Correcting Biased Observation Model Error in Data Assimilation

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Biased observation model error problems

All the Kalman based DA method assumes unbiased observation model error, e.g.,

$$y_i = h(x_i) + \eta_i, \quad \eta_i \sim \mathcal{N}(0, R).$$

Suppose the operator h is un known. Instead, we are only given $\tilde{h}$, then

$$y_i = \tilde{h}(x_i) + b_i,$$

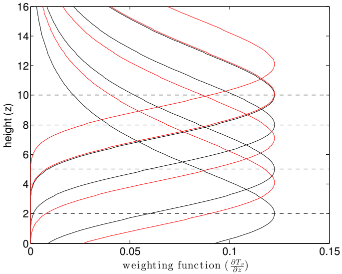
where we introduce a biased model error, $b_i = h(x_i) - \tilde{h}(x_i) + \eta_i$.

Example: Basic radiative transfer model

Consider solutions of the stochastic cloud model[1], $\{T(z), \theta_{eb}, q, f_d, f_s, f_c\}$.
Based on this solutions, define a basic radiative transfer model as follows,

$$h_\nu(x) = \theta_{eb} T_\nu(0) + \int_0^\infty T(z) \frac{\partial T_\nu}{\partial z}(z)\, dz,$$

where T_ν is the transmission between heights z to ∞ that is defined to depend on q.
The weighting function, $\frac{\partial T_\nu}{\partial z}$ are defined as follows:



[1] Khouider, Biello, Majda 2010

Example: Basic radiative transfer model

Suppose the deep and stratiform cloud top height is $z_d = 12$km, while the cumulus cloud top height is $z_c = 3$km. Define $f = \{f_d, f_c, f_s\}$ and $x = \{T(z), \theta_{eb}, q\}$. Then the cloudy RTM is given by,

$$\begin{aligned} h_\nu(x, f) &= (1 - f_d - f_s)\left[\theta_{eb} T_\nu(0) + \int_0^{z_d} T(z) \frac{\partial T_\nu}{\partial z}(z)\, dz\right] \\ &\quad + (f_d + f_s) T(z_t) T_\nu(z_d) + \int_{z_d}^{\infty} T(z) \frac{\partial T_\nu}{\partial z}(z)\, dz \end{aligned}$$

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&\quad + (f_d + f_s) T(z_t) T_\nu(z_d) + \int_{z_d}^{\infty} T(z)\frac{\partial T_\nu}{\partial z}(z)\, dz \\
&= (1 - f_d - f_s)\Big[(1 - f_c)\Big(\theta_{eb} T_\nu(0) + \int_0^{z_c} T(z)\frac{\partial T_\nu}{\partial z}(z)\, dz\Big) \\
&\quad + f_c T(z_c) T_\nu(z_c) + \int_{z_c}^{z_d} T(z)\frac{\partial T_\nu}{\partial z}(z)\, dz\Big] \\
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One can check that $h_\nu(x,0)$ corresponds to cloud-free RTM.

Observation model error in data assimilation

Suppose the observation is generated with

$$y_\nu = h_\nu(x, f) + \eta, \qquad \eta \sim \mathcal{N}(0, R)$$

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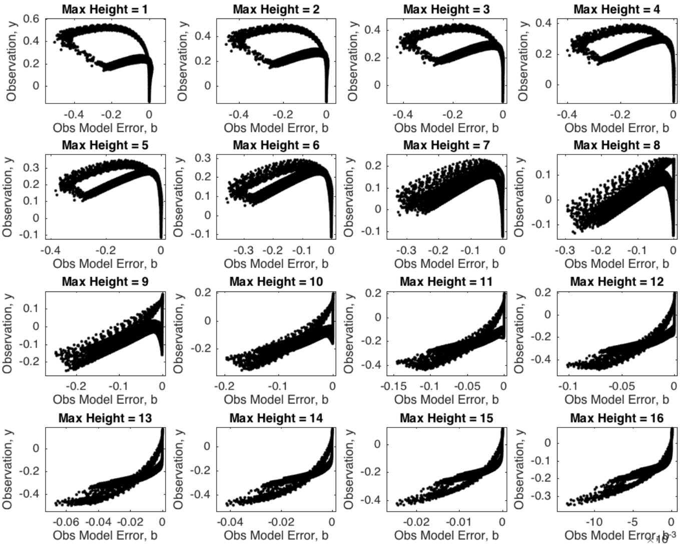
In an extreme case, we consider filtering with a cloud-free RTM:

$$y_\nu = h_\nu(x, 0) + b_\nu$$

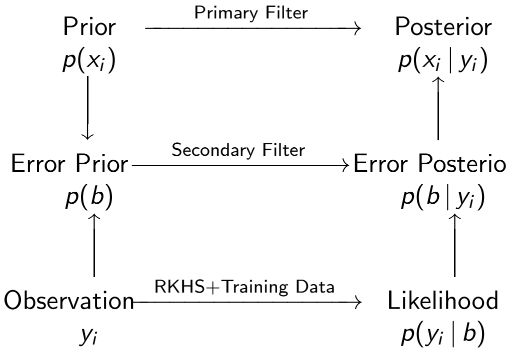
where $b_\nu = h_\nu(x, f) - h_\nu(x, 0) + \eta$ is model error with bias.

Observations (y_ν) v Model error (b_ν)



State estimation of the model error

We propose a secondary filter to estimate the statistics for b_i as follows:

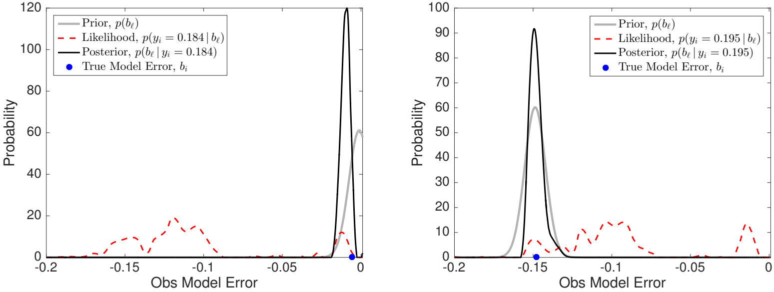


A machine learning technique, kernel embedding of conditional distribution[2], is employed to train a nonparametric likelihood function.

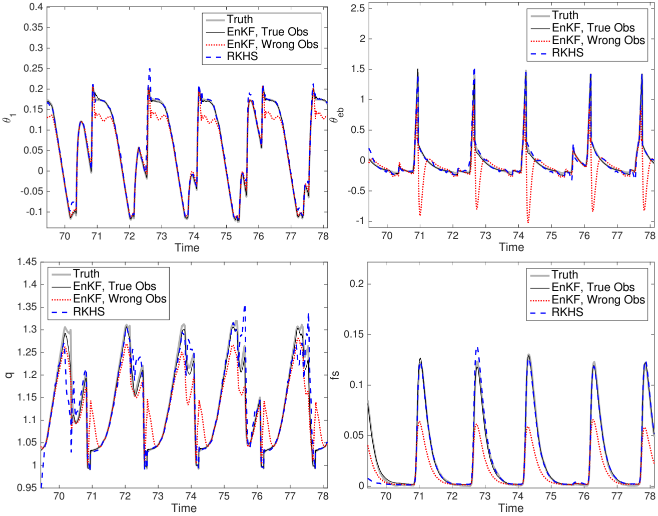
[2] Song, Fukumizu, Gretton, 2013.

Secondary Bayesian filter

$$p(b|y_i) \propto p(b)p(y_i|b)$$

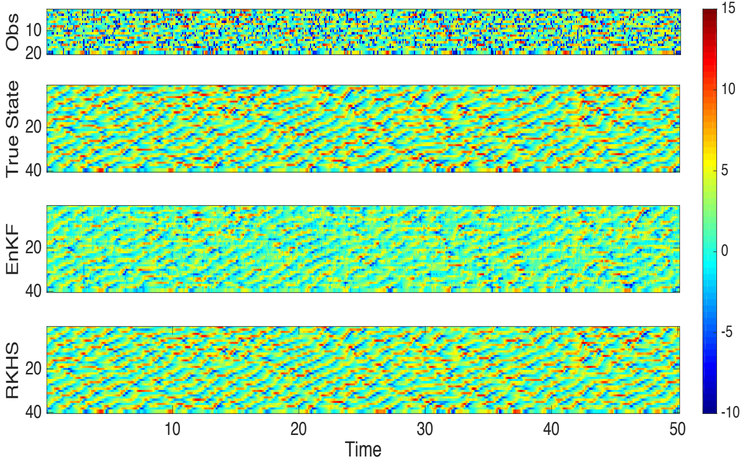


Filter estimates (with adaptive tuning of R and Q).



Example: Lorenz-96

Biased occurs random in space and times.



Nonparametric likelihood functions

We will use the kernel embedding of conditional distribution.[3]

Let X be a r.v on $\mathcal{M}$ and distribution $P(X)$. Given a kernel $K : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$, the Moore-Aronszajn theorem states that there exists a Reproducing Kernel Hilbert Space (RKHS) $L^2(\mathcal{M}, q)$.

This means that we can evaluate any function $f \in L^2(\mathcal{M}, q)$ as follows:

$$f(x) = \langle f, K(x, \cdot) \rangle_q.$$

[3] Song, Fukumizu, Gretton, 2013.

Nonparametric likelihood functions

The kernel embedding of conditional distribution $P(Y|B)$ is defined as,

$$\mu_{Y|b} = \mathbb{E}_{Y|b}[\tilde{K}(Y,\cdot)] = \int_{\mathcal{N}} \tilde{K}(y,\cdot)dP(y|b).$$

Given $g \in L^2(\mathcal{N}, \tilde{q})$,

$$\begin{aligned} \mathbb{E}_{Y|b}[g(Y)] &= \int_{\mathcal{N}} g(y)dP(y|b) = \int_{\mathcal{N}} \langle g, \tilde{K}(y,\cdot)\rangle_{\tilde{q}} dP(y|b) \\ &= \Big\langle g, \int_{\mathcal{N}} \tilde{K}(y,\cdot)dP(y|b)\Big\rangle_{\tilde{q}} = \langle g, \mu_{Y|b}\rangle_{\tilde{q}}. \end{aligned}$$

One can verify that

$$\mu_{Y|b} = q\mathcal{C}_{YB}\mathcal{C}_{BB}^{-1}K(b,\cdot),$$

where

$$\mathcal{C}_{BY} = \int_{\mathcal{M}\times\mathcal{N}} K(b,\cdot)\otimes\tilde{K}(y,\cdot)\, dP(b,y)$$

is the kernel embedding of $P(B,Y)$ on appropriate Hilbert spaces.

Data-driven nonparametric likelihood functions

Given $\{b_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$, apply the *diffusion maps*[4] to learn the data-driven orthonormal basis functions $\varphi_j(b) \in L^2(\mathcal{M}, q)$ and $\tilde{\varphi}_k(y) \in L^2(\mathcal{M}, \tilde{q})$. Let

$$p(y|b) = \sum_k \mu_{Y|b,k} \tilde{\varphi}_k(y) \tilde{q}(y)$$

where

$$\begin{aligned} \mu_{Y|b,k} &= \langle p(\cdot|b), \tilde{\varphi}_k \rangle = \mathbb{E}_{Y|b}[\tilde{\varphi}_k] = \langle \mu_{Y|b}, \tilde{\varphi}_k \rangle_{\tilde{q}} \\ &= \langle q \mathcal{C}_{YB} \mathcal{C}_{BB}^{-1} K(b, \cdot), \tilde{\varphi}_k \rangle_{\tilde{q}} \\ &= \ldots \\ &= \sum_j \varphi_j(x) [C_{YB} C_{BB}^{-1}]_{kj} \end{aligned}$$

where

$$[C_{YB}]_{jk} = \langle \mathcal{C}_{YB}, \tilde{\varphi}_j \otimes \varphi_k \rangle_{\tilde{q} \otimes q} \approx \frac{1}{N} \sum_{i=1}^N \tilde{\varphi}_j(y_i) \varphi_k(b_i),$$

$$[C_{BB}]_{jk} = \langle \mathcal{C}_{BB}, \varphi_j \otimes \varphi_k \rangle_q \approx \frac{1}{N} \sum_{i=1}^N \varphi_j(b_i) \varphi_k(b_i),$$

[4] Coifman & Lafon 2006, Berry & H, 2016.

A manifold learning algorithm: Diffusion Maps[6]

Given $\{x_i\} \in \mathcal{M} \subset \mathbb{R}^n$ with a sampling measure q, the diffusion maps algorithm is a kernel based method that produces orthonormal basis functions on the manifold, $\varphi_k \in L^2(\mathcal{M}, q)$.

[5] Berry & H, 2016

[6] Coifman & Lafon 2006

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These basis functions are solutions of an eigenvalue problem,

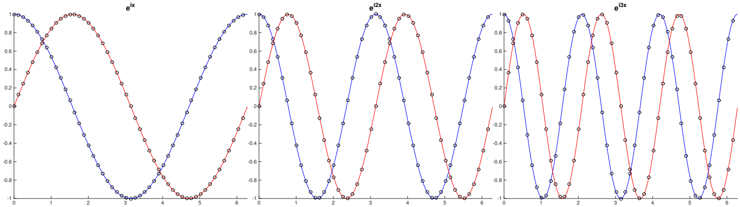
$$q^{-1}\text{div}\Big(q\nabla\varphi_k(x)\Big) = \lambda_k\varphi_k(x),$$

where the weighted Laplacian operator is approximated with an integral operator using a variable bandwidth kernel[5]
with appropriate normalizations.

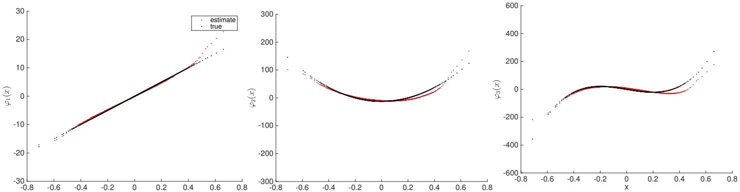
[5] Berry & H, 2016
[6] Coifman & Lafon 2006

Examples:

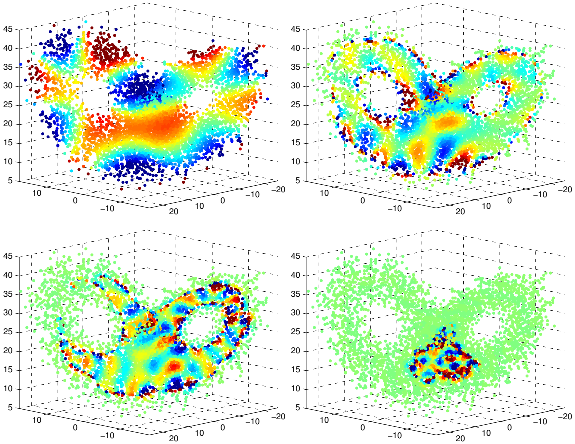
Example: Uniformly distributed data on a circle, we obtain the Fourier basis.



Example: Gaussian distributed data on a real line, we obtain the Hermite polynomials.



Example: Nonparametric basis functions estimated on nontrivial manifold



Remark: Essentially, one can view the DM as a method to learn generalized Fourier basis on the manifold.

References:

1. T. Berry & H, "Correcting biased observation model error in data assimilation", Mon. Wea. Rev. (in press).
2. T. Berry & H, "Variable bandwidth diffusion kernels", Appl. Comput. Harmon. Anal. 40, 68-96, 2016.
3. H, "An introduction to data-driven methods for stochastic modeling of dynamical systems", Springer (submitted for a review).