UNIVERSITY OF WISCONSIN–MADISON
Department of Mathematics

FINITE-TIME BRAIDING EXPONENTS

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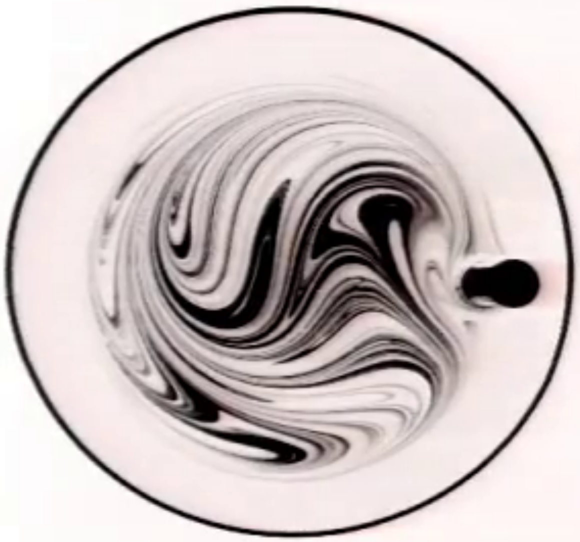
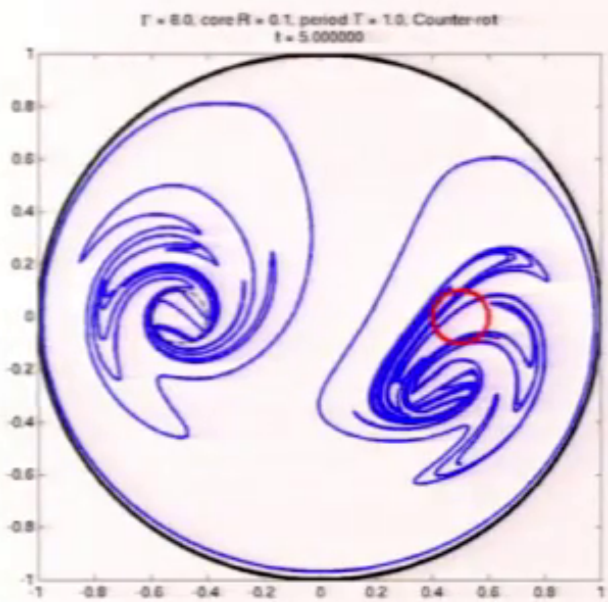
Complexity of the material transport



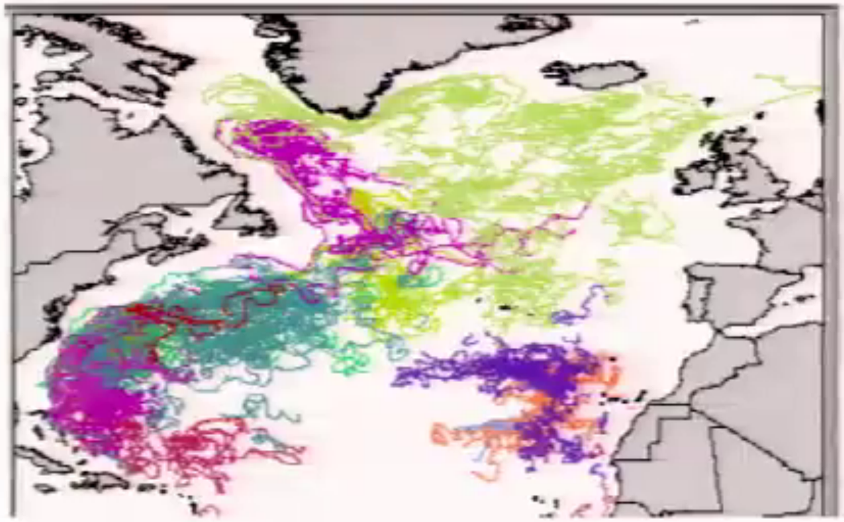
Complexity of the flow – rate of growth of material interfaces. Chaotic advection increases the rate of diffusion and/or reactions.

Newhouse, Pignataro (1993):
Fastest growth of material line is given by the topological entropy of the flow.



Growth of material lines is computed by front tracking – requires velocity fields and delicate numerical algorithms.

Sparse Lagrangian Sampling



(WHOI)

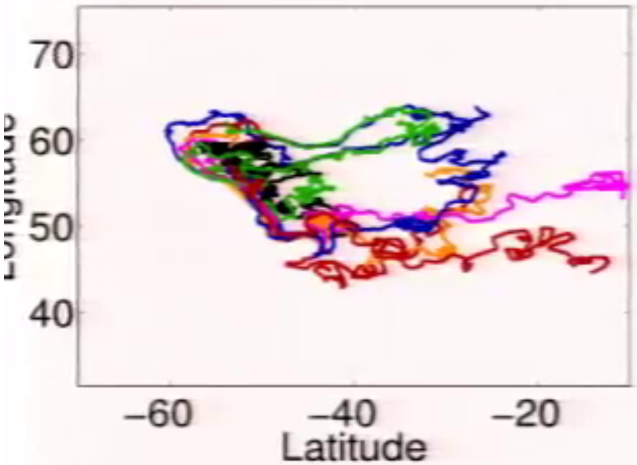
- What if we have no access to velocity fields?
- **Lagrangian trajectories** the only data available: ocean drifters, granular flows, crowds
- Fast characterization of mixing:
  - no need to re-seed tracer
  - computation is cheaper

(Rocky DEM)

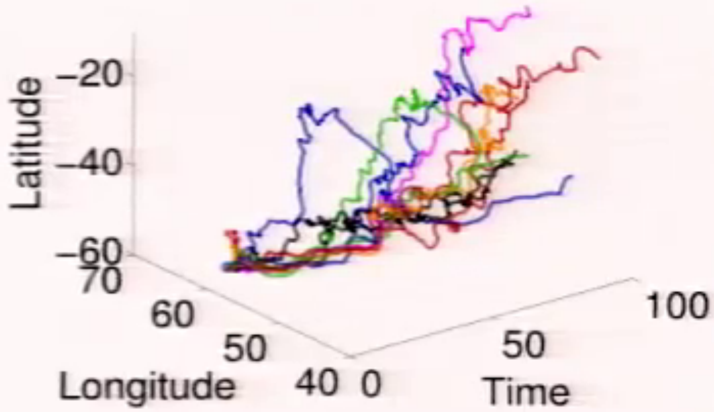
(Wilderness Films India Ltd.)

Trajectories are represented by braids.

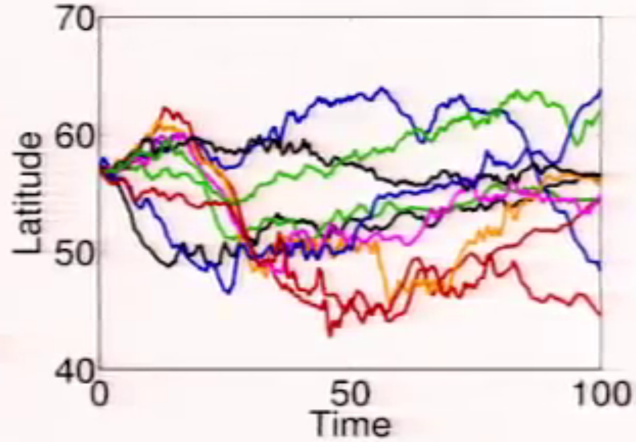
"Spaghetti plot" of **planar** trajectories



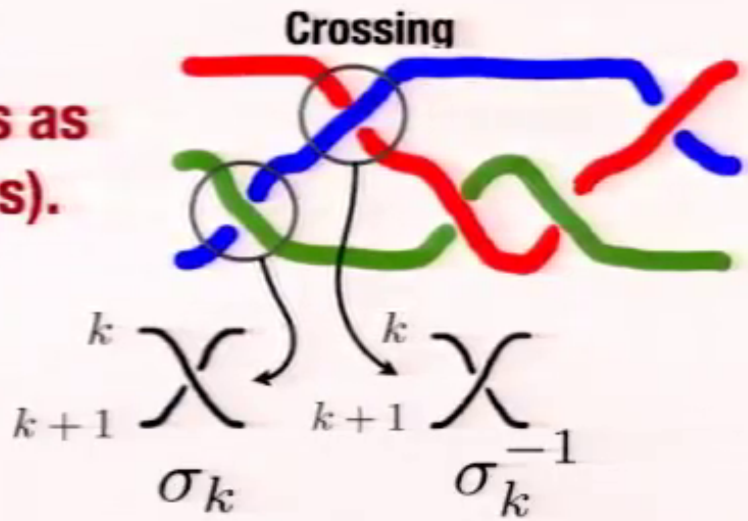
Trajectories in space-time, **"physical braid"**

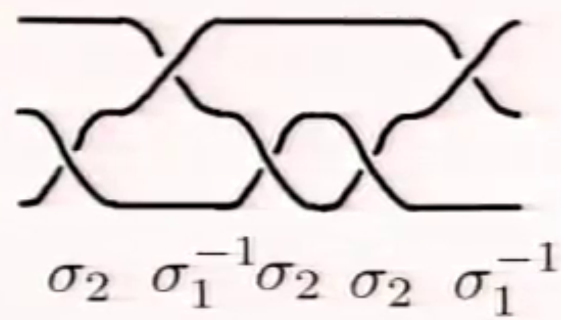


Project onto a plane and monitor exchanges



Braid encodes trajectory crossings as symbols (generators).



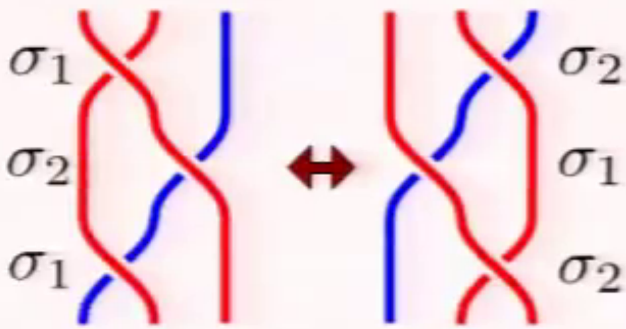


Topology retained, geometry discarded.

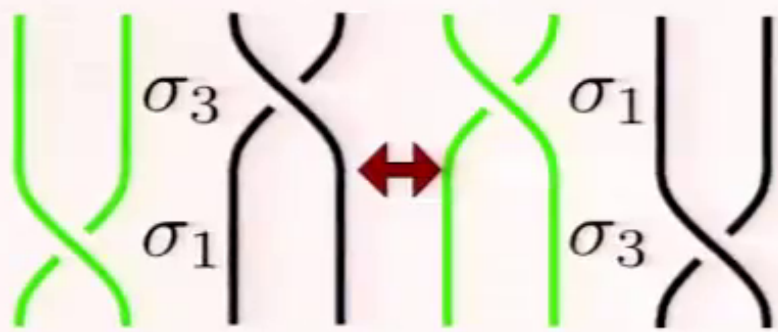
Braids of punctured disks in dynamical systems

- **More generally: braids are "labels" for flow maps (homeomorphisms)**
- **We use Artin generators: easy to deduce from data (other generator sets are also useful)**
- **braids form a non-commutative group with additional relations**

Braid relation



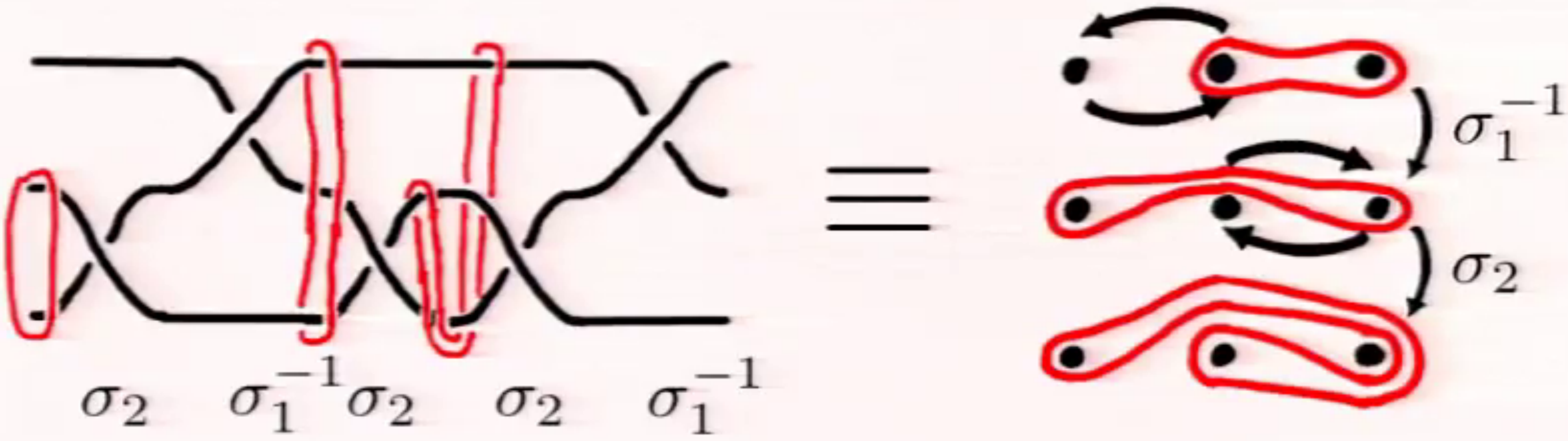
Partial commutativity



Takeaway: Number of generators is not a good measure of complexity.

$$\sigma_1\sigma_1^{-1}\sigma_1\sigma_1^{-1}\cdots = e$$

Advected material is represented by loops – “rubber bands”.



	Full model	Reduced model
Dynamics	Non-autonomous ODEs	Braids of N trajectories
Material	Detailed curves	Loops pulled tight
Rate of mixing	Material line growth	Braid entropy?
Advection	Front/interface tracking	P/w linear maps on $\mathbb{R}^{2N-4}$

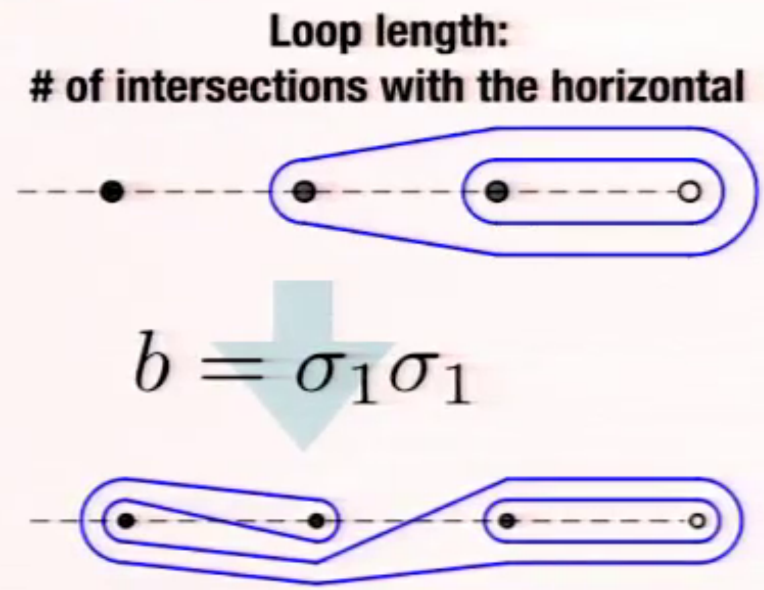
[Dynnikov, 2002]
[Hall, Yurtas, 2009]

Topological entropy of a braid (braid entropy)

Topological entropy:
exponential rate of loop growth
under iterated braid.

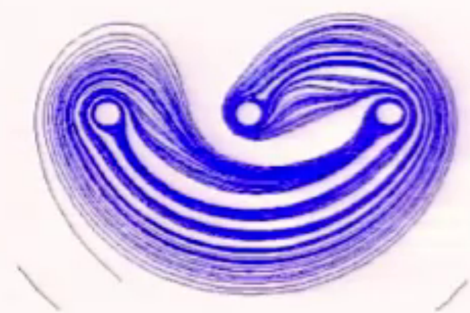
$$|\ell|, |b\ell|, \ldots, |b^n\ell| \sim e^{h(b)n}$$

$$h(b) = \lim_{n\to\infty} \frac{1}{n} \log |b^n \ell|$$



Braid t. entropy ≤ Flow t. entropy
independent of the choice of trajectories

Specifying stirrer trajectories according to a high-entropy braid, forces the increase in flow entropy.
(Aref, Boyland, Finn, Stremler, Thiffeault,…)



Finite-Time Braiding Exponent

$$FTBE = \frac{1}{T} \log(|b_T \ell| / |\ell|)$$

↖ Duration of recorded data instead of iterate no.

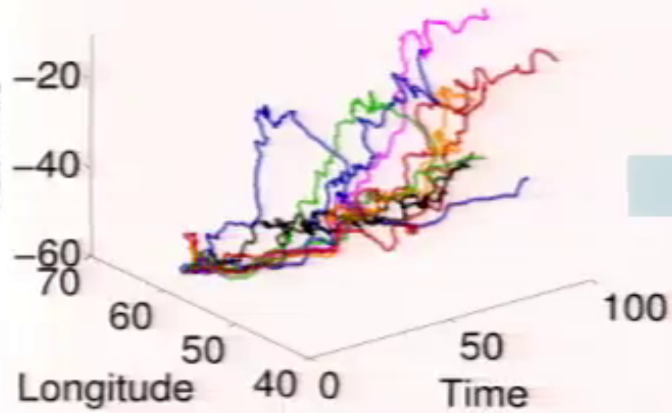
What's in a name?

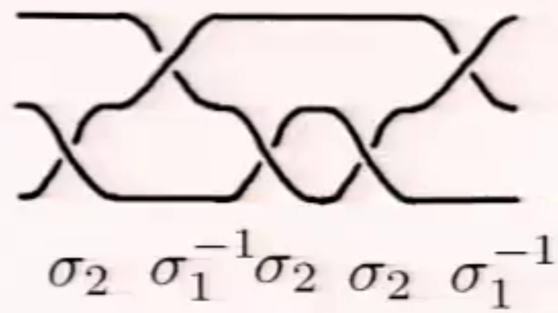
FTBE to braid entropy is parallel to FTLE relative to Lyapunov exponents.

Why not stick with braid entropy?

- iteration of a braid justified only for periodic trajectories
- note: avoiding iteration removes the need for braid "closure"
- coarse: non-exponential regions all assigned zero braid entropy

Dependence of FTBE on parameters





FTBE

"Significant" parameters:

- <u>**number N of strands sampled**</u>
- **duration T of trajectories**
- **loop used to compute FTBE**
- **locations of initial conditions**

↓

FTLyapE is a scalar field over the flow domain.
FTBraidE depends on N>1 initial conditions.

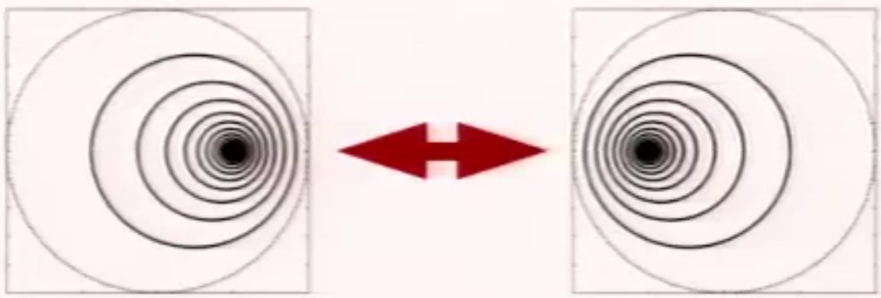
"Nuisance" parameters:

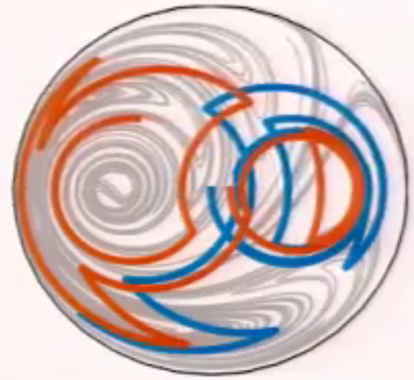
- **trajectory integration step**
- **angle of projection of trajectories (rotational frame of trajectories)**

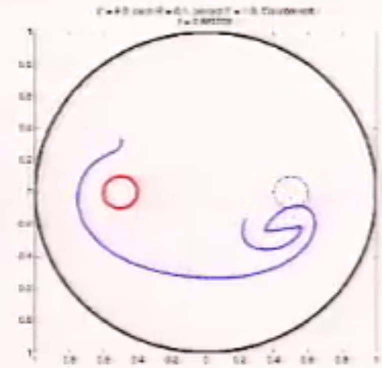
Thiffeault (2005), (2010) has preliminary statistics

Numerical study: mixing Aref Blinking Vortex

Circular domain, periodic alternation
between two integrable Rankine vortices







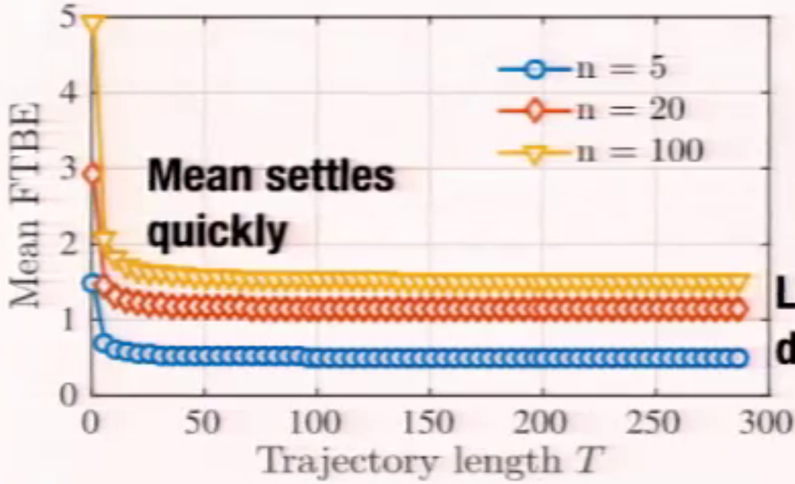
Circulation varied within mixing regime.
Mixing zone is the entire domain.

Classical expectations from ergodic theory:
- locations of initial conditions "forgotten" with time
- flow entropy depends on mag. of circulation

Braids (and FTBEs) are not "classical" observables – they "live" on a configuration space of particles, instead of the state space.

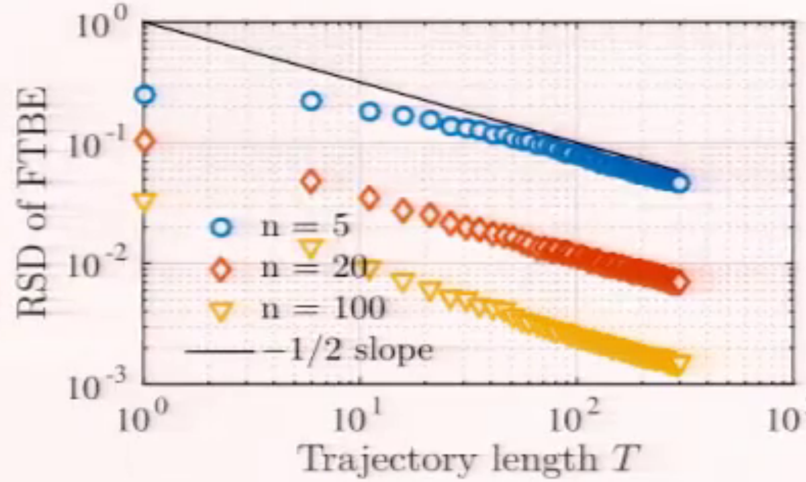
In mixing flows, initial conditions quickly stop mattering.

- at each n we seed 100 n-tuples of trajectories
- compute mean and relative standard deviation



Mean settles quickly

Limits are different



Variance decay consistent with Central Limit Theorem (mixing dynamics)

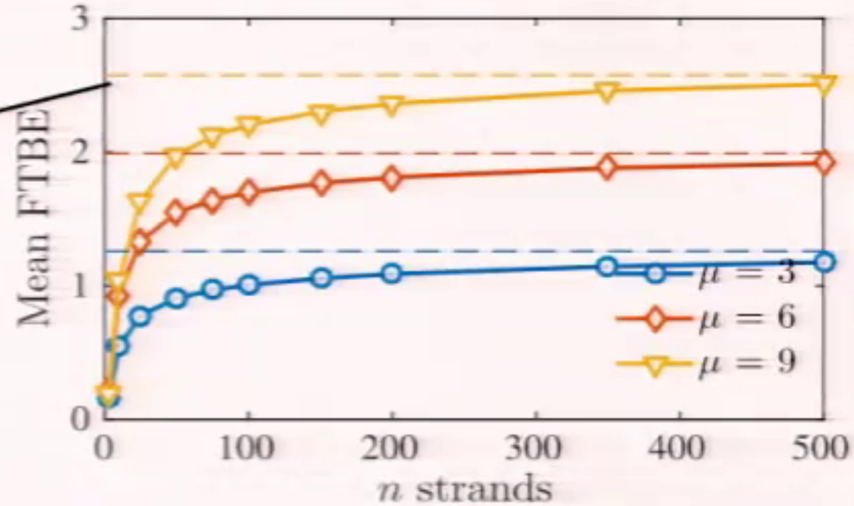
Mixing CLT carries through braid & FTBE calculations.
What is the significance of the value of mean FTBE.

Mean FTBE → flow entropy trajectories added.

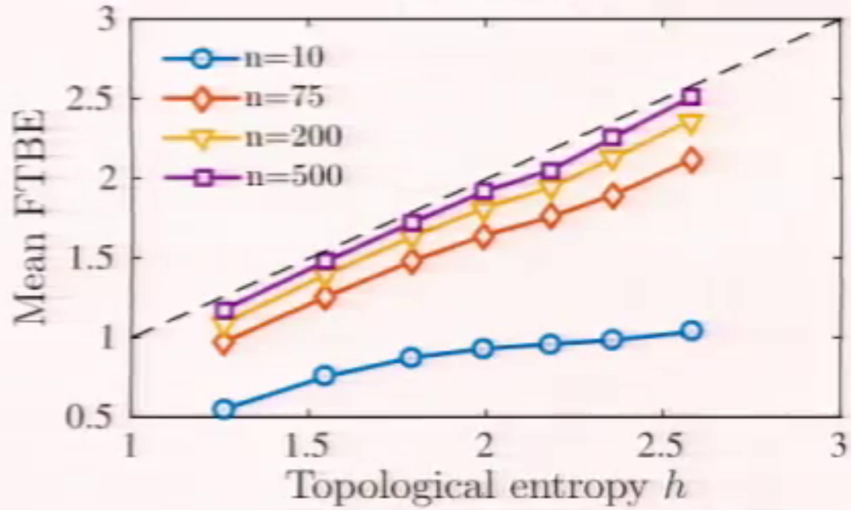
Initial conditions drawn uniformly at random (100 ensembles for each point).
All trajectories long enough for mean FTBE to stabilize.

Spatial mean FTBE – colors are different circulation strengths

Topological entropy estimated by material advection.



Mean FTBE approaches flow entropy as strands added



Limit mean FTBEs and flow entropy correlate across different circulations.

Rebraiding saves simulation time.

Indepedent sampling:
Each braid gets its "fresh" set of simulated trajectories.

Initial conditions

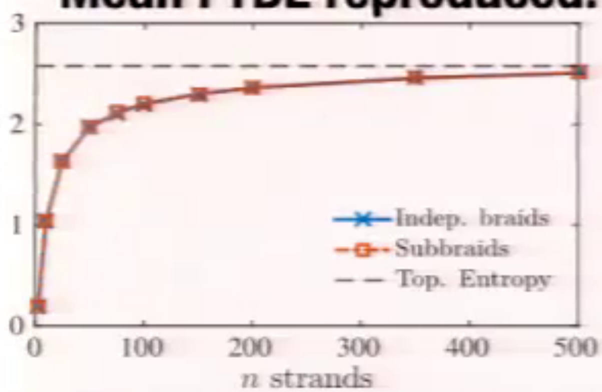
N trajectories:
1 N-strand braid

Resampling:
From a pool of strands (gray) we choose a subset (red) to form into braid.

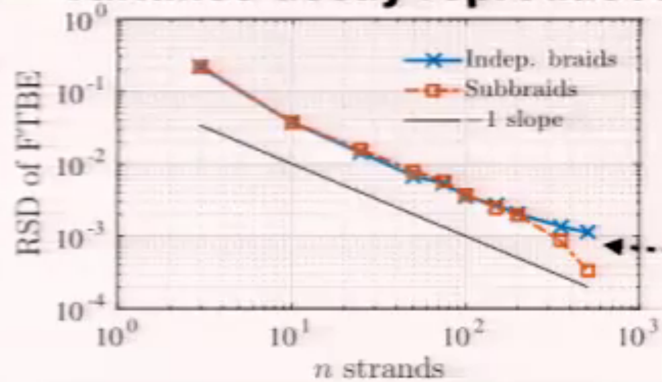
2N trajectories:
$\binom{2N}{N} \sim 4^N$ N-strand braids

Rebraiding of 550 strands vs. naive sampling using 150k strands (200x more!):

Mean FTBE reproduced.



Variance decay reproduced.



FTBE dist. at worst point