Homeostasis as a Network Phenomenon

Minisynposium on
Structure-Dynamics Relation in Networks of Coupled Dynamical Systems
2015 SIAM Conference on Dynamical Systems
May 18, 2015

Marty Golubitsky
Mathematical Biosciences Institute
and
Department of Mathematics
Ohio State University

Homeostasis

Question motivated by **Mike Reed**; Joint work with **Ian Stewart**

$$\dot{X} = F(X, I) \qquad X = (x_1, \ldots, x_N) \in \mathbf{R}^N, I \in \mathbf{R}$$

- Assume there exists a stable equilibrium at X_0 when $I = I_0$
- There exists $X(I) = (x_1(I), \ldots, x_N(I))$ with $X(I_0) = X_0$

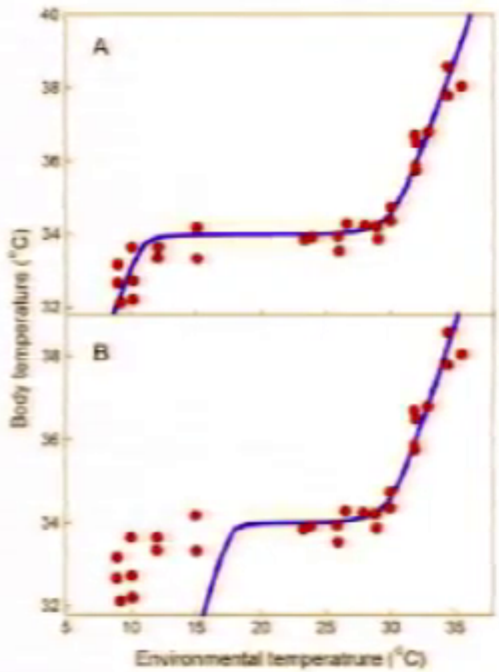
$$F(X(I), I) \equiv 0$$

- Equilibrium is **homeostatic** in coordinate j if
  $x_j(I)$ is approximately constant on a neighborhood of I_0
- Equilibrium is **infinitesimally homeostatic** in coord j if $\frac{dx_j}{dI}(I_0) = 0$
- Homeostasis: opposite of bifurcation; compute with bifur. calculations
- Homeostasis is **not** an invariant of changes of coordinates

Thm: Homeostasis can be an invariant of network preserving diffeo's

The Chair

- **Nijhout, Best, Reed**: Escape from Homeostasis (*Math Biosci*, 2014)

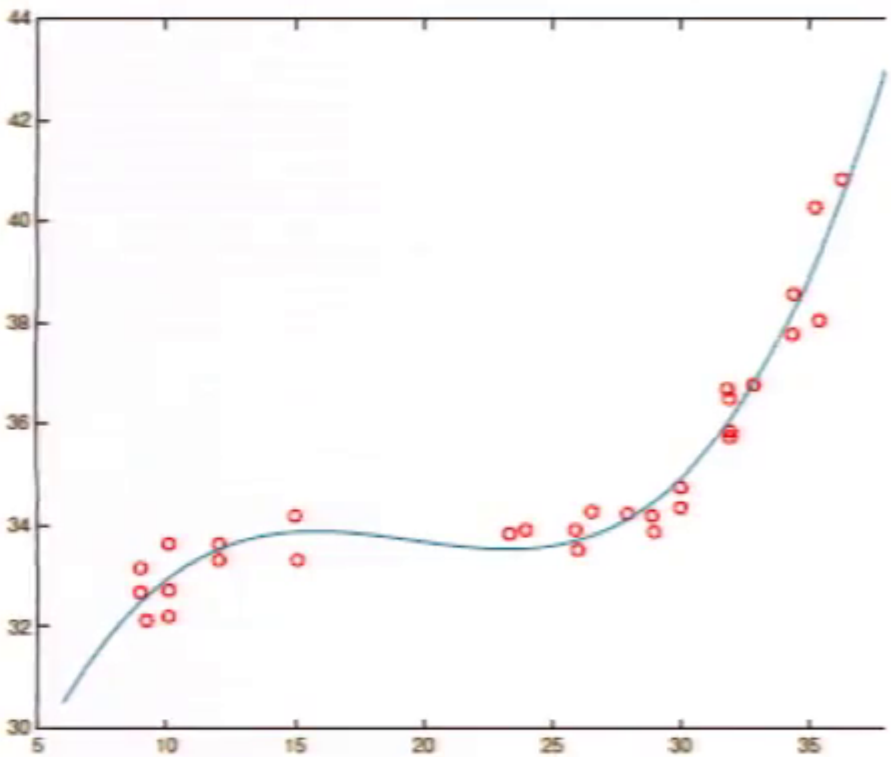


Thermoregulatory homeostasis in brown opossum.
A. Data shown by circles; model calculations shown by chair-shaped curve.
B. Reduction in efficacy of heater narrows range of environmental temperatures over which body temperature can be maintained.

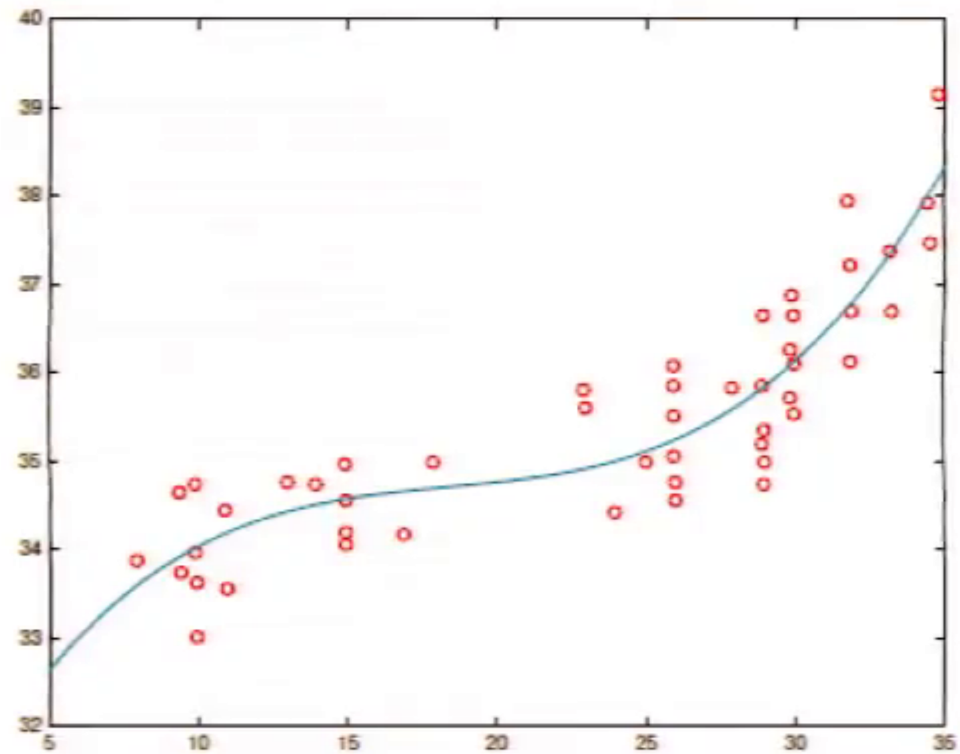
- **The Chair**: $\frac{dx_j}{dI}(I_0) = \frac{d^2x_j}{dI^2}(I_0) = 0; \frac{d^3x_j}{dI^3}(I_0) > 0$

- Chair can be network invariant

Chair 2: Brown and Eten Opossum



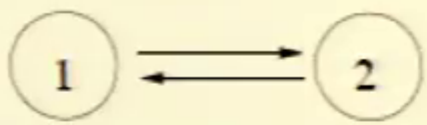
Brown Opossum



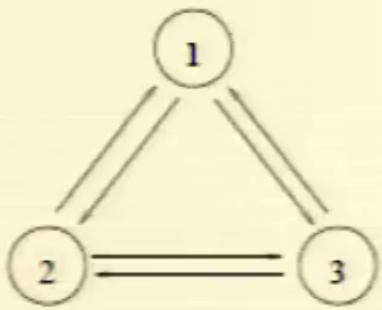
Tan Opossum

P.R. Morrison. Temperature regulation in three Central American mammals, *J. Cell. Compar. Physiol.* **27** (1946) 125–137.

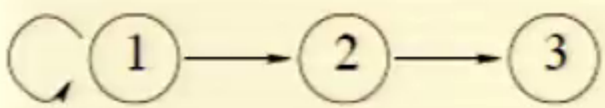
Homogeneous Networks and Coupled Systems



$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1)\end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$



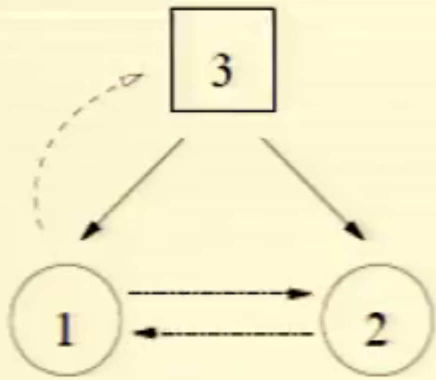
$$\begin{aligned}\dot{x}_1 &= f(x_1, \overline{x_2, x_3}) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_3}) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2})\end{aligned} \qquad f(x, \overline{y, z}) = f(x, \overline{z, y}) \qquad x_1, x_2, x_3 \in \mathbf{R}^k$$



$$\begin{aligned}\dot{x}_1 &= f(x_1, x_1, \lambda) \\ \dot{x}_2 &= f(x_2, x_1, \lambda) \\ \dot{x}_3 &= f(x_3, x_2, \lambda)\end{aligned} \qquad x_1, x_2, x_3 \in \mathbf{R}^k$$

Used network architecture and symmetry to discover rigid synchrony, rigid phase-shift synchrony, and unusual bifurcations

Networks, Coupled Systems, and Synchrony



$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2, x_3) & x_1 \in \mathbf{R}^k \\ \dot{x}_2 &= f(x_2, x_1, x_3) & x_2 \in \mathbf{R}^k \\ \dot{x}_3 &= g(x_3, x_1) & x_3 \in \mathbf{R}^\ell\end{aligned}$$

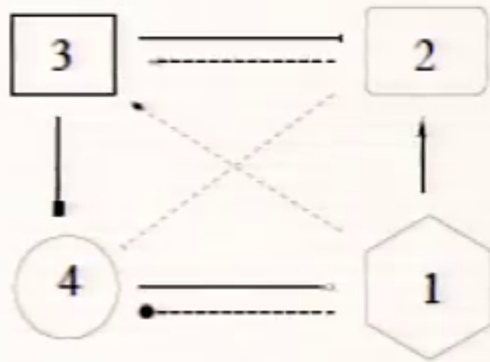
- $Y = \{x : x_1 = x_2\}$ is flow-invariant
- General theorm for classifying all flow-invariant subspaces

  Use balanced colorings

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

Admissible Vector Fields for Coupled Cell Systems

- All nodes are different; all arrows are different



$$F(x) = \begin{bmatrix} f_1(x_1, x_4) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \\ f_4(x_1, x_2, x_3, x_4) \end{bmatrix}$$

- **Definition**: A diffeomorphism Φ is **network preserving** if the pullback of every admissible vector field is admissible
- Use 'network preserving' to discuss the invariance of homeostasis

Infinitesimal Homeostasis and Chairs are Network Invariants

- Suppose $\Phi = (\varphi_1, \ldots, \varphi_n)$ is network preserving and $\varphi_j(X) = \varphi_j(x_j)$
- Then $x_j(I) \to \varphi_j(x_j(I))$ and

  j-homeostasis and j-chairs are invariants of network preserving changes of coordinates since

$$\frac{d}{dI}\varphi_j(x_j(I))\bigg|_{I=I_0} = \varphi_j'(X_0)\frac{dx_j}{dI}(I_0)$$

- If $\frac{dx_j}{dI}(I_0) = 0$, then

$$\frac{d^2}{dI^2}\varphi_j(x_j(I))\bigg|_{I=I_0} = \varphi_j'(X_0)\frac{d^2x_j}{dI^2}(I_0)$$

Network Preserving A

- A diffeomorphism $\Phi = (\varphi_1, \ldots, \varphi_n)$ is **left network preserving** iff

$$F \text{ admissible} \Longrightarrow \Phi F \text{ admissible}$$

- A diffeomorphism $\Phi = (\varphi_1, \ldots, \varphi_n)$ is **right network preserving** iff

$$F \text{ admissible} \Longrightarrow F\Phi \text{ admissible}$$

Note: Φ left or right network preseving implies Φ admissible

Proposition: Left & right network preserving diffeo's form a group

Network Preserving B

- Φ is **network preserving** iff F admissible $\Longrightarrow (D\Phi)^{-1}F\Phi$ admissible

Lemma: Left and right network preserving implies network preserving

Proof: Let $F(X)$ be admissible. Then $I + tF$ is admissible. Note:

$$\frac{d}{dt}\Phi^{-1}(I + tF)\Phi(X)\bigg|_{t=0} = (D\Phi)_X^{-1}F(\Phi(X))$$

Since LHS is admissible, so is RHS. Hence vector field change of coordinates by Φ preserves admissibility

Network Preserving C

- **Extended input set** $J(i) = i$ plus j such that there exists $j \to i$
- **Extended output set** $O(i) = i$ plus j such that there exists $i \to j$

$$
\begin{aligned}
R(i) &\equiv \{j \in J(i) : O(j) \supseteq O(i)\} \\
L(i) &\equiv \{j \in J(i) : J(j) \subseteq J(i)\} \\
LR(i) &\equiv R(i) \cap L(i)
\end{aligned}
$$

- $j \in R(i)$ if either $i = j$ or for every diagram in $\mathcal{G}$ of the form

$$
\begin{array}{ccc} j & \Longrightarrow & i \\ & & \downarrow \\ & & k \end{array}
\quad \text{there exists an arrow such that} \quad
\begin{array}{ccc} j & \Longrightarrow & i \\ & \searrow & \downarrow \\ & & k \end{array}
$$

- $j \in L(i)$ if either $i = j$ or for every diagram in $\mathcal{G}$ of the form

$$
\begin{array}{ccc} k & & \\ \downarrow & & \\ j & \Longrightarrow & i \end{array}
\quad \text{there exists an arrow such that} \quad
\begin{array}{ccc} k & & \\ \downarrow & \searrow & \\ j & \Longrightarrow & i \end{array}
$$

Network Preserving D

Theorem:

1) A diffeomorphism $\Phi = (\varphi_1, \ldots, \varphi_n)$ is **left network preserving** iff

$$F \text{ admissible} \Longrightarrow \Phi F \text{ admissible iff}$$

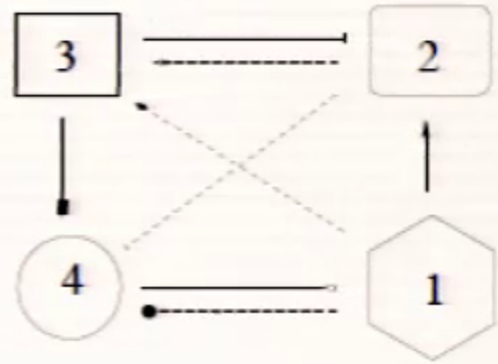
$$\varphi_i(x) = \varphi_i(x_{L(i)}) \quad \forall i$$

2) A diffeomorphism $\Phi = (\varphi_1, \ldots, \varphi_n)$ is **right network preserving** iff

$$F \text{ admissible} \Longrightarrow F\Phi \text{ admissible iff}$$

$$\varphi_i(x) = \varphi_i(x_{R(i)}) \quad \forall i$$

Example



$$\begin{aligned} R(1) &= \{1\} \\ R(2) &= \{1,2,3\} \\ R(3) &= \{1,2,3\} \\ R(4) &= \{1,4\} \end{aligned}$$

$$\begin{aligned} L(1) &= \{1\} \\ L(2) &= \{2,3\} \\ L(3) &= \{2,3\} \\ L(4) &= \{1,2,3,4\} \end{aligned}$$

$$\begin{aligned} LR(1) &= \{1\} \\ LR(2) &= \{2,3\} \\ LR(3) &= \{2,3\} \\ LR(4) &= \{1,4\} \end{aligned}$$

Automorphisms

Definition: A **network** automorphism is a permutation of nodes σ such that if $i \to j$, then $\sigma(i) \Rightarrow \sigma(j)$.

$Aut =$ group of network automorphisms

- Every network automorphism is network preserving; that is,

$$\sigma^{-1}F(\sigma X)$$

  is admissible if F is admissible

- Network automorphisms need not be admissible; whereas every $\Phi \in LR$ is admissible

Theorem: The group of network preserving diffeomorphisms is $LR + Aut$