

Analysis of a nudging-based algorithm for data assimilation

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(Joint works with C. Foias and E. S. Titi)

Outline

- General idea of Data Assimilation (DA).
- Feedback-control (nudging) algorithm.
 - Discrete in time DA with systematic errors.
 - Numerical approximation – Postprocessing Galerkin.
- Summary.
- Remarks/Future work.

Question: How to make a weather forecast?

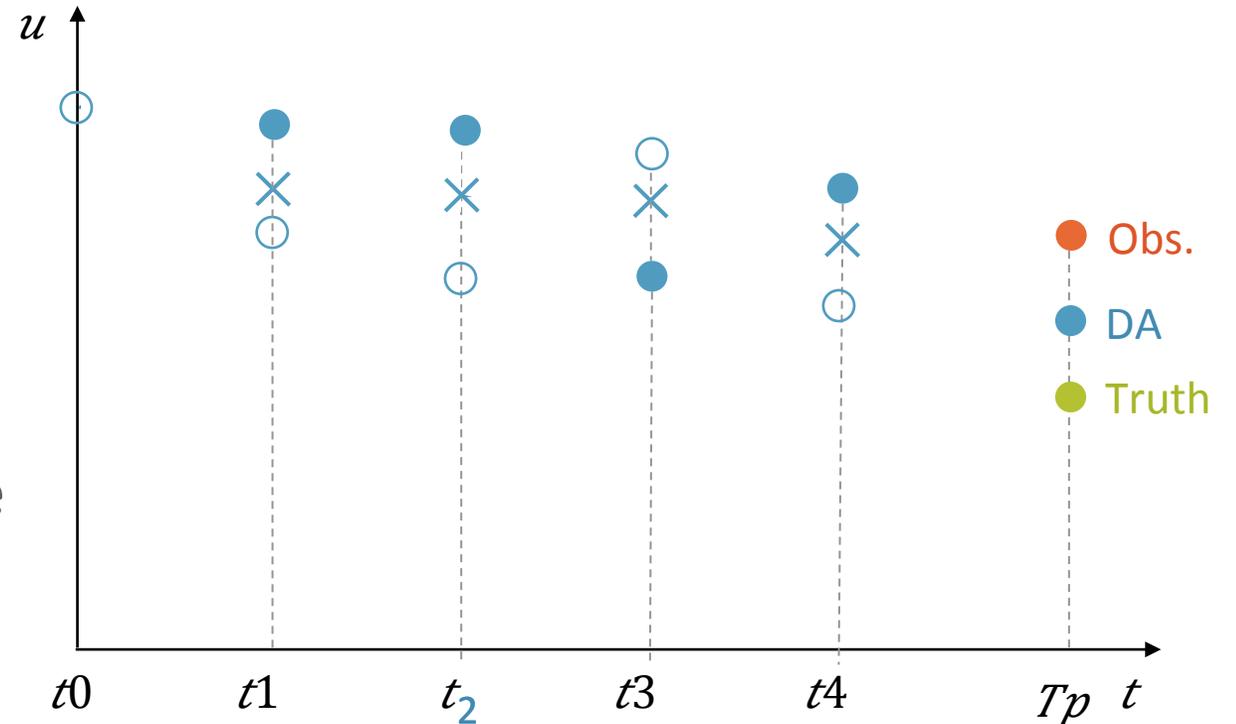
You will need...

- A theoretical model:

$$du/dt = F(t, u(t))$$

u : unknown variable representing the state of the atmosphere (velocity field, temperature, pressure, ...).

- Observational measurements.

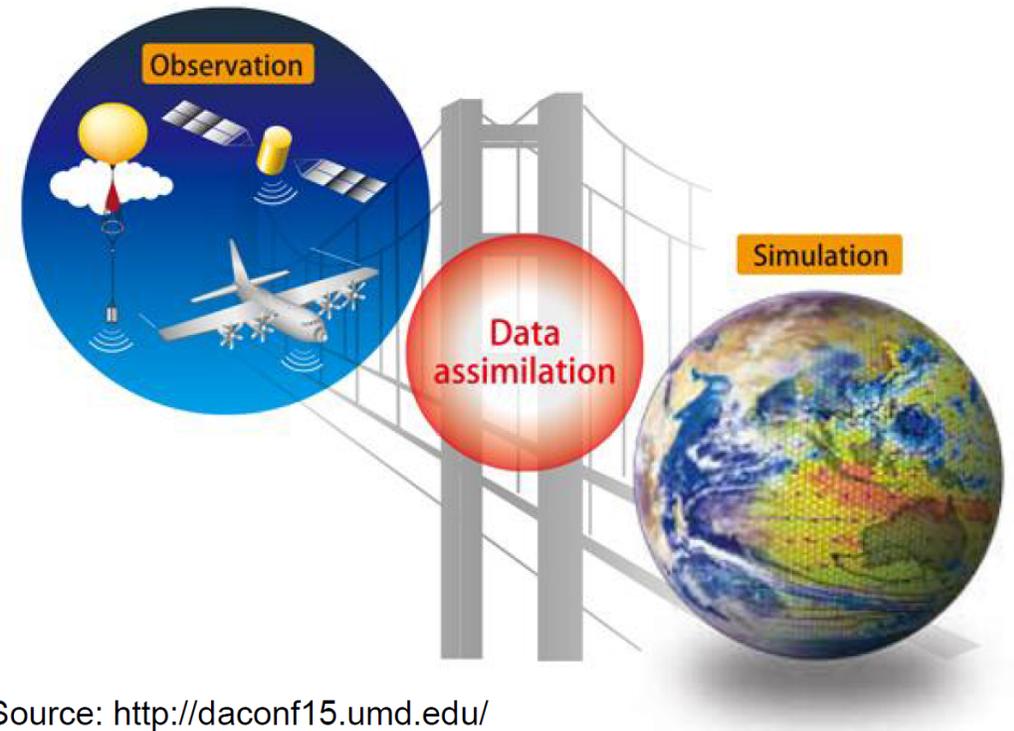


● Background (model)

○ Observations

× Analysis

- **Data Assimilation** combines the theoretical model with information from observations in order to obtain a good approximation of the state of the physical system at a certain future time.
- Numerous applications: meteorology, oceanography, oil industry, neuroscience, etc.
- Several approaches:
 - Nudging.
 - Kalman Filter (KF).
 - Ensemble Kalman Filter (EnKF).
 - Local Ensemble Transform Kalman Filter (LETKF).
 - 3DVAR.
 - 4DVAR.

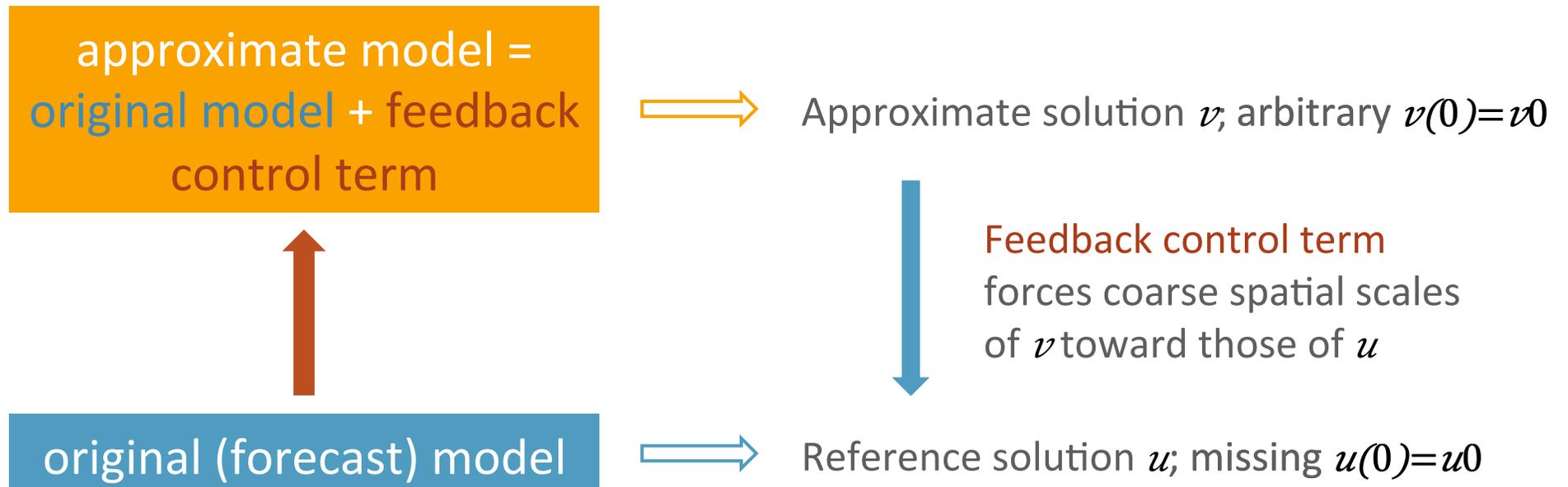


Source: <http://daconf15.umd.edu/>

Feedback-control (nudging) approach

(Azouani-Olson-Titi, '14)

- Combine model and measurements by adding a feedback-control term to the equations.



Background idea

- Long-time behavior of solutions to dissipative evolution equations is determined by only a **finite** number of degrees of freedom.
 - Fourier modes, 2D-NSE (Foias-Prodi, '67):

Let P_N be the projection operator onto the first N Fourier modes.
 $\exists N \gg 1$ s.t. if $\mathbf{u}_1, \mathbf{u}_2$ are two solutions of 2D-NSE with

$$\|P_N \mathbf{u}_1 - P_N \mathbf{u}_2\|_{L^2} \rightarrow 0, \quad t \rightarrow \infty$$

then

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2} \rightarrow 0, \quad t \rightarrow \infty.$$

- Spatial nodes, 2D-NSE (Foias-Temam, '84).
- Finite volume elements, 2D-NSE (Foias-Titi, '91; Jones-Titi, '92).
- Other dissipative evolution eqs. (Cockburn-Jones-Titi, '97).

Example

- Consider the forecast (theoretical) model given by the **2D incompressible Navier-Stokes equations**:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad (2D\text{-NSE})$$

\mathbf{u} : velocity field

ν : kinematic viscosity

p : pressure

\mathbf{f} : density of volume forces

- Assume:
 - No model error.
 - Continuous in time and error-free measurements.

Approximate model

controls
small scales

controls large scales

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla \pi = \mathbf{f} - \beta [I_h(\mathbf{v}) - I_h(\mathbf{u})], \quad \nabla \cdot \mathbf{v} = 0.$$

ν, \mathbf{f} : same as for the 2D-NSE

β : relaxation parameter

π : modified pressure

I_h : linear interpolant operator in space

h : resolution of spatial mesh

- Denote $\mathbf{w} = \mathbf{v} - \mathbf{u}$.

$$\begin{aligned} \frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + [(\mathbf{u} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{u} - (\mathbf{w} \cdot \nabla) \mathbf{w}] + \nabla(\pi - p) &= -\beta I_h(\mathbf{w}) \\ &= -\beta [I_h(\mathbf{w}) - \mathbf{w}] - \beta \mathbf{w} \end{aligned}$$

$$\Rightarrow \frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + \beta \mathbf{w} + \nabla(\pi - p) = [(\mathbf{u} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{u} - (\mathbf{w} \cdot \nabla) \mathbf{w}] - \beta [I_h(\mathbf{w}) - \mathbf{w}]$$

- Assume

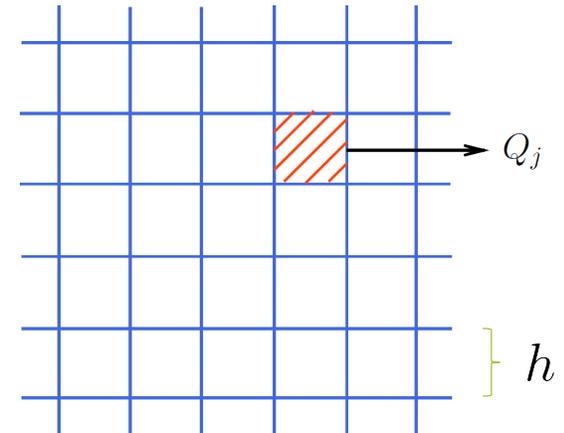
$$\|I_h(\varphi) - \varphi\|_{L^2} \leq c_0 h \|\nabla \varphi\|_{L^2} \quad \forall \varphi \in (H^1)^2.$$

Ex.:

- Low modes projector: $I_h(\varphi) = P_N \varphi$, $N \in \mathbb{N}$.

- Finite volume elements: $\Omega = \bigcup_{j=1}^N Q_j$.

$$I_h(\varphi) = \sum_{j=1}^N \bar{\varphi}_j \chi_{Q_j}, \quad \text{where } \bar{\varphi}_j = \frac{1}{|Q_j|} \int_{Q_j} \varphi dx.$$

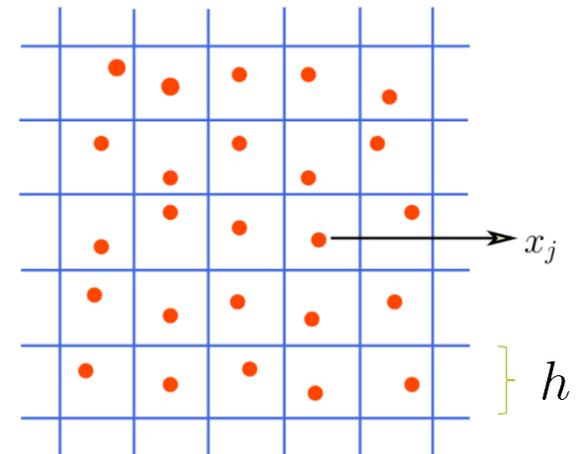


OR: $\|I_h(\varphi) - \varphi\|_{L^2} \leq c_0 h \|\varphi\|_{H^1} + c_1 h^2 \|\varphi\|_{H^2} \quad \forall \varphi \in (H^2)^2.$

Ex.:

- Nodal values: $x_j \in Q_j, j = 1, \dots, N$.

$$I_h(\varphi) = \sum_{j=1}^N \varphi(x_j) \chi_{Q_j}.$$



Theorem (Azouani-Olson-Titi, '14)

If $\beta \gg \nu \lambda_1^2$ and $h \lesssim \nu^{1/2} / \beta^{1/2}$, then $\|\mathbf{v}(t) - \mathbf{u}(t)\| \leq O(e^{-\beta t})$.

Some related works

- Other models: 3D NS-alpha (Albanez-Nussenzveig Lopes-Titi, '16), 3D Brinkman-Forchheimer-extended Darcy (Markowich-Titi-Trabelsi, '16), 2D-SQG (Jolly-Martinez-Titi, '17).
- Partial observations of the state variables:
 - 2D Bénard, only velocity (Farhat-Jolly-Titi, '15).
 - 2D-NSE, one velocity component (Farhat-Lunasin-Titi, '16).
 - 3D planetary geostrophic model, only temperature (Farhat-Lunasin-Titi, '16).
 - 2D Bénard, only horizontal velocity component (Farhat-Lunasin-Titi, '17).
 - 3D Bénard in porous media, only temperature (Farhat-Lunasin-Titi, '17).
 - 3D Leray-alpha, only two components of velocity (Farhat-Lunasin-Titi, 17).

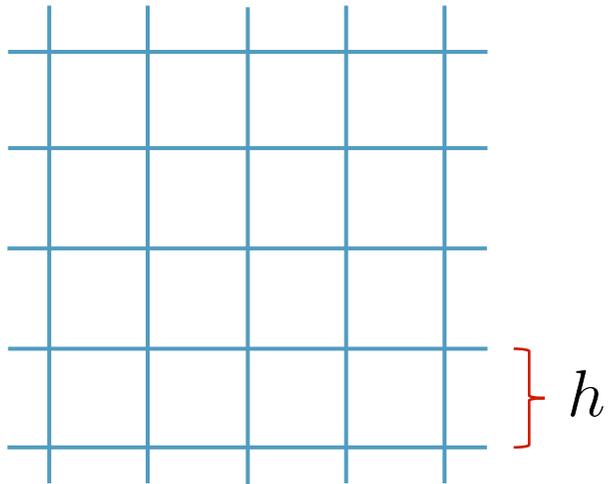
Some related works (cont'd)

- Higher order convergence, Gevrey class and L^∞ (Biswas-Martinez, '17).
- Measurements with stochastic errors (Blomker-Law-Stuart-Zygalakis, '13; Bessaih-Olson-Titi, '15).
- Time-averaged meas.: 2D-SQG (Jolly-Olson-Titi-Martinez), Lorenz (Blocher-Olson-Martinez).
- Discrete in time meas. with syst. errors, 2D-NSE (Foias-M-Titi, '16).
- Numerical computations:
 - 2D-NSE (Gesho-Olson-Titi, '16).
 - 2D Bénard (Altaf-Titi-Gebrael-Knio-Zhao-McCabe-Hoteit, '16).
- Numerical approximation by PPGM, 2D-NSE (M-Titi).

Discrete in time Data Assimilation

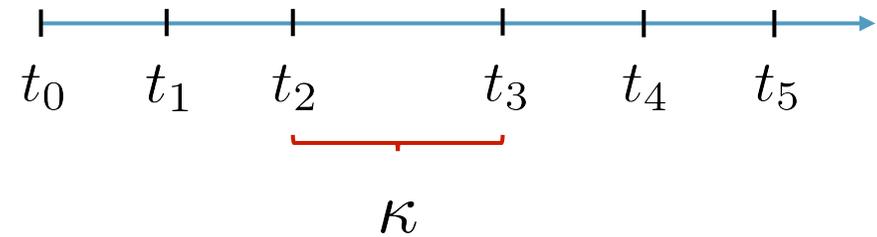
Measurements are...

- Discrete in space.



Spatial mesh with resolution of size h .

- Discrete in time.



$$|t_{n+1} - t_n| \leq \kappa, \quad n = 0, 1, 2, \dots$$

- May contain errors.
Denote by η_n the error at time t_n ,
 $n = 0, 1, 2, \dots$

$$\text{Measurement at time } t_n: I_h(\mathbf{u}(t_n)) + \eta_n$$

Approximate Model

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla \pi = \mathbf{f} - \beta \sum_{n=0}^{\infty} \{I_h(\mathbf{v}(t_n)) - [I_h(\mathbf{u}(t_n)) + \eta_n]\} \chi_{[t_n, t_{n+1})}$$

- Assume $I_h : (L^2)^2 \rightarrow (L^2)^2$ is a linear operator satisfying:

$$\|\varphi - I_h(\varphi)\|_{L^2} \leq c_0 h \|\varphi\|_{H^1}, \quad \forall \varphi \in (H^1)^2.$$

$$\|I_h(\varphi)\|_{L^2} \leq c_1 \|\varphi\|_{L^2}, \quad \forall \varphi \in (L^2)^2.$$

- Examples: low Fourier modes projector, finite volume elements.

Theorem (Foias-M.-Titi, '16)

Assume:

$$\|\eta_n\|_{H^1} \leq E \quad \forall n.$$

If $\beta \gg \nu \lambda_1^2$, $\kappa \lesssim (\nu \lambda_1)^2 / \beta^3$ and $h \lesssim \nu^{1/2} / \beta^{1/2}$, then

$$\limsup_{t \rightarrow \infty} \|\mathbf{v}(t) - \mathbf{u}(t)\|_{H^1} \leq cE.$$

Moreover, if $E = 0$, then

$$\|\mathbf{v}(t) - \mathbf{u}(t)\|_{H^1} \leq O(e^{-\beta t}).$$

Numerical Approximation

- In practice, numerical models can only compute *finite-dimensional* approximations.
- **Goal:** Obtain an analytical estimate of the error between a numerical approximation of \mathbf{v} and the (full) reference solution \mathbf{u} .
- For simplicity, assume: continuous in time and error-free measurements.
- Setting:
 - Phase space of 2D-NSE: $H = \{\mathbf{u} \in (L^2)^2 \mid \nabla \cdot \mathbf{u} = 0 + b.c.\}$.
 - Apply projector $P_\sigma : (L^2)^2 \rightarrow H$ to the feedback-control equation:

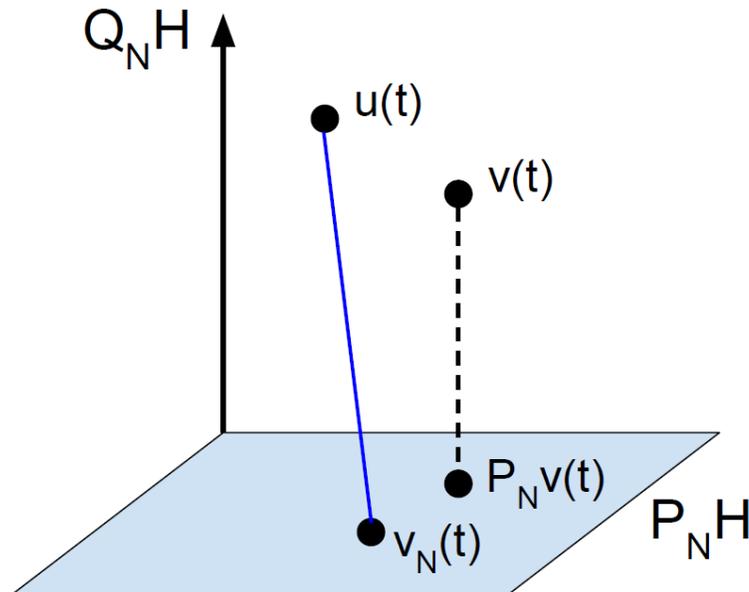
$$\frac{d\mathbf{v}}{dt} + \nu A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) = \mathbf{f} - \beta P_\sigma I_h(\mathbf{v} - \mathbf{u}),$$

- Eigenvectors of $A = P_\sigma(-\Delta) : \{\mathbf{w}_j\}_j$, with eigenvalues $\{\lambda_j\}_j$.
- Finite-dimensional space: $\text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_N\} = P_N H$.

Galerkin spectral method

Find $\mathbf{v}_N \in P_N H$ satisfying

$$\frac{d\mathbf{v}_N}{dt} + \nu A \mathbf{v}_N + P_N B(\mathbf{v}_N, \mathbf{v}_N) = P_N \mathbf{f} - \beta P_N P_\sigma I_h(\mathbf{v}_N - \mathbf{u}).$$



Notation: $Q_N = I - P_N$.

Theorem (M.-Titi)

If $\beta \gg \nu \lambda_1^2$ and $h \lesssim \nu^{1/2} / \beta^{1/2}$, then $\exists \theta = \theta(\beta) \in [0, 1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1-1} \|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C \frac{L_N}{\lambda_{N+1}}.$$

Thus, $\exists T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{L^2} \leq C \frac{L_N}{\lambda_{N+1}}, \quad \forall t \geq T,$$

where

$$L_N = \left[1 + \log \left(\frac{\lambda_N}{\lambda_1} \right) \right]^{1/2}.$$

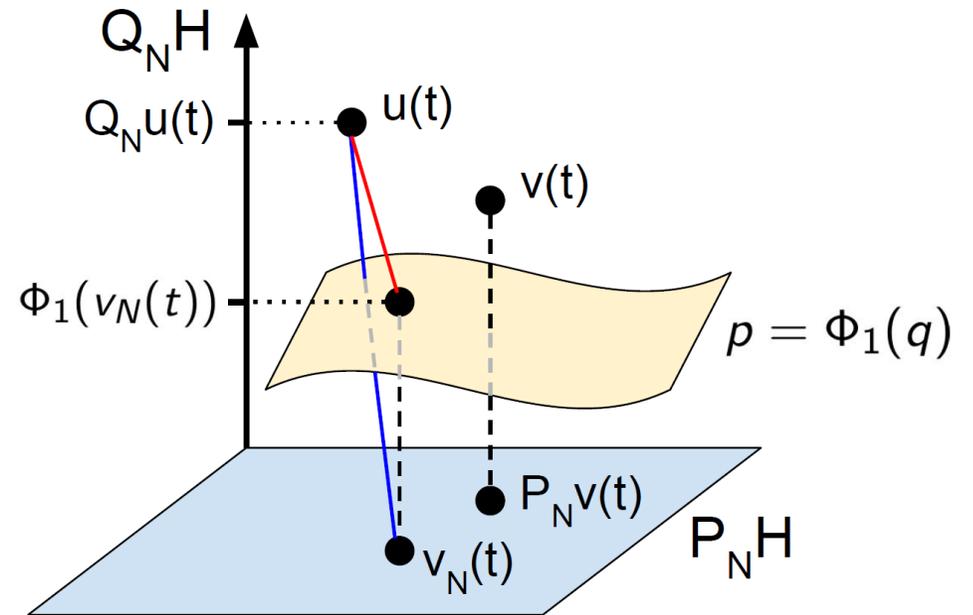
A Postprocessing of the Galerkin method

(‘García-Archilla’-Novo-Titi, ‘98)

- Idea: Add to the Galerkin approximation of \mathbf{v} a suitable approximation of \mathbf{q} :

$$\mathbf{q} \approx \Phi_1(\mathbf{p}) = (\nu A)^{-1} Q_N[\mathbf{f} - B(\mathbf{p}, \mathbf{p})]$$

(Approximate inertial manifold, Foias-Manley-Temam, ‘88)



Notation: $\mathbf{p} = P_N \mathbf{u}$, $\mathbf{q} = Q_N \mathbf{u}$
($\mathbf{u} = \mathbf{p} + \mathbf{q}$)

Postprocessing Galerkin Algorithm

For obtaining an approximation of \mathbf{v} , and thus \mathbf{u} , at a certain time $T > t_0$:

1. Integrate the Galerkin system over $[t_0, T]$ to obtain $\mathbf{v}_N(T)$.
 2. Obtain \mathbf{q}_N satisfying $\nu A\mathbf{q}_N = Q_N[\mathbf{f} - B(\mathbf{v}_N(T), \mathbf{v}_N(T))]$.
 3. Compute $\mathbf{v}_N(T) + \mathbf{q}_N$.
- Information on the high modes (fine spatial scales) is only used at the final time T ! This is one of the reasons for the efficiency of the Postprocessing Galerkin method (compared to, e.g., the Nonlinear Galerkin method).

Particular case: $I_h = P_K$, $K \in \mathbb{N}$

Theorem (M.-Titi)

If $\beta \gg \nu \lambda_1^2$ and $\lambda_K \gtrsim \beta/\nu$, then $\exists \theta = \theta(\beta) \in [0, 1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1-1} \|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C \frac{L_N^4}{\lambda_{N+1}^{3/2}}.$$

Thus, $\exists T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq C \frac{L_N^4}{\lambda_{N+1}^{3/2}}, \quad \forall t \geq T.$$

General case

- Assume $I_h : (L^2)^2 \rightarrow (L^2)^2$ is a linear operator satisfying:

- $\exists c_0 > 0$ s.t.

$$\|\varphi - I_h(\varphi)\|_{L^2} \leq c_0 h \|\varphi\|_{H^1}, \quad \forall \varphi \in H^1(\Omega)^2.$$

- $\exists c_{-1} > 0$ s.t.

$$\|\varphi - I_h(\varphi)\|_{H^{-1}} \leq c_{-1} h \|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\Omega)^2.$$

- $\exists \tilde{c}_0 > 0$ s.t.

$$\|I_h(\mathbf{q})\|_{L^2} \leq \tilde{c}_0 \frac{|\Omega|^{3/4}}{h^2 \lambda_{N+1}^{1/4}} \|\mathbf{q}\|_{L^2}, \quad \forall \mathbf{q} \in Q_N H.$$

- Examples: low Fourier modes projector; finite volume elements.

Theorem (M.-Titi)

If $\beta \gg \nu \lambda_1^2$ and $h \lesssim \nu^{1/2} / \beta^{1/2}$, then $\exists \theta = \theta(\beta) \in [0, 1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1-1} \|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C \frac{L_N}{\lambda_{N+1}^{5/4}}.$$

Thus, $\exists T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq C \frac{L_N}{\lambda_{N+1}^{5/4}}, \quad \forall t \geq T.$$

Comparison

- Error using the Galerkin method (both types of I_h):

$$\|\mathbf{v}_N - \mathbf{u}\|_{L^2} \leq O(L_N \lambda_{N+1}^{-1}).$$

- Error using the Postprocessing Galerkin method:
 - Case $I_h = P_K$:

$$\|(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u}\|_{L^2} = O(L_N^4 \lambda_{N+1}^{-3/2}).$$

- General class of I_h :

$$\|(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u}\|_{L^2} = O(L_N \lambda_{N+1}^{-5/4}).$$

Summary

- Original feedback-control data assimilation algorithm (Azouani-Olson-Titi, '14): continuous in time and error-free measurements.
 - Foias-M.-Titi, '16: *discrete in time measurements with errors*.
 - Exponential convergence of the approximate solution to the reference solution, up to an upper bound on the errors.
- Numerical approximations of v , and thus u (M.-Titi):
 - Postprocessing Galerkin method has a better convergence rate than the Galerkin method, with respect to the numerical resolution.
 - Error estimates are *uniform in time* – feedback-control term stabilizes the large scales of the difference $v - u$, resulting in a globally asymptotically stable system.

Remarks/Future work

- Theoretical condition on the spatial resolution of the measurements, h , is far from being valid for real flows.
 - Numerical simulations done in, e.g. [Gesho-Olson-Titi, '16] and [Altaf et al., '16] show that a much less restrictive condition on h is sufficient for exponential convergence.
 - Analyze ensemble averages of the physical quantities associated to the approximate flow – less restrictive conditions on the parameters? Provide rigorous analytical proofs.
- Other types of numerical methods (e.g., finite volume elements) need to be considered for approximating v . This may yield better convergence rates with respect to the numerical resolution.



Thank you!