Maximum Persistency via Iterative Relaxed Inference with Graphical Models

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Persistency (Partial Optimality)

ILP	LP
$\min c^{\mathsf{T}} x$	$\min c^{\mathsf{T}} x$
$Ax \leq b$	$Ax \leq b$
$x \in \{0,1\}^n$	$x \in [0,1]^n$

1. When the solution to LP is integer?

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2. Is the integer part of an optimal solution to LP optimal for ILP?

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4. Sufficient conditions for a part of an optimal solution to LP to be optimal for ILP?

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3. Is a part of the integer part is optimal for ILP?
4. Sufficient conditions for a part of an optimal solution to LP to be optimal for ILP?
5. Find the largest part satisfying sufficient conditions.

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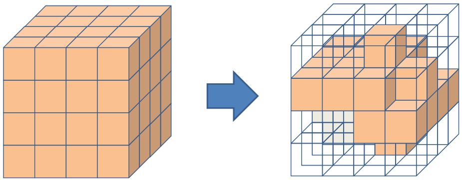
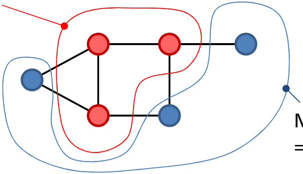


Outline

- Introduction
  - ✓ Persistency (Partial Optimality)
  - Vertex Packing, QPBO, Energy Minimization
  - Optimization-Based Methods for Persistency
- Generalized Sufficient Conditions
  - Improving Substitution
  - Relaxed-Improving Substitution
  - Generality
- Maximizing Persistency
  - Optimization-Based Formulation
  - Discrete Cutting Plane
  - OpenGM Benchmark

Vertex Packing / Maximum Independent Set



Maximum Weighted Vertex Packing

- $(\mathcal{V}, \mathcal{E})$ – an undirected graph;
- *Vertex Packing* is a subset $P \subset \mathcal{V}$ for which $u, v \in P \Rightarrow (u, v) \notin \mathcal{E}$;
- Weights $c\colon \mathcal{V} \to \mathbb{R}$;
- Problem:

$$\max_x \sum_{v \in \mathcal{V}} c_v x_v \tag{VP}$$
$$(\forall uv \in \mathcal{E})\ x_u + x_v \leq 1,$$
$$(\forall v \in \mathcal{V})\ x_v \in \{0, 1\}.$$

Vertex Packing / Maximum Independent Set

Relaxing the integrality constraints:

$$\max_{\mu} \sum_{v \in \mathcal{V}} c_v \mu_v \qquad \text{(VPL)}$$
$$(\forall uv \in \mathcal{E}) \ \mu_u + \mu_v \leq 1,$$
$$(\forall v \in \mathcal{V}) \ \mu_v \geq 0.$$

Theorems

- (Balinski, 1965; Lorentzen, 1966): Any basic feasible solution to (VLP) is $\{0, \frac{1}{2}, 1\}$-valued.
- (Edmonds and Pulleyblank) (VLP) reduces to a maxflow problem on a related symmetric bipartite graph;
- (Nemhauser and Trotter, 1975): Variables which assume binary values in an optimum (VLP) solution retain the same values in an optimum (VP) solution.
- (Picard and Queyranne, 1977): There exists a unique maximum set of variables that are integer valued in an optimal solution to (VLP).

QPBO

Quadratic pseudo-Boolean Optimization (QPBO)

- $(\mathcal{V}, \mathcal{E})$ – an undirected graph;
- Weights $a\colon \mathcal{V} \cup \mathcal{E} \to \mathbb{R}$;
- Problem:

$$\min_x \sum_{v\in\mathcal{V}} a_v x_v + \sum_{uv\in\mathcal{E}} a_{uv} x_u x_v$$

$$(\forall v \in \mathcal{V})\ x_v \in \{0, 1\}.$$

- Generalizes Vertex Packing (let $a_{uv} = B$, a big number; $a = -c_v$).

QPBO

Natural linear relaxation: $x_s \to \mu_s \in [0, 1]$, $x_s x_t \to \mu_{st} \in [0, 1]$ (lifting)

$$\min_{\mu\colon \mathcal{V}\cup\mathcal{E}\to[0,1]} \sum_{v\in\mathcal{V}} a_v \mu_v + \sum_{uv\in\mathcal{E}} a_{uv}\mu_{uv} \qquad \text{(LP)}$$

s.t. $(\forall uv \in \mathcal{E})\ \mu_u + \mu_v - 1 \leq \mu_{uv} \leq \min(\mu_u, \mu_v)$ (local convex hulls).

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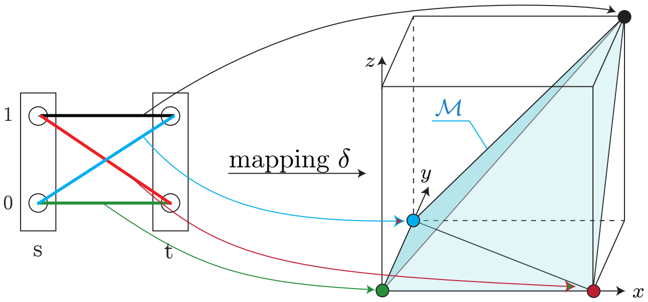

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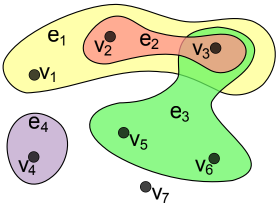
Theorems

- Each extreme point of the feasible set is $\{0, \frac{1}{2}, 1\}$-valued.
- (Hammer et al., 1984; Boros et al., 1991): LP reduces to a maxflow problem;
- **Weak Persistency** (Hammer et al., 1984): Variables μ_v which assume binary values in an optimum (LP) solution retain the same values in an ILP solution.
- **Strong Persistency** (Hammer et al., 1984): Variables μ_v which assume binary values in **all** optimal (LP) solutions retain the same values in **all** optimal ILP solutions.

0-1 Polynomial Programming



A hypergraph (courtesy of wikipedia).

0-1 Polynomial Programming / pseudo-Boolean Optimization

- $(\mathcal{V}, \mathcal{E})$ – a hypergraph, $\mathcal{E} \subset 2^{\mathcal{V}}$;
- Weights $f \colon \mathcal{E} \to \mathbb{R}$;
- Problem:

$$\min_{x \in \{0,1\}^{\mathcal{V}}} \sum_{\mathrm{C} \in \mathcal{E}} f_{\mathrm{C}} \prod_{v \in \mathrm{C}} x_v. \tag{PP}$$

- Any pseudo-Boolean function can be represented as a multilinear polynomial.

0-1 Polynomial Programming

- Quadratization techniques
  - + 0-1 PP can be reduced to QPBO with auxiliary variables Boros and Hammer (2001), Ishikawa (2011), Fix et al. (2011)
  - + Can apply roof dual relaxation (combinatorial, persistency)
  - – Relaxation of reduced problem is looser, multiple reductions

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- Special Relaxations: (bi)submodular relaxations (Kolmogorov, 2012)
  - \+ extreme feasible solutions are half-integral;
  - \+ reduces to sum of (bi)submodular functions minimization (combinatorial);
  - \+ all integer variables are persistent;
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- Special Relaxations: (bi)submodular relaxations (Kolmogorov, 2012)
  - + extreme feasible solutions are half-integral;
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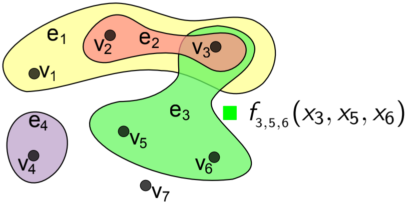
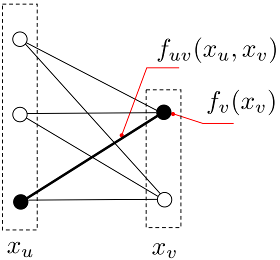
  - + all integer variables are persistent;
  - – Relatively loose, multiple choices
- Tighter relaxations, *e.g.* relaxation of Sherali and Adams (1990)
  - – optimal solutions are not half-integral in general;
  - – no combinatorial method to solve;
  - – not persistent in general;

Energy Minimization / Graphical Model





Energy Minimization / Weighted Constraint Satisfaction

- $(\mathcal{V}, \mathcal{E})$ - a hypergraph;
- $\mathcal{X}_v$ - a finite set of *labels*, $v \in \mathcal{V}$;
- Costs $f_{\mathrm{C}} \colon \prod_{v \in \mathrm{C}} \mathcal{X}_v \to \mathbb{R}$, $\mathrm{C} \in \mathcal{E}$;
- Energy: $E_f(x) = \sum_{\mathrm{C} \in \mathcal{E}} f_{\mathrm{C}}(x_{\mathrm{C}})$;
- Probloem: $\min_{x \in \mathcal{X}} E_f(x)$;

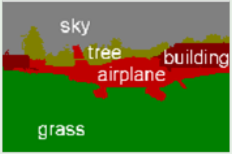
Energy Minimization

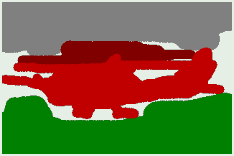
Example: Potts Model for Object Class Segmentation

- $\mathcal{V}$ - set of pixels; $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ neighboring pixels;
- $\mathcal{X}_s = \{1, \dots K\}$ – class label;
- $E_f(x) = \sum_{s \in \mathcal{V}} f_s(x_s) + \sum_{st \in \mathcal{E}} \lambda_{st} [\![x_s \neq x_t]\!]$.

Image

Ground Truth





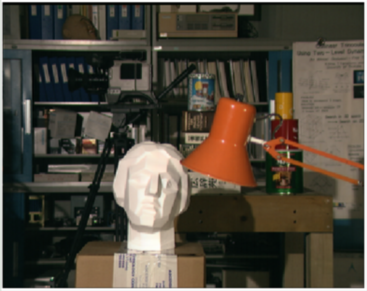
(MSRC object class segmentation)

Energy Minimization

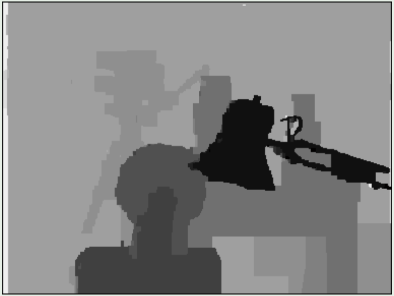
Example: Potts Model for Stereo

- $\mathcal{V}$ - set of pixels; $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ neighboring pixels;
- $\mathcal{X}_s = \{1, \dots K\}$ – disparity value;
- $E_f(x) = \sum_{s \in \mathcal{V}} f_s(x_s) + \sum_{st \in \mathcal{E}} \lambda_{st} [\![x_s \neq x_t]\!]$.

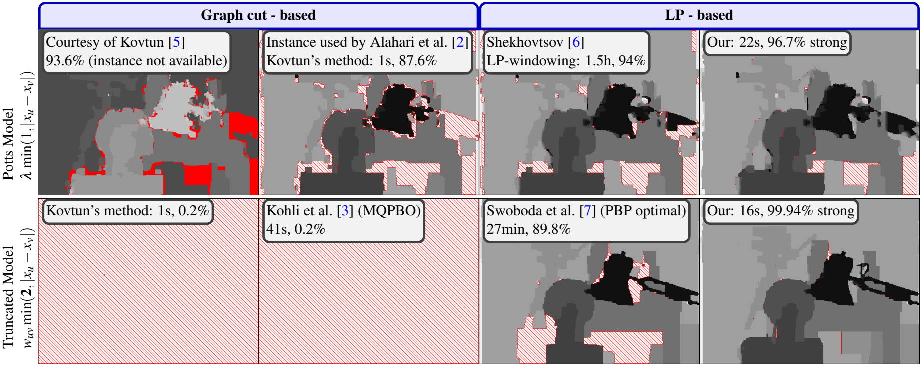
Reference (Left) Image



Depth Reconstruction



Persistency - Sufficient Conditions



- Model 1 (Kovtun'03, Alahari et al.'10): Potts, strong unaries with window aggregation
- Model 2 (Szeliski et al., 2008): Nearly Potts, per-pixel unaries

Generalized Sufficient Conditions for Persistency

Improving Substitution

- $E_f : \mathcal{X} \to \mathbb{R}$, $x \in \mathcal{X}$
- Substitution: $(x_1, x_2, \ldots, x_v, \ldots x_n) \to (x_1, x_2, \ldots, \alpha, \ldots x_n)$
- Denote as $x[v \leftarrow \alpha]$
- If $E_f(x[v \leftarrow \alpha]) \leq E_f(x)$ for all x then $x_v = \alpha$ is optimal!

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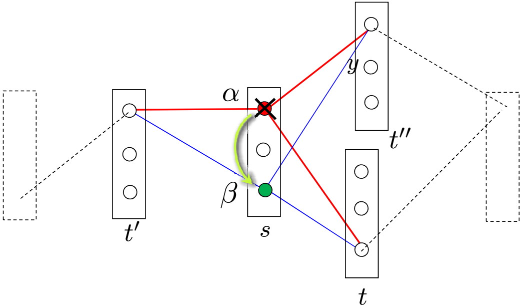
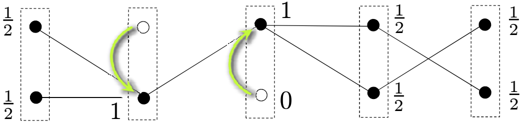
- Denote as $x[v \leftarrow \alpha]$
- If $E_f(x[v \leftarrow \alpha]) \leq E_f(x)$ for all x then $x_v = \alpha$ is optimal!
- If $E_f(x[v \leftarrow \beta]) \leq E_f(x[v \leftarrow \alpha])$ for all x then $x_v = \alpha$ can be thrown away!



- Substitutability (constraint programming) dominance (valued constraint satisfaction) dead end elimination (bioinformatics)

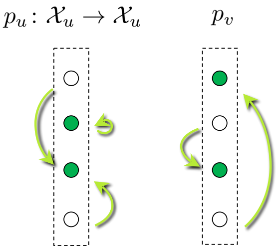
Improving Substitution

- Substitute simultaneously:
- Let $x[\mathcal{A} \leftarrow y_{\mathcal{A}}]_v = y_v$ for $v \in \mathcal{A}$ and x_v for $v \in \mathcal{V} \backslash \mathcal{A}$.
- If $E_f(x[\mathcal{A} \leftarrow y_{\mathcal{A}}]) \leq E_f(x)$ for all x then $y_{\mathcal{A}}$ is a part of an optimal assignment.



- Autarky in QPBO
- Verifying whether $y_{\mathcal{A}}$ satisfies condition is NP-hard.

Simultaneous Improving Substitution

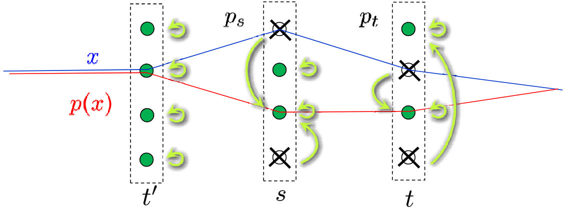


$p\colon \mathcal{X} \to \mathcal{X}$ node-wise

Definition

Substitution p is improving if
$(\forall x) \quad E_f(p(x)) \leq E_f(x)$.

- If x is optimal then $p(x)$ is optimal. Search space can be reduced.



Sufficient Conditions for Peristency

- Improving mapping: $(\forall x \in \mathcal{X})\ E_f(p(x)) \leq E_f(x)$ – NP hard to verify

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- Lift: $\min_{\mu \in \delta(\mathcal{X})} (\langle f, \mu \rangle - \langle f, P\mu \rangle) \geq 0$

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- Relax: $\min_{\mu \in \Lambda} \langle f - P^{\mathsf{T}} f, \mu \rangle \geq 0$, Λ - any tractable polytope containing $\delta(\mathcal{X})$

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Definition:

Substitution p is *relaxed-improving* if $\min_{\mu \in \Lambda} \langle (I - P^{\mathsf{T}}) f, \mu \rangle \geq 0$

- Polynomial to verify.

Sufficient Conditions for Peristency

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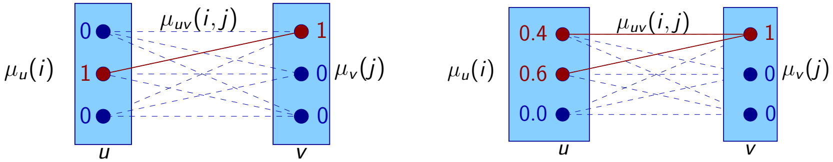
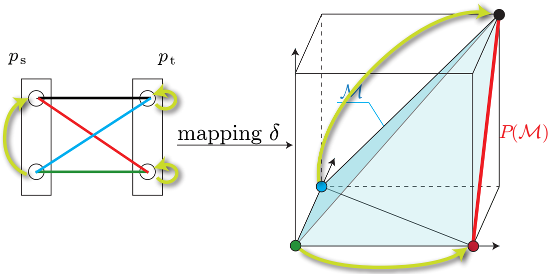


Relaxed Improving Substitution

- Substitution $p\colon \mathcal{X} \to \mathcal{X}$ can be represented in the lifted space:



- Linear mapping P is the *extension* of $p\colon \mathcal{X} \to \mathcal{X}$,
- An oblique projection onto a facet.

Generality of Sufficient Conditions

- Sufficient condition for persistency
- Can be verified by solving LP over $\Lambda \supset \delta(\mathcal{X})$
- Tightens with relaxation:
  For $\Lambda' \subset \Lambda$, if p is improving on Λ then it is improving on Λ'.

Generality of Sufficient Conditions

Theorems (Shekhovtsov (2014, 2015))

Relaxed-improving condition with natural (local) relaxations are satisfied for a.o.f.:

pairwise multilabel	Simple DEE (Goldstein, 1994)	✓
	MQPBO (Kohli et al., 2008)	✓
	Kovtun (2003) one-agains-all	✓
	Kovtun (2011) iterative	✓
	Swoboda et al. (2014)*	✓
	Roof dual / QPBO Hammer et al. (1984)	✓
higher order pseudo-Boolean	Reductions: HOCR (Ishikawa, 2011), (Fix et al., 2011)	FLP
	Bisubmodular relaxations (Kolmogorov, 2010)**	BLP
	Generalized Roof Dualilty (Kahl and Strandmark, 2011)	FLP
	Persistency by Adams et al. (1998)	FLP

BLP = Basic LP Relaxation Werner (2007); Thapper and Živný (2013);
FLP = Full Local LP Relaxation, equivalent to Sherali and Adams (1990);

*Swoboda et al. (2014) is higher order but the comparison proof is for pairwise case. **Result holds for sum of bisubmodular functions over the same hypergraph as the BLP relaxation.

Maximizing Persistency

Maximum Persistency

- Given that verification problem is polynomially solvable,
- which method is better?

Maximum Persistency

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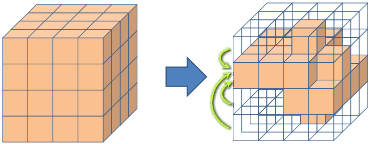
Maximum Persistency Problem

Find the substitution $p\colon \mathcal{X} \to \mathcal{X}$ that delivers the maximum problem reduction:

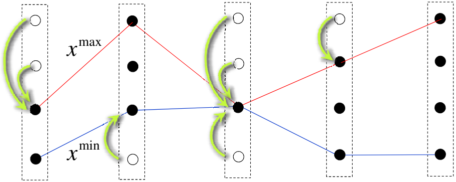
$$\min_{p \in \mathcal{P}} \sum_{u \in \mathcal{V}} |p(\mathcal{X}_u)| \quad \text{s.t. } p \text{ is relaxed-improving,}$$

$\mathcal{P}$ - class of mappings.

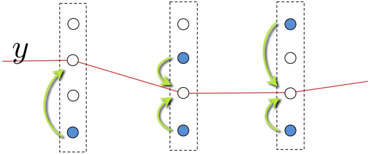


Restricted Class of Mappings

- Clamping to an interval. Order-dependent



- Fix a test labeling y and substitute any subset $\mathcal{Y}_v \subset \mathcal{X}_v$ with y_v. Order independent



- Lattice (nesting) of substitutions in both cases

Restricted Class of Mappings

- Can find the maximum (eliminating most of variables) (strictly) Λ- improving substitution in these cases for any Λ!

Subsets substituting class covers

pairwise multilabel	Simple DEE (Goldstein, 1994)	
	MQPBO (Kohli et al., 2008)	
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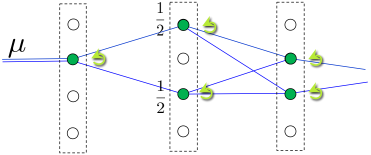
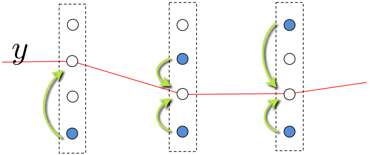
Discrete Cutting Plane

Theorem

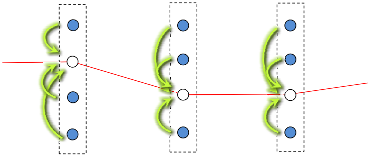
Let μ be a solution to LP-relaxation: $\mu \in \operatorname{argmin}_{\mu\in\Lambda}\langle f, \mu\rangle$ and $p\colon \mathcal{X} \to \mathcal{X}$ be (strictly) relaxed-improving. Then $P\mu = \mu$.



- Initialize test labeling y from μ



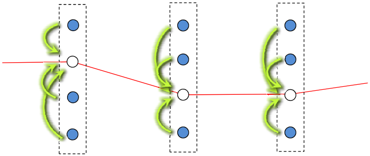
Discrete Cutting Plane



Algorithm

- Start with a mapping p that substitutes *everything* with y

Discrete Cutting Plane

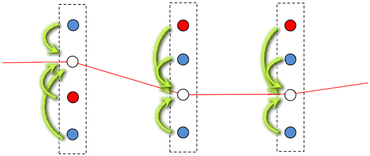


Algorithm

- Start with a mapping p that substitutes *everything* with y
- Auxiliary problem $g = (I - P^{\mathsf{T}})f$
- Check relaxed-improving conditions by solving LP:

$$\min_{\mu \in \Lambda} \langle g, \mu \rangle \overset{?}{\geq} 0$$

Discrete Cutting Plane



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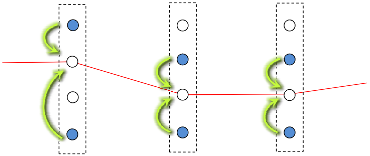
- If not satisfied, determine the most violating solution μ

Discrete Cutting Plane



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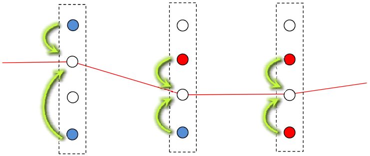
Discrete Cutting Plane



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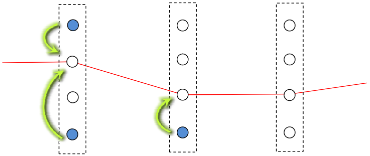

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Correctness and Optimality

Main Properties:

- Runs in polynomial time;
- Finds the maximum relaxed improving substitution when the LP solver is e.g. the interior point method (uses strict complementarity).
- Correct even with sub-optimal (no convergence guarantees) LP solvers
- Correct with dual suboptimal solvers (we use TRW-S by Kolmogorov (2006))
- Can be implemented as incremental

Efficiency

- solving relaxed inference approximately even once is slow
- Fast block-coordinate algorithms TRW-S not finitely converging

How can we iterate such relaxed inference?

Fast implementation with TRW-S

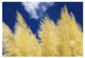
- Incremental: reuse reparametrizations φ
- Guaranteed to prune something even after 1 iteration of TRW-S (there is a blocking constraint not yet pruned)
- An optimal pruning is often possible before the dual is solved (cuts)
- Problem reductions preserving the sufficient condition
- Fast message passing for $(I - P^{\mathsf{T}})f$ with reductions

Combined Effect of Speedups

Instance	Initialization (1000 it.)	Extra time for persistency				
		no speedups	+reduction	+node pruning	+labeling pruning	+fast msgs
Protein folding `1CKK`	8.5s	268s (26.53%)	168s (26.53%)	2.0s (26.53%)	2.0s (26.53%)	2.0s (26.53%)
colorseg-n4 `pfau-small`	9.3s	439s (88.59%)	230s (93.41%)	85s (93.41%)	76s (93.41%)	19s (93.41%)

Experiments

OpenGM Benchmark



ColorSegmentation (N8)
J. Lellmann et.al.
converted by J. Lellmann and J.H. Kappes

Object Segmentation
K. Alahari et.al.
converted by J.H. Kappes

MRF Stereo
R. Szeliski et.al.
converted by J.H. Kappes

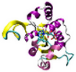
ColorSegmentation
K. Alahari et.al.
converted by J.H. Kappes

MRF Photomontage
R. Szeliski et.al.
converted by J.H. Kappes

MRF Inpainting
R. Szeliski et.al.
converted by J.H. Kappes

Chinese Characters
S. Nowozin et.al.
converted by S. Nowozin and J. H. Kappes

Scene Decomposition
Gould et.al.
converted by S. Nowozin and J. H. Kappes



Protein Folding
Yanover et. al.
converted by Joerg Kappes

Brain 3mm
J. H. Kappes et.al.
converted by J. H. Kappes

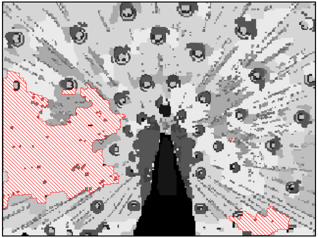
Geometric Surface Labeling (3)
Gallagher et.al.
converted by D. Batra and J. H. Kappes

Problem family	#I	#L	#V	MQPBO		MQPBO-10		Kovtun		[29]-TRWS		Our-TRWS	
mrf-stereo	3	16-60	> 100000	†		†		†		2.5h	13%	117s	**73.56%**
mrf-photomontage	2	5-7	≤ 514080	93s	22%	866s	16%	†		3.7h	16%	483s	**41.98%**
color-seg	3	3-4	≤ 424720	22s	11%	87s	16%	**0.3s**	98%	1.3h	**>99%**	61.8s	**99.95%**
color-seg-n4	9	3-12	≤ 86400	22s	8%	398s	14%	**0.2s**	67%	321s	90%	4.9s	**99.26%**
ProteinFolding	21	≤ 483	≤ 1972	685s	2%	2705s	2%	†		48s	18%	9.2s	**55.70%**
object-seg	5	4-8	68160	3.2s	0.01%	†		**0.1s**	93.86%	138s	98.19%	2.2s	**100%**

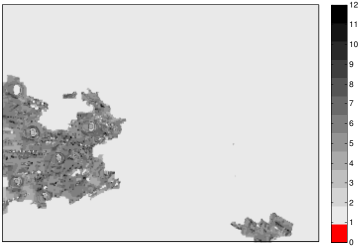
OpenGM Benchmark

Input image (Potts model Color Segmentation)

Proved optimal part



Reminder (number of labels)

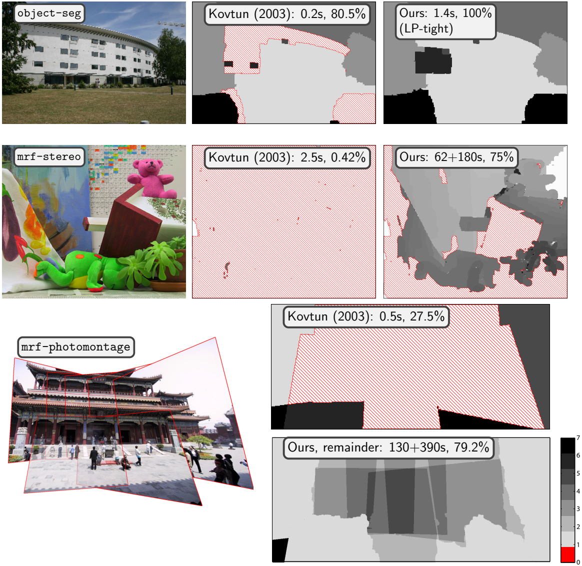


OpenGM Benchmark

Easy

Hard

Very hard



Conclusion

- We find a part of an optimal solution in polynomial time
- New general sufficient condition
- Covers many methods in the literature
- Developed an efficient algorithm (implementation available, matlab interface)
- In a sense, we converted a method without guarantees (TRW-S) into a method with guarantees at a reasonable overhead

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