A rigorous error analysis framework for slender body theory

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-
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What is a slender body?

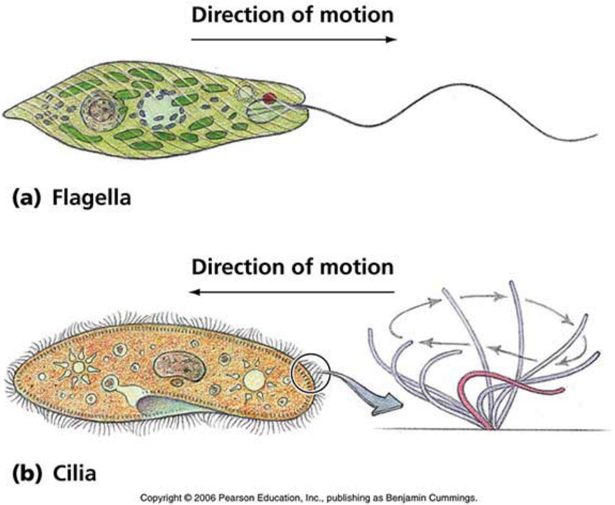
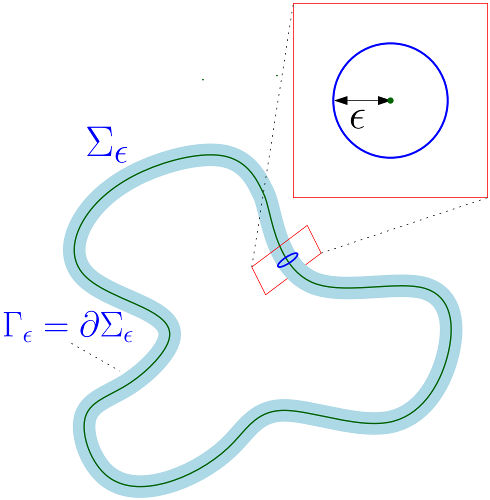


Image source: Pearson Education, Inc.

Slender body theory: setup

Hancock (1953), Cox (1970), Batchelor (1970); Keller/Rubinow (1976), Johnson (1980), Götz (2000)



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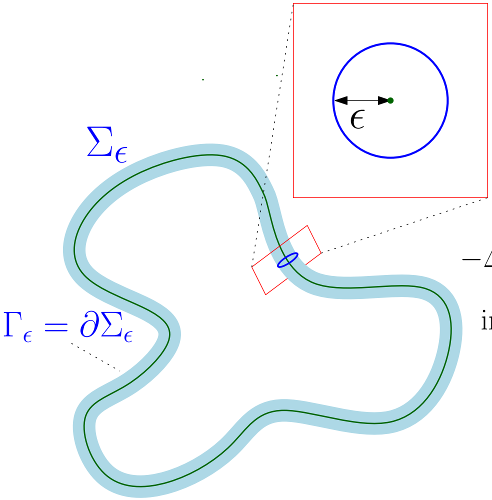

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$$\operatorname{div} \boldsymbol{u} = 0$$
$$\text{in } \Omega_\epsilon = \mathbb{R}^3 \setminus \overline{\Sigma_\epsilon}$$

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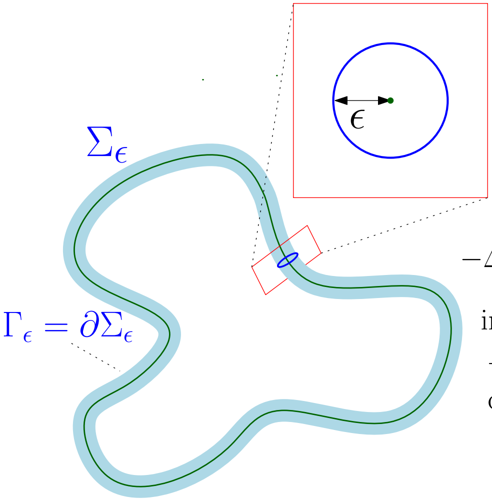

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+ boundary conditions ...

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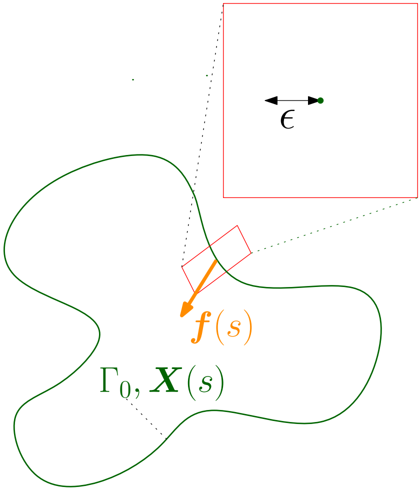

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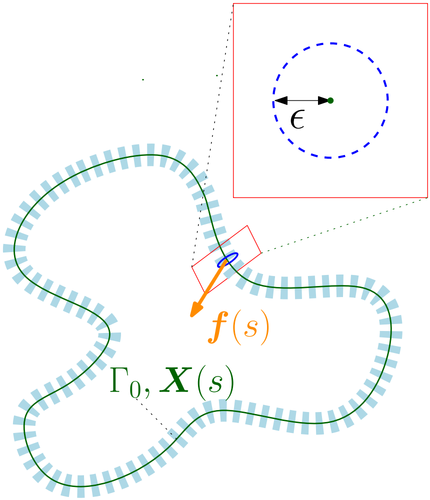

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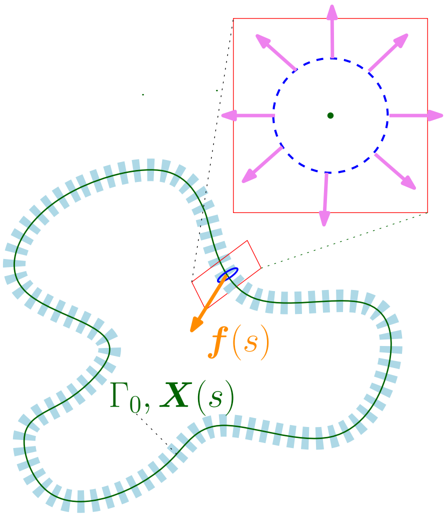

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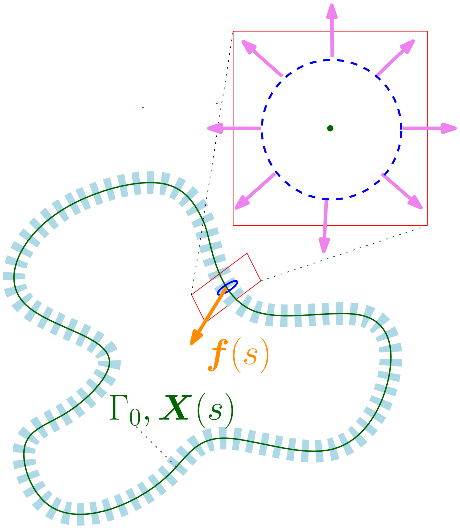

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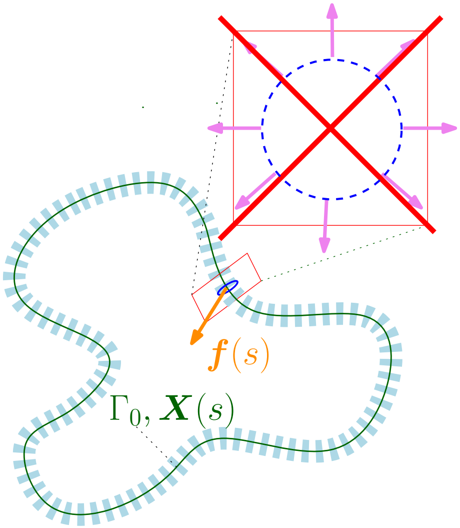

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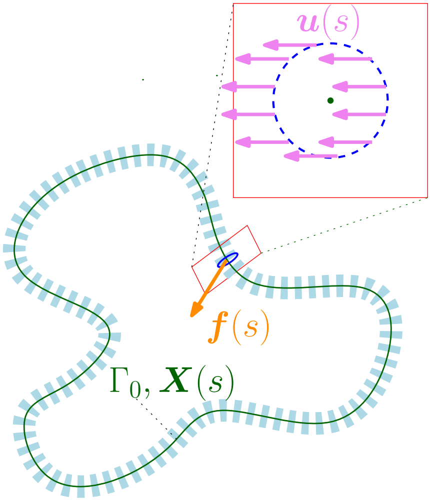

Slender body theory: building blocks

Stokes flow in $\mathbb{R}^3$ resulting from point source at $\boldsymbol{x}_0$ of strength $\boldsymbol{f}$:

$$\boldsymbol{u} = \frac{1}{8\pi}\mathcal{S}(\boldsymbol{x} - \boldsymbol{x}_0)\boldsymbol{f},$$

where the Stokeslet $\mathcal{S}$ is defined by

$$\mathcal{S}(\boldsymbol{x}) = \frac{\mathbf{I}}{|\boldsymbol{x}|} + \frac{\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}}{|\boldsymbol{x}|^3}.$$

SBT arises as 1D force density along curve $\boldsymbol{X}(s)$:

$$8\pi\,\boldsymbol{u}^{\mathrm{SB}}(\boldsymbol{x}) = \int_{\mathbb{T}} \mathcal{S}(\boldsymbol{x} - \boldsymbol{X}(s))\boldsymbol{f}(s)\,ds + \dots$$

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Fiber integrity condition?

Add doublet as correction:

$$\mathcal{D}(\boldsymbol{x}) = \frac{1}{2}\Delta\mathcal{S} = \frac{\mathbf{I}}{|\boldsymbol{x}|^3} - \frac{3\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}}{|\boldsymbol{x}|^5}$$

Full slender body approximation:

$$8\pi\,\boldsymbol{u}^{\mathrm{SB}}(\boldsymbol{x}) = \int_{\mathbb{T}} \left(\mathcal{S}(\boldsymbol{x} - \boldsymbol{X}(s)) + \frac{\epsilon^2}{2}\mathcal{D}(\boldsymbol{x} - \boldsymbol{X}(s)) \right) \boldsymbol{f}(s)\, ds$$

Shelley-Ueda (2000), Tornberg-Shelley (2004), Smith-Gaffney-Blake (2007), Lauga-Powers(2008), Spagnolie-Lauga (2011), Smith-Smith-Blake (2010), Pak-Spagnolie-Lauga (2012), Cortez-Nicholas (2012), Olson-Lim-Cortez (2013), Buchmann, et al. (2015)

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Questions

How **close** is the slender body approximation to a **true** solution?

Objective: Given $\boldsymbol{f}(s)$, approximate the slender body velocity $\boldsymbol{u}^{\mathrm{SB}}(s)$ and the flow $\boldsymbol{u}^{\mathrm{SB}}(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_\epsilon$

Main difficulty: not immediately clear how to formulate the underlying "true solution" as a well-posed PDE.

To what, exactly, is the slender body expression

$$8\pi\, \boldsymbol{u}^{\mathrm{SB}}(\boldsymbol{x}) = \int_{\mathbb{T}} \left(\mathcal{S}(\boldsymbol{x} - \boldsymbol{X}(s)) + \frac{\epsilon^2}{2} \mathcal{D}(\boldsymbol{x} - \boldsymbol{X}(s)) \right) \boldsymbol{f}(s)\, ds$$

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Slender Body PDE

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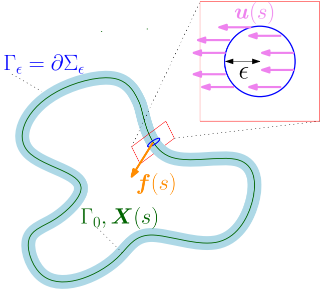

On $\Gamma_\epsilon = \partial\Sigma_\epsilon$:

$$\int_0^{2\pi} \boldsymbol{\sigma}\boldsymbol{n}\, \mathcal{J}_\epsilon(s,\theta)\, d\theta = \boldsymbol{f}(s), \quad \boldsymbol{\sigma} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}} - p\mathbf{I}$$

$$\boldsymbol{u}\big|_{\Gamma_\epsilon} = \boldsymbol{u}(s) \quad \text{(unknown but independent of } \theta\text{)}$$

At ∞:

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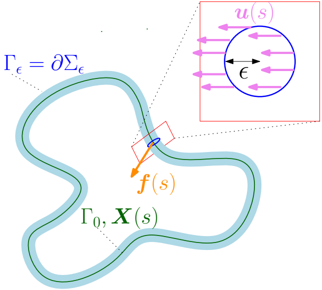

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Why is this the right formulation?

- Force balance:

$$\int_{\Omega_\epsilon} 2|\mathcal{E}(\boldsymbol{u})|^2 \, d\boldsymbol{x} = \int_{\Gamma_\epsilon} \boldsymbol{u}(s)\boldsymbol{\sigma}\boldsymbol{n}\, \mathcal{J}_\epsilon(s,\theta)\, d\theta ds$$
$$= \int_{\mathbb{T}} \boldsymbol{u}(s)\boldsymbol{f}(s)\, ds, \quad \mathcal{E}(\boldsymbol{u}) = \frac{\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{T}}}{2}.$$

- Gives rise to natural solution theory (Lax-Milgram)
- Energy estimate:

$$\|\boldsymbol{u}\|_{D^{1,2}(\Omega_\epsilon)} + \|p\|_{L^2(\Omega_\epsilon)} \leq C\|\boldsymbol{f}\|_{L^2(\mathbb{T}^1)}$$

Another difficulty: what is C?

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Geometric considerations

Need to determine ϵ-dependence in:

- Korn inequality

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega_\epsilon)} \leq C\|\mathcal{E}(\boldsymbol{u})\|_{L^2(\Omega_\epsilon)}$$

- Trace inequality

$$\|\mathrm{Tr}(\boldsymbol{u})\|_{L^2(\mathbb{T}^1)} \leq C\|\nabla \boldsymbol{u}\|_{L^2(\Omega_\epsilon)}$$

- Pressure estimate

$$\|p\|_{L^2(\Omega_\epsilon)} \leq C\|\mathcal{E}(\boldsymbol{u})\|_{L^2(\Omega_\epsilon)}$$

- Higher regularity

$$\sup_{0<h<1} \frac{1}{h}\left(\|\frac{1}{\rho}\delta_h^\theta \boldsymbol{u}\|_{H^1} + \|\delta_h^s \boldsymbol{u}\|_{H^1}\right) \leq C\left(\|\nabla \boldsymbol{u}\|_{L^2(\Omega_\epsilon)} + \|p\|_{L^2(\Omega_\epsilon)}\right)$$

Geometric considerations

We show:

- Korn inequality

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega_\epsilon)} \le c_\kappa \|\mathcal{E}(\boldsymbol{u})\|_{L^2(\Omega_\epsilon)}$$

- Trace inequality

$$\|\mathrm{Tr}(\boldsymbol{u})\|_{L^2(\mathbb{T}^1)} \le c_\kappa |\log \epsilon|^{1/2} \|\nabla \boldsymbol{u}\|_{L^2(\Omega_\epsilon)}$$

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Well-posedness of slender body PDE

Theorem (Mori, O., Spirn 2018)

Let $\Omega_\epsilon = \mathbb{R}^3 \backslash \overline{\Sigma_\epsilon}$*. Given* $\boldsymbol{f} \in L^2(\mathbb{T})$*, there exists a unique weak solution* $(\boldsymbol{u}, p) \in D^{1,2}(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$ *to the SB PDE satisfying*

$$\|\boldsymbol{u}\|_{D^{1,2}(\Omega_\epsilon)} + \|p\|_{L^2(\Omega_\epsilon)} \le |\log \epsilon|^{1/2} c_\kappa \|\boldsymbol{f}\|_{L^2(\mathbb{T})}, \tag{1}$$

where c_κ *depends only on the shape of the fiber centerline* $\boldsymbol{X}(s)$*.*

If $\boldsymbol{X}(s)$ *is at least* C^4 *and* $\boldsymbol{f}(s) \in H^{1/2}(\mathbb{T})$*, then* $(\boldsymbol{u}, p)$ *is a strong solution; i.e.* $(\boldsymbol{u}, p)$ *is in* $D^{2,2}(\Omega_\epsilon) \times H^1(\Omega_\epsilon)$ *and satisfies the SB PDE pointwise almost everywhere, and*

$$\|\boldsymbol{u}\|_{D^{2,2}(\Omega_\epsilon)} + \|p\|_{H^1(\Omega_\epsilon)} \le \epsilon^{-1} |\log \epsilon|^{1/2} c_\kappa \|\boldsymbol{f}\|_{H^{1/2}(\mathbb{T})}. \tag{2}$$

How does SBT compare to the true solution?

Need to use integral expression

$$8\pi\, \boldsymbol{u}^{\rm SB}(\boldsymbol{x}) = \int_{\mathbb{T}^1} \frac{\boldsymbol{f}(t)}{|\boldsymbol{R}|} + \frac{\boldsymbol{R}\boldsymbol{R}^{\rm T}\boldsymbol{f}(t)}{|\boldsymbol{R}|^3} + \frac{\epsilon^2}{2}\left(\frac{\boldsymbol{f}(t)}{|\boldsymbol{R}|^3} - \frac{3\boldsymbol{R}\boldsymbol{R}^{\rm T}\boldsymbol{f}(t)}{|\boldsymbol{R}|^5}\right) dt$$

to compute

1. θ-dependence in velocity on Γ_ϵ:

$$\boldsymbol{u}^{\rm r}(s,\theta) = \boldsymbol{u}^{\rm SB}\big|_{\Gamma_\epsilon}(s,\theta) - \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{u}^{\rm SB}\big|_{\Gamma_\epsilon}(s,\phi)\, d\phi$$

2. Total force over Γ_ϵ:

$$\boldsymbol{f}^{\rm SB}(s) = \int_0^{2\pi} \boldsymbol{\sigma}^{\rm SB}\boldsymbol{n}\, \mathcal{J}_\epsilon(s,\theta)\, d\theta$$

(in terms of $\boldsymbol{f}$) on the actual fiber surface Γ_ϵ

For straight centerline/constant force/infinitely long fiber, SBT exactly recovers $\boldsymbol{f}$

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Error sources

For a fiber with $C^{2,\alpha}$ centerline $\boldsymbol{X}(s)$, given $\boldsymbol{f} \in C^1(\mathbb{T})$ we show:

1. θ-dependence in velocity on Γ_ϵ:

$$|\boldsymbol{u}^{\mathrm{r}}| \le \epsilon |\log \epsilon| \, c_\kappa \, \|\boldsymbol{f}\|_{C^1(\mathbb{T})};$$

2. Total force over Γ_ϵ:

$$\left|\boldsymbol{f}^{\mathrm{SB}} - \boldsymbol{f}\right| \le \epsilon \, c_\kappa \, \|\boldsymbol{f}\|_{C^1(\mathbb{T})},$$

where c_κ depends only on the shape of the fiber centerline.

PDE for error

Errors $\boldsymbol{u}_e = \boldsymbol{u}^{\mathrm{SB}} - \boldsymbol{u}$, $p_e = p^{\mathrm{SB}} - p$, and $\boldsymbol{\sigma}_e = -p_e \mathbf{I} + 2\mathcal{E}(\boldsymbol{u}_e) = \boldsymbol{\sigma}^{SB} - \boldsymbol{\sigma}$ satisfy

$$\begin{aligned} -\Delta \boldsymbol{u}_e + \nabla p_e &= 0 \qquad \text{in } \Omega_\epsilon \\ \operatorname{div} \boldsymbol{u}_e &= 0 \\ \int_0^{2\pi} \boldsymbol{\sigma}_e \boldsymbol{n} \, \mathcal{J}_\epsilon(s,\theta) \, d\theta &= \boldsymbol{f}^{\mathrm{SB}}(s) - \boldsymbol{f}(s) \quad \text{on } \Gamma_\epsilon \\ \boldsymbol{u}_e|_{\Gamma_\epsilon} &= \bar{\boldsymbol{u}}_e(s) + \boldsymbol{u}^{\mathrm{r}}(s,\theta) \\ \boldsymbol{u}_e \to 0 \text{ as } &|\boldsymbol{x}| \to \infty \end{aligned} \tag{3}$$

How does SBT compare to the true solution?

Theorem (Mori, O., Spirn 2018)

Let $\Omega_\epsilon = \mathbb{R}^3 \setminus \overline{\Sigma_\epsilon}$ *for* Σ_ϵ *with centerline* $\boldsymbol{X}(s) \in C^{2,\alpha}(\mathbb{T})$. *Given* $\boldsymbol{f} \in C^1(\mathbb{T})$, *the difference* $\boldsymbol{u}^{\rm SB} - \boldsymbol{u}$, $p^{\rm SB} - p$ *satisfies*

$$\|\boldsymbol{u}^{\rm SB} - \boldsymbol{u}\|_{D^{1,2}(\Omega_\epsilon)} + \|p^{\rm SB} - p\|_{L^2(\Omega_\epsilon)} \le \epsilon|\log\epsilon|\, c_\kappa\, \|\boldsymbol{f}\|_{C^1(\mathbb{T})}, \tag{4}$$

where c_κ *depends only on the shape of the fiber centerline* $\boldsymbol{X}(s)$.

The L^2 *trace of the error* $\boldsymbol{u}^{\rm SB} - \boldsymbol{u}$ *along* Γ_ϵ, *scaled by* $|\Gamma_\epsilon|^{-1/2} \sim \frac{1}{\sqrt{\epsilon}}$, *satisfies*

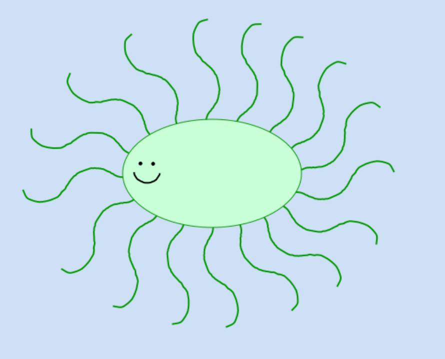
$$\frac{1}{|\Gamma_\epsilon|^{1/2}}\|{\rm Tr}(\boldsymbol{u}^{\rm SB} - \boldsymbol{u})\|_{L^2(\Gamma_\epsilon)} \le \epsilon|\log\epsilon|^{3/2}\, c_\kappa\, \|\boldsymbol{f}\|_{C^1(\mathbb{T})}. \tag{5}$$

Current work

- **Free ends**: special consideration needed at endpoints
- **Rotating fibers**: another admissible motion we have not yet considered
- **Inextensible fibers**: Easy to write down mathematical formulation: Lagrange multiplier (tension) $\lambda(s)$ in force term, but SB approximation is cumbersome
- **Numerical methods**: numerical verification of error estimate by discretizing layer potential formulation of SB PDE

Thanks

Thank you for your attention!



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