

# Control of Thin-Layer Flows with Patterned Surfaces

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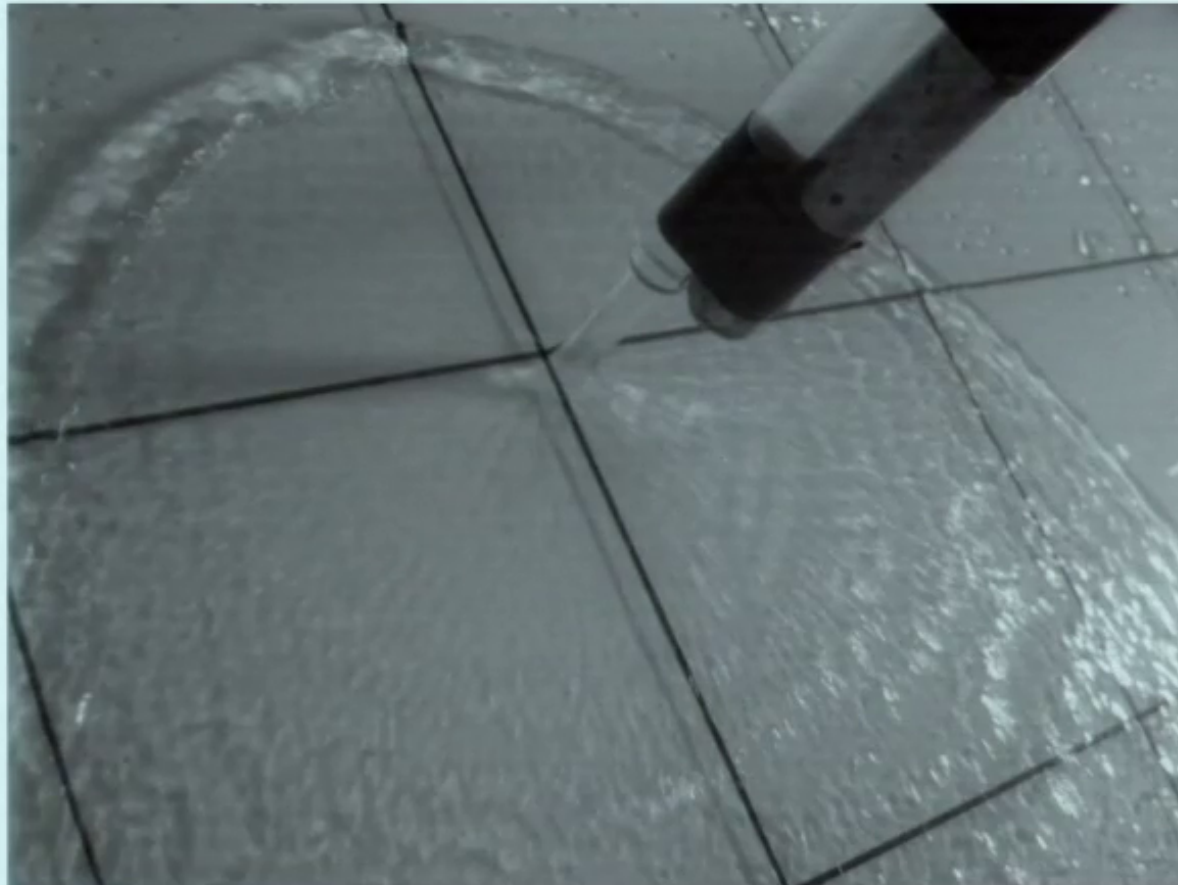
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# A jet hitting an inclined plane



Plane inclined at  $45^\circ$ . The flow rate is  $Q \simeq 120 \text{ cm}^3 \text{ s}^{-1}$ .

[with Andrew Belmonte in Claudia Cenedese and Karl Helfrich's lab at Woods Hole, GFD 2008]



Try steady potential flow:  $\mathbf{u} = \nabla\varphi$ , with

$$\nabla^2\varphi = 0, \quad \text{mass conservation;}$$
$$\frac{1}{2}|\nabla\varphi|^2 + \frac{p}{\rho} - \mathbf{g} \cdot \mathbf{r} = H, \quad \text{Bernoulli's law;}$$



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Boundary conditions:

$$\begin{aligned}\partial_z\varphi &= 0 & \text{at } z = 0, & \text{no-throughflow at substrate;} \\ \nabla\varphi \cdot \nabla h &= \partial_z\varphi & \text{at } z = h, & \text{kinematic condition at free surface;} \\ p &= 0 & \text{at } z = h, & \text{constant pressure at free surface.}\end{aligned}$$

Here  $z$  is normal to the substrate,  $x_1$  and  $x_2$  are parallel to it.

# Small-parameter expansion



Expand Bernoulli's law in the small fluid depth  $\varepsilon$ :

$$\sum_{j=1}^2 (\partial_j \varphi)^2 + \varepsilon^{-2} (\partial_z \varphi)^2 + \frac{2p}{\rho} - 2\mathbf{g} \cdot (\mathbf{X} + \varepsilon z \hat{\mathbf{e}}_3) = 2H,$$

where  $\mathbf{X} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2$ . Also expand  $\varphi$ :

$$\varphi(x_1, x_2, z) = \varphi_{(0)} + \varepsilon \varphi_{(1)} + \varepsilon^2 \varphi_{(2)} + \dots,$$

to obtain at leading order  $\partial_z \varphi_{(0)} = 0$ , so that

$$\varphi_{(0)} = \Phi(x_1, x_2).$$



At next order:

$$\sum_{j=1}^2 (\partial_j \Phi)^2 + (\partial_z \varphi_{(1)})^2 + \frac{2p}{\rho} - 2\mathbf{g} \cdot \mathbf{X} = 2H,$$

Evaluate at  $z = h$  and use the boundary conditions:

$$\sum_{j=1}^2 (\partial_j \Phi)^2 - 2\mathbf{g} \cdot \mathbf{X} = 2H,$$

# Solution in terms of characteristics



Differentiate to get rid of constant:

$$\sum_{j=1}^2 \partial_j \Phi \partial_{ij} \Phi = \mathbf{g} \cdot \partial_i \mathbf{X}, \quad i = 1, 2.$$

Introduce the characteristics  $x_1(\tau)$ ,  $x_2(\tau)$ :

$$\dot{x}_1 = \partial_1 \Phi(\mathbf{x}), \quad \dot{x}_2 = \partial_2 \Phi(\mathbf{x}),$$

We have  $\partial_{ij} \Phi = \partial_i \dot{x}_j$  and  $\ddot{x}_i = (\partial_j \dot{x}_i) \dot{x}_j = \partial_j \Phi \partial_{ij} \Phi$ , so that

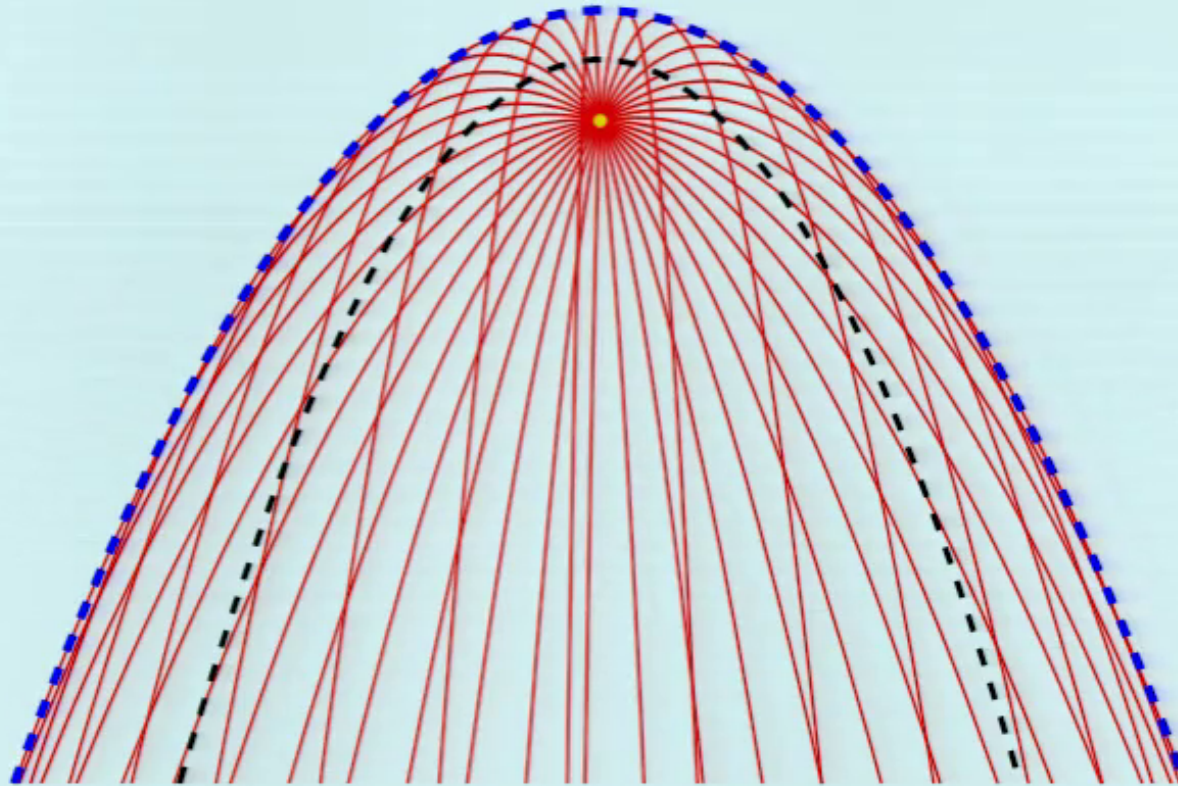
$$\ddot{x}_i = \mathbf{g} \cdot \hat{\mathbf{e}}_i, \quad i = 1, 2.$$

[Rienstra (1996)]

# Characteristics for a jet striking an inclined plane



The characteristics have a parabolic envelope (blue dashed):

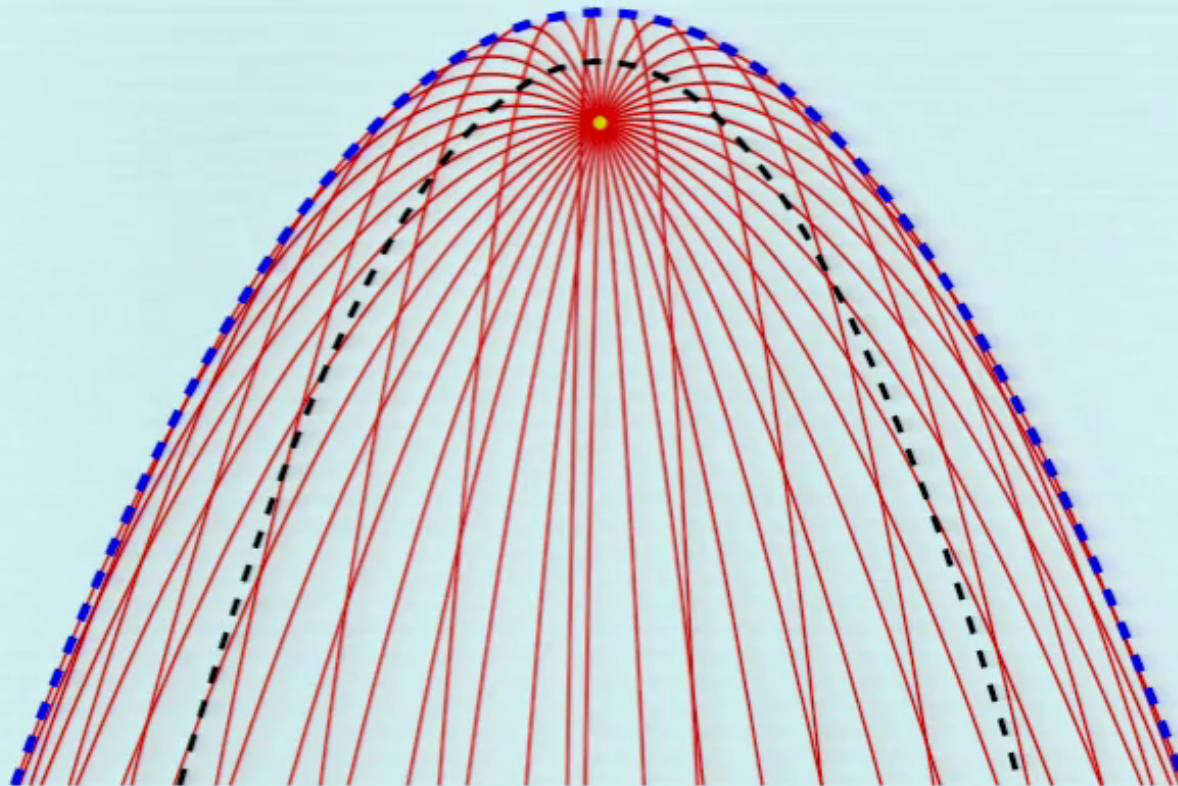




# Characteristics for a jet striking an inclined plane



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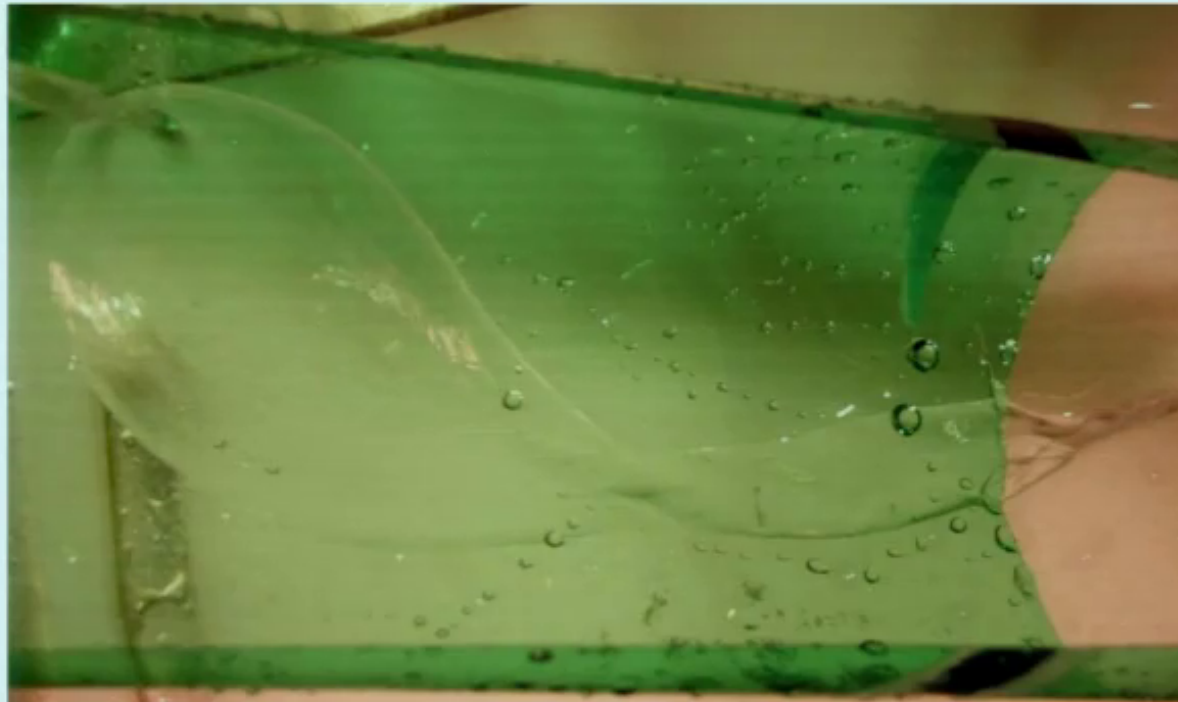


Edwards *et al.* (2008) used the 'delta-shock' framework to account for characteristics crossing: this lowers the rise distance by  $5/9$ , and the profile remains essentially parabolic (black dashes).

# Curved substrates



Rienstra (1996) also applied his inviscid model to curved surfaces (spheres, cylinders). Here's my attempt at an experiment [Thiffeault & Kamhawi (2008)]:



Compare to characteristics on a cylinder:



# General curved substrates



Rienstra (1996) treated surfaces with global orthogonal coordinates (plane, cylinder, sphere).

What about more general surfaces?

Write  $x^1, x^2$  for general 2D coordinates that locate a point on the substrate.



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What about more general surfaces?

Write  $x^1, x^2$  for general 2D coordinates that locate a point on the substrate. A small-thickness expansion similar to Rienstra's yields for the characteristics [Thiffeault & Kamhawi (2008)]:

$$\ddot{x}^\sigma + \Gamma_{\alpha\beta}^\sigma \dot{x}^\alpha \dot{x}^\beta = \mathbf{g} \cdot \mathbf{e}^\sigma$$

where  $\Gamma_{\alpha\beta}^\sigma$  are the **Christoffel symbols** for the shape of the substrate.

This is the **geodesic equation** with a gravitational forcing. The fluid particles (characteristics) are trying to follow straight lines, but their trajectories are bent by the substrate curvature and gravity.



The geodesic equations are actually a **fourth-order** autonomous system.

Hence, chaos is a possibility, **as long as the substrate does not possess a continuous symmetry!** (Ruled out for plane, cylinder, sphere.)

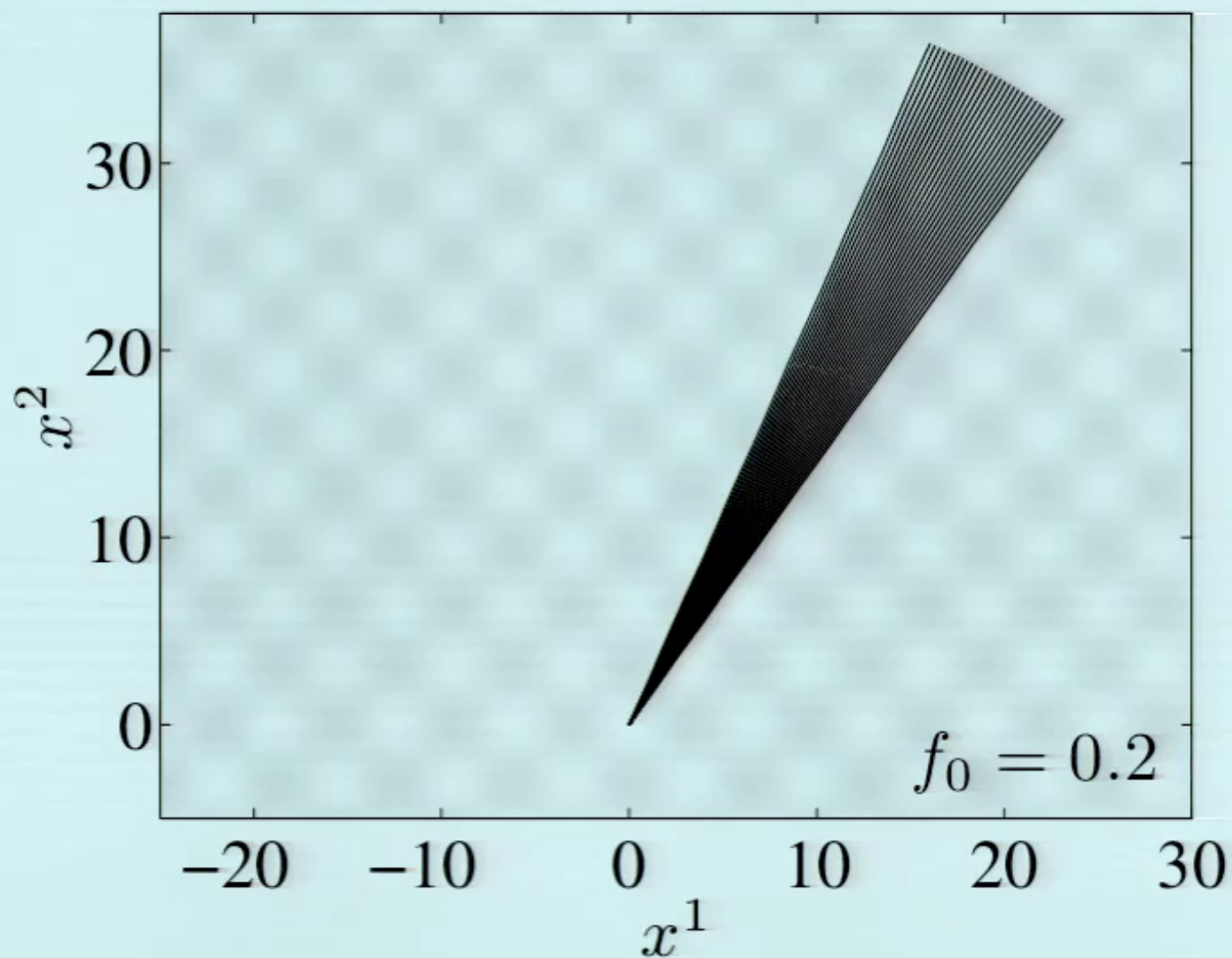
Consider a simple substrate shape parametrized by:

$$f(x^1, x^2) = f_0 \cos x^1 \cos x^2$$

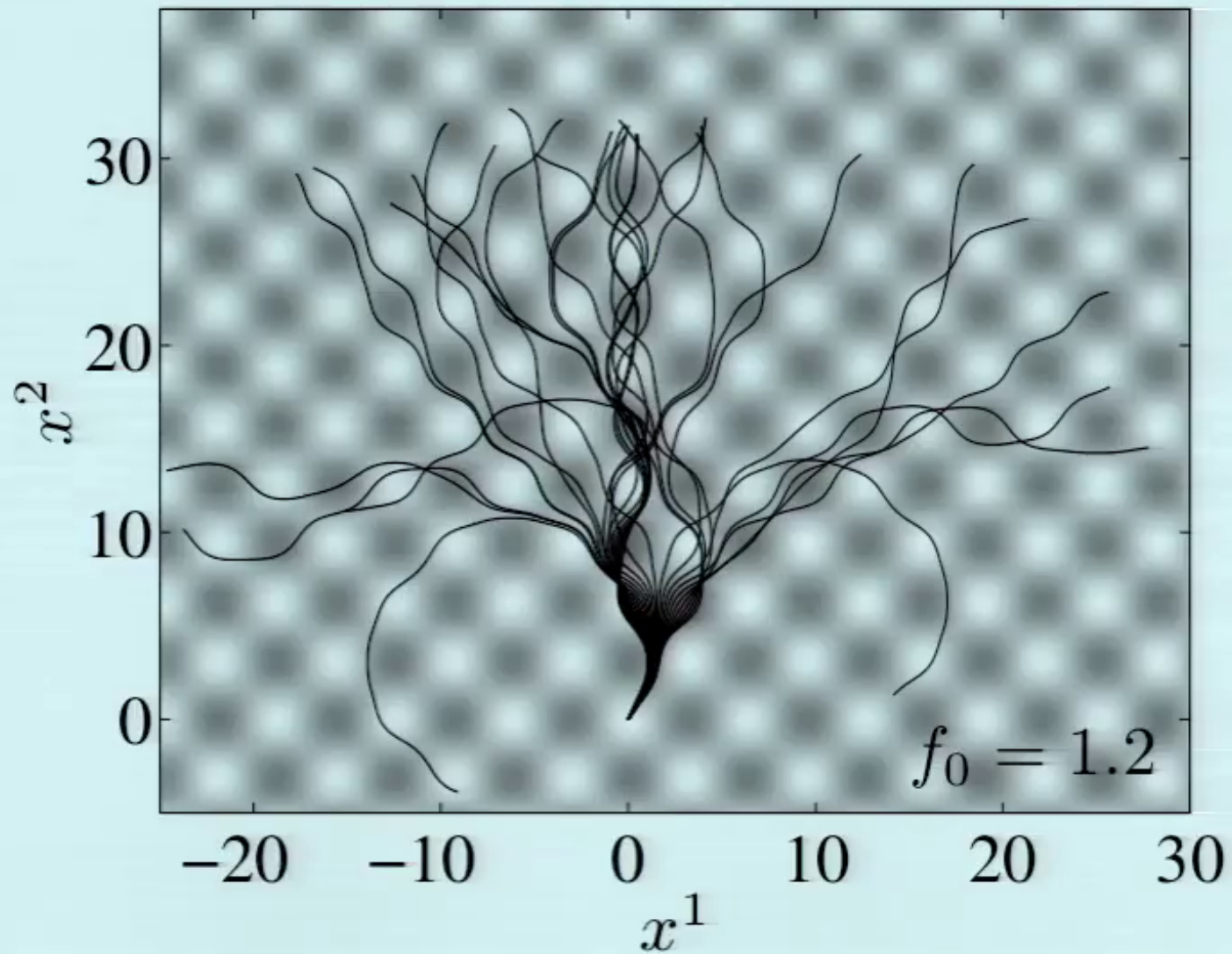
# Horizontal substrate: $f_0 = 0.2$



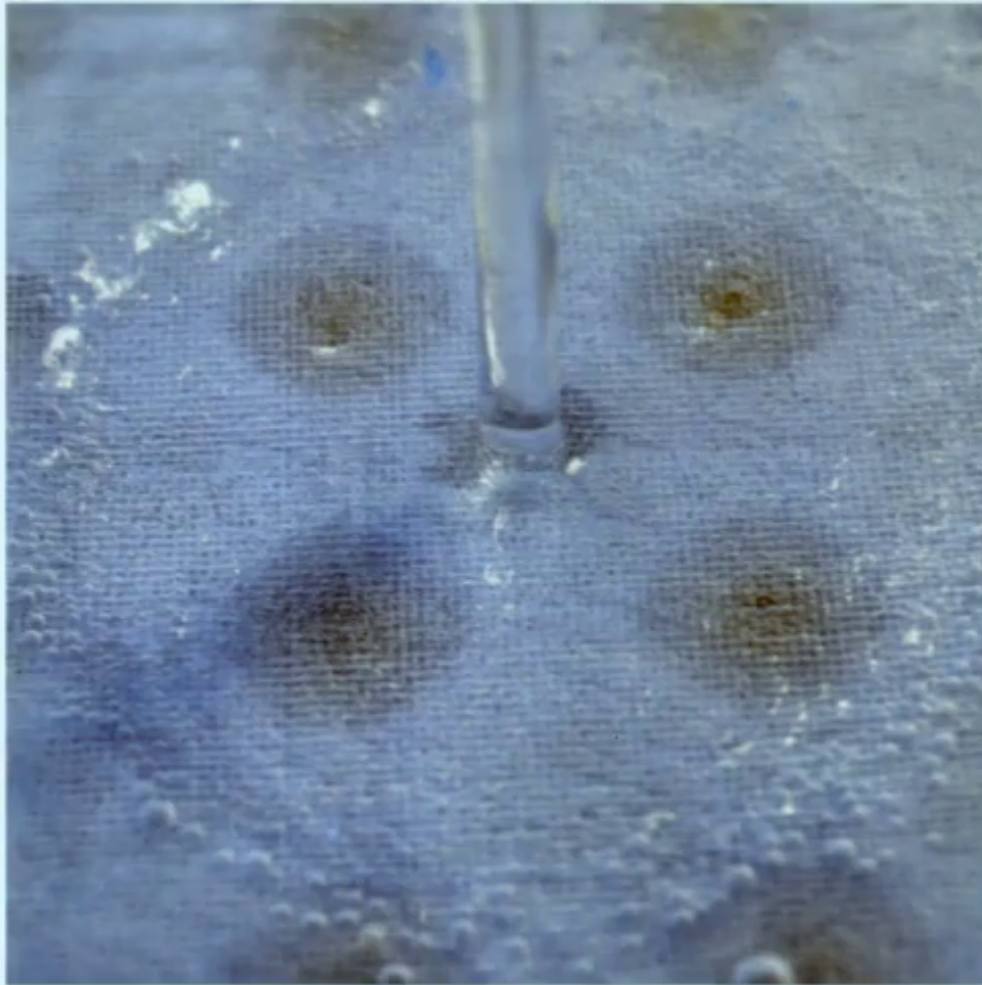
First take  $g = 0$  and keep the surface horizontal.



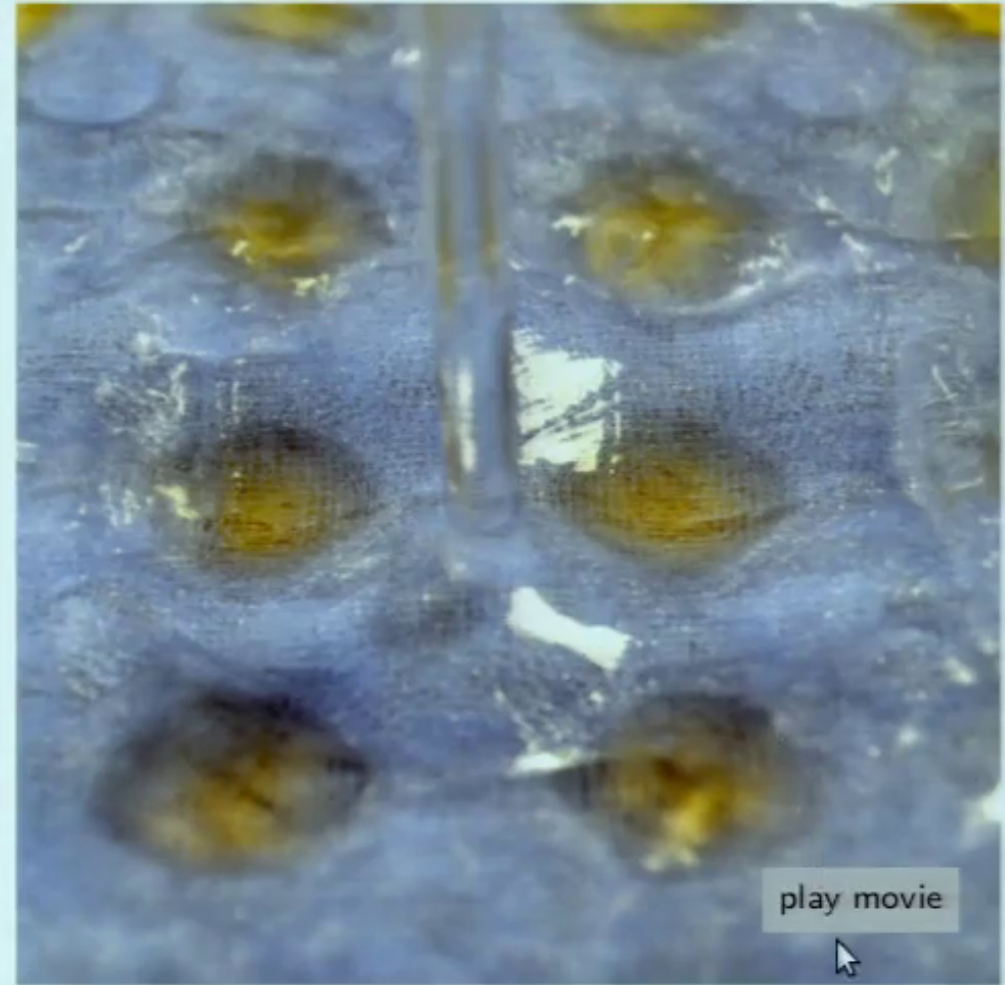
Horizontal substrate:  $f_0 = 1.2$



# Experiments with 3D-printed substrate



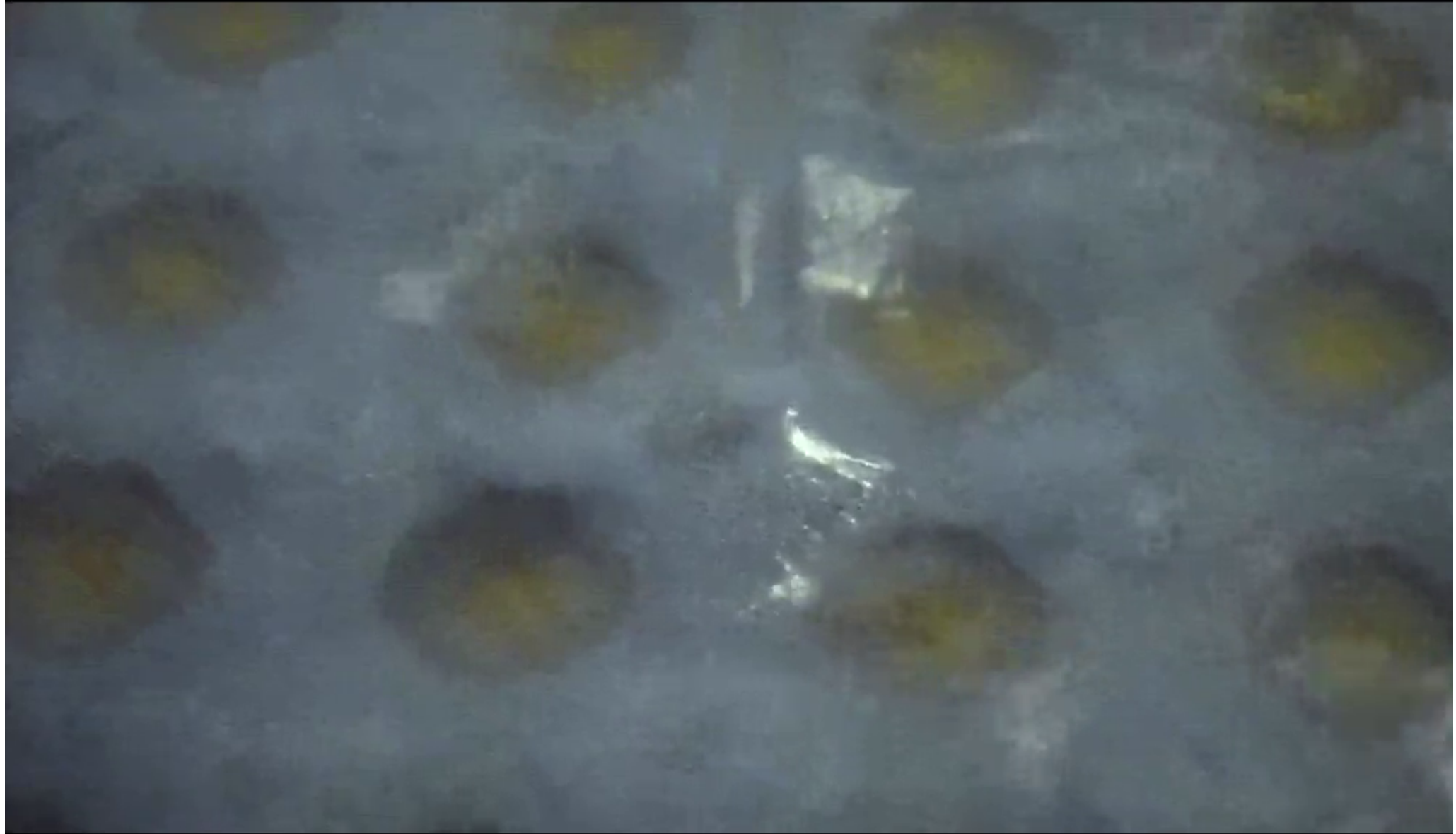
Flat substrate



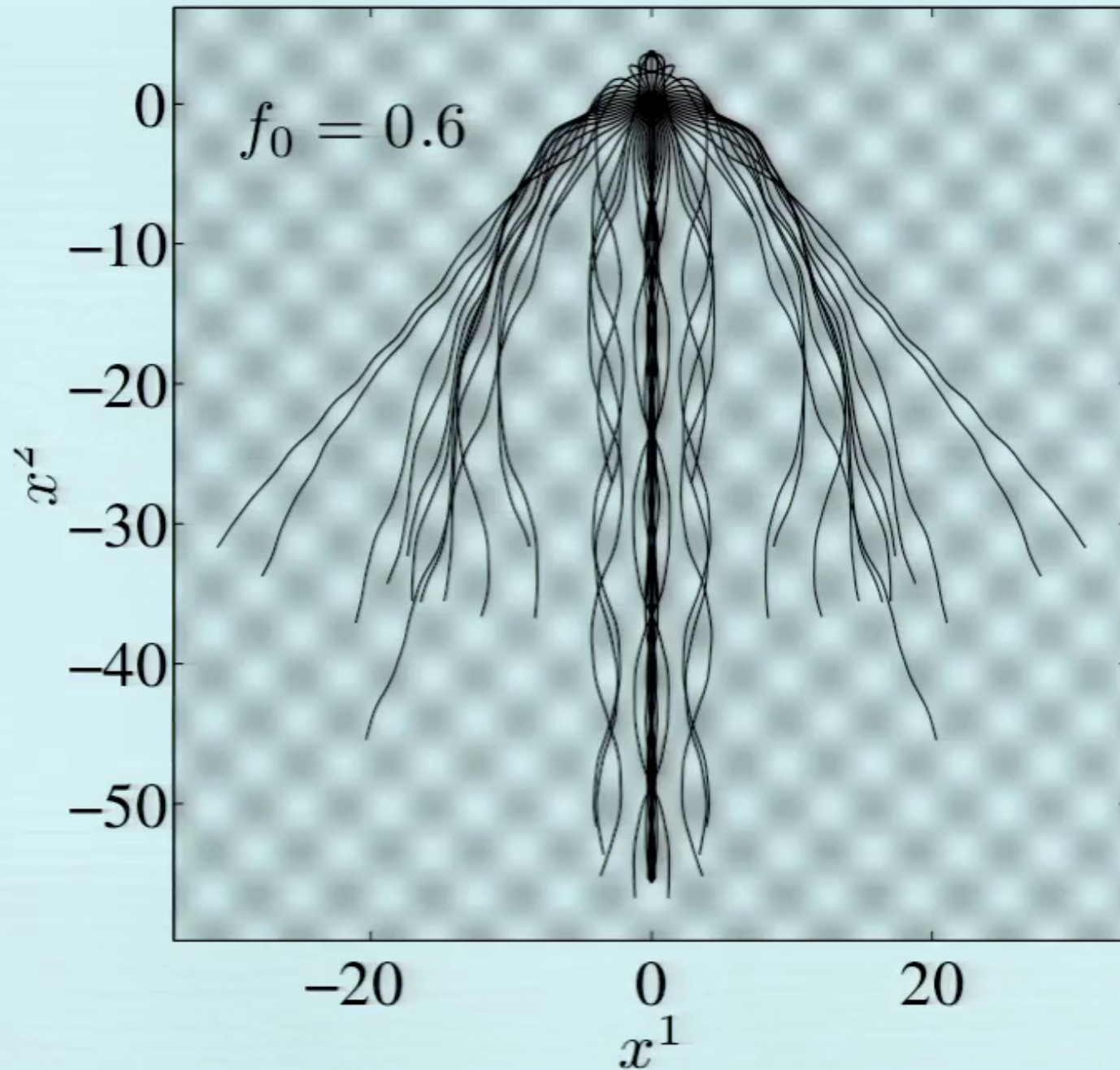
Patterned substrate

Jump is about 50% larger for a flat substrate. [Experiments with Jay Johnson.]

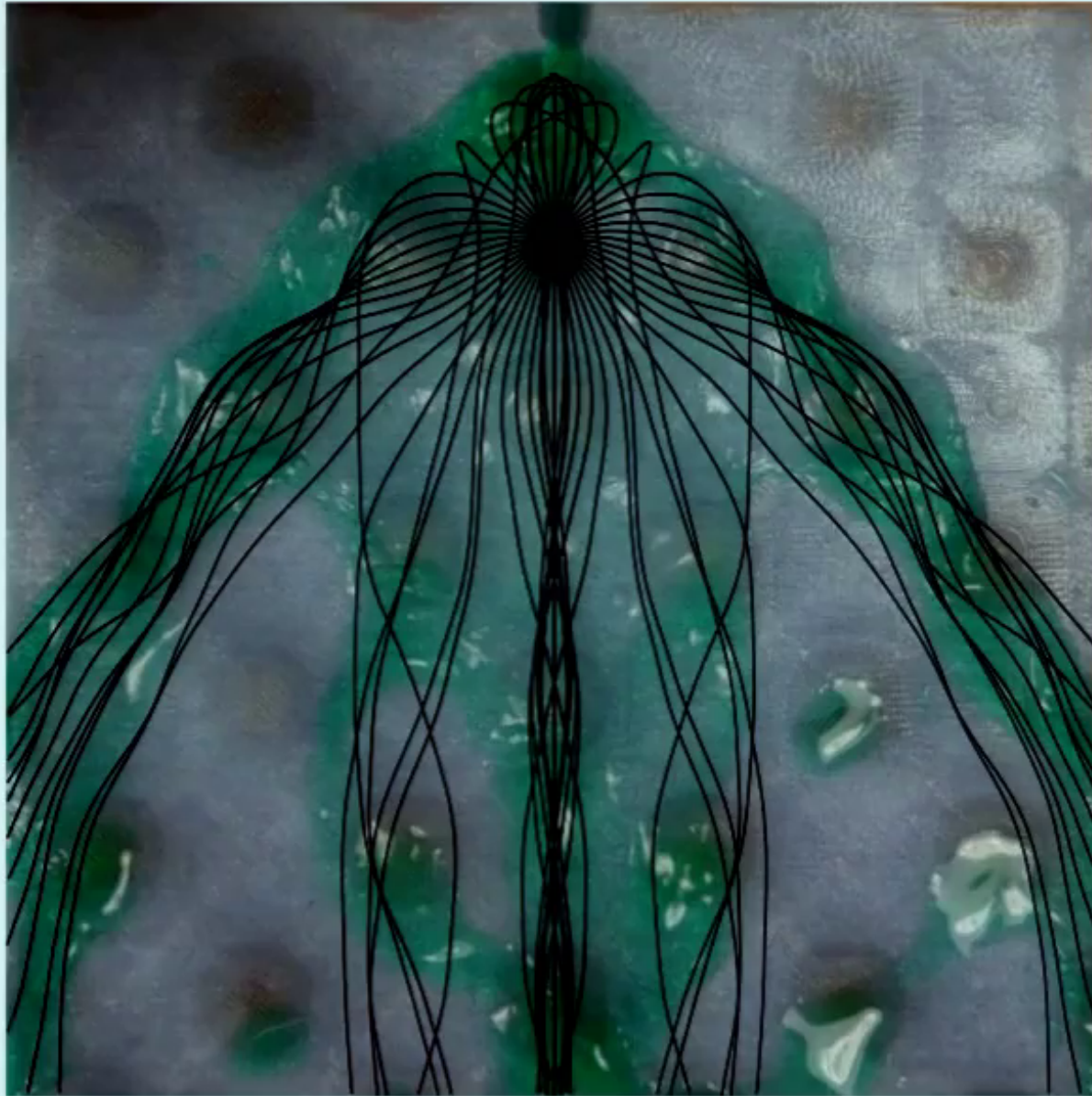




# Inclined substrate: $f_0 = 0.6$



# Experiments: an inclined substrate



The simple model correctly predicts the multiple 'paths'.