Stability of Nozaki-Bekki holes near the nonlinear Schrödinger limit

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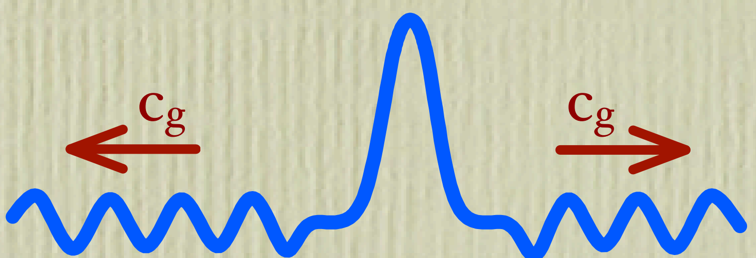
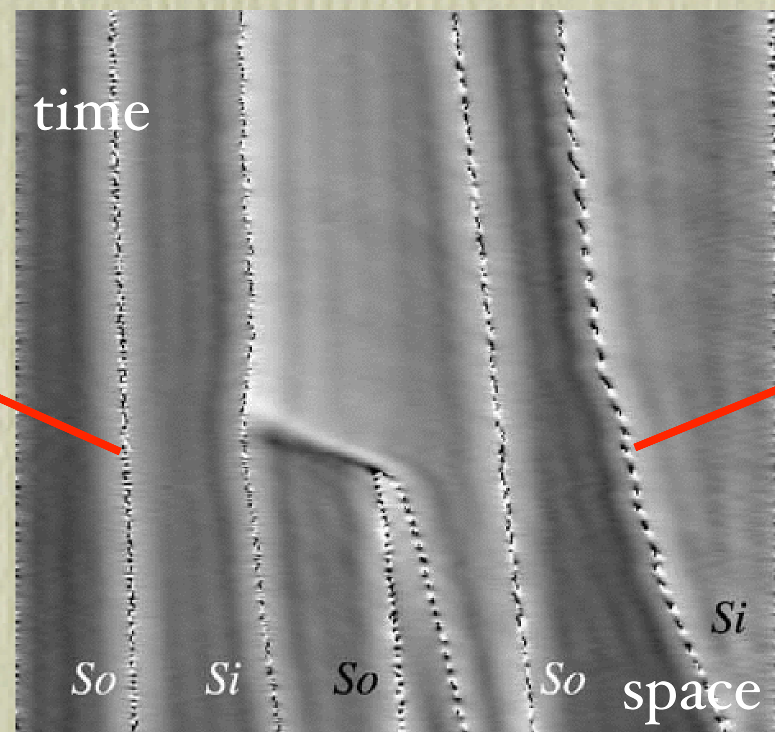
Sources in experiments

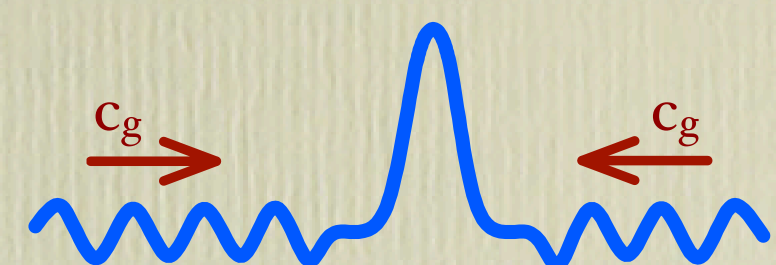


sources

are active: select unique wave numbers



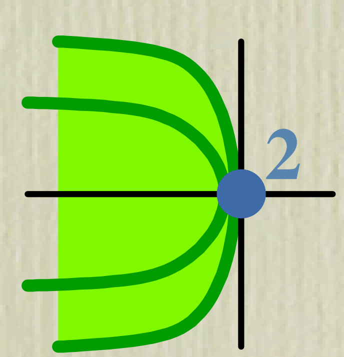
Surface waves [Pastur et al.]



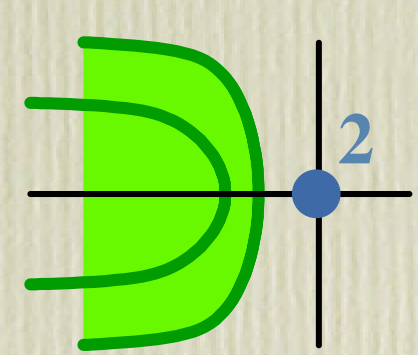
sinks

are passive: exist for all wave numbers

- Sources are time-periodic patterns
- Floquet spectra of generic spectrally stable sources [S. & Scheel]:



L^2 space



weighted L^2 space

Spectral stability implies nonlinear stability [Beck, Nguyen, S., Zumbrun]

Nozaki-Bekki holes

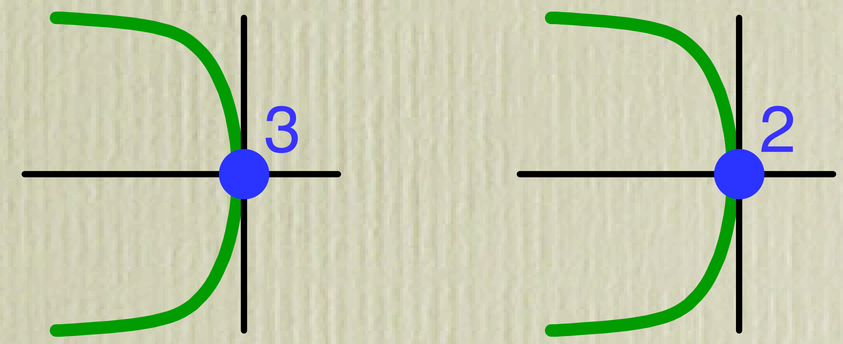
- Complex cubic-quintic Ginzburg-Landau equation:

$$iA_t = (1 + i\alpha)A_{xx} + (\omega + i\mu)A - (1 + i\gamma)|A|^2 A - (\delta_1 - i\delta_2)|A|^4 A$$

- Nozaki-Bekki holes:
  - temporal frequency is ω
  - explicit one-parameter family of sources for fixed parameter values (α,γ,μ) with δ=0, parametrized by wave speed: not generic! [Doelman]
  - standing holes persist for δ≠0
- Profiles:

$$A(x) = r(x)e^{i\varphi(x)} \qquad r(x) \longrightarrow \pm r_\infty, \quad \varphi'(x) \longrightarrow \pm\kappa \text{ as } x \longrightarrow \pm\infty$$

- Expected spectra:



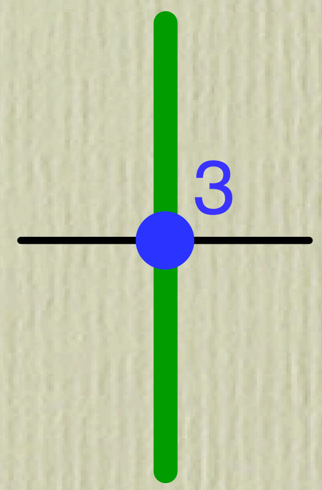
δ=0 δ≠0

Dark solitons for nonlinear Schrödinger equation

- Nonlinear Schrödinger equation:

$$iA_t = A_{xx} + \omega A - |A|^2 A$$

- Dark solitons: explicit one-parameter family of solitons, parametrized by ω
- Spectra:



Evans function E(λ): multiplicity of λ=0 is
2 for even eigenfunctions, and
1 for odd eigenfunctions

Plan: attempt perturbation analysis near this limit and trace roots of Evans functions!

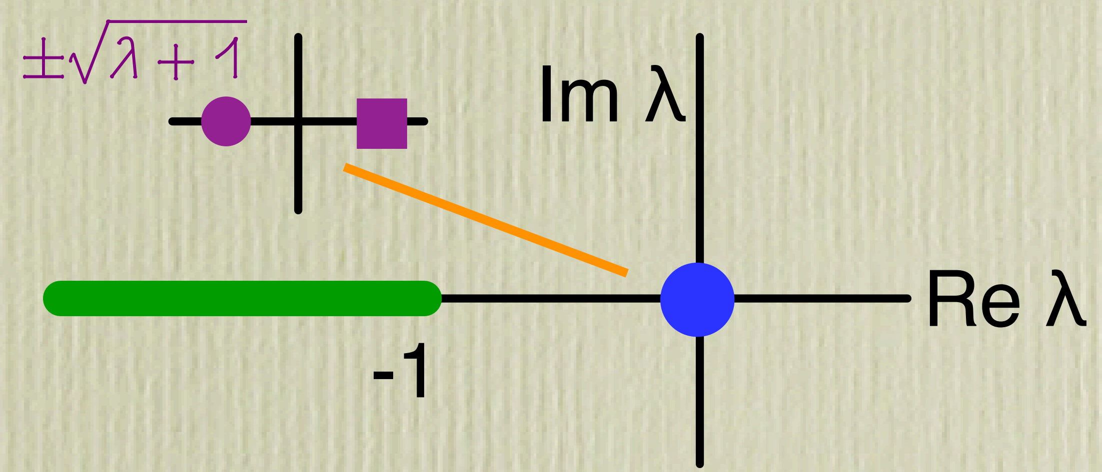
$$iA_t = (1 + i\alpha)A_{xx} + (\omega + i\mu)A - (1 + i\gamma)|A|^2 A - (\delta_1 - i\delta_2)|A|^4 A$$

(α,γ,μ,δ) := O(ε) for 0<ε<<1

[Lega & Fauve], [Kapitula & Rubin]

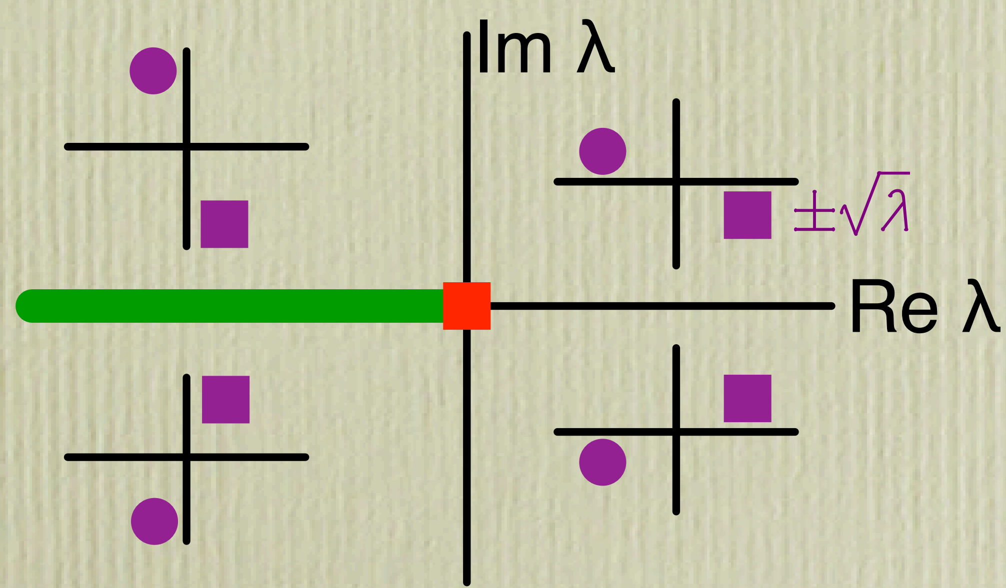
Example 1: heat equation with potential

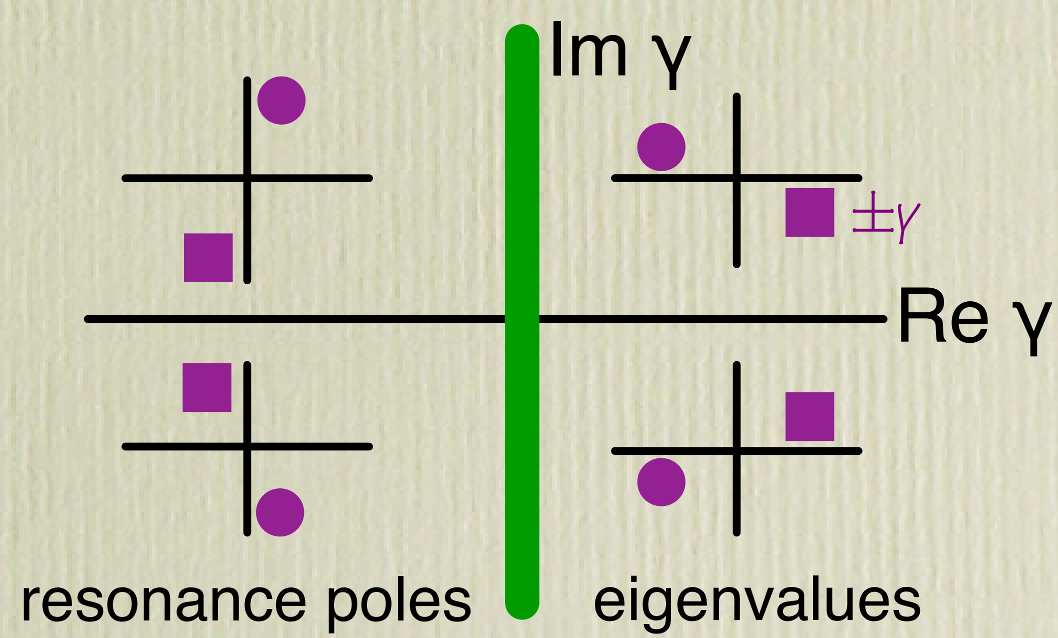
- Heat equation with localized potential: $u_t = u_{xx} - u + V(x)u$
- Asymptotic eigenvalue problem: $\lambda u = u_{xx} - u$
- Need to construct solutions of the form $u(x) \approx e^{\pm\sqrt{\lambda+1}x}$ as $x \longrightarrow \mp\infty$ for $\lambda > 0$
- E(λ) = Wronskian of these solutions: will be analytic in λ near λ=0



Example 2: heat equation with potential

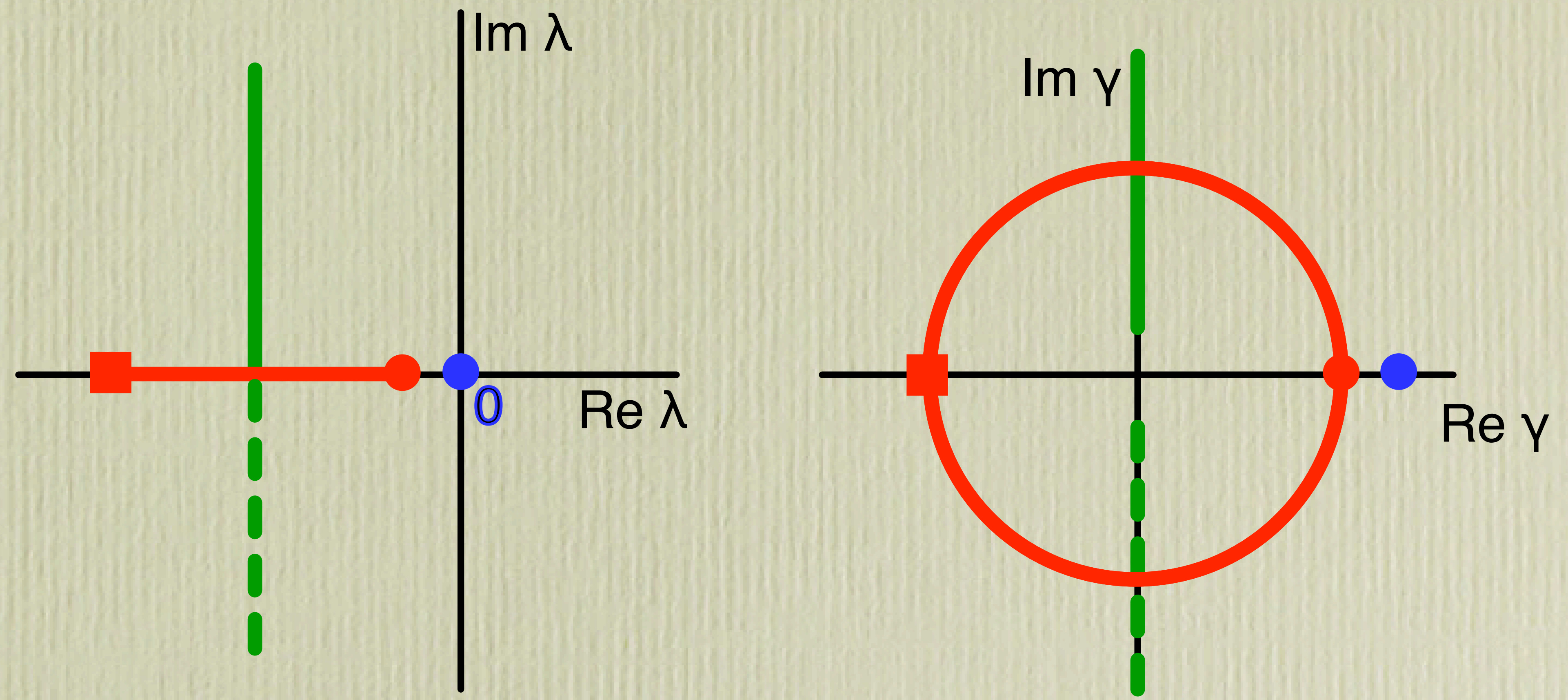
- Heat equation with localized potential: $u_t = u_{xx} + V(x)u$
- Asymptotic eigenvalue problem: $\lambda u = u_{xx}$
- Need to construct solutions of the form $u(x) \approx e^{\pm\sqrt{\lambda}x}$ as $x \to \mp\infty$ for $\lambda > 0$
- $E(\lambda)$ = Wronskian of these solutions: will not be analytic in λ near $\lambda=0$
- Define $\lambda=\gamma^2$, then $u(x) \approx e^{\pm\gamma x}$ as $x \to \mp\infty$ for $\gamma > 0$
- $E(\gamma)$ = Wronskian of these solutions: will be analytic in γ near $\gamma=0$





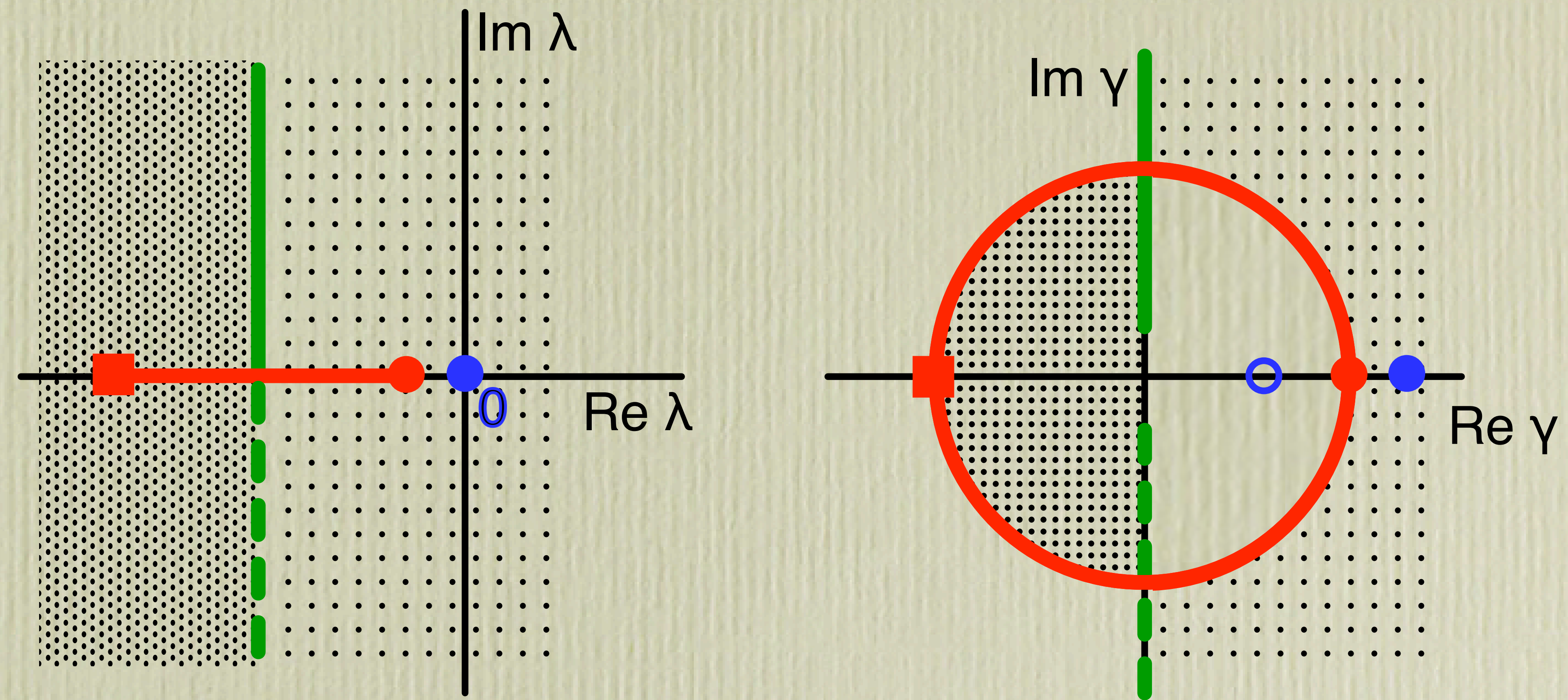
branch point ■ : spatial eigenvalues collide

Perturbation from NLS to CGL



- Two branch points $O(\varepsilon^3)$ and $O(\varepsilon)$ away from $\lambda=0$
- Choose coordinate transformation that lifts λ-plane to a Riemann surface covering: blows up absolute spectrum to a circle
- Resulting Evans function $E(\gamma,\varepsilon)$ has 6 roots for $\varepsilon\approx 0$ (4 with even and 2 with odd eigenfunctions)

Perturbation from NLS to CGL



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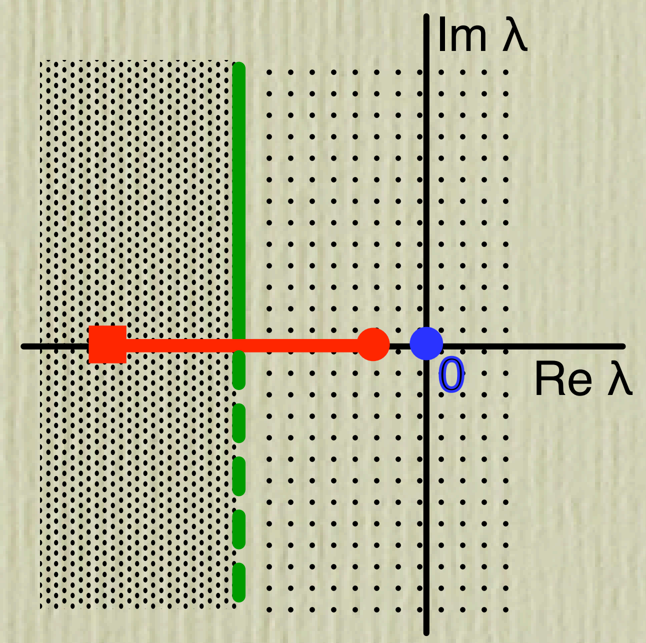
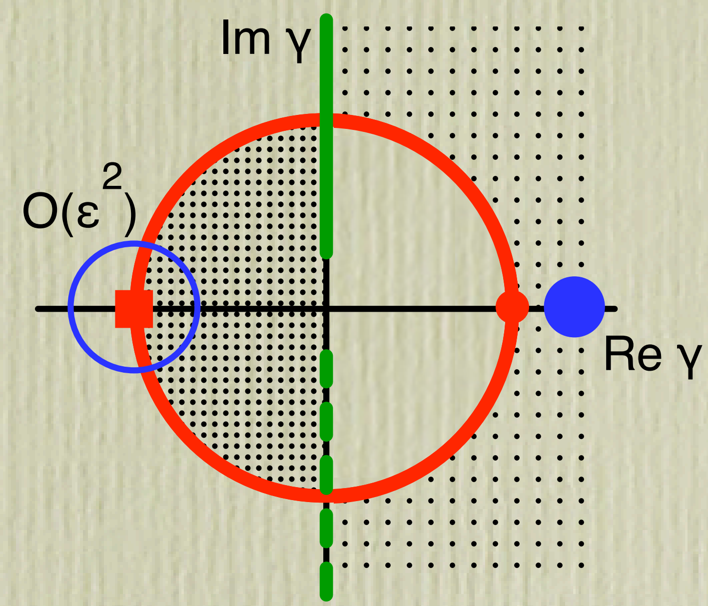
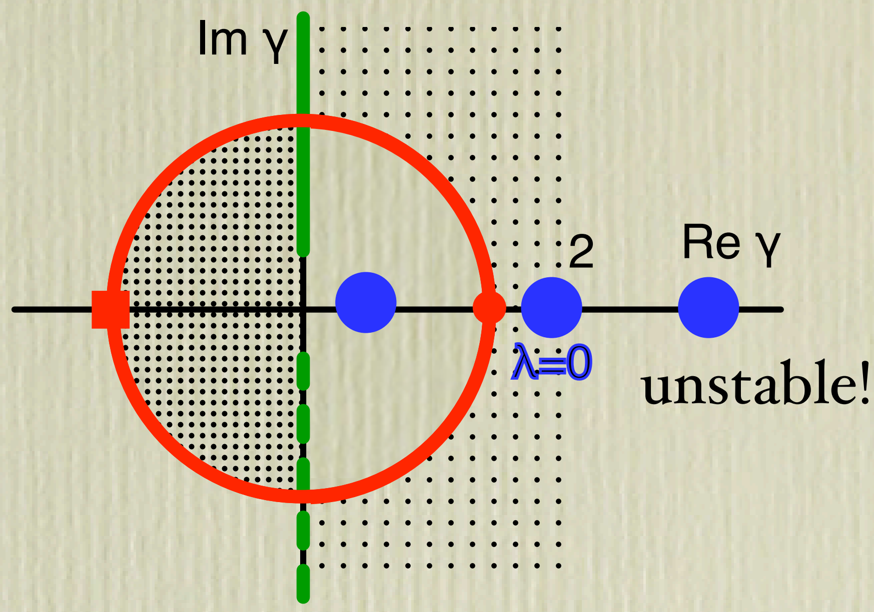


λ-plane



odd Evans function

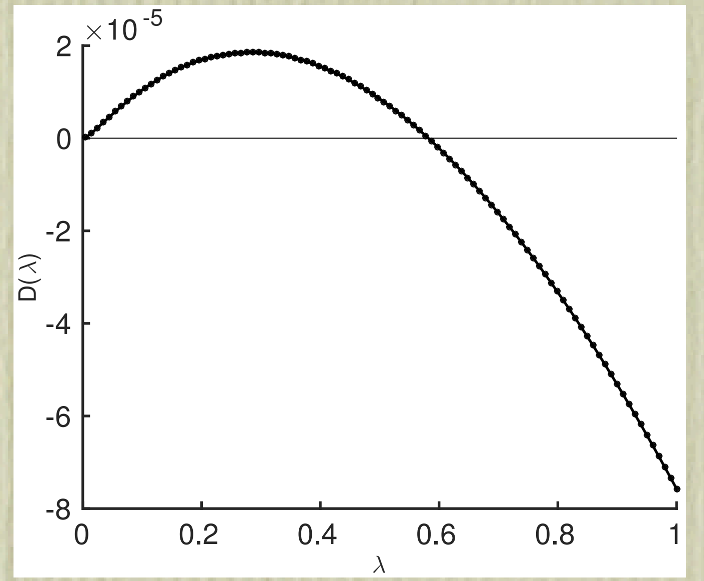


even Evans function

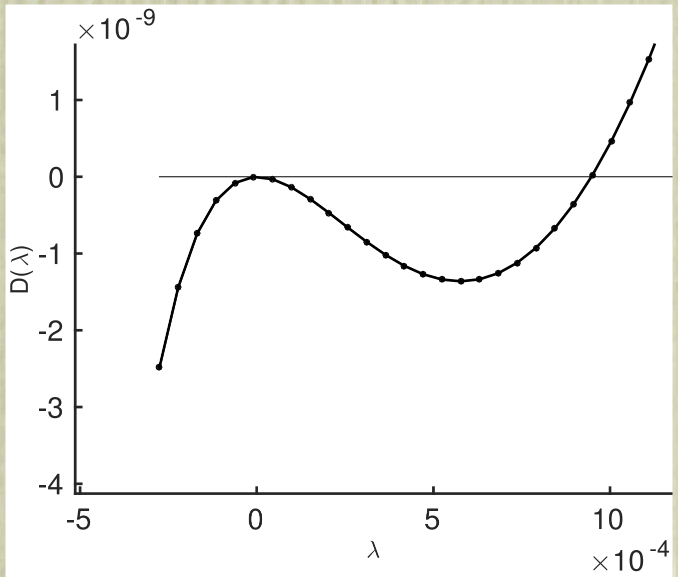
- Odd Evans function: one root at λ=0 plus one resonance pole/stable eigenvalue
- Even Evans function: two roots at λ=0 plus one unstable root and a resonance pole

Conclusion: Nozaki-Bekki holes are unstable near the NLS limit

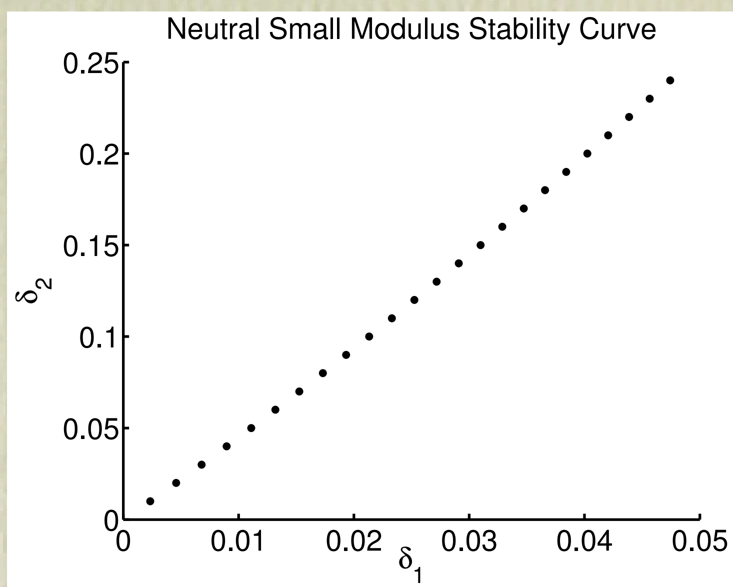
Numerical results



Evans function



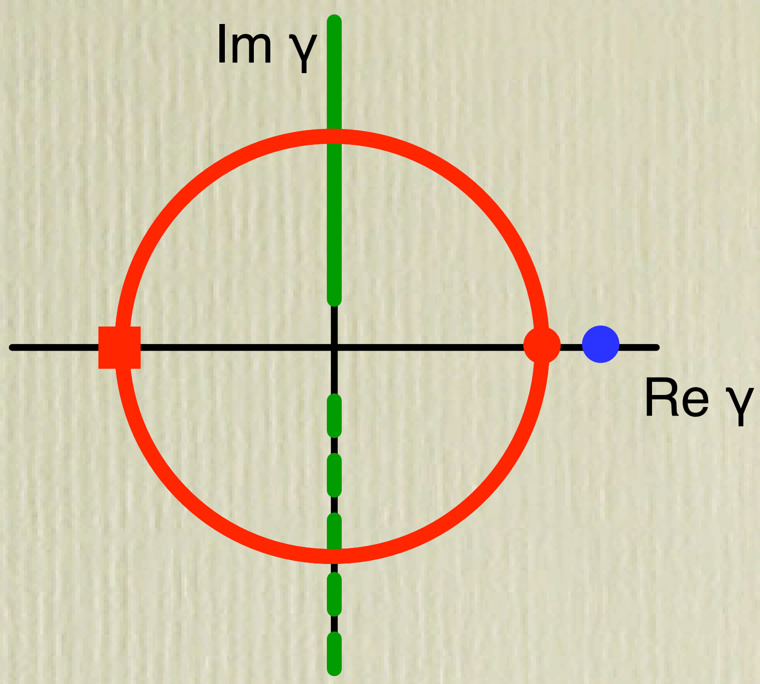
Evans function (zoom-in)



double root at $\lambda=0$

Numerical computations confirm analytical results near NLS limit

Summary + Outlook



Summary:

- Proved that Nozaki-Bekki holes are unstable near NLS limit
- Provided strategy for tackling similar problems

Outlook:

- Finalize expansion of double root at $\lambda=0$ for $\delta\neq 0$

Previous results:

- Numerical results: [Sakaguchi], [Chate & Manneville], [Stiller et al.], ...
- Soliton perturbation theory: [Lega & Fauve]
- Evans-function analysis: [Kapitula & Rubin]
- Survey article: [Lega]

Thank you!