The Multiscale Perturbation Method for Elliptic Equations

A. Alsadig[1], H. Mankad[1], F. Pereira[1], F. S. Sousa[2]

[1] The University of Texas at Dallas, USA (UTD)
[2] University of São Paulo, Brazil (USP)

March 12, 2019

Two Phase flow problem

Consider a two-phase oil-water flow problem:

$$-\lambda(s_w)\kappa\nabla p = \mathbf{u}$$
$$\nabla\cdot\mathbf{u} = q$$
$$\phi\frac{\partial s_w}{\partial t} + \nabla\cdot\left(\frac{\lambda_w(s_w)}{\lambda(s_w)}\mathbf{u}\right) = 0$$

- $\mathbf{u}(\mathbf{x},t)$, $p(\mathbf{x},t)$ − velocity and pressure fields, $s_w(\mathbf{x},t)$ − water saturation, $q(\mathbf{x},t)$ − source term, $\phi(\mathbf{x})$, $\kappa(\mathbf{x})$ − porosity, absolute permeability, $\lambda(s_w) = \lambda_w(s_w) + \lambda_o(s_w)$ − phase mobilities, no capillary pressure or gravity effects. Boundary and initial conditions have to be provided.

Operator Splitting

- One of the popular techniques to solve this problem is Operator Splitting ($s_w = s$):

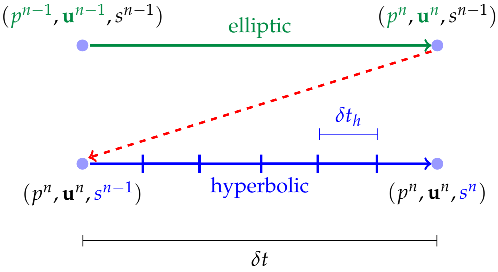


Figure: The operator splitting technique.

Operator Splitting: References

- **Two-phase:** J. Douglas, Jr., F. Furtado, F. P. (1997), On the numerical simulation of waterflooding of heterogeneous petroleum reservoirs, Computational Geosciences, v. 1, # 2, pp. 155-190.

- **Three-phase:** E. Abreu, J. Douglas Jr., F. Furtado and F. P. (2008), Operator Splitting Based on Physics for Flow in Porous Media, International Journal of Computational Science, v. 2, 3, pp. 315-335.

- **Compositional:** M. Akbarabadi, M. Borges, A. Jan, F. P. and M. Piri (2015), A Bayesian Framework for the Validation of Models for Subsurface Flows: Synthetic Experiments, Computational Geosciences, Volume 19, Issue 6, pp. 1231-1250.

Two Phase flow contd.

- Computational efficiency: elliptic time step $>$ saturation time step
- Key point: total velocity $\mathbf{u}^{n-1} \approx \mathbf{u}^n$ because

$$\lambda(s_w^{n-1})\kappa(\mathbf{x}) \approx \lambda(s_w^n)\kappa(\mathbf{x})$$

  - $\frac{\kappa(\mathbf{x})_{MAX}}{\kappa(\mathbf{x})_{MIN}} \approx 10^\alpha, \quad \alpha = 5, 6, 7\ldots$
  - $\frac{\lambda(s_w)_{MAX}}{\lambda(s_w)_{MIN}} \approx 4$

The velocity-pressure problem

Conclusion: The two-phase flow problem discussed above requires the solution of several problems of the following form

Find the velocity $\mathbf{u}$ and pressure p fields satisfying

$$\begin{aligned} \mathbf{u} &= -K\nabla p && \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= q && \text{in } \Omega \\ p &= g^p && \mathbf{x}\in\partial\Omega_p \\ \mathbf{u}\cdot\mathbf{n} &= g^{\mathbf{u}} && \mathbf{x}\in\partial\Omega_u \end{aligned} \tag{1}$$

where $K(\mathbf{x}) = \lambda(s_w(\mathbf{x}))\kappa(\mathbf{x})$ (a scalar in our discussion), q is the source term, g^p is the Dirichlet boundary data and $g^{\mathbf{u}}$ is the Neumann boundary data.

The Multiscale Perturbation Method (MPM)

MPM combines:

- The Multiscale Mixed Method (MuMM)
- Classical Perturbation Theory
- HPC: Multi-core devices

References for the MuMM

- **Original DDM:** J. Douglas, Jr., P.J. Paes Leme, J.E. Roberts, and Junping Wang (1993), A parallel iterative procedure applicable to the approximate solution of second order partial differential equations by mixed finite element methods, Numer. Math. 65, 95-108.

- **MuMM:** A. Francisco, V. Ginting, F. P., J. Rigelo (2014), Design and Implementation of a Multiscale Mixed Method for Porous Media Flows, Mathematics and Computers in Simulation, 99, pp. 125-138.

MuMM : Formulation

- Three spatial scales enter in the definition of the MuMM: $h \leq \overline{H} \leq H$

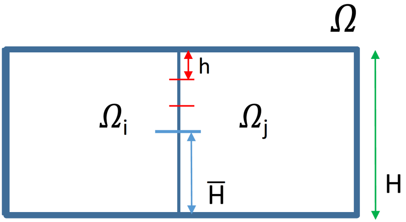


Figure: The three spatial scales for the definition of MuMM and MPM.

The MuMM (cont.)

- The solution of a local Robin BVP (with given Dirichlet and Neumann data on $\partial\Omega$) defined in Ω_i with Robin conditions (r_{ij}^{ℓ}), $\ell = 1, 2, ..., N$ is, obtained as

$$\{\mathbf{u}_h^i, p_h^i\} = \sum_{\Gamma_{ij} \subset \partial\Omega_i} \sum_{\ell=1}^{N} r_{ij}^{\ell} \Psi_{ij}^{\ell} + \Phi_i \qquad (2)$$

- The global solution of MuMM is based on a red-black iteration over subdomains.

The MuMM (cont.)

- The divergence-free Multiscale Basis Functions (MSBF) $\boldsymbol{\Psi}_{ij}^{\ell} = \{\psi_{ij}^{\ell}, \phi_{ij}^{\ell}\}$ are defined as the solution of the following Robin boundary value problem (BVP)

$$\boldsymbol{\Psi}_{ij}^{\ell} : \begin{cases} \psi_{ij}^{\ell} = -K\nabla\phi_{ij}^{\ell} & \text{in } \Omega_i \\ \nabla\cdot\psi_{ij}^{\ell} = 0 & \text{in } \Omega_i \\ \phi_{ij}^{\ell} = 0 & \mathbf{x}\in\partial\Omega_p \\ \psi_{ij}^{\ell}\cdot\mathbf{n}^i = 0 & \mathbf{x}\in\partial\Omega_{\mathbf{u}} \\ -\beta\psi_{ij}^{\ell}\cdot\mathbf{n}^i + \phi_{ij}^{\ell} = g_{ij}^{\ell} & \mathbf{x}\in\Gamma_{ij} \\ \text{such that } g_{ij}^{\ell} = 1 \text{ in } \Gamma_{ij}^{\ell} \text{ and } 0 \text{ in } \partial\Omega_i\backslash\Gamma_{ij}^{\ell} & \end{cases}$$

The MuMM (cont.)

- Non-trivial Dirichlet/Neumann boundary conditions and source terms are incorporated in the local BVP solutions through an auxiliary basis function (that takes trivial Robin conditions) $\Phi_i = \{\chi_i, \varphi_i\}$ defined as

$$\Phi_i : \begin{cases} \chi_i = -K\nabla\varphi_i & \text{in } \Omega_i \\ \nabla\cdot\chi_i = q & \text{in } \Omega_i \\ \varphi_i = g^p & \mathbf{x}\in\partial\Omega_p \\ \chi_i\cdot\mathbf{n}^i = g^{\mathbf{u}} & \mathbf{x}\in\partial\Omega_{\mathbf{u}} \\ -\beta\chi_i\cdot\mathbf{n}^i + \varphi_i = 0 & \mathbf{x}\in\Gamma_{ij} \end{cases}$$

The Multiscale Perturbation Method (MPM)

Consider the global problem (1) along with two fields, $K(\mathbf{x}) = K_0(\mathbf{x})$ and $K(\mathbf{x}) = K_t(\mathbf{x})$

- $K_0(\mathbf{x})$ corresponds to $t_0 = 0$
- $K_t(\mathbf{x})$ corresponds to $t_1 > 0$

To identify a small parameter that reflects the change in $K(\mathbf{x})$ from t_0 to t_1, write

$$K_t(\mathbf{x}) = K_0(\mathbf{x}) + \epsilon K_p(\mathbf{x}), \quad \mathbf{x} \in \Omega \qquad (3)$$

where $\epsilon = ||K_t(\mathbf{x}) - K_0(\mathbf{x})||$ and $K_p(\mathbf{x}) = \frac{K_t(\mathbf{x}) - K_0(\mathbf{x})}{||K_t(\mathbf{x}) - K_0(\mathbf{x}||}$;

Here ϵ is a dimensionless parameter with its value < 1.

The MPM scheme (cont.)

The target global BVP can be written as

$$P_t : \begin{cases} \mathbf{u} = -(K_0 + \epsilon K_p)\nabla p & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = q & \text{in } \Omega \\ p = g^p & \mathbf{x} \in \partial\Omega_p \\ \mathbf{u} \cdot \mathbf{n} = g^{\mathbf{u}} & \mathbf{x} \in \partial\Omega_{\mathbf{u}} \end{cases}$$

- Let $\mathbf{u}_{ref}, p_{ref}$ be a guess to the solution of the target problem, such that,

$$\mathbf{u} = \mathbf{u}_{ref} + \delta\mathbf{u}$$
$$p = p_{ref} + \delta p$$

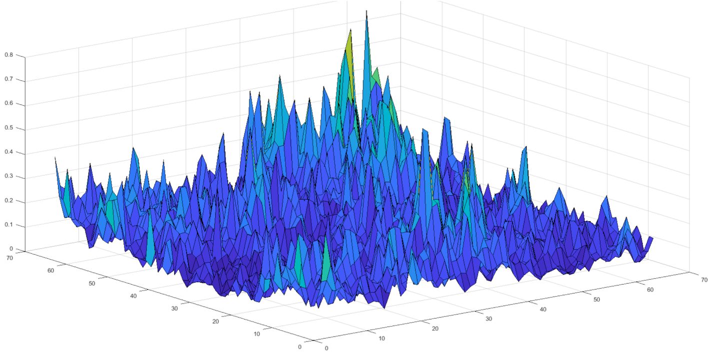
The MPM scheme contd.

- Considering the expansion,

$$\delta \mathbf{u} = \delta \mathbf{u} + \epsilon \delta \mathbf{u} + \epsilon^2 \delta \mathbf{u} + \cdots + \epsilon^r \delta \mathbf{u}$$
$$\delta p = \delta p_0 + \epsilon \delta p_1 + \epsilon^2 \delta p_2 + \cdots + \epsilon^r \delta p_r$$

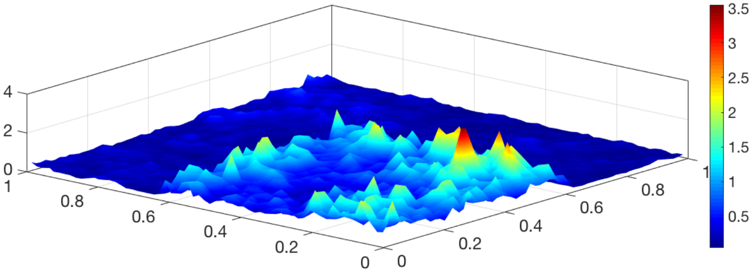
  we derive a hierarchy of second order elliptic equations that refer to K_0 and not K_t.
- These BVPs can be efficiently solved using the MuMM in multi-core computers.
- The multiscale basis functions computed at time $t = 0$ can be reused for all BVPs, once they are defined in terms of $K_0(\mathbf{x})$.

The Model Problem



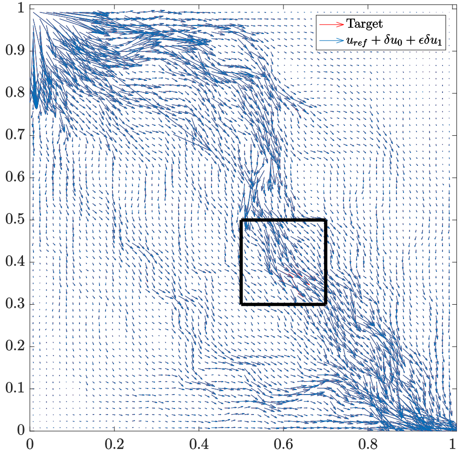
The absolute permeability field. $\frac{\kappa_{max}}{\kappa_{min}} \approx 10^3$

The Model Problem (cont.)



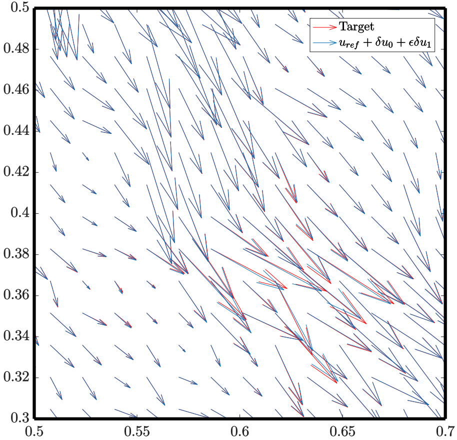
The target permeability field showing the area that has been perturbed from $t = 0$ to $t = t_1$. The region with larger variability mimics an invading water front.

Vector field: $\epsilon = 0.05$



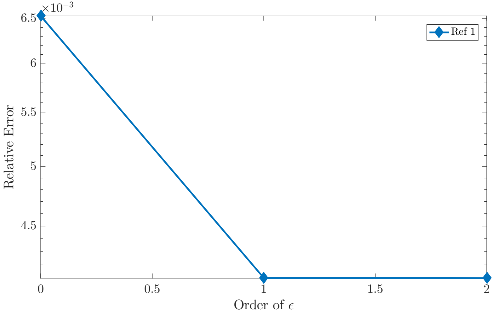
Comparing the fine scale solution with the first order ϵ

Vector field: $\epsilon = 0.05$



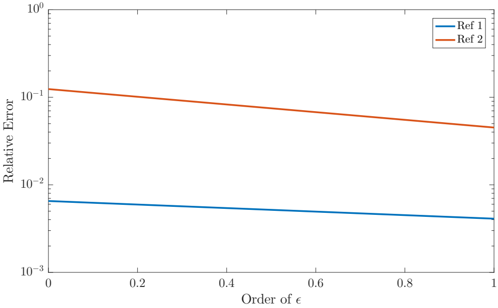
Comparing the fine scale solution with the first order ϵ

Vector field: $\epsilon = 0.05$



Relative Error for flux vs several orders of ϵ

Vector field: $\epsilon = 0.05$



Relative Error for flux vs several orders of ϵ

Conclusion/Future Work

- MPM has been introduced and tested on simple problems.
- The solution of a two phase flow problem obtained using techniques like Operator splitting and IMPES (IMplicit Pressure, Explicit Saturation) [1] can be obtained much more efficiently without recomputing the basis functions at every time step.

[1] P. Jenny and S.H. Lee and H.A. Tchelepi (2006), Adaptive fully implicit multi-scale finite-volume method for multi-phase flow and transport in heterogeneous porous media, Journal of Computational Physics, V. 217, pp 627-641

Thank You.