converges weakly in $H^1(\omega)$ to some Q. This obliges $\bar{\nabla}\varphi$ to gain one degree of regularity.

Which object to work on?

- usual bending: 2nd fundamental form $(\bar\nabla\varphi)^T\bar\nabla n$, 2×2, symmetric,
- here: $(\bar\nabla\varphi)^T\bar\nabla b$, 2×2, b given at level 0 in terms of a $G_{2\times 2}$-isometry φ by

$$[\bar\nabla\varphi|b]^T[\bar\nabla\varphi|b]=G,\quad \det[\bar\nabla\varphi|b]>0.$$

Expect D^2W to enter the picture, $D^2W(\mathrm{Id})(H,H)=D^2W(\mathrm{Id})(\mathrm{sym}\,H,\mathrm{sym}\,H)$.

For $H^\sharp$, 2×2 matrix, define

$$W_2(\bar x,H^\sharp)=\min\{D^2W(Id)(A^{-1}(\bar x)\,H\,A^{-1}(\bar x))^{(2)},\ H\in\mathbb{M}_3,\ H_{2\times 2}=H^\sharp\}.$$

Again, W_2 acts on $\mathrm{sym}(H^\sharp)$.

$$\frac{I^h}{h^2}\xrightarrow{\Gamma-H^1(\Omega)} I_2,\ I_2(\Phi)=\begin{cases}\frac{1}{4!}\int_\omega W_2\left(\bar x,(\bar\nabla\varphi^T\bar\nabla b)(\bar x)\right)d\bar x, & \Phi=\varphi\in H^2(\omega;\mathbb{R}^3),\text{iso},\\ +\infty \text{ otherwise.}\end{cases}$$

Order 4 model: Generalized von Kármán enegy

$$I_2(\Phi) = \frac{1}{2}\int_\omega W_2\left(\bar x, (\bar\nabla\varphi^T\bar\nabla b)(\bar x)\right)d\bar x,\ \Phi = \varphi \in H^2(\omega;\mathbb{R}^3), \text{iso.}$$

If the min is 0, further information should be sought for.

Start from $\min I_2 = 0$, i.e. $\mathcal{Q}_{1212} = \mathcal{Q}_{1213} = \mathcal{Q}_{1223} = 0$,

$$\min I_2 = 0 \Leftrightarrow \exists\varphi \in H^2(\omega;\mathbb{R}^3),\ \bar\nabla\varphi^T\bar\nabla b \text{ skew}.$$

First finding. Then $\inf I^h$ is indeed smaller: $\inf I^h \leq Ch^4$

- if exists, then unique, because its 2nd fundamental form, in addition to its ... in the basis $(\partial_1\varphi, \partial_2\varphi, n)$ as

Hint: find ... Choose simply ... $\varphi(\bar x) + x_3 b(\bar x) + \frac{h^2 x_3^2}{2} d(\bar x)$ with d as follows.

Letting $Q = [\bar\nabla\varphi|b]$, $QA^{-1} \in SO(3)$, $B = [\bar\nabla b|d]$,

$$\nabla_h \Phi^h A^{-1}(\bar x, x_3) = QA^{-1}(\mathrm{Id} + hx_3 A^{-1}Q^T BA^{-1} + h^2 x_3^2 T),\ G^{ij}.$$

- computations using the decomposition ... show that

$$W(\nabla_h\Phi^h A^{-1}) = W(\mathrm{Id} + hx_3 A^{-1}Q^T BA^{-1} + h^2 x_3^2 T).$$

Make $Q^T B = \begin{pmatrix} \bar\nabla\varphi^T\bar\nabla b & \bar\nabla\varphi^T d \\ b^T\bar\nabla b & b\cdot d\end{pmatrix}$ skew (to kill the h^2 term in $\int D_2 W(\mathrm{Id})$).

$\min I_2 = 0 \Leftrightarrow \mathcal{Q}_{1212} = \mathcal{Q}_{1213} = \mathcal{Q}_{1223} = 0$

First block is skew, then choose d: $Q^T d = (-b\cdot\partial_1 b, -b\cdot\partial_2 b, 0)^T$.

which does not mean that $\mathcal{Q} = 0$; there may be some locking in the 3d-body that does not show up at the bending level.

Limit model. We already know that $\Phi^h \overset{H^1}{\rightarrow} \varphi$, $\frac{1}{h}\partial_3\Phi^h \overset{L^2}{\rightarrow} b$. Now,

$$u^h(\bar{x}) := \frac{1}{h}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\Phi^h - (\varphi + hx_3 b)\right) \mathrm{d}x_3 \overset{H^1}{\rightarrow} u^1, \ \mathrm{sym}\left(\bar{\nabla}\varphi^T \bar{\nabla} u^1\right) = 0,$$

$$\frac{1}{h}\mathrm{sym}\left(\bar{\nabla}\varphi^T \bar{\nabla} u^h\right) \rightarrow e^2 \in L^2(\omega; \mathbb{S}_2),$$

$$\begin{aligned} I_4(u^1, e^2) &= \int_\omega |e^2 + \frac{1}{2}(\bar{\nabla}u^1)^T \bar{\nabla} u^1 + \frac{1}{4!}\bar{\nabla} b^T \bar{\nabla} b|^2 \\ &+ \int_\omega |\bar{\nabla}\varphi^T \bar{\nabla} p^1 + (\bar{\nabla}u^1)^T \bar{\nabla} b|^2 \\ &+ \int_\omega |\mathrm{sym}(\bar{\nabla}\varphi^T \bar{\nabla} d) + \bar{\nabla} b^T \bar{\nabla} b|^2 \end{aligned}$$

where $p^1(u^1)$.

Link with usual case:

$$\begin{aligned} \partial_\alpha u^1_\beta + \partial_\beta u^1_\alpha &= 0 \\ e^2 + \frac{1}{2}(\bar{\nabla}u^1)^T \bar{\nabla} u^1 &= \frac{1}{2}(\partial_\alpha u^2_\beta + \partial_\beta u^2_\alpha + \partial_\alpha u^1_3 \partial_\beta u^1_3) \\ \bar{\nabla}\varphi^T \bar{\nabla} p^1 &= -\partial_{\alpha\beta} u^1_3. \end{aligned}$$

Order 4 model: Generalized von Kármán enegy

Start from $\min I_2 = 0$, *i.e.* $\mathscr{R}_{1212} = \mathscr{R}_{1213} = \mathscr{R}_{1223} = 0$,

i.e. $\exists! \varphi \in H^2(\omega; \mathbb{R}^3)$, $\bar{\nabla}\varphi^T \bar{\nabla}\varphi = G_{2\times 2}$ and $\bar{\nabla}\varphi^T \bar{\nabla} b$ skew.

First finding. Then $\inf I^h$ is indeed smaller: $\inf I^h \leq Ch^4$.

Hint: Choose simply $\Phi^h(\bar{x}, x_3) = \varphi(\bar{x}) + hx_3 b(\bar{x}) + \frac{h^2 x_3^2}{2} d(\bar{x})$ with d as follows.

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Can be interpreted as

$$\begin{aligned} I_4(u^1, e^2) &= \int_\omega |\text{change in metric departing from } \varphi|^2 \\ &+ \int_\omega |\text{change in curvature departing from } \varphi|^2 \\ &+ \int_\omega |\text{sym}(\bar{\nabla}\varphi^T \bar{\nabla} d) + \bar{\nabla} b^T \bar{\nabla} b|^2. \end{aligned}$$

Remark: the third term is constant and can be written as

$$\text{sym}(\bar{\nabla}\varphi^T \bar{\nabla} d + \bar{\nabla} b^T \bar{\nabla} b) = \begin{bmatrix} \mathscr{R}_{1313} & \mathscr{R}_{1323} \\ \mathscr{R}_{1323} & \mathscr{R}_{2323} \end{bmatrix} = [\text{remaining entries}].$$

Therefore, the third term is 0 iff $\mathscr{R} = 0$, *i.e*, the 3d metric is flat. All minima including those of the 3d-problem are 0.

The story ends. But,...

Order 4 model: Generalized von Kármán enegy

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Make $Q^TB = \begin{pmatrix} \bar\nabla\varphi^T\bar\nabla b & \bar\nabla\varphi^T d \\ b^T\bar\nabla b & b\cdot d\end{pmatrix}$ skew (to kill the h^2 term in $\int D_2W(\mathrm{Id})$).

First block is skew, then choose d: $Q^Td = (-b\cdot\partial_1 b, -b\cdot\partial_2 b, 0)^T$.

Order 4 model: Generalized von Kármán enegy

Start from $\min I_2 = 0$, *i.e.* $\mathscr{R}_{1212} = \mathscr{R}_{1213} = \mathscr{R}_{1223} = 0$,

i.e. $\exists! \varphi \in H^2(\omega;\mathbb{R}^3)$, $\bar\nabla\varphi^T\bar\nabla\varphi = G_{2\times 2}$ and $\bar\nabla\varphi^T\bar\nabla b$ skew.

First finding. Then $\inf I^h$ is indeed smaller: $\inf I^h \le Ch^4$.

Hint: Choose simply $\Phi^h(\bar x, x_3) = \varphi(\bar x) + hx_3 b(\bar x) + \frac{h^2 x_3^2}{2} d(\bar x)$ with d as follows.

Letting $Q = [\bar\nabla\varphi | b]$, $QA^{-1} \in SO(3)$, $B = [\bar\nabla b | d]$,

$$\nabla_h \Phi^h A^{-1}(\bar x, x_3) = (QA^{-1})(\mathrm{Id} + hx_3 A^{-1} Q^T B A^{-1} + h^2 x_3^2 T),$$

$$W(\nabla_h \Phi^h A^{-1}) = W(\mathrm{Id} + hx_3 A^{-1} Q^T B A^{-1} + h^2 x_3^2 T).$$

Make $Q^T B = \begin{pmatrix} \bar\nabla\varphi^T\bar\nabla b & \bar\nabla\varphi^T d \\ b^T\bar\nabla b & b\cdot d \end{pmatrix}$ skew (to kill the h^2 term in $\int D_2 W(\mathrm{Id})$).

First block is skew, then choose d: $Q^T d = (-b\cdot\partial_1 b, -b\cdot\partial_2 b, 0)^T$.

Limit model. We already know that $\Phi^h \stackrel{H^1}{\rightharpoonup} \varphi$, $\frac{1}{h}\partial_3\Phi^h \stackrel{L^2}{\rightharpoonup} b$. Now,

$$u^h(\bar{x}) := \frac{1}{h}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\Phi^h - (\varphi + hx_3 b)\right) \mathrm{d}x_3 \stackrel{H^1}{\rightharpoonup} u^1, \ \mathrm{sym}\left(\bar{\nabla}\varphi^T \bar{\nabla} u^1\right) = 0,$$

$$\frac{1}{h}\mathrm{sym}\left(\bar{\nabla}\varphi^T \bar{\nabla} u^h\right) \rightharpoonup e^2 \in L^2(\omega; \mathbb{S}_2),$$

$$\begin{aligned} I_4(u^1, e^2) &= \int_\omega |e^2 + \frac{1}{2}(\bar{\nabla} u^1)^T \bar{\nabla} u^1 + \frac{1}{4!}\bar{\nabla} b^T \bar{\nabla} b|^2 \\ &+ \int_\omega |\bar{\nabla}\varphi^T \bar{\nabla} p^1 + (\bar{\nabla} u^1)^T \bar{\nabla} b|^2 \\ &+ \int_\omega |\mathrm{sym}(\bar{\nabla}\varphi^T \bar{\nabla} d) + \bar{\nabla} b^T \bar{\nabla} b|^2 \end{aligned}$$

where $p^1(u^1)$.

Link with usual case:

$$\begin{aligned} \partial_\alpha u^1_\beta + \partial_\beta u^1_\alpha &= 0 \\ e^2 + \frac{1}{2}(\bar{\nabla} u^1)^T \bar{\nabla} u^1 &= \frac{1}{2}(\partial_\alpha u^2_\beta + \partial_\beta u^2_\alpha + \partial_\alpha u^1_3 \partial_\beta u^1_3) \\ \bar{\nabla}\varphi^T \bar{\nabla} p^1 &= -\partial_{\alpha\beta} u^1_3. \end{aligned}$$