Algebraic Vision

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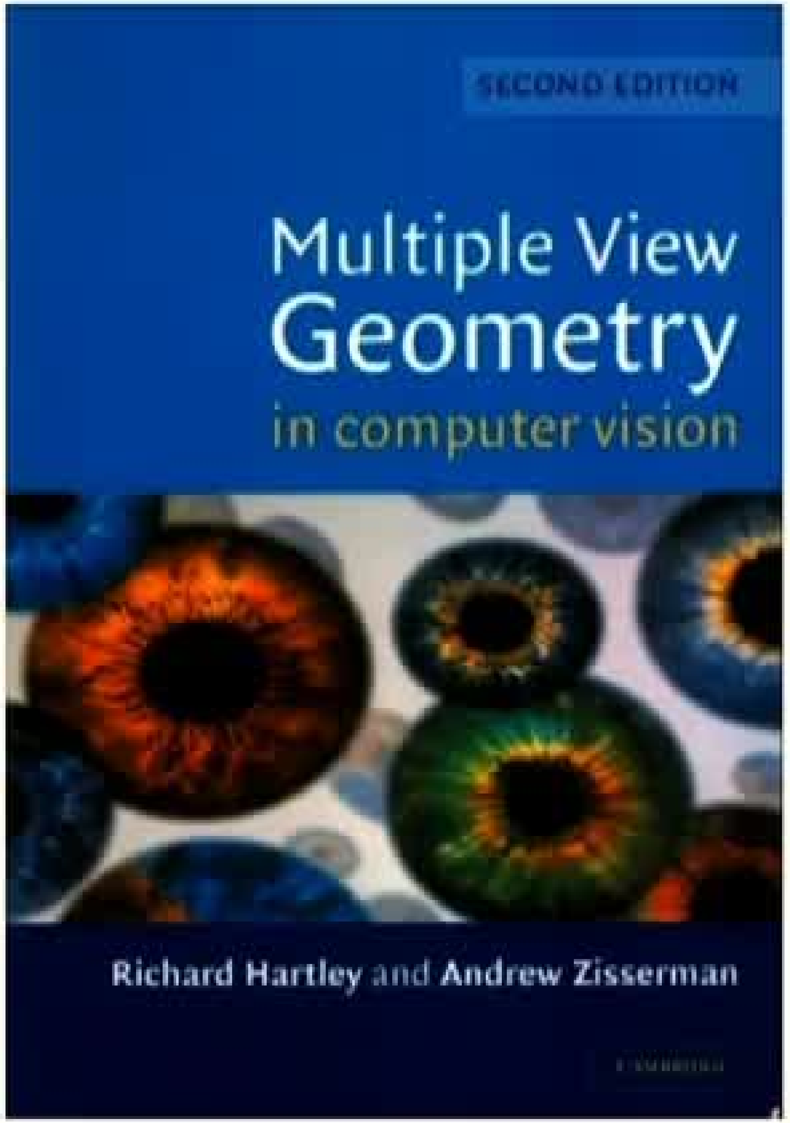


Multiple View Geometry in Computer Vision

Richard Hartley and Andrew Zisserman,
Cambridge University Press, 2000

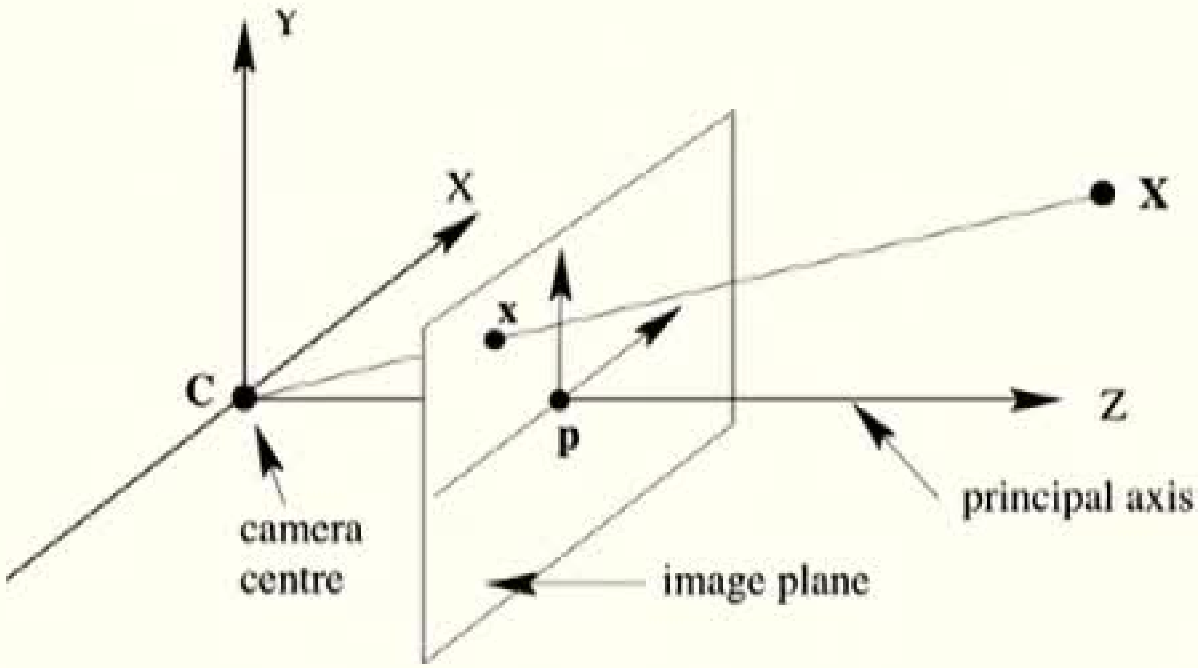
An Invitation to 3D Vision

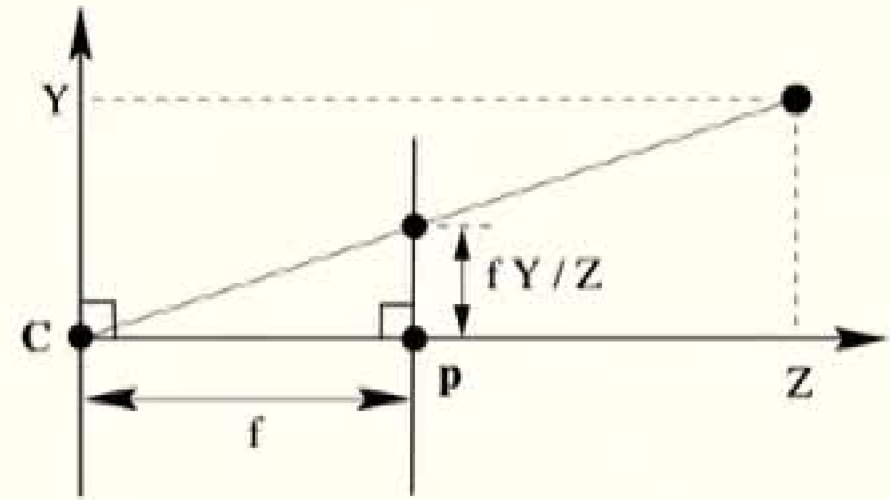
Yi Ma, Stefano Soatto, Jana Kosecka, Shankar Sastry,
Springer-Verlag 2004

Theory of Reconstruction from Image Motion

Stephen Maybank, Springer-Verlag 1993

The basic pinhole camera





$$\begin{array}{ccc} \mathbb{R}^3 & \rightarrow & \mathbb{R}^2 \\ (X, Y, Z) & \mapsto & \left(\frac{fX}{Z}, \frac{fY}{Z}\right) \\ \mathbf{X} & & \mathbf{x} \end{array}$$

$$\begin{array}{ccc} \mathbb{P}^3 & \rightarrow & \mathbb{P}^2 \\ (X, Y, Z, 1) & \mapsto & \underbrace{\left(\frac{fX}{Z}, \frac{fY}{Z}, 1\right)} \end{array}$$

central projection from
world to image coordinates

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

camera matrix

general projective camera

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$$
$$\mathbf{X} \mapsto A\mathbf{X} =: \mathbf{x}$$

$A = [B \mid b] \in \mathbb{R}^{3\times 4}$
rank $(A) = 3$

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camera center $\quad c \in \mathbb{P}^3 \, : \, Ac = 0$

---

finite camera $\quad B$ nonsingular $\Rightarrow c = \left(-B^{-1}b, 1\right)$

A finite $\Rightarrow A = K[R \;\; t]$

K upper triangular w/ positive diagonal

R rotation matrix

$t \in \mathbb{R}^3$

K contains the internal parameters of the camera; focal length, skew etc

---

calibrated camera $\quad A = K[R \; t] \quad K$ known *linear model of image formation*

$$\mathbf{x} \sim A\mathbf{X}$$

image — *camera* — *world point*

Reconstruction Problems

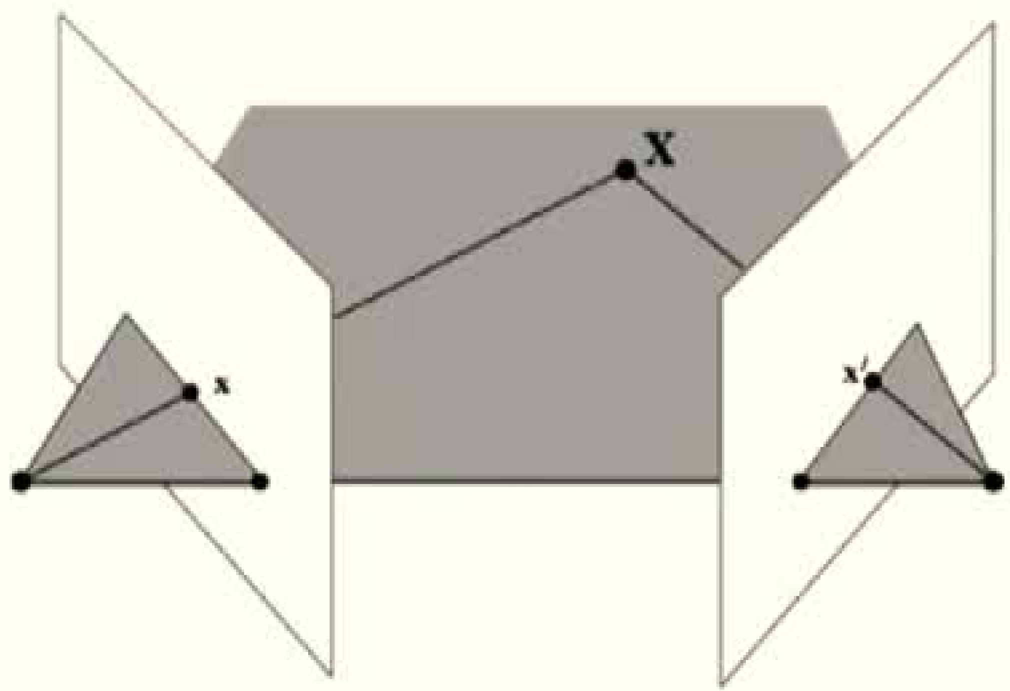
	Given	*Find*
Resectioning	$\mathbf{X}_i \leftrightarrow \mathbf{x}_i$	$A \,:\, \mathbf{x}_i \sim A\mathbf{X}_i$
Triangulation	$A_i,\ \mathbf{x}_i$	$\mathbf{X} \,:\, \mathbf{x}_i \sim A_i\mathbf{X}$
Reconstruction	$\mathbf{x}_i^1 \leftrightarrow \mathbf{x}_i^2 \cdots \leftrightarrow \mathbf{x}_i^n$	$A_j, \mathbf{X}_i \,:\, \mathbf{x}_i^j \sim A_j\mathbf{X}_i$

in practice, images are noisy and need to find MLE

Triangulation

Want to recover a 3D point $\mathbf{X}$ *from noisy images* $\hat{\mathbf{x}}_i$ *in n cameras*

$$\mathcal{A} = (A_1, \ldots, A_n)$$



Polynomial Optimization

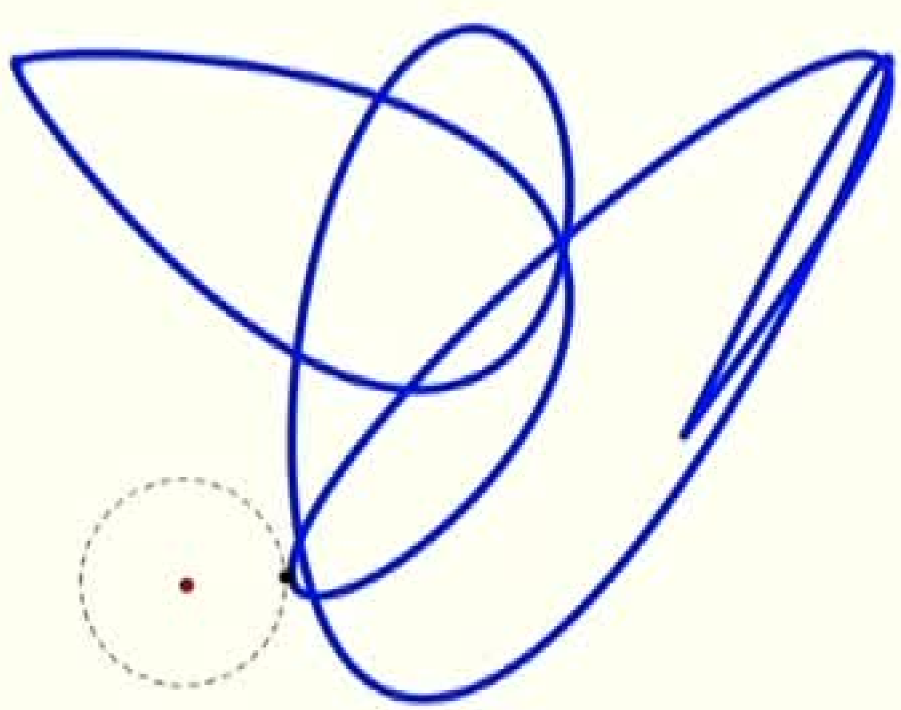
$$\begin{array}{ll} \min & \sum \|x_i - \hat{\mathbf{x}}_i\|^2 \\ \text{s.t.} & (x_1, \ldots, x_n) \in \overline{\varphi(\mathbb{P}^3)} \end{array}$$

Algebraic Geometry

$$\begin{array}{llll} \varphi : & \mathbb{P}^3 & \rightarrow & \mathbb{P}^2 \times \cdots \times \mathbb{P}^2 \\ & X & \mapsto & (A_1 X, \ldots, A_n X) \end{array}$$

$$\overline{\varphi(\mathbb{P}^3)} \subset (\mathbb{P}^2)^n$$

multiview variety of $\mathcal{A}$



$J_{\mathcal{A}}$ *multiview ideal* — *vanishing ideal of the multiview variety*

$$A_i \mathbf{X} = \lambda_i \mathbf{x}_i \;\; \forall i = 1, \ldots, n \;\; \Leftrightarrow \begin{bmatrix} A_1 & \mathbf{x}_1 & 0 & 0 & \cdots & 0 \\ A_2 & 0 & \mathbf{x}_2 & 0 & \cdots & 0 \\ \vdots & & & & & \\ A_n & 0 & 0 & 0 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{pmatrix} \mathbf{X} \\ \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = 0$$

so all maximal minors of this matrix lie in $J_{\mathcal{A}}$

b_{ij}	t_{ijk}	q_{ijkl}
bilinear/epipolar constraints	*trilinear/trifocal constraints*	*quadrilinear/quadrifocal constraints*

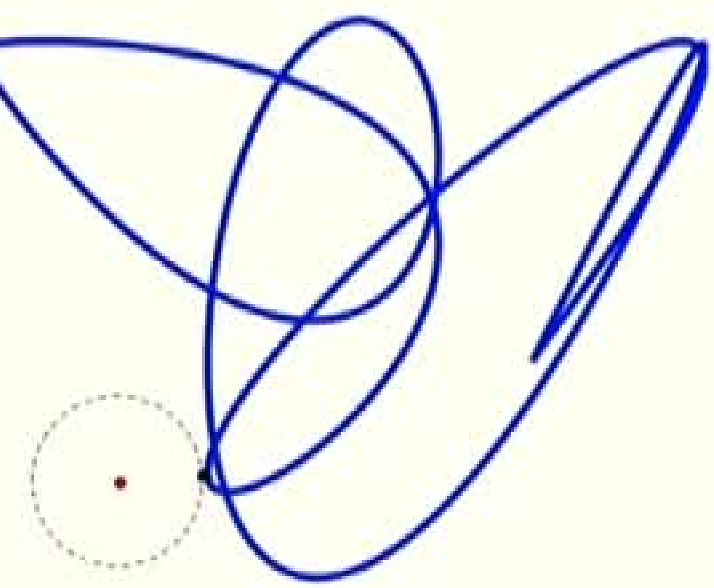
Aholt, Sturmfels, T. (2011) *strengthens* Heyden-Åström (1997)

$\{b_{ij}, t_{ijk}, q_{ijk}\}$ *form a universal Gröbner basis for* $J_{\mathcal{A}}$ *generically*

… *Hilbert schemes in vision* AST (2011), Lieblich-van Meter (2018)

Polynomial Optimization

$$\begin{array}{ll} \min & \|x - \hat{\mathbf{x}}\|^2 \\ \text{s.t.} & x \in \overline{\varphi(\mathbb{P}^3)} \end{array} \quad = \quad \begin{array}{ll} \min & \|x - \hat{\mathbf{x}}\|^2 \\ \text{s.t.} & b_{ij} = 0 \\ & t_{ijk} = 0 \end{array}$$

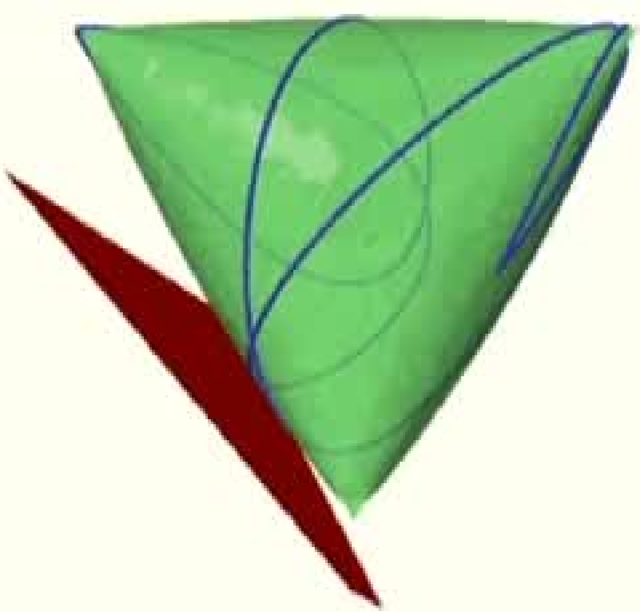


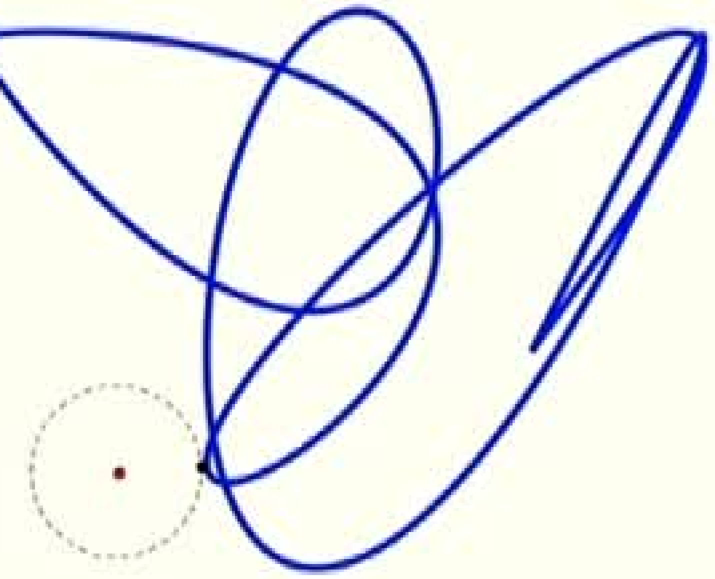
$$\begin{array}{ll} \min & \langle G, Y \rangle \\ \text{s.t.} & \langle B_{ij}, Y \rangle = 0 \\ & Y \succeq 0 \end{array}$$

semidefinite programming
(SDP)
relaxation

Polynomial Optimization

$$\begin{array}{ll} \min & \|x - \hat{\mathbf{x}}\|^2 \\ \text{s.t.} & x \in \overline{\varphi(\mathbb{P}^3)} \end{array} \quad = \quad \begin{array}{ll} \min & \|x - \hat{\mathbf{x}}\|^2 \\ \text{s.t.} & b_{ij} = 0 \\ & t_{ijk} = 0 \end{array}$$





Polynomial Optimization

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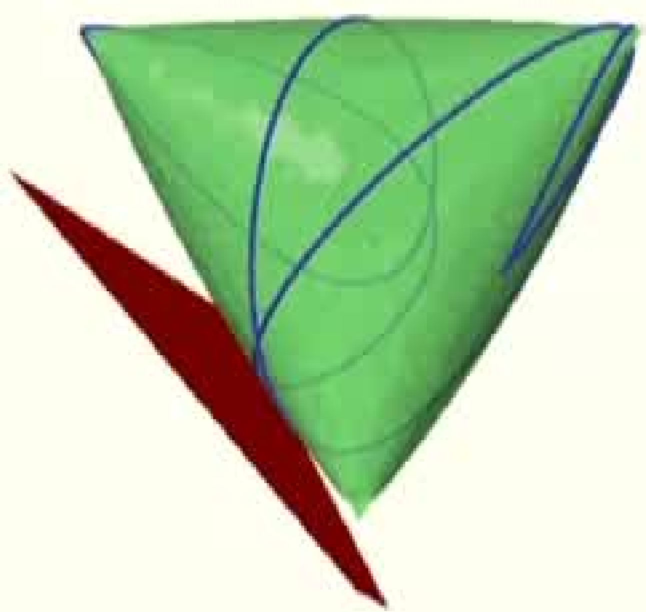
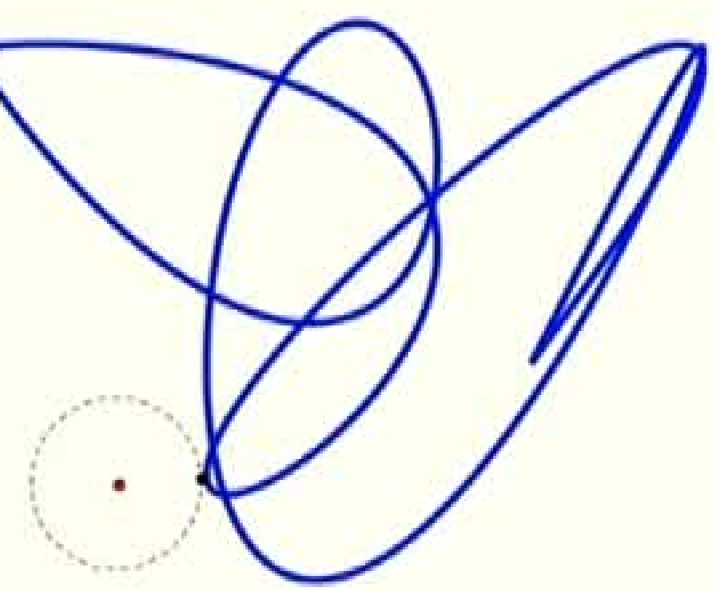
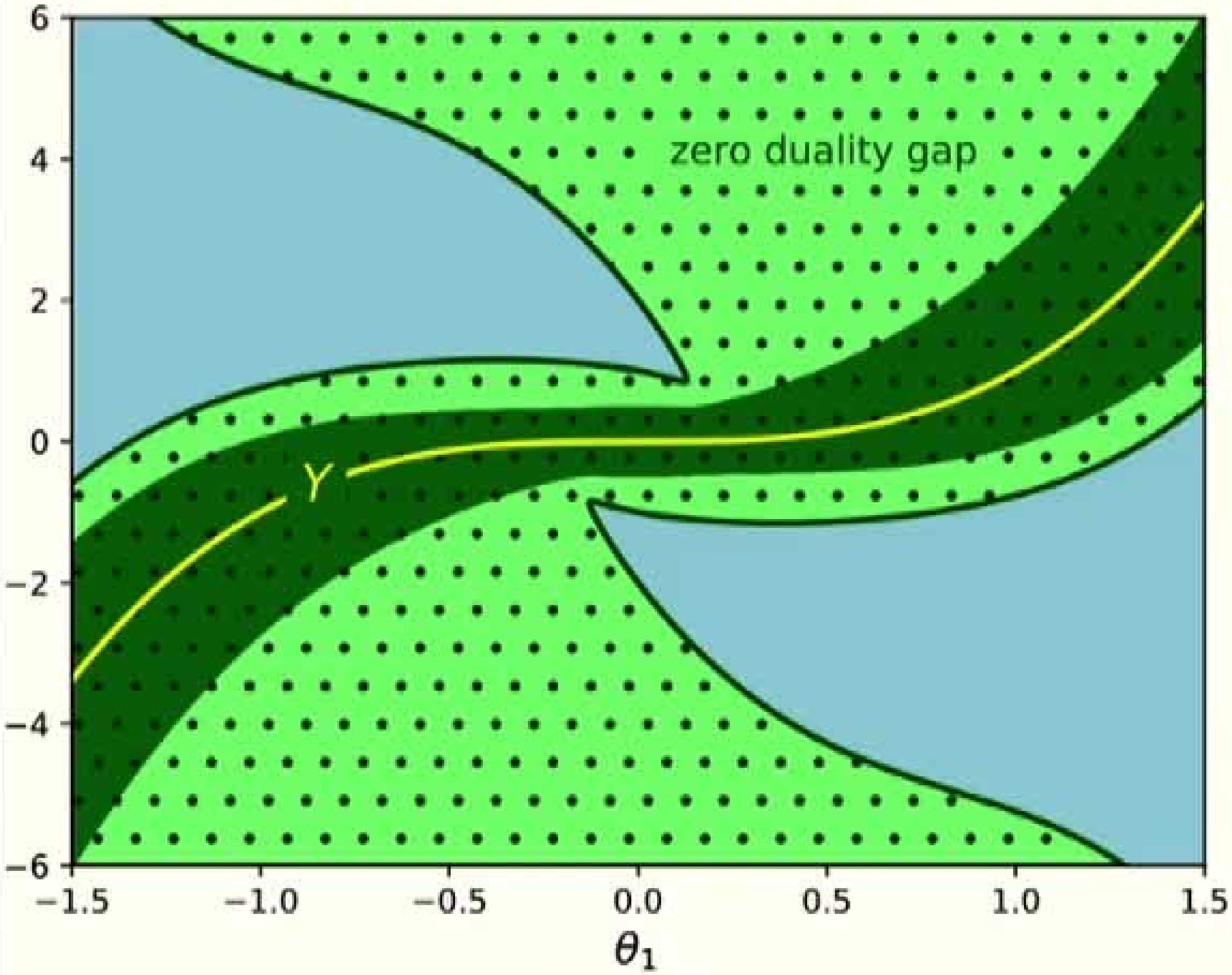




Theorem: **Aholt, Agarwal, T. (2012)** *This SDP relaxation solves the original triangulation problem exactly, under low noise.*

Local stability of SDP relaxations of QCQPs

Cifuentes, Agarwal, Parrilo, T. (2017)

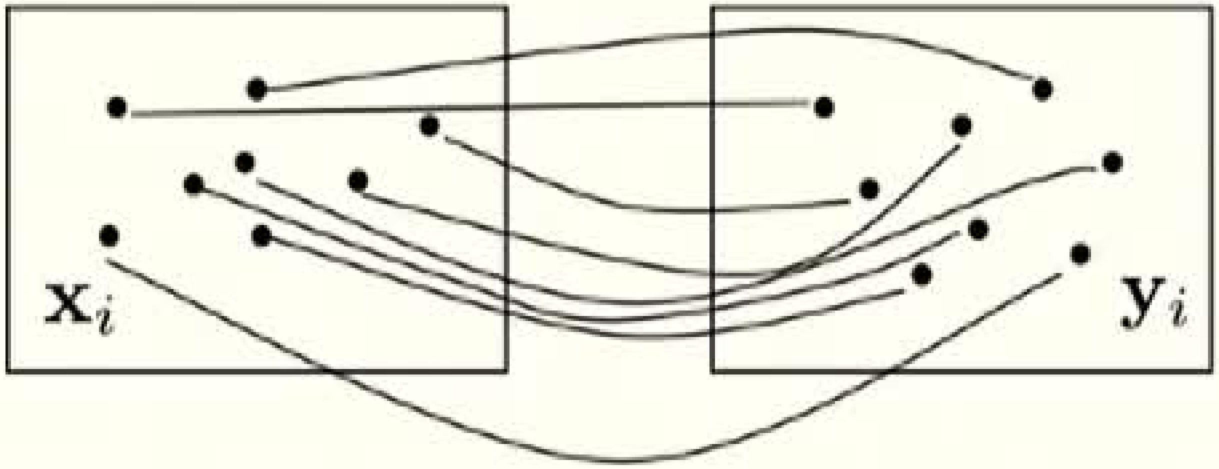


$$\begin{array}{ll} \min & \|y-\theta\|^2 \\ \text{s.t.} & y_2 = y_1^2,\ y_3 = y_1 y_2 \end{array}$$

A simple theorem: Suppose $\bar{\theta} \in Y$ (quadratic variety) is smooth. Then SDP relaxation solves the QCQP $\forall \theta$ sufficiently close to $\bar{\theta}$.

Two View Geometry

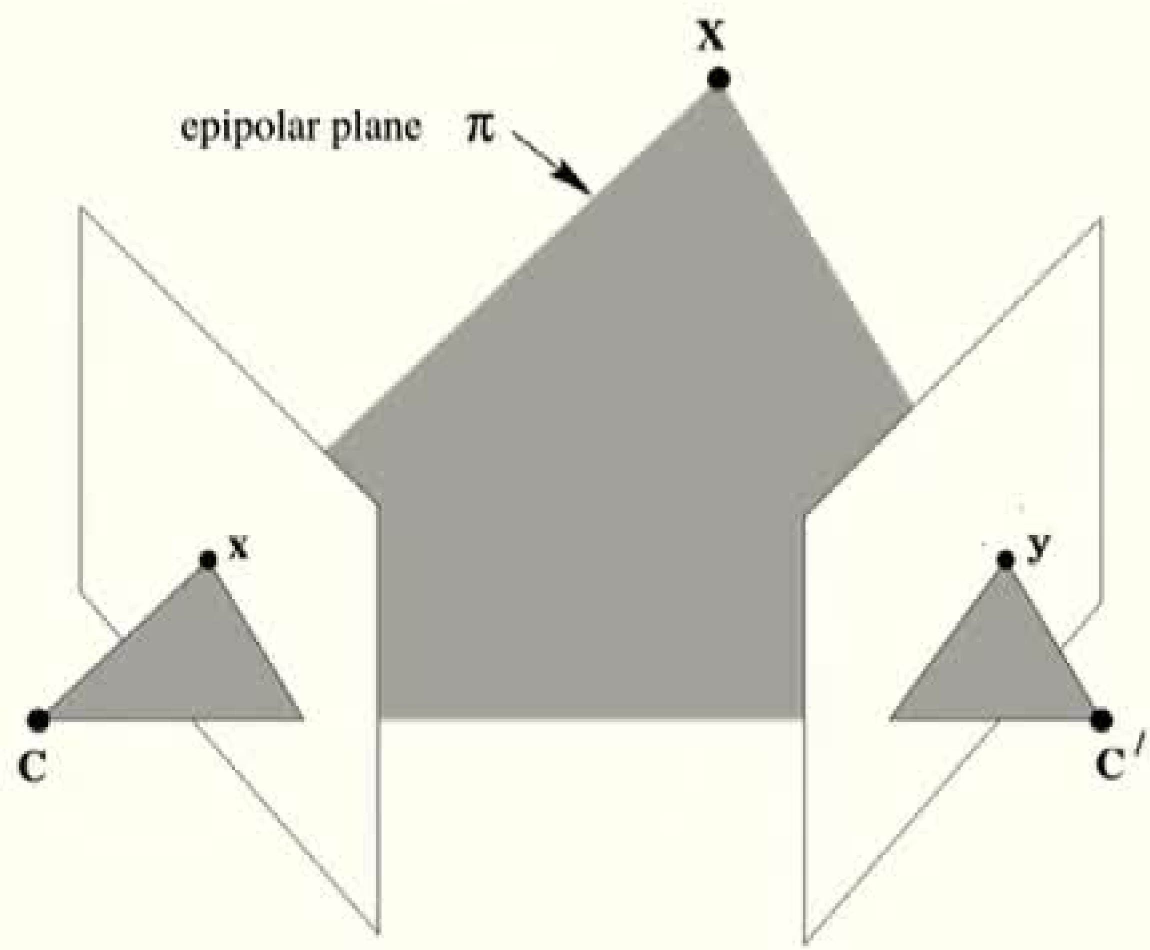
Given m correspondences: $\{(\mathbf{x}_i, \mathbf{y}_i) \in \mathbb{R}^2 \times \mathbb{R}^2,\ i = 1, \ldots, m\}$



Are these images of m world points in 2 cameras?
If yes, find the cameras and reconstruct the 3D point cloud.

Two View Geometry

$$\mathbf{x} \sim A\mathbf{X}, \;\; \mathbf{y} \sim A'\mathbf{X} \qquad \Rightarrow \qquad \mathbf{X}, \mathbf{x}, \mathbf{y}, C, C' \text{ all coplanar}$$

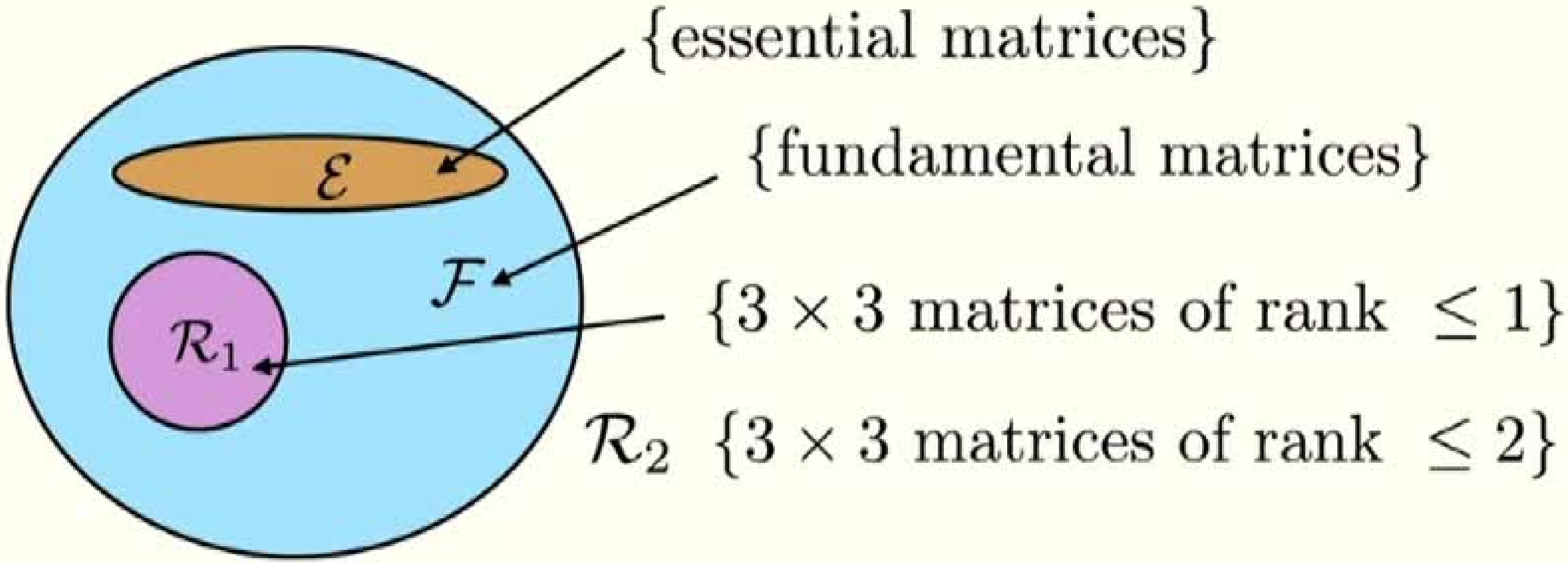


$$\Rightarrow \qquad \exists\, 3 \times 3 \text{ matrix } M, \; \text{rank}(M) = 2 \text{ s.t.}$$

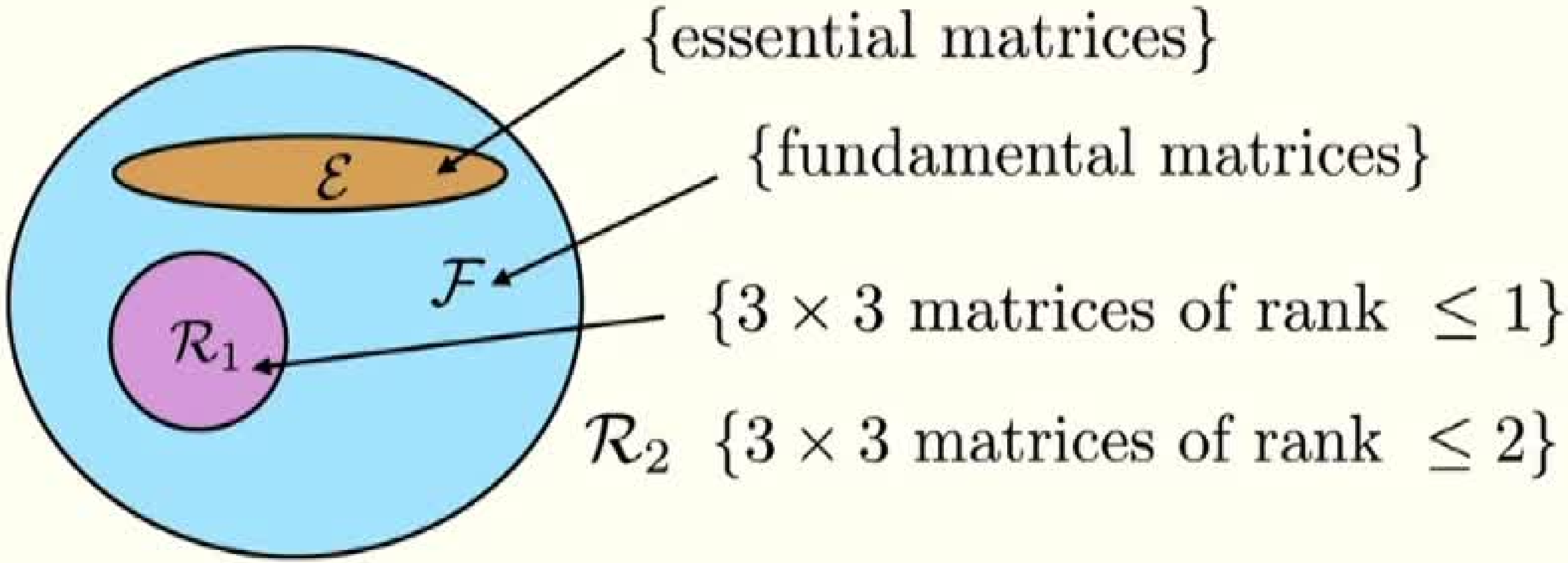
$$\underbrace{\mathbf{y}^\top M \mathbf{x} = 0}$$

epipolar equation

uncalibrated cameras	calibrated cameras
$M = F$	$M = E$
fundamental matrix	*essential matrix*
	$\sigma_1(E) = \sigma_2(E) > 0$



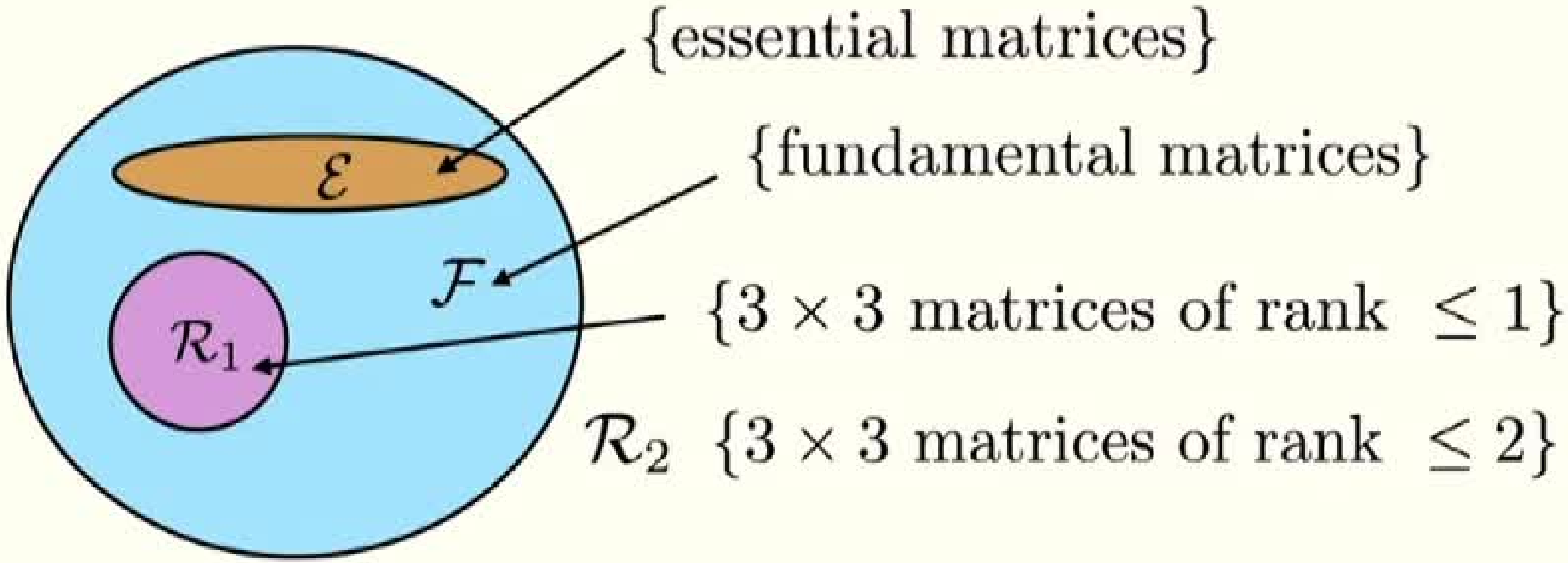
m point correspondences $\leftrightarrow$ subspace L of codim m



m point correspondences $\leftrightarrow$ subspace L of codim m

$\exists\ F \ \Leftrightarrow\ L \cap \mathcal{F} \neq \emptyset \qquad\qquad \exists\ E \ \Leftrightarrow\ L \cap \mathcal{E} \neq \emptyset$

$\mathcal{F} = \mathcal{R}_2 \backslash \mathcal{R}_1$ *matrices of rank exactly 2*	$\mathcal{E}$ *real algebraic variety cut out by* $2EE^\top E - \mathrm{trace}(EE^\top)E = 0,\ \det(E) = 0$ *(10 cubics)* **Demazure (1988)**



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$\exists\ F \ \Leftrightarrow\ L \cap \mathcal{F} \neq \emptyset$ $\qquad$ $\exists\ E \ \Leftrightarrow\ L \cap \mathcal{E} \neq \emptyset$

Theorems:

Agarwal, Lee, Sturmfels, T. (2017)

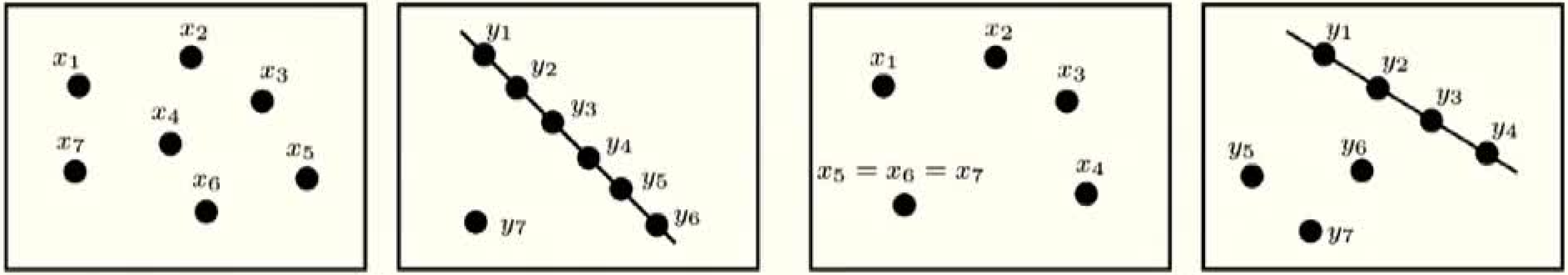
$m \leq 5 \ \Rightarrow\ L \cap \mathcal{F} \neq \emptyset$

$m \leq 4 \ \Rightarrow\ L \cap \mathcal{E} \neq \emptyset$

Fundamental Matrices

$$m = 6, 7, 8$$

There is always a real 3x3 matrix in the intersection but it may not have rank two.



(a) (b)

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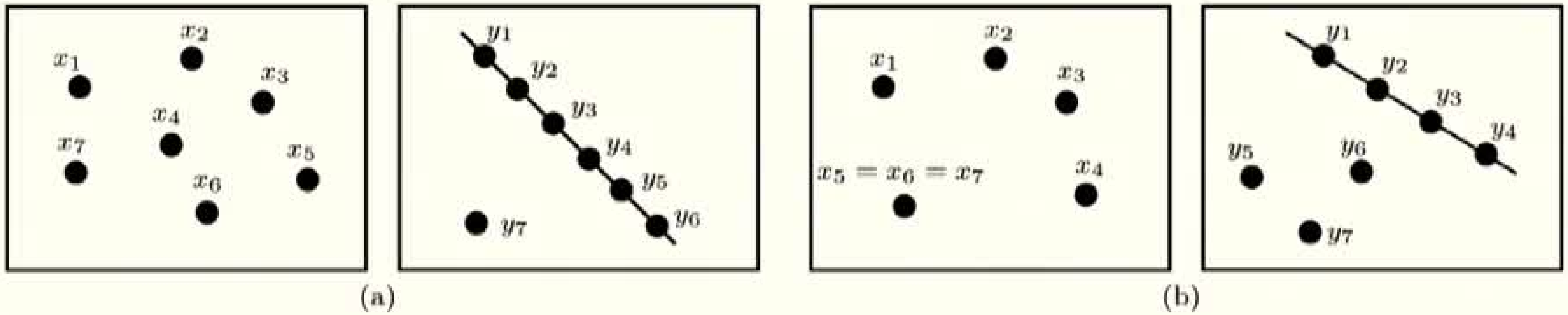


Essential Matrices

$m = 5, 6$ ***open*** *There maybe no real points in the intersection.*

$$X = \begin{bmatrix} 3 & 0 & 1 \\ 9 & 1 & 1 \\ 1 & 2 & 1 \\ 8 & 8 & 1 \\ 4 & 8 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 2 & 0 & 1 \\ 5 & 4 & 1 \\ 9 & 6 & 1 \\ 2 & 5 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$L \cap \mathcal{E} = \emptyset$$

$$L \cap \mathcal{E}_{\mathbb{C}} = 10$$

5-point algorithm for E Nistér (2004)

1. *Plug* $sA_1 + tA_2 + uA_3 + A_4 \in L$ *into the 10 Demazure cubics.*

2. *Gauss-Jordan elimination yields:*

$\mathcal{A}$	s^3	t^3	s^2t	st^2	s^2u	s^2	t^2u	t^2	stu	st	s	t	1
(a)	1										[2]	[2]	[3]
(b)		1									[2]	[2]	[3]
(c)			1								[2]	[2]	[3]
(d)				1							[2]	[2]	[3]
(e)					1						[2]	[2]	[3]
(f)						1					[2]	[2]	[3]
(g)							1				[2]	[2]	[3]
(h)								1			[2]	[2]	[3]
(i)									1		[2]	[2]	[3]
(j)										1	[2]	[2]	[3]

Kukelova's thesis('12)

~10.6 micro secs

many such solvers
Pajdla's group @ Czech TU

3. *Define:*

$$\begin{aligned}(k) &:= (e) - u\cdot(f)\\ (l) &:= (g) - u\cdot(h)\\ (m) &:= (i) - u\cdot(j)\end{aligned}$$

and arrange into a 3x3 matrix:

$\mathcal{B}$	s	t	1
(k)	[3]	[3]	[4]
(l)	[3]	[3]	[4]
(m)	[3]	[3]	[4]

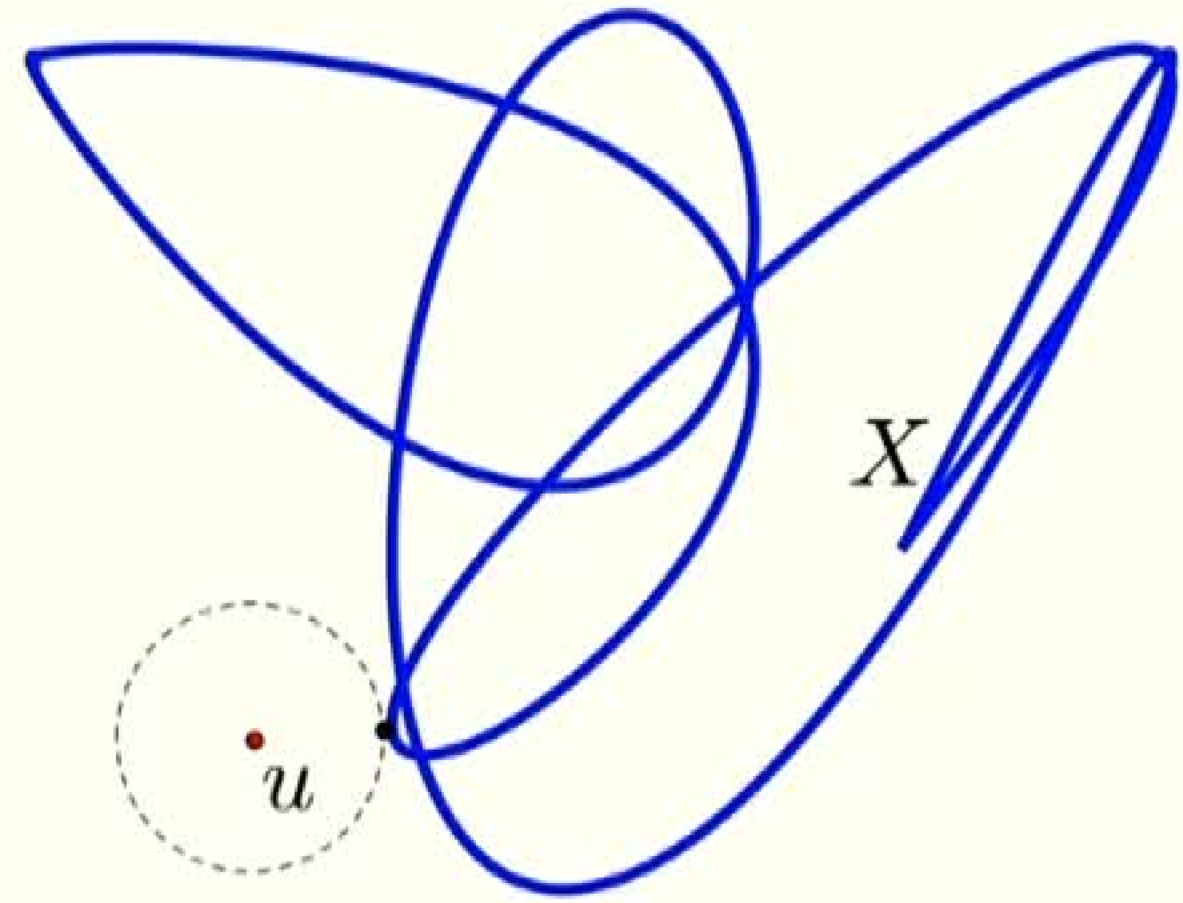
4. $\det(\mathcal{B})$ *is a univariate polynomial in u of degree 10.*
Solve using Sturm sequences/eigenvalue methods

Euclidean distance degree of an algebraic variety

$u \in \mathbb{R}^n$ *data point*

$X \subseteq \mathbb{R}^n$ *real algebraic variety*

$$\min_{x \in X} \|x - u\|^2$$



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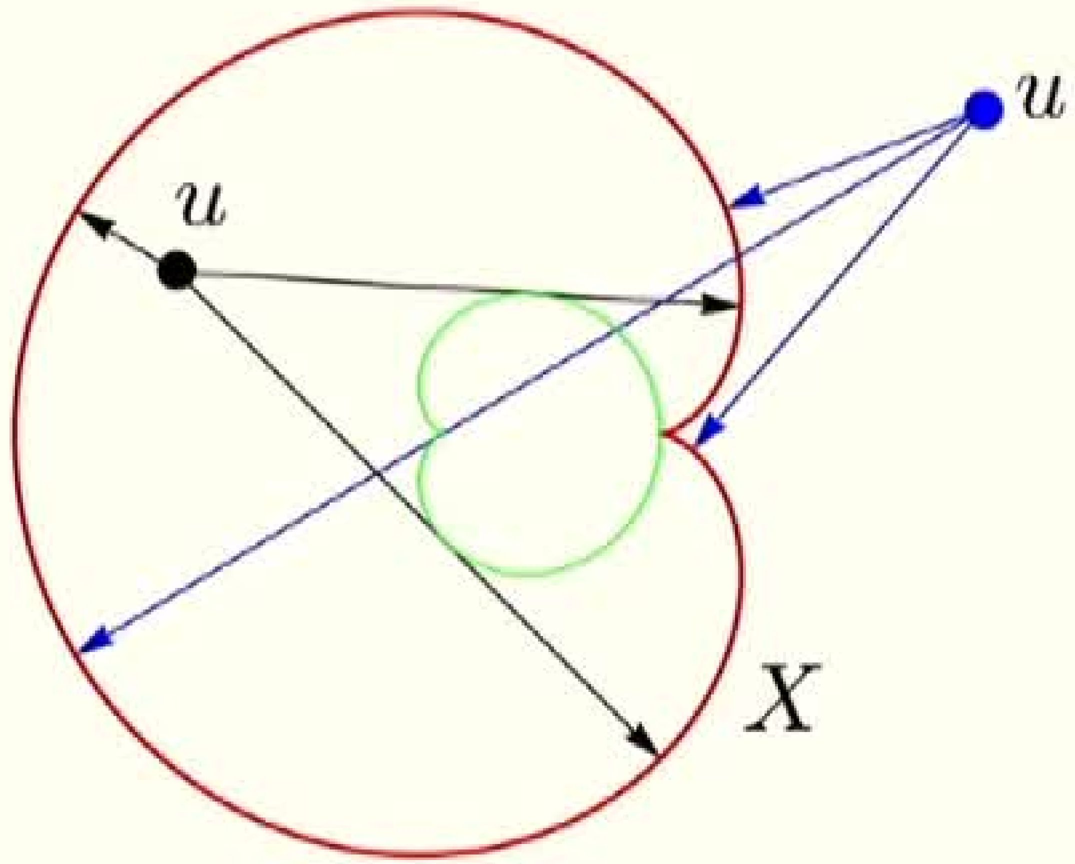


Theorem: Draisma, Horobet, Ottaviani, Sturmfels, T. (2016)

smooth critical points of this problem is a constant for generic u

Euclidean distance degree of X =

smooth critical points of this problem

Euclidean distance degree of the essential variety

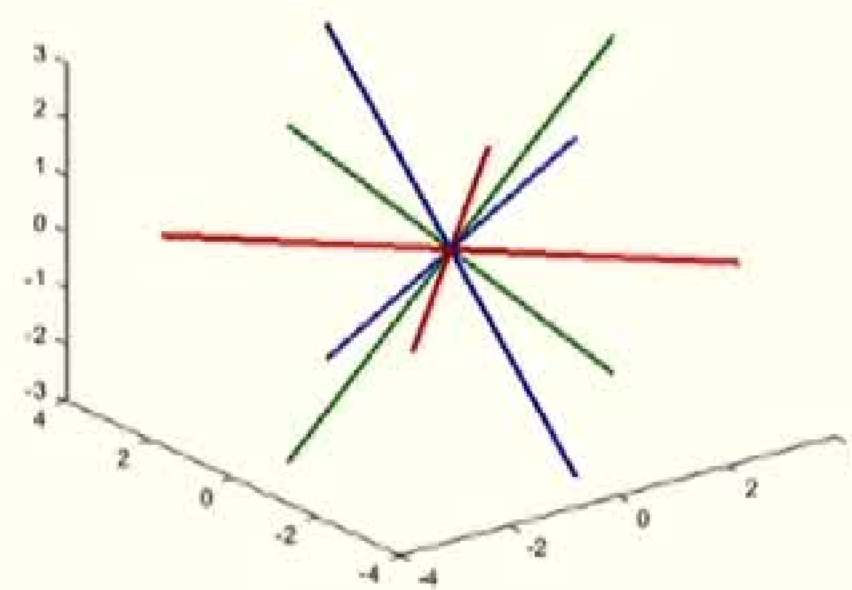
$$\mathcal{E} = \{E \in \mathbb{R}^{3\times 3} \, : \, \mathrm{rank}(E) = 2, \; \sigma_1(E) = \sigma_2(E) > 0\}$$

Theorem: Drusvyatskiy, Lee, Ottaviani, T. (2017)

$$\mathrm{EDdegree}(\mathcal{E}_{\mathbf{C}}) = 6$$

A transfer principle:

the EDdegree of orthogonally invariant matrix varieties equals the EDdegree of their singular values variety after symmetrization.



Euclidean distance degree of triangulation

X_n *multiview variety from n cameras*

Stewenius, Shaffalitzky, Nistér (2005):

n	2	3	4	5	6	7
$\text{EDdegree}(X_n)$	6	47	148	336	638	1081

Conjecture: $$\text{EDdegree}(X_n) = \frac{9}{2}n^3 - \frac{21}{2}n^2 + 8n - 4$$

A complex variant with a slightly different (cubic) formula was proven by **Harris & Lowengrub (2017)**