Well posedness of initial and boundary value problems for the inviscid linear and non-linear shallow water equations

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Articles / Collaborations

[HPT] A. Huang, M. Petcu and R. Temam, The nonlinear 2D supercritical inviscid shallow water equations in a rectangle, Asymptotic Analysis, 93, 2015, 187-218.

[HT1] A. Huang and R. Temam, The linearized 2D inviscid shallow water equations in a rectangle: boundary conditions and well-posedness, *Archives for Rational Mechanics and Analysis*, **211**, 2014, 1027-1063

[HT2] A. Huang and R. Temam, The nonlinear 2D subcritical inviscid shallow water equation with periodicity in one direction, Communications on Pure and Applied Analysis (CPAA), 13, No. 5, 2014, 2005-2038.

[HT3] A. Huang and R. Temam, The linear hyperbolic initial and boundary value problems in a domain with corners, Discrete and Continuous Dynamical Systems (DCDS-B), 19, No. 6, 2014, 1627-1665.

[HT4] A. Huang and R. Temam, The 2D nonlinear fully hyperbolic inviscid shallow water equations in a rectangle, J. of Dynamics and Differential Equations, to appear.

The shallow water equations

We are interested in this lecture in issues concerning the well-posedness of the initial-boundary value problem for the inviscid Shallow Equations (SW) in a rectangle, a problem closely related to the Primitive Equations.
We consider the linearized inviscid SW equations.

$$(1) \qquad \begin{cases} u_t + u_0 u_x + v_0 u_y + g\phi_x - fv = 0, \\ v_t + u_0 v_x + v_0 v_y + g\phi_y + fu = 0, \\ \phi_t + u_0 \phi_x + v_0 \phi_y + \phi_0 (u_x + v_y) = 0; \end{cases}$$

and the fully nonlinear inviscid SW equations

$$(2) \qquad \begin{cases} u_t + u u_x + v u_y + g\phi_x - fv = 0, \\ v_t + u v_x + v v_y + g\phi_y + fu = 0, \\ \phi_t + u \phi_x + v \phi_y + \phi (u_x + v_y) = 0. \end{cases}$$

For (1) we assume without loss of generality that $u_0 >, v_0 > 0$, $\Phi_0 > 0$ and we exclude the non generic cases where

$$(3) \qquad u_0^2 = g\Phi_0, \quad or \quad v_0^2 = g\Phi_0, \quad or \quad u_0^2 + v_0^2 = g\Phi_0.$$

In the linearized case, the issue was fully discussed in [HT1] using **the linear semi group theory.** For that purpose we write (1) in the form

$$U_t + \mathcal{E}_1 U_x + \mathcal{E}_2 U_y + BU = 0, \tag{4}$$

where $U = (u, v, \phi)^t$, $BU = (-fv, fu, 0)^t$ and

$$\mathcal{E}_1 = \begin{pmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \\ \phi_0 & 0 & u_0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & g \\ 0 & \phi_0 & v_0 \end{pmatrix}.$$

We observe that (4) is *Friedrichs symmetrizable*, i.e. $\mathcal{E}_1$, $\mathcal{E}_2$ admit a symmetrizer $S_0 = \text{diag}(1, 1, g/\phi_0)$.

By direct calculation with the help of Matlab, we find that $\mathcal{E}_1^{-1}\mathcal{E}_2$ is diagonalizable:

(5) $$\hat{P}^{-1}\cdot\mathcal{E}_1^{-1}\mathcal{E}_2\cdot\hat{P} = diag(\lambda_1,\lambda_2,\lambda_3),$$

where $\hat{P}$ has a complicated expression, whereas

$$\hat{P}^{-1} = \begin{pmatrix} \frac{v_0}{2\kappa_0} & -\frac{u_0}{2\kappa_0} & \frac{1}{2} \\ -\frac{v_0}{2\kappa_0} & \frac{u_0}{2\kappa_0} & \frac{1}{2} \\ \frac{u_0 v_0}{u_0^2+v_0^2} & \frac{v_0^2}{u_0^2+v_0^2} & \frac{g v_0}{u_0^2+v_0^2} \end{pmatrix},$$

where $\kappa_0 = \sqrt{g(u_0^2+v_0^2-g\phi_0)/\phi_0}$, and

(6) $$\lambda_1 = \frac{u_0 v_0 + \phi_0\kappa_0}{u_0^2 - g\phi_0},\ \ \lambda_2 = \frac{u_0 v_0 - \phi_0\kappa_0}{u_0^2 - g\phi_0},\ \ \lambda_3 = \frac{v_0}{u_0}.$$

Therefore, we can conclude that $\mathcal{E}_1^{-1}\mathcal{E}_2$ is diagonalizable over $\mathbb{C}$. The diagonalization over $\mathbb{R}$ depends on the sign of $u_0^2 + v_0^2 - g\Phi_0$.

The study conducted in [HT1] shows that there are 5 cases, essentially 3.

$$(7) \qquad \begin{cases} j = 1,2,3,4, & u_0^2 > \text{ or } < g\phi_0, \quad v_0^2 > \text{ or } < g\phi_0 \\ & \text{but } u_0^2 + u_0^2 - g\phi_0 > 0, \\ j = 5 & u_0^2 + v_0^2 - g\phi_0 < 0 \text{ implying} \\ & u_0^2 < g\phi_0, v_0^2 < g\phi_0. \end{cases}$$

(i) In the case where $u_0^2 + v_0^2 - g\Phi_0 < 0$, the equation is partly hyperbolic and partly parabolic.

(ii) In the other cases, the time independent part of the equation is fully hyperbolic, and the boundary conditions are determined by the direction of the characteristics.

(iii) We also have partial results for the nonlinear SW equations:

- The case $u_0^2 > g\Phi_0, v_0 > g\Phi_0$, was studied in [HPT].
- The other cases $u_0^2 >$ or $< g\Phi_0$, $v_0^2 >$ *or* $< g\Phi_0$, but $u_0^2 + v_0^2 - g\Phi_0 > 0$, raise additional difficulties. They were studied in the recent article [HT4].

The inviscid fully nonlinear shallow water equations (SWE) read

$$
\begin{cases} u_t + uu_x + vu_y + g\phi_x - fv = 0, \\ v_t + uv_x + vv_y + g\phi_y + fu = 0, \\ \phi_t + u\phi_x + v\phi_y + \phi(u_x + v_y) = 0. \end{cases} \tag{8}
$$

Setting $U = (u, v, \phi)^t$, we write (25) in compact form

$$
U_t + \mathcal{E}_1(U)U_x + \mathcal{E}_2(U)U_y + \ell(U) = 0, \tag{9}
$$

where $\ell(U) = (-fv, fu, 0)^t$, and

$$
\mathcal{E}_1(U) = \begin{pmatrix} u & 0 & g \\ 0 & u & 0 \\ \phi & 0 & u \end{pmatrix}, \quad \mathcal{E}_2(U) = \begin{pmatrix} v & 0 & 0 \\ 0 & v & g \\ 0 & \phi & v \end{pmatrix}.
$$

The assumptions and difficulties

(i) We assume that $u^2 < g\Phi, \;\; v^2 > g\phi$ and $u^2 + v^2 > g\phi$ with $u, v, \phi > 0$. More precisely

$$
(10) \quad \begin{cases} c_0 \le u,\, v,\, \phi \le c_1, \\ u^2 + v^2 - g\phi \ge c_2^2, \quad u^2 - g\phi \le -c_2^2, \quad v^2 - g\phi \ge c_2^2, \end{cases}
$$

for some given positive constants $c_0, c_1, c_2 > 0$.
In this case the flow in subsonic in the x direction and supersonic in the y direction. This will produce characteristics entering different sides of the rectangle and this will raise some **compatibility issues** at the corners and at $t = 0$.

(ii) The boundary conditions are not directly related to the linearized case. Their linearization is totally different from the boundary conditions for the linearized equation, and they do not produce a well-posed problem for the linearized equations.

(iii) We establish the short term existence of solutions in the vicinity of a stationary solution as done in [BS07] in the case of smooth domains.

[BS07]S. Benzoni-Gavage, and D. Serre, **Multidimensional hyperbolic partial differential equations**, Oxford Mathematical Monographs, First-order systems and applications, The Clarendon Press, Oxford University Press, Oxford, 2007.

Stationary Solutions independant of y

We can construct a stationary solution $U_s = (u_s, v_s, \Phi_s)$, independent of y and satisfying (10):

$$(11) \qquad \begin{cases} uu_x + g\phi_x - fv = 0, \\ uv_x + fu = 0, \\ (u\phi)_x = 0. \end{cases}$$

and consequently

$$(12) \qquad \begin{cases} u\phi = \kappa_1, \\ v = -fx + \kappa_2, \\ u^2 + 2g\phi = -f^2x^2 + 2f\kappa_2 x + \kappa_3, \end{cases}$$

where $\kappa_1, \kappa_2, \kappa_3$ are constants.

More generally, we assume that a stationary solution $U_s(x, y)$ exists for all $(x, y) \in (0, 1)_x \times \mathbb{R}_y$ and satisfies

$$\mathcal{E}_1(U_s)U_{s,x} + \mathcal{E}_2(U_s)U_{s,y} + \ell(U_s) = 0, \qquad \forall\,(x, y) \in (0, 1)_x \times \mathbb{R}_y. \tag{13}$$

The reason why we assume U_s exists for all $y \in \mathbb{R}_y$ instead of $y \in (0, 1)_y$ is that, in relation with the compatibility issue, we are going to extend the problem into the channel domain $(0, 1)_x \times \mathbb{R}_y$ and the assumption that U_s exists for all $y \in \mathbb{R}_y$ will simplify our presentation.

Technicalities

We choose $\kappa_{0,1}, \kappa_{0,2}, \kappa_{0,3} > 0$ and $\delta > 0$ such that

(14)
$$\begin{cases} c_0 \le \kappa_{0,1} \pm c_3\delta < c_1, \; c_0 \le \kappa_{0,2} \pm c_3\delta < c_1, \; c_0 \le \kappa_{0,3} \pm c_3\delta < c_1, \\ (\kappa_{0,1} + c_3\delta)^2 - g(\kappa_{0,3} - c_3\delta) \le -c_2^2, \\ (\kappa_{0,2} + c_3\delta)^2 - g(\kappa_{0,3} - c_3\delta) \ge c_2^2, \end{cases}$$

where $c_0, c_1, c_2 > 0$ are as in (10) and c_3 is a constant appearing in the proof.

In what follows, we think of the stationary solution U_s in a more general form (i.e. U_s depends on both x and y), and we choose $U_s = (u_s, v_s, \phi_s)$ such that

$$\text{(15)} \qquad | u_s - \kappa_{0,1} | \le \delta/4, \quad | v_s - \kappa_{0,2} | \le \delta/4, \quad | \phi_s - \kappa_{0,3} | \le \delta/4,$$

and by (14), U_s satisfies the *mixed hyperbolic condition* (10).

For convenience, we write

$$\text{(16)} \qquad | U_s - \boldsymbol{\kappa}_0 | \le \delta/4, \qquad \forall\, (x, y) \in (0,1)_x \times \mathbb{R}_y,$$

to stand for (15), where $\boldsymbol{\kappa}_0 = (\kappa_{0,1}, \kappa_{0,2}, \kappa_{0,3})$, and the $\kappa_{0,i}$ ($i = 1, 2, 3$) are positive constants satisfying (14).

We set $U = U_s + \widetilde{U}$ and substitute these values into (9); we obtain a new system for $\widetilde{U}$, and dropping the tildes, our new system reads:

$$L_{U_s+U}\, U = -L_{U_s+U}\, U_s, \tag{17}$$

where the operator L is defined by

$$L_W\, U = U_t + \mathcal{E}_1(W)U_x + \mathcal{E}_2(W)U_y + \ell(U). \tag{18}$$

We supplement (17) with the following initial and boundary conditions:

$$U = U_0(x,y), \text{ on } t=0, \quad U = \mathrm{G}(x,t), \text{ on } y=0, \quad b(U_s+U) = \Pi(y,t), \tag{19}$$

where

$$b(U_s+U) = \begin{cases} u + u_s + 2\sqrt{g(\phi+\phi_s)} = \pi_1(y,t), & \text{on } x=0, \\ v + v_s = \pi_2(y,t), & \text{on } x=0, \\ u + u_s - 2\sqrt{g(\phi+\phi_s)} = \pi_3(y,t), & \text{on } x=1, \end{cases} \quad \Pi = \begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\end{pmatrix}, \quad \mathrm{G} = \begin{pmatrix} g_1\\ g_2\\ g_3\end{pmatrix}.$$

We regard the initial condition $U_0 = U_s + \widetilde{U_0}$ as a small perturbation of the stationary solution, and after dropping the tilde, we choose the small perturbation U_0 satisfying

$$| U_0 | \leq \epsilon\delta, \tag{20}$$

for some $\epsilon > 0$ small enough.

Compatibility conditions on the data (1)

In order to be able to solve the system (17) we need to introduce some technical conditions (see [BS07]). First we require that $U = 0$ is a solution of the special IBVP (17) with zero initial data and boundary data $\Pi(y, t = 0)$ and $\mathrm{G}(x, t = 0)$, which amounts to asking that the following compatibility conditions are satisfied by U_s:

$$b(U_s) = \begin{cases} u_s + 2\sqrt{g\phi_s} = \pi_1(y, 0), & \text{on } x = 0, \\ v_s = \pi_2(y, 0), & \text{on } x = 0, \\ u_s - 2\sqrt{g\phi_s} = \pi_3(y, 0), & \text{on } x = 1, \end{cases} \tag{21}$$

$$U_s = \mathrm{G}(x, 0), \text{ on } y = 0.$$

Compatibility conditions on the data (2)

The other compatibility conditions are classically obtained by writing that the time derivatives of the equation (9) are satisfied at $t = 0$, where all quantities are (can) be compared in terms of the data(Rauch-Massey, Smale, Temam)

Rewrite (9), (17) as

$$U_t = H(U + U_s) - \mathcal{E}_1(U + U_s)U_x - \mathcal{E}_2(U + U_s)U_y - \ell(U), \tag{22}$$

where we denote by $H(U + U_s)$ the right-hand side of (17), that is $-L_{U_s+U}U_s$.
Now, if U is continuous, then necessarily at $t = 0$, there should holds

$$b(U_s + U_0) = \Pi(y, 0), \qquad \mathrm{G}(x, 0) = U_0|_{y=0}. \tag{23}$$

If U is $\mathcal{C}^1$ up to the boundary, then at $t = 0$,

$$\begin{aligned}
\partial_t \Pi(y,0) &= \mathrm{d}b(U_s + U_0)\cdot \partial_t U(x,0) \\
&= \mathrm{d}b(U_s + U_0)\cdot \big(H(U_0 + U_s) - \mathcal{E}_1(U_0 + U_s)U_{0,x} \\
&\qquad - \mathcal{E}_2(U_0 + U_s)U_{0,y} - \ell(U_0)\big), \\
\partial_t \mathrm{G}(x,0) &= \partial_t U(x,0) = H(U_0 + U_s) - \mathcal{E}_1(U_0 + U_s)U_{0,x} \\
&\qquad - \mathcal{E}_2(U_0 + U_s)U_{0,y} - \ell(U_0),
\end{aligned}$$

where $\mathrm{d}b(U_s + U)$ is a matrix-valued function, the gradient of the function $b(U_s + U)$ with respect to the variable U.

Similarly, additional conditions are required if U is C^{m-1} up to the boundary.

Compatibility conditions on the data (3)

We also need some similar compatibility conditions at $y = 0$.

The main result

Theorem 1

We are given a rectangular domain $\Omega = (0,1)_x \times (0,1)_y$*, a real number* $T > 0$*, an integer* $m \geq 3$*, the stationary solution* $U_s \in H^{m+1}(\Omega)$ *satisfying* (15) *(i.e. the mixed hyperbolic condition* (10)*), the initial data* $U_0 = (u_0, v_0, \phi_0)$ *belonging to* $H^{m+1/2}(\Omega)$*, the boundary data* $\mathrm{G} = (g_1, g_2, g_3)$ *belonging to* $H^{m+1/2}((0,1)_x \times (0,T))$ *and* $\Pi = (\pi_1, \pi_2, \pi_3)$ *belonging to* $H^{m+1/2}((0,1)_y \times (0,T))$*. We assume the condition* (21) *and the suitable conditions which are necessary to obtain a smooth solution in* $H^m(\Omega \times (0,T))$*. We also assume that the initial data* U_0 *is small enough in the space* $H^m(\Omega)$*. Then there exists* $T^* > 0$ *(* $T^* \leq T$*) such that the system* (17)-(19) *admits a unique solution* $U \in H^m(\Omega \times (0,T^*))$*.*

Idea of the proof

1) Make the initial and boundary conditions homogeneous by subtracting a suitable lifting U_g of the data (classical procedure, see e.g. [BS07]).

2) Extend the problem from $\Omega \times (0, T)$ to $\mathcal{Q} \times (0, T), \Omega = (0,1)_x \times (0,1)_y, \mathcal{Q} = (0,1)_x \times \mathbb{R}_y$.
Here we use the classical Babitch extension procedure in such a way that the extension of the initial and boundary values satisfy the compatibility conditions.

3) For the extended problems in $\mathcal{Q}$, the domain is **smooth** and the results of [BS07] directly apply.

Remarks

(i) The compatibility conditions are explicitly written in the article [HT4] for the case where $m = 3$ (solutions in $H^m(\Omega \times (o, T_*))$).

(ii) A basic principle in physics is that the physical laws should be independent of the reference frame chosen, which gives us the so-called invariance property of SW equations. The invariance property enables us to solve the fully hyperbolic case (that is $u^2 + v^2 > g\phi$) completely.

(iii) One key is our proof is to extend original IBVP (9) in a non-smooth domain (rectangle) into a new IBVP problem in a smooth domain (channel) and then apply the results in [BS07]. We could also solve the IBVP (9) in some other non-smooth domains as long as the non-smooth domain could be extend to a smooth domain in a suitable way.

The invariance property of SW equations

Let T be 2×2 orthogonal matrix and set

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} u' \\ v' \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}, \tag{24}$$

then (u', v', ϕ) also satisfies the SW equations:

$$\begin{cases} u'_t + u' u'_{x'} + v' u'_{y'} + g\phi_{x'} - f v' = 0, \\ v'_t + u' v'_{x'} + v' v'_{y'} + g\phi_{y'} + f u' = 0, \\ \phi_t + u' \phi_{x'} + v' \phi_{y'} + \phi(u'_{x'} + v'_{y'}) = 0. \end{cases} \tag{25}$$

We also have

$$u^2 + v^2 = u'^2 + v'^2.$$

Hence, in the fully hyperbolic condition $u^2 + v^2 > g\phi$, with a suitable coordinate transformation, we are able to find

$$v'^2 > g\phi.$$

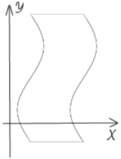
The choice of the domain

The 2d nonlinear inviscid SWE are said to be **supercritical** in the direction $\vec{l} = (\alpha, \beta)$ with $\alpha^2 + \beta^2 = 1$ (α, β are constants) if the following holds

$$(u\alpha + v\beta)^2 > g\phi. \tag{26}$$

Note that in our case $u^2 < g\phi, v^2 > g\phi$ with domain $(0,1)_x \times (0,1)_y$, the SWE is **supercritical** in the direction $(0,1)$ and hence the boundary conditions only need to be assigned at $y = 0$. This enables us to extend the rectangular domain into a channel (smooth) domain.

Some other non-smooth domain may also have such property. For example, we could solve the IBVP (9) in the following curvilinear polygonal domain.



Thank you for your attention!

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