# A contemporary view of high-dimensional Quasi Monte Carlo


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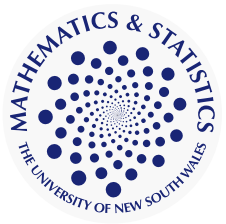
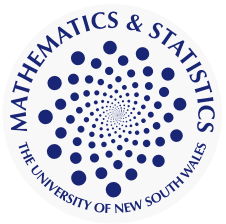


In UQ we often meet integrals with hundreds or thousands of variables.

Let's define the $s$-dimensional integral,

$$I_s(F) := \int_0^1 \cdots \int_0^1 F(y_1, \ldots, y_s) \mathrm{d}y_1 \cdots \mathrm{d}y_s,$$

where $s$ may be large.



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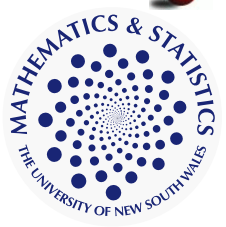
The main classes of methods for $\int_{[0,1]^s} F(\mathbf{y}) \mathbf{dy}$ are:

- Monte Carlo (MC)
- Sparse grids
- Quasi Monte Carlo (QMC)



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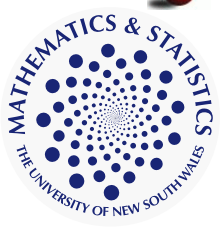


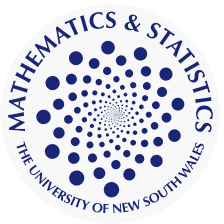

And when $s$ is **very** large, only MC and QMC remain.

# MC and QMC

Both MC and QMC approximate $I_s(F)$ by an equal-weight sum:

$$Q_{N,s}(F) := \frac{1}{N} \sum_{k=1}^{N} F(\mathrm{t}_k),$$

where $\mathrm{t}_1, \ldots, \mathrm{t}_N$ are points in $[0,1]^s$.



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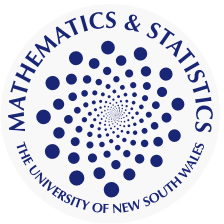

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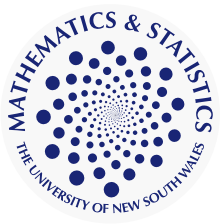

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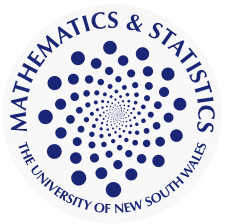
But how to choose good QMC points $\mathbf{t}_1, \ldots, \mathbf{t}_N$?

And how to prove that your choice is good?



# The traditional view e.g. Wikipedia

The famous **Koksma-Hlawka inequality** says,

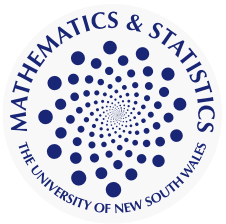
$$|I_s(F) - Q_{N,s}(F)| \leq D_N^*(\mathrm{t}_1, \ldots, \mathrm{t}_N)\ V(F),$$

– $V(F)$ is the **variation of $F$ in the sense of Hardy-Krause**

– $D_N^*$ is the **star discrepancy**,

$$D_N^*(\mathrm{t}_1, \ldots, \mathrm{t}_N) = \sup_B \Big|(\text{fraction of points in} B) - (\text{volume of} B)\Big|.$$

where the supremum is over all rectangular boxes $B \subseteq [0,1]^s$ with bottom left-hand corner at the origin.

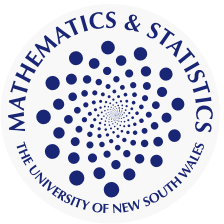


# The traditional view (continued)

"Good" point sets are the first $N$ members of a **low-discrepancy sequence**:

**Definition:** A **low-discrepancy sequence** $\mathbf{t_1, t_2, \ldots}$ (such as the **Sobol** sequence) satisfies

$$D_N^* \leq C_s \frac{(\log N)^s}{N}$$



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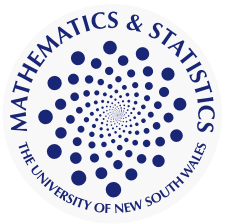
$$\Longrightarrow \quad |I_s(F) - Q_{N,s}(F)| \leq C_s \frac{(\log N)^s}{N} V(F) \, .$$

Cf. $\sqrt{\mathbb{E}[(I_s(F) - Q_{N,s}(F)^{\mathrm{MC}})^2]} \leq \frac{1}{\sqrt{N}} \sqrt{I_s(F^2) - (I_s(F))^2}.$



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This is fine if $s$ is not too large, but if $s$ is large ...

# The KH inequality isn't useful if $s$ is large!

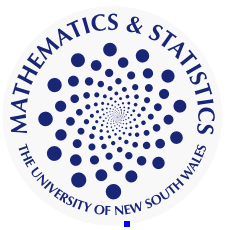
It isn't useful because for fixed $s$ **the bound is increasing with** $N$ – it continues to increase until $N \approx e^s$.

And by the time $N = e^s$ the bound can be truly astronomical:

$$\frac{(\log N)^s}{N} = \left(\frac{s}{e}\right)^s.$$

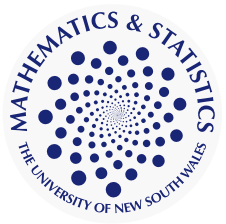
For example, if $s = 100$ then

$$\left(\frac{s}{e}\right)^s \approx 10^{150}.$$



# The contemporary point of view on QMC

is that the dependence on $s$ is as important as the dependence $N$!



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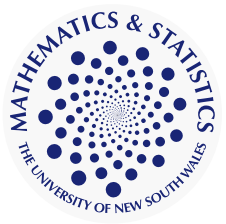
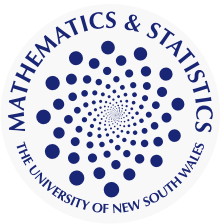
But how should we characterise the dependence on $s$?

How much harder does a problem become as the dimension increases?



# Applications are the key

The dependence on $s$ should be driven by **applications**.



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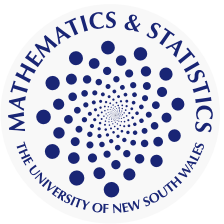
In applications (an example is coming!), we often have a **sequence** of integrands $F_s = F_s(y_1, \ldots, y_s)$, with $s \to \infty$. So we may want a sequence of QMC rules for ever increasing dimension $s$.



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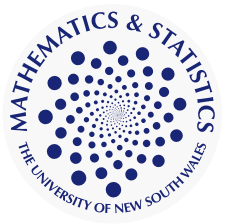
Given an application, how to decide what would be **good** QMC rules?

A convincing way is to **make some error bound small**.



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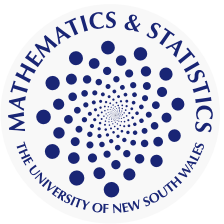


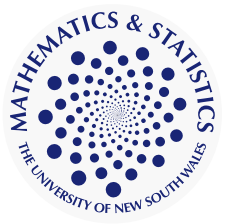
# What sort of error bound would we wish for?

What sort of error bound do we want? An ideal result might look like:

$$\boxed{|I_s(F_s) - Q_{N,s}(F_s)| \leq \frac{C}{N^\kappa}\|F_s\| \leq \frac{C'}{N^\kappa}},$$

where

- $\|\cdot\|$ is some norm that depends on the smoothness of $F_s$,
- $\kappa$ is as large as possible,
- and $C$ and $\|F_s\|$ (and their product $C'$) are independent of $s$.



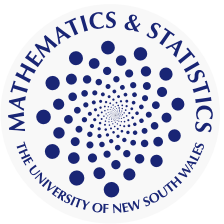
But if we want to have

$$|I_s(F_s) - Q_{N,s}(F_s)| \leq \frac{C}{N^\kappa}\|F_s\|_s \leq \frac{C'}{N^\kappa},$$

with $C, C'$ independent of $s$, then clearly

- we cannot use star discrepancy, because of its $(\log N)^s$ factor
- the norms $\|F_s\|$ must be bounded, so

  **the norm $\|\cdot\|$ MUST depend on the application**
- and we may need an essentially different QMC rule for each $s$.



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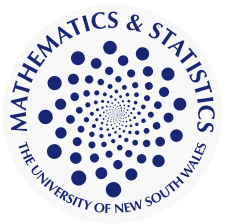
For guidance, let's turn to an important class of problems:

**PDE with random coefficients**.



# PDE with random coefficients

A motivating example: flow through a porous medium

Darcy's law is $\vec{q}(\mathbf{x}) = -a(\mathbf{x})\nabla p(\mathbf{x}),$

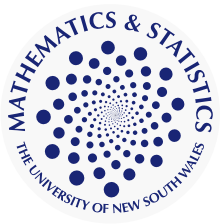
where

$p(\mathbf{x})$ is pressure of the fluid

$\vec{q}(\mathbf{x})$ is velocity of the fluid

$a(\mathbf{x})$ is "permeability" of the medium

Incompressibility: $\nabla \cdot \vec{q} = 0$



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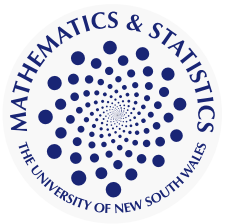
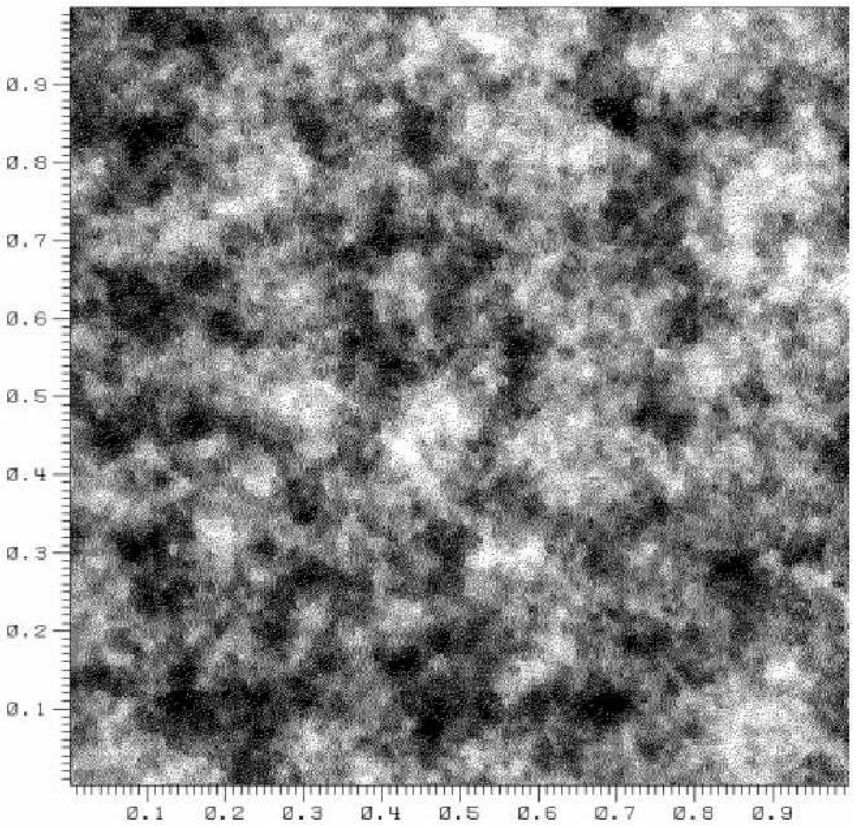
Together these give a second order elliptic PDE:

$$\nabla \cdot (a(\mathbf{x})\nabla p(\mathbf{x})) = 0$$



# Modeling the permeability

Describing all the microscopic pores and channels in a real material is commonly considered much too hard. So it is common engineering practice to model the permeability as a random field:

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# A simple model problem – the "uniform" case

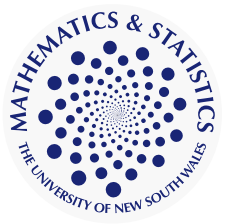
$$-\nabla \cdot (a(\mathrm{x},\mathrm{y})\,\nabla u(\mathrm{x},\mathrm{y})) \;=\; f(\mathrm{x}) \quad \text{for } \mathrm{x} \in D\,,$$

$$u(\mathrm{x},\mathrm{y}) \;=\; 0 \quad \text{on} \quad \partial D, \quad \mathrm{y} \in U := [0,1]^{\mathbb{N}},$$

with $D$ a bounded Lipschitz domain in $\mathbb{R}^d$, with $d = 1, 2,$ or $3$ and

$$a(\mathrm{x},\mathrm{y}) \;=\; \overline{a} + \sum_{j=1}^{\infty} (y_j - \tfrac{1}{2})\,\psi_j(\mathrm{x}), \qquad \mathrm{x} \in D, \quad \mathrm{y} \in U\,,$$

where $y_1, y_2, \ldots$ are parameters representing independent random variables uniformly distributed on $[0, 1]$;



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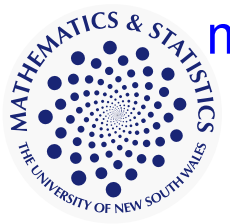
where $y_1, y_2, \ldots$ are parameters representing independent random variables uniformly distributed on $[0,1]$; with $\overline{a}, \psi_j$ such that $\sum_j \|\psi_j\|_\infty < \infty$, and

$$a_{\max} \geq a(\mathrm{x},\mathrm{y}) \geq a_{\min} > 0,$$

making the PDE **strongly elliptic** for every $\mathrm{y}$.



# A simple model problem – the "uniform" case

In practice it is necessary to truncate the infinite sum to $s$ terms:

$$-\nabla \cdot (a(\mathrm{x},\mathrm{y})\,\nabla u(\mathrm{x},\mathrm{y})) \;=\; f(\mathrm{x}) \quad \text{in} \quad D\,,$$

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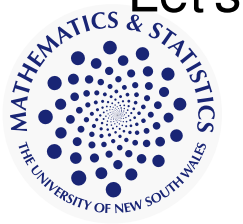
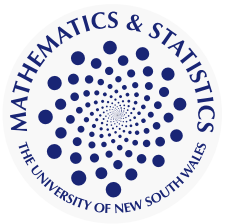
where $y_1, y_2, \ldots, y_s$ are independent random variables uniformly distributed on $[0,1]$.

Let's call the solution of the truncated system $u_s$.



# What might we want to compute?

- The mean pressure at a particular point or over a particular small region
- The effective permeability
- The mean “breakthrough time”
- . . .



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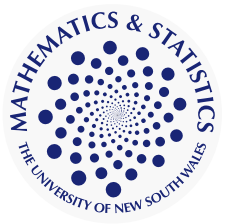
**All are expected values – and expected values are integrals.**

**And since there are $s$ uniform random variables, the expected value is an integral over $[0, 1]^s$.** And it is easy for $s$ to be large!



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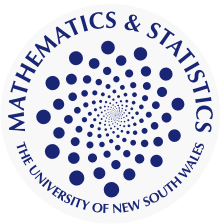
where $G$ is the **mean pressure $\overline{u}$ over some specified subregion** or some other linear functional.



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The expected value is then an infinite-dimensional **integral**

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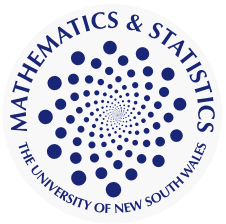


Suppose the problem is to compute the expected value of

$$F(\mathrm{y}) := G(u(\cdot;\mathrm{y})),$$

where $G$ is the **mean pressure $\overline{u}$ over some specified subregion** or some other linear functional.

The expected value is then an infinite-dimensional **integral**

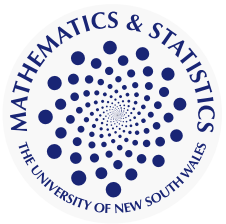
$$I(F) := \int_{[0,1]^{\mathbb{N}}} F(\mathrm{y})\mathrm{d}\mathrm{y}.$$

This is now approximated by

$$I_s(F_s) = \int_{[0,1]^s} F_s(y_1,\ldots,y_s)\mathrm{d}y_1\ldots\mathrm{d}y_s,$$

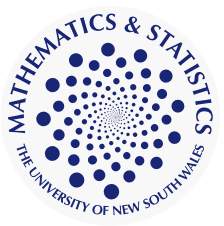
where

$$F_s(y_1,\ldots,y_s) := G(u_s(\cdot;y_1,\ldots,y_s,\tfrac{1}{2},\tfrac{1}{2},\cdots)),$$



# A bound of the desired shape

For this random PDE problem a recent paper [Kuo/Schwab/IHS, SINUM 2012] obtained a first error bound of the desired form:



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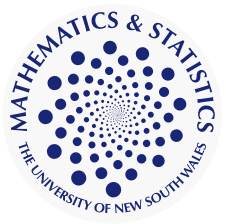
**Theorem** [KSS12] For all $\delta > 0$,

$$\boxed{|I_s(F_s) - Q_{N,s}(F_s)| \leq \frac{C_\delta}{N^{1-\delta}} \|F_s\| \leq \frac{C'_\delta}{N^{1-\delta}},}$$

with $C_\delta$ independent of $s$; AND $\sup_s \|F_s\| < \infty$

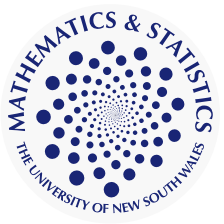
**if** $\sum_{j=1}^{\infty} \|\psi_j\|_\infty^{2/3} < \infty$ and other technical conditions.



# So what was the norm?

So what was the norm $\|\boldsymbol{F_s}\|$ in [KSS12]?

Later!



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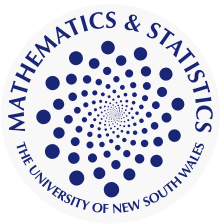

# So what was the norm?

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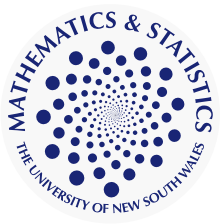
And what was the QMC rule in [KSS12]?

The QMC rule in that work was a **lattice rule** (more precisely, a **randomly shifted lattice rule**).

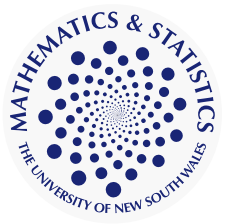


# Lattice rules

Lattice rules were introduced by number theorists in the late 1950s and 1960s, especially:

**Korobov, Hlawka, Zaremba, L. K. Hua**

They were designed for **periodic** functions.



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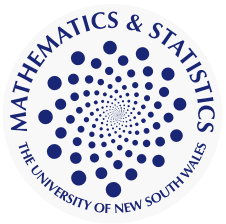
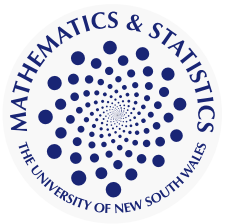
Most integrands are **not** periodic, but lattice rules can still be useful.



The simplest kind of lattice rule has the form

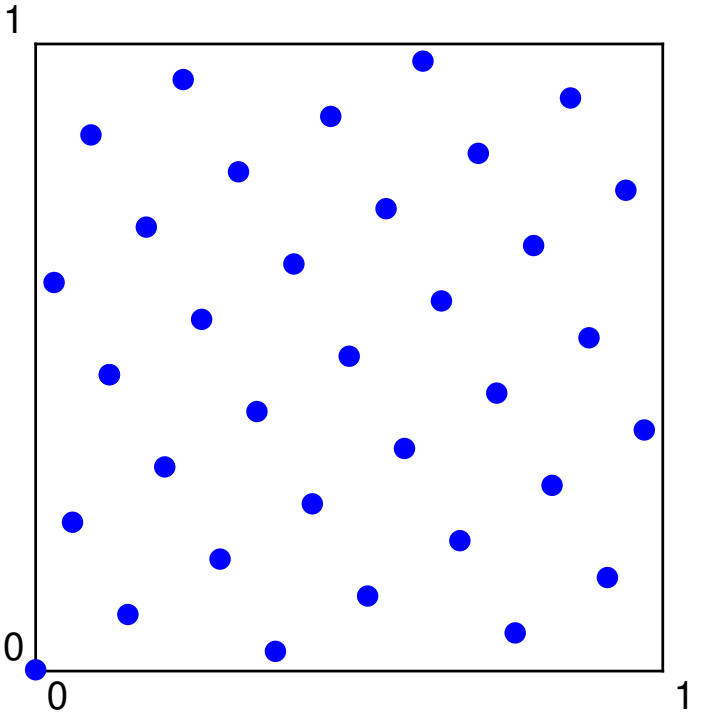
$$Q_{N,s}(\mathbf{z}; F) = \frac{1}{N} \sum_{k=0}^{N-1} F\left(\left\{k\frac{\mathbf{z}}{N}\right\}\right),$$

where $\mathbf{z} \in \{1, \ldots, N-1\}^s$, and the braces mean that each component of the $s$-vector in the braces is to be replaced by its fractional part.



# Example of a (good) 2-D lattice rule

$$s = 2, N = 34, \mathbf{z} = (1, 21)$$

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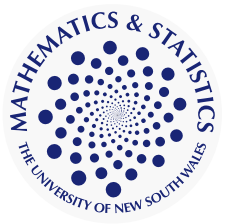
For the lattice rule

$$Q_{N,s}(\mathbf{z}; F) = \frac{1}{N} \sum_{k=0}^{N-1} F\left(\left\{k\frac{\mathbf{z}}{N}\right\}\right),$$

the corresponding **shifted lattice rule** is

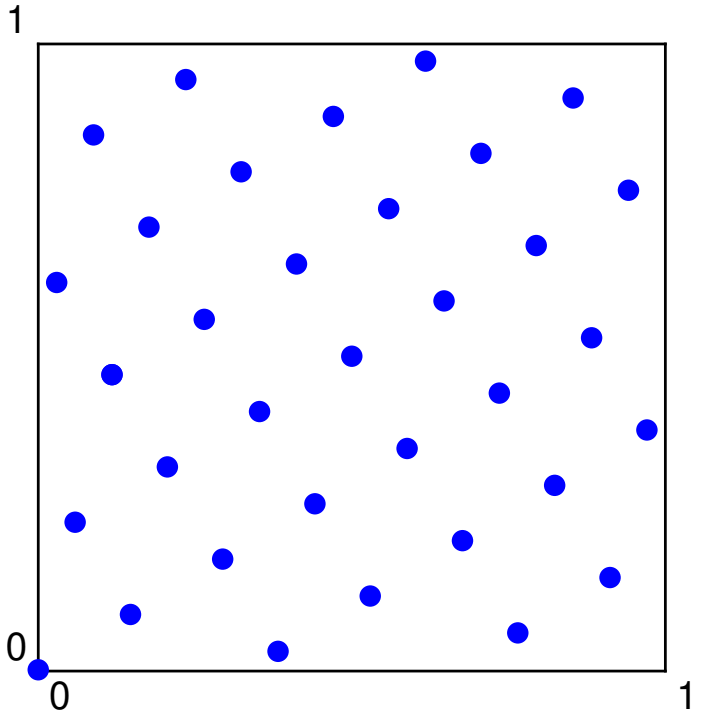
$$Q_{N,s}(\mathbf{z}, \boldsymbol{\Delta}; F) = \frac{1}{N} \sum_{k=0}^{N-1} F\left(\left\{k\frac{\mathbf{z}}{N} + \boldsymbol{\Delta}\right\}\right),$$

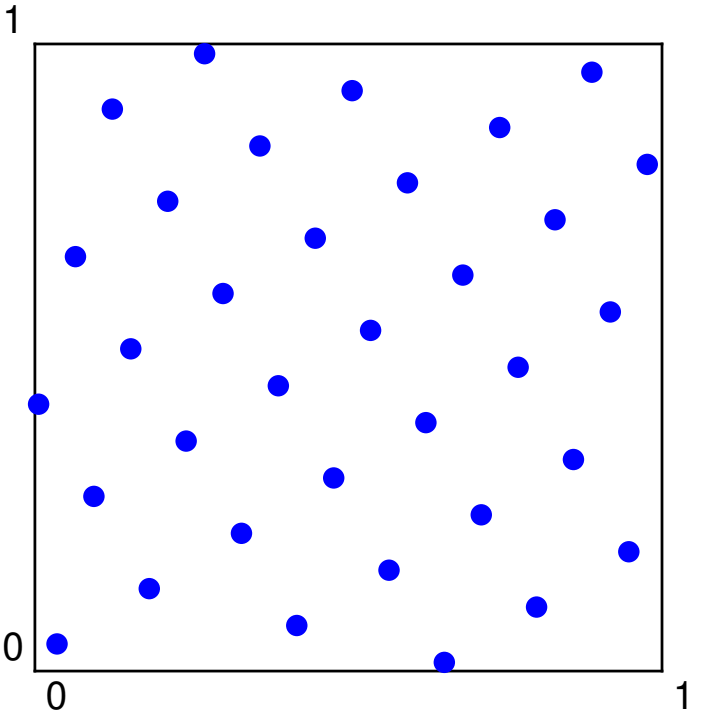
where $\boldsymbol{\Delta} \in [0,1)^s$ is the " **shift**".



# Lattice rule & shifted lattice rule

$N = 34, \mathbf{z} = (1, 21)$ $N = 34, \mathbf{z} = (1, 21), \boldsymbol{\Delta} = (0.8, 0.1)$

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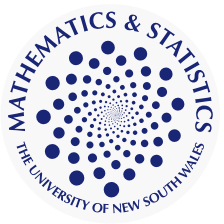
# Randomly shifted lattice rule

In a **randomly shifted** lattice rule $\boldsymbol{\Delta}$ is a **random** vector, uniformly distributed over $[0,1]^s$. In principle we now compute **expected values**.



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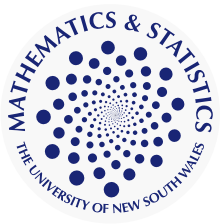

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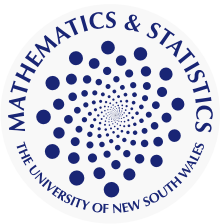
In a randomly shifted lattice rule the only thing to decide is how to choose $\mathbf{z}$!

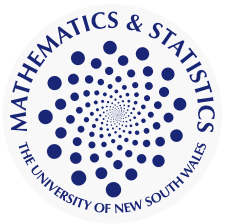


# The true error bound

The true error bound found in [KSS12] (no fake news here!) was for a randomly shifted lattice rule with a special choice of $\mathbf{z}$, and was a bound on the **root mean square expected error averaged over shifts**:

For all $\delta < \infty$,

$$\sqrt{\mathbb{E}_{\boldsymbol{\Delta}}[|I_s(F_s) - Q_{N,s}(\mathbf{z}, \boldsymbol{\Delta}; F)|^2]} \leq \frac{C_\delta}{N^{1-\delta}} \|F_s\| \leq \frac{C'_\delta}{N^{1-\delta}},$$



# How to choose $\mathbf{z}$?

Recall: the lattice rule for the integral over $[0,1]^s$ is

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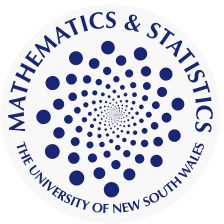
So we need to choose $\mathbf{z}$. However, **there is no known formula for a "good" $\mathbf{z}$, beyond $s = 2$.**



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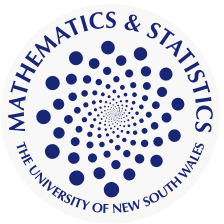

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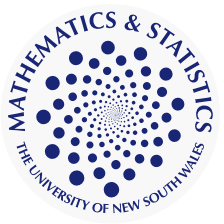
We choose a convenient Hilbert space $H$ to which $F$ belongs.



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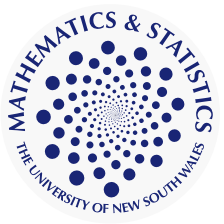
Then choose $\mathbf{z}$ to make the **worst-case error** for that space small.

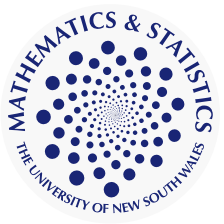


# Worst-case error

**Definition**: The **worst-case error** in the space $H$ of a given QMC rule $Q_{N,s}(\mathbf{z};\cdot)$ is

$$\boxed{e_{N,s,\mathbf{z}}(H) := \sup_{\|F\|_H \le 1} |I_s(F) - Q_{N,s}(\mathbf{z};F)|},$$

i.e. it is the largest error of $Q_{N,s}(\mathbf{z};F)$ for $F$ in the unit ball of $H$.



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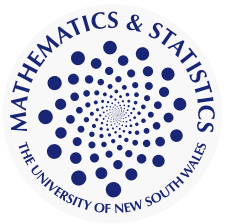
But in practice we are given $F$, not $H$. For a given $F$ we deduce

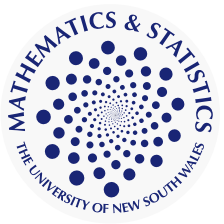
$$|I_s(F) - Q_{N,s}(\mathbf{z};F)| \le e_{N,s,\mathbf{z}}(H)\|F\|_H.$$

The **shift averaged worst-case error** has a similar error bound.



# So how to choose the function space $H$?

- We choose $H$ to be a Hilbert space of smooth enough functions, for which the worst-case error is easily computable.



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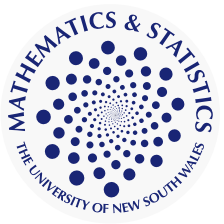

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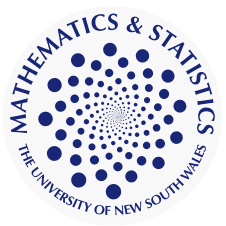
Specifically, we take $H$ to be a space of functions with **square-integrable mixed first derivatives**, and the following norm:



# The space $H$ for the uniform case

The norm squared in $H$ is

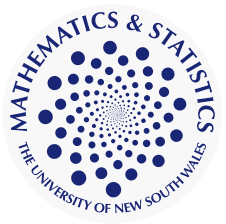
$$\|F\|_{s,\boldsymbol{\gamma}}^2 := \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|} F}{\partial \mathbf{y}_{\mathfrak{u}}}(\mathbf{y}_{\mathfrak{u}}; \tfrac{\mathbf{1}}{\mathbf{2}}) \right|^2 \mathrm{d}\mathbf{y}_{\mathfrak{u}},$$

where

$$(\mathbf{y}_{\mathfrak{u}}; \tfrac{\mathbf{1}}{\mathbf{2}})_j = \begin{cases} y_j \text{ if } j \in \mathfrak{u}, \\ \frac{1}{2} \text{ if } j \notin \mathfrak{u}. \end{cases}$$

For example, for $\mathfrak{u} = \{1, 3\}$ the corresponding term is

$$\frac{1}{\gamma_{1.3}} \int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial y_1 \partial y_3}(y_1, \tfrac{1}{2}, y_3, \tfrac{1}{2}, \tfrac{1}{2} \ldots) \right|^2 \mathrm{d}y_1 \mathrm{d}y_3.$$



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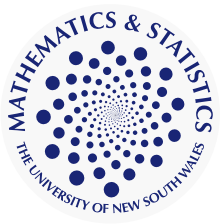
The $\gamma_{\mathfrak{u}}$ are **weights** - positive numbers chosen to fit the problem.

# Weights

Weights are in most applications **essential**. They play a big role in the contemporary QMC story.

Nowadays there are many kinds of weights ("order-dependent weights", "finite-order weights", "POD" weights, "SPOD weights", ...).



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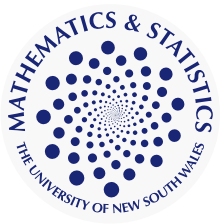
But the original weights, the simplest to handle, and still very important, are PRODUCT WEIGHTS.

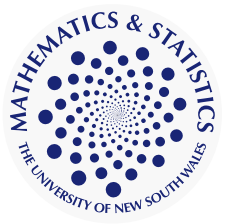


# Product weights

**Product weights** take the form

$$\gamma_{\boldsymbol{u}} = \prod_{j \in \boldsymbol{u}} \alpha_j, \quad \boldsymbol{u} \subseteq \{1, \ldots, s\},$$

where $\alpha_1 \geq \alpha_2 \geq \ldots > 0$ (also called weights!) are positive numbers chosen to quantify the decreasing importance of successive variables.

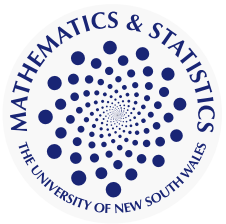


# Product weights are nice!

For product weights $\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \alpha_j$, it has been known for $\mathbf{20}$ years that:

**Theorem** (IHS/Woźniakowski 98). There exist QMC points for which the worst case error is bounded independently of $s$ **iff**

$$\sum_{j=1}^{\infty} \alpha_j < \infty.$$



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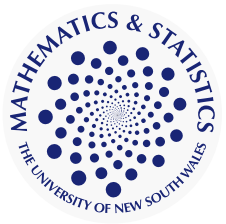
$$\sum_{j=1}^{\infty} \alpha_j < \infty.$$

This is satisfied, for example, by $\alpha_j = 1/j^2$.

It is not satisfied by $\alpha_j = 1$.



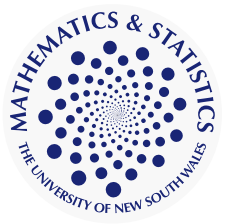
# The convergence rate can be $N^{-1+\delta}$

**Theorem** (IHS/Woźniakowski 01). If $N$ is prime, and if

$$\sum_{j=1}^{\infty} \alpha_j^{1/2} < \infty,$$

then for each $s$ there exists a **shifted lattice rule** $Q^{\text{lattice}}_{N,s,\mathbf{z},\Delta}$ such that

$$e_{N,s,\gamma}(Q^{\text{lattice}}_{N,s,\mathbf{z},\Delta}) \le \frac{C_{\gamma,\delta}}{N^{1-\delta}} \qquad \forall \delta > 0.$$



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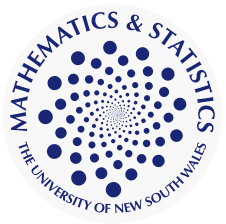
The original proofs of both theorems were non-constructive! They were existence proofs only. (But now ...)



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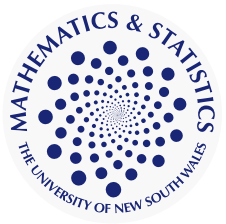
So now to choose $\mathbf{z}$:

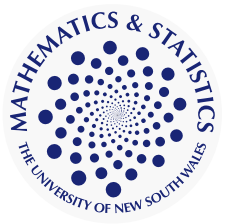


# Finding a good $z$ for product weights

For product weights the shift-averaged WCE of a lattice rule is easily computed:

$$e_{N,s,\mathbf{z}}(H)^2 = \frac{1}{N}\sum_{k=0}^{N-1}\prod_{j=1}^{s}\left(1+\alpha_j\left[B_2\left(\left\{\frac{kz_j}{N}\right\}\right)+\frac{1}{12}\right]\right)-\prod_{j=1}^{s}\left(1+\frac{\alpha_j}{12}\right),$$

where $B_2(x) := x^2 - x + 1/6$.



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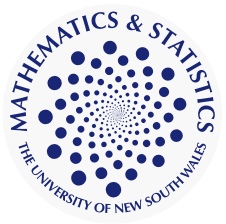
So we can compute the WCE for all possible choices of $\mathbf{z}$, and choose the best. Problem solved? No! For an exhaustive search would take $(N-1)^s$ evaluations of the worst-case error.



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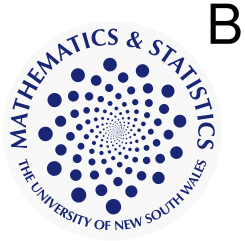
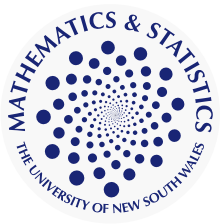
But there is a way: the **CBC construction**.



# CBC construction of a good $\mathbf{z}$

In the component-by-component (or CBC) algorithm for product weights (Korobov 1950s, IHS/Kuo/Joe 2002), a good generator $\mathbf{z} = (z_1, \ldots, z_s)$ is constructed one component at a time:



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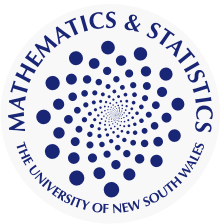
- choose $z_1 = 1$
- choose $z_2$ to minimise $e_{N,2}(z_1, z_2)$, then
- choose $z_3$ to minimise $e_{N,3}(z_1, z_2, z_3)$, then
- ...



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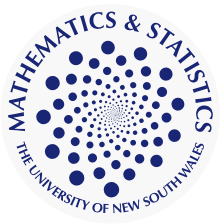
- ...

so that at each step there are only (at most) $N - 1$ choices.

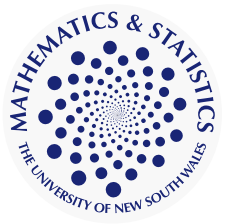
A naive implementation costs $O(s^2 N^2)$ operations.



# Fast CBC for product weights weights and shift-averaged WCEs

For product weights Cools and Nuyens (2006) developed **Fast CBC** – requires a time of order only $O(s\,N\log N)$.

The Nuyens and Cools implementation allows the CBC algorithm for product weights to be run with $s$ in thousands, $N$ in millions.



# The CBC algorithm has optimal convergence!

THEOREM Frances Kuo, J. Complexity, (2003)

*Let $N$ be prime, and let $z_1, z_2, \ldots, z_s$ be chosen by the CBC algorithm. Assume product weights, with*

$$\sum_{j=1}^{\infty} \alpha_j^{1/2} < \infty.$$

*Then* $\forall \delta > 0$

$$e_{N,s,\gamma}(\mathbf{z}) \leq \frac{C_{\gamma,\delta}}{N^{1-\delta}}.$$

Thus the optimal rate is achieved by the CBC algorithm!

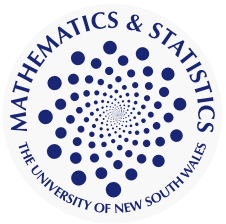
# All that remains is to choose the weights $\gamma_u$

Recall that we can bound the error in terms of the **worst case error**:

$$|I_s(F) - Q_{N,s}(\mathbf{z}; F)| \leq e_{N,s,\mathbf{z},\gamma} \|F\|_{s,\gamma},$$

or more properly,

$$\sqrt{\mathbb{E}_{\boldsymbol{\Delta}}[|I_s(F) - Q_{N,s}(\mathbf{z}, \boldsymbol{\Delta}; F)|^2]} \leq e_{N,s,\mathbf{z},\gamma} \|F\|_{s,\gamma},$$



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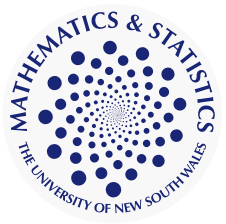
**Both factors on the right depend on the weights $\gamma_u$.**

**In [KSS12] we chose weights to minimise the product.**

Or rather, to minimise the product of known upper bounds on the two factors.



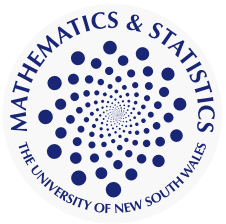
# The best weights weren't product weights!

The weights found in [KSS12] by the minimising strategy above turned out to be of the form

$$\gamma_{\mathfrak{u}} = (|\mathfrak{u}|!)^{(\text{known power bigger than 1})} \prod_{j=1}^{s} \alpha_j,$$

with known exponent, and known $\alpha_j$.

Such weights are called **POD weights** (for "product and order dependent").



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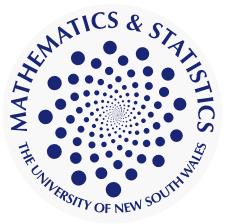
And fast CBC also exists for POD weights (Kuo/Schwab/IHS 11).

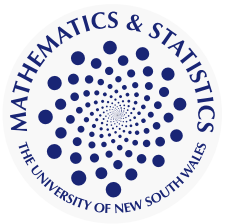


# There exist many extensions

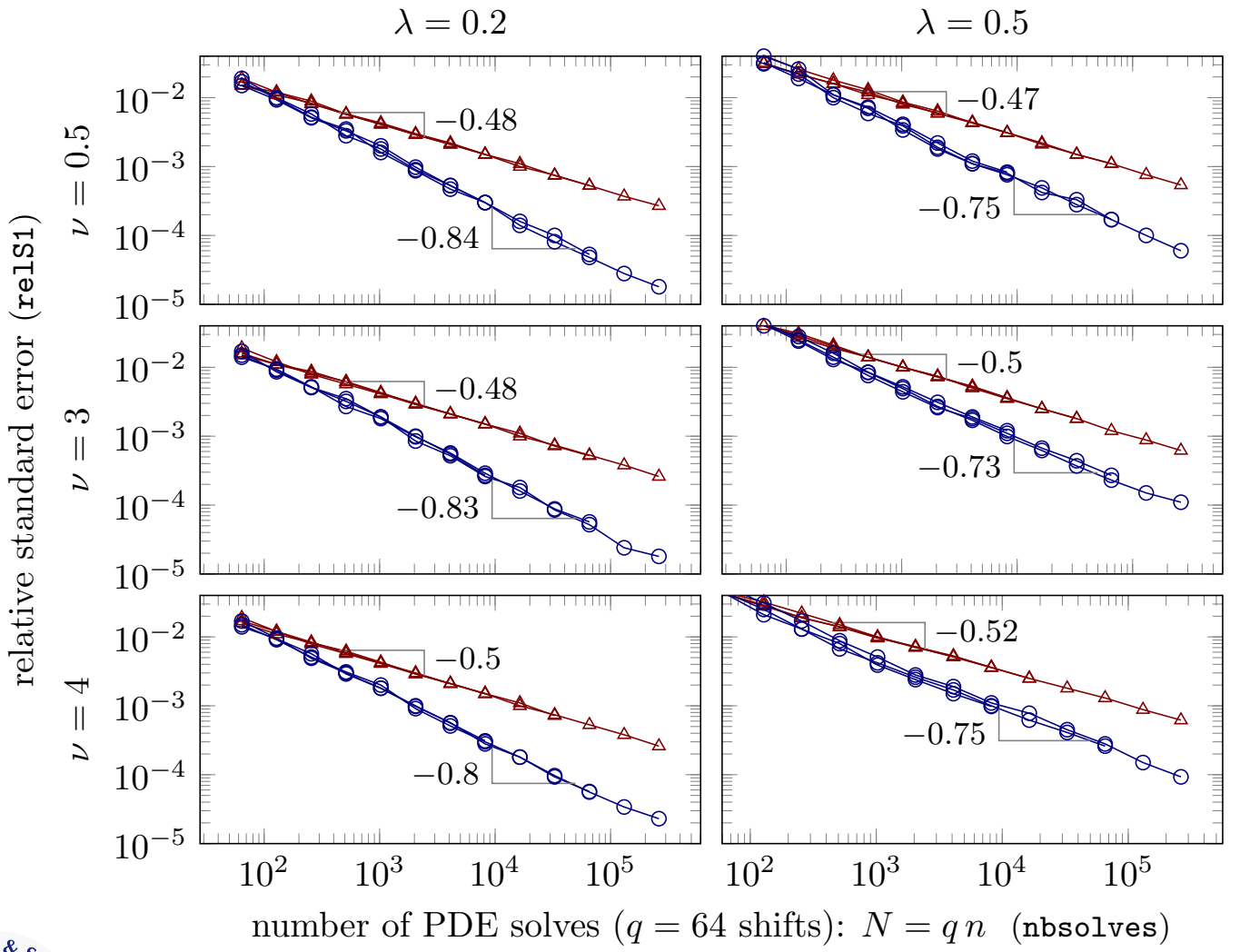
- Similar analysis of the **lognormal** case (i.e $\log a(x)$ is a Gaussian random field). (This is more difficult.)
- Analysis of **multilevel** QMC for both uniform and lognormal cases.
- **Higher order** QMC rules (Josef Dick): e.g. for the uniform case we now have

$$|I_s(F) - Q_{N,s}(F)| \le \frac{C}{N^2}\|F\|_s \le \frac{C'}{N^2}.$$

Lattice rules are now replaced by **interlaced polynomial lattice rules**, and POD weights by **SPOD** weights (standing for "smoothness-driven product and order-dependent weights"). And now the norm needs mixed **second** derivatives; and so on, to arbitrary order.



# QMC in practice

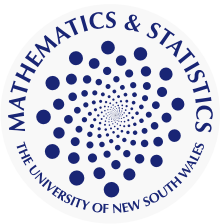



Graham/Kuo/Nuyens/Scheichl/IHS, Numer.Math. 2018

# Summarising the contemporary view of QMC:

- dependence on $s$ is as important as dependence on $N$
- applications should drive the study of dimension dependence
- that we should try to construct methods with provably good error bounds that are independent of dimension

  (or grow only slowly with dimension)



# Surveys and software

*Surveys*:

J. Dick, F. Kuo and I. Sloan, *High dimensional integration – the Quasi-Monte Carlo Way*, Acta Numerica **22** (2013).

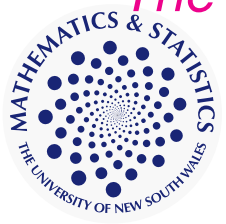
F. Kuo and D Nuyens, *Application of quasi-Monte Carlo methods to PDEs with random coefficients - a survey of analysis and implementation*, Found. Comput. Math. **16** (2016).

*The qmc4pde software from Leuven*:

http://people.cs.kuleuven.be/ dirk.nuyens/qmc4pde

*The gMLQMC software from ETH*:

http://www.sam.math.ethz.ch/HOQMC/gMLQMC



# Contributors to the contemporary view of QMC

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Andrei Reztsov
Rob Scheichl, Bath
Christoph Schwab, ETH
Elisabeth Ullmann, Hamburg
Grzegorz Wasilkowski, Kentucky
Henryk Woźniakowski, Warsaw and Columbia

And apologies to those I missed!