Localized pattern formation

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Margaret Beck

Jason Bramburger

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ICERM

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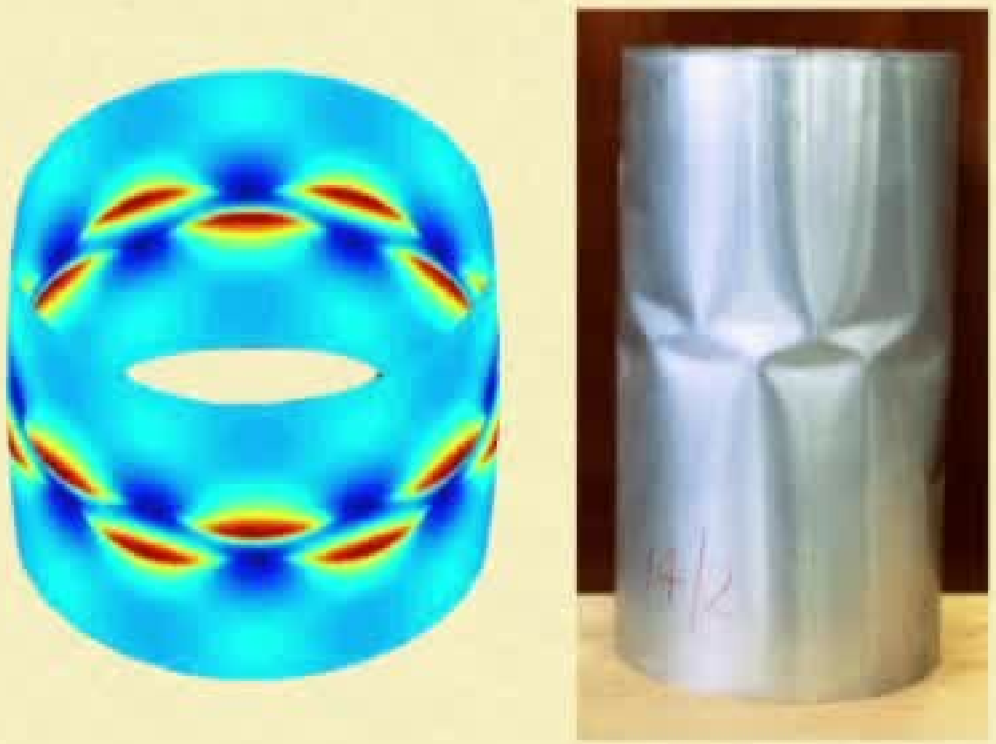
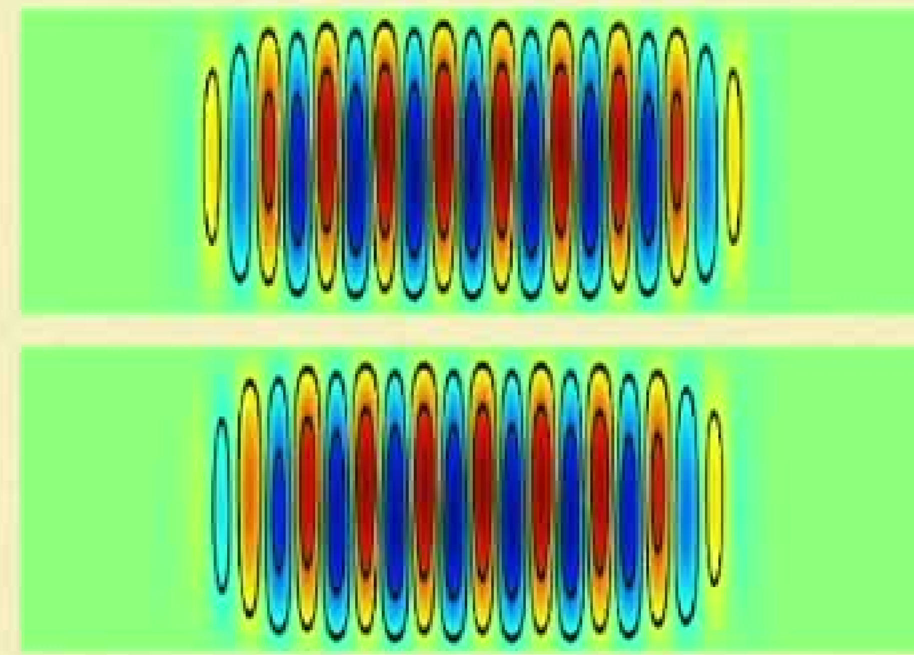
Localized roll patterns

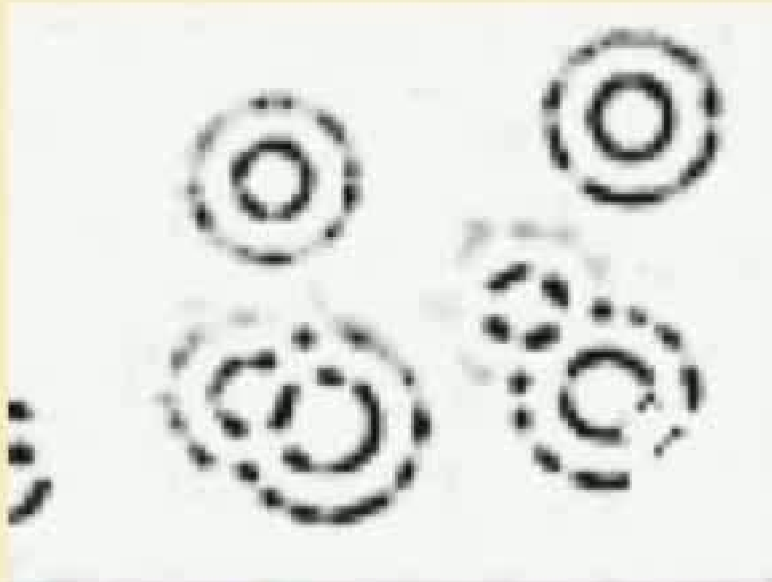


Buckling of cylindrical shells
[Lord et al.]



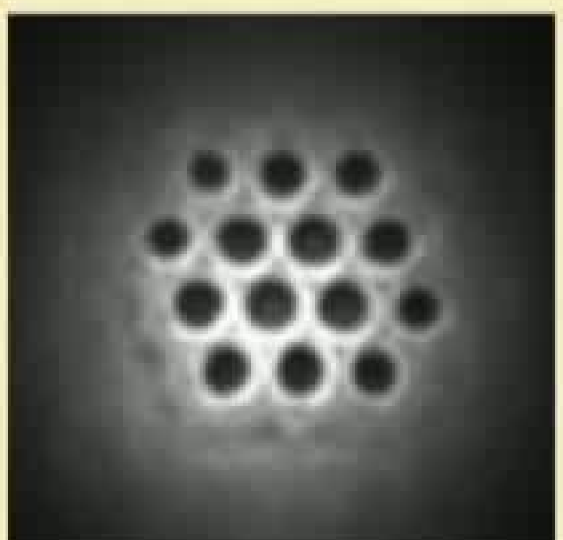
Convectons in binary fluid convection
[Batiste et al.]

Localized planar roll and hexagons patterns

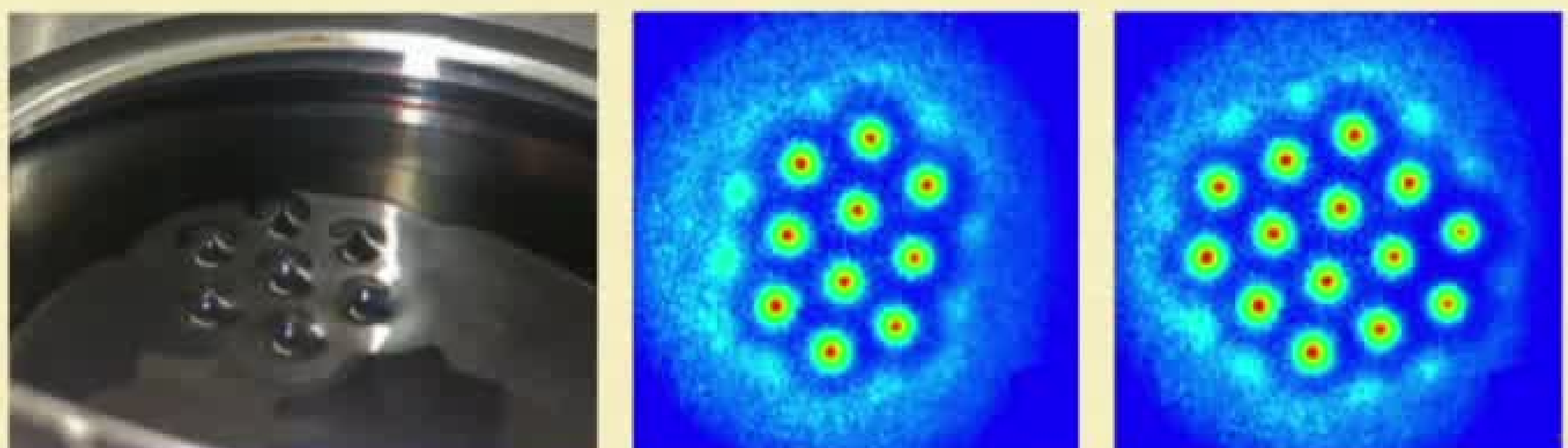


Belousov–Zhabotinsky reaction
[Vanag & Epstein]

Vegetation patches [Sheffer et al.]



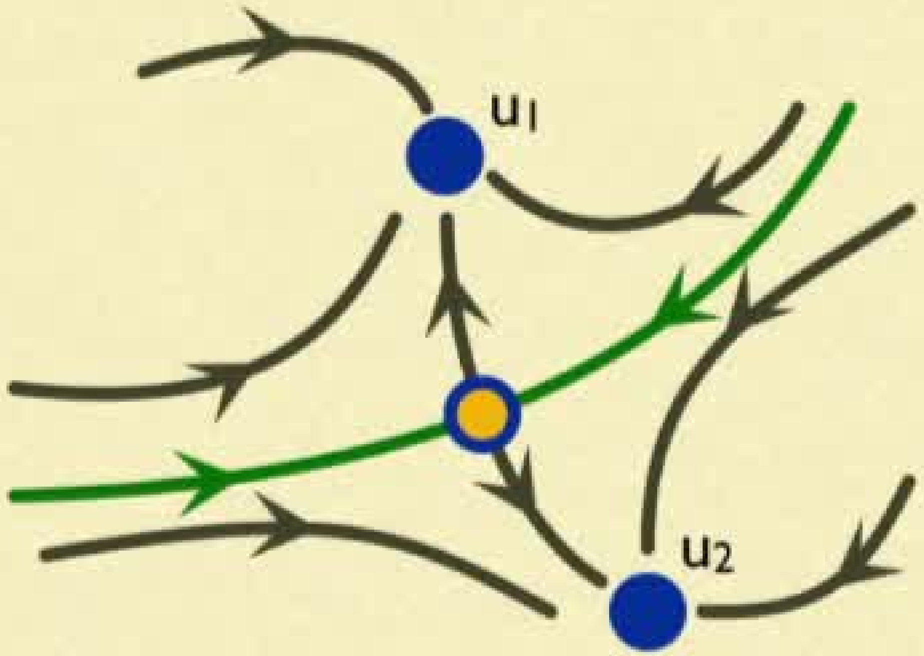
Sodium vapor
[Ackemann]



Ferro fluids [Lloyd, Gollwitzer, Rehberg, Richter]

Bistability

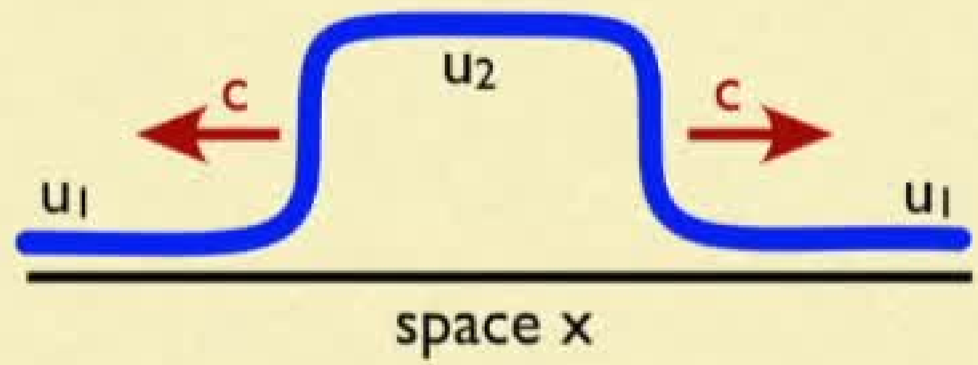
Bistable ODE: $u_t = f(u,\mu)$

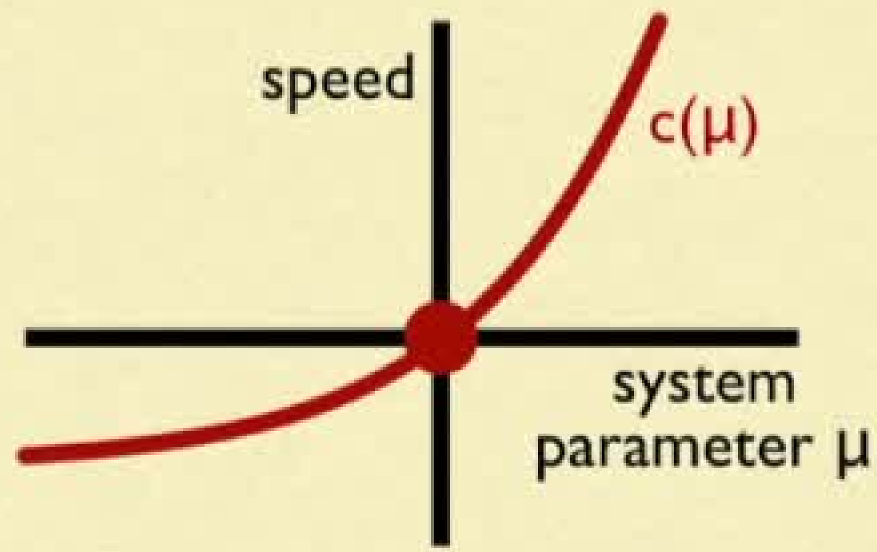


Invasion fronts:

- chemical reactions
- ecology
- epidemiology

Bistable PDE: $u_t = Du_{xx} + f(u,\mu)$





Expect unique parameter μ at which rest states coexist (c=0)

Bistability

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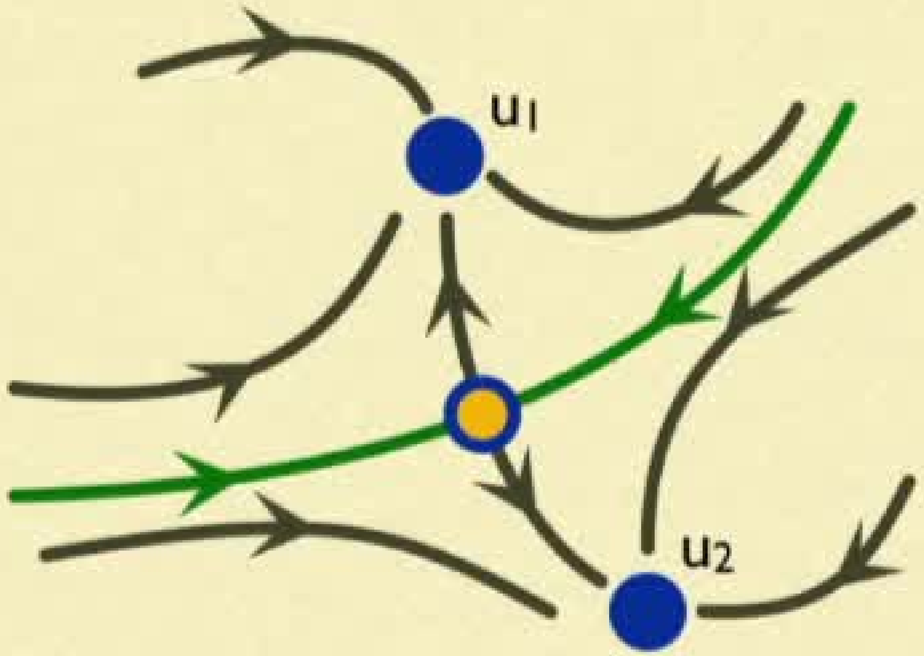

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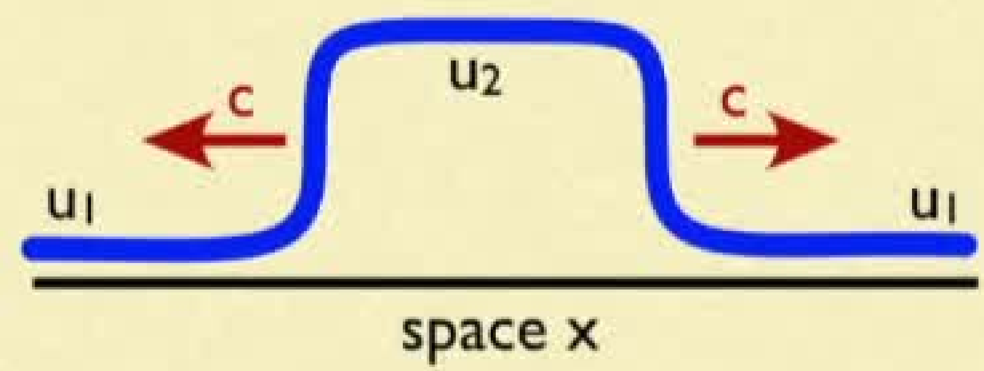
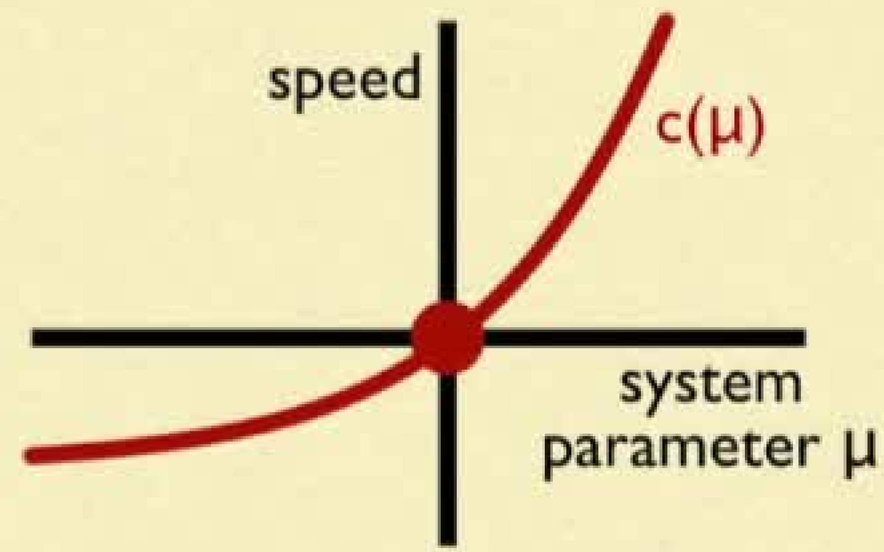


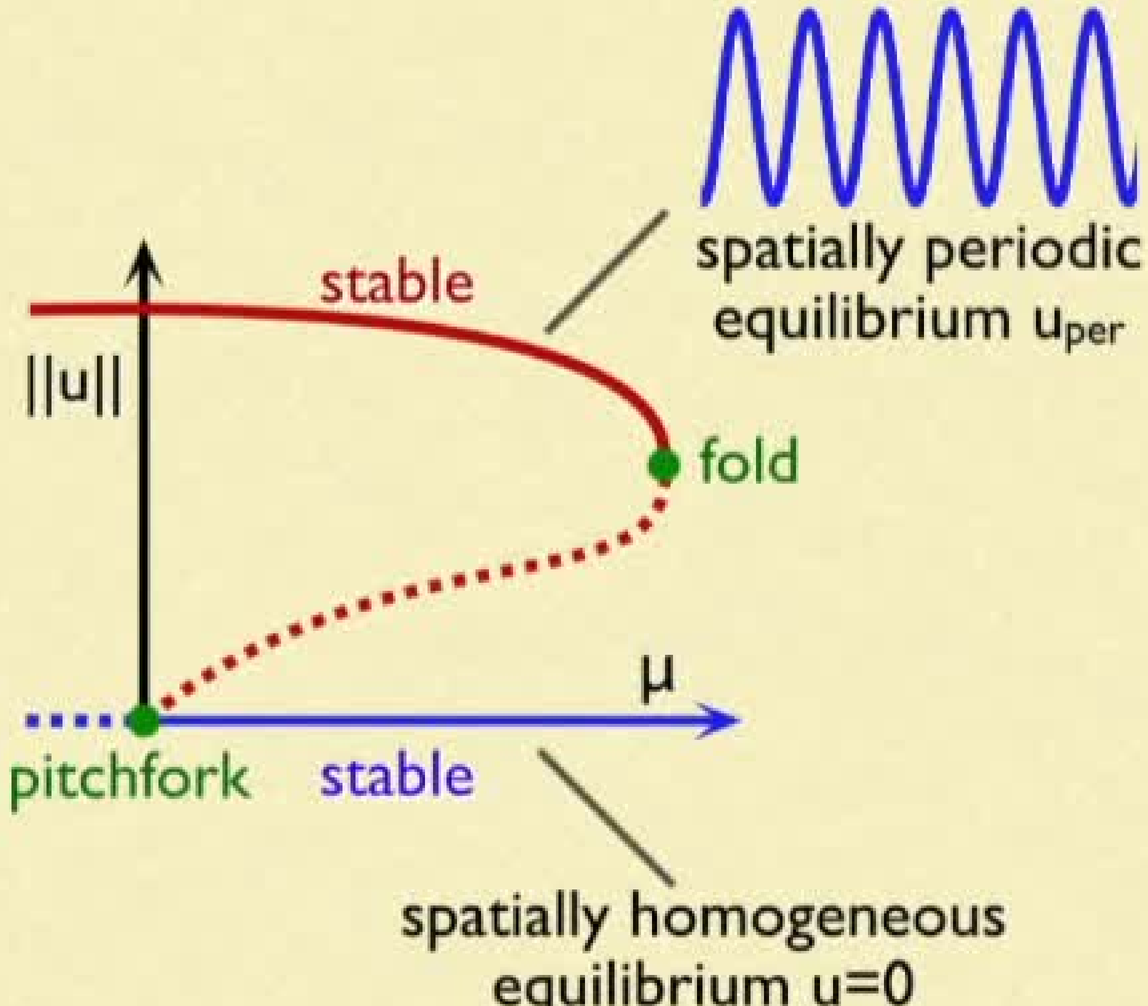


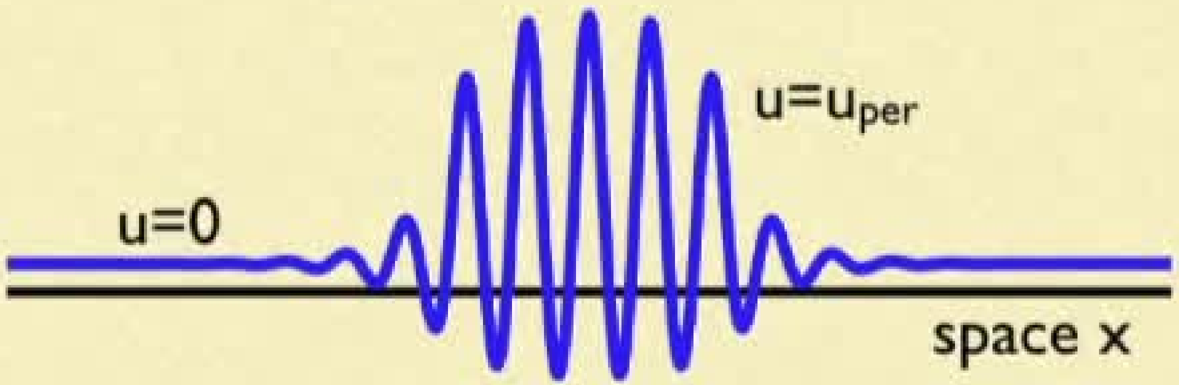
Expect unique parameter μ at which rest states coexist ($c=0$)

Bistability

Swift-Hohenberg equation

$$u_t = -[1 + \partial_x^2]^2 u - \mu u + \nu u^2 - u^3$$





Examples

- buckling patterns
- convectons

Bistability

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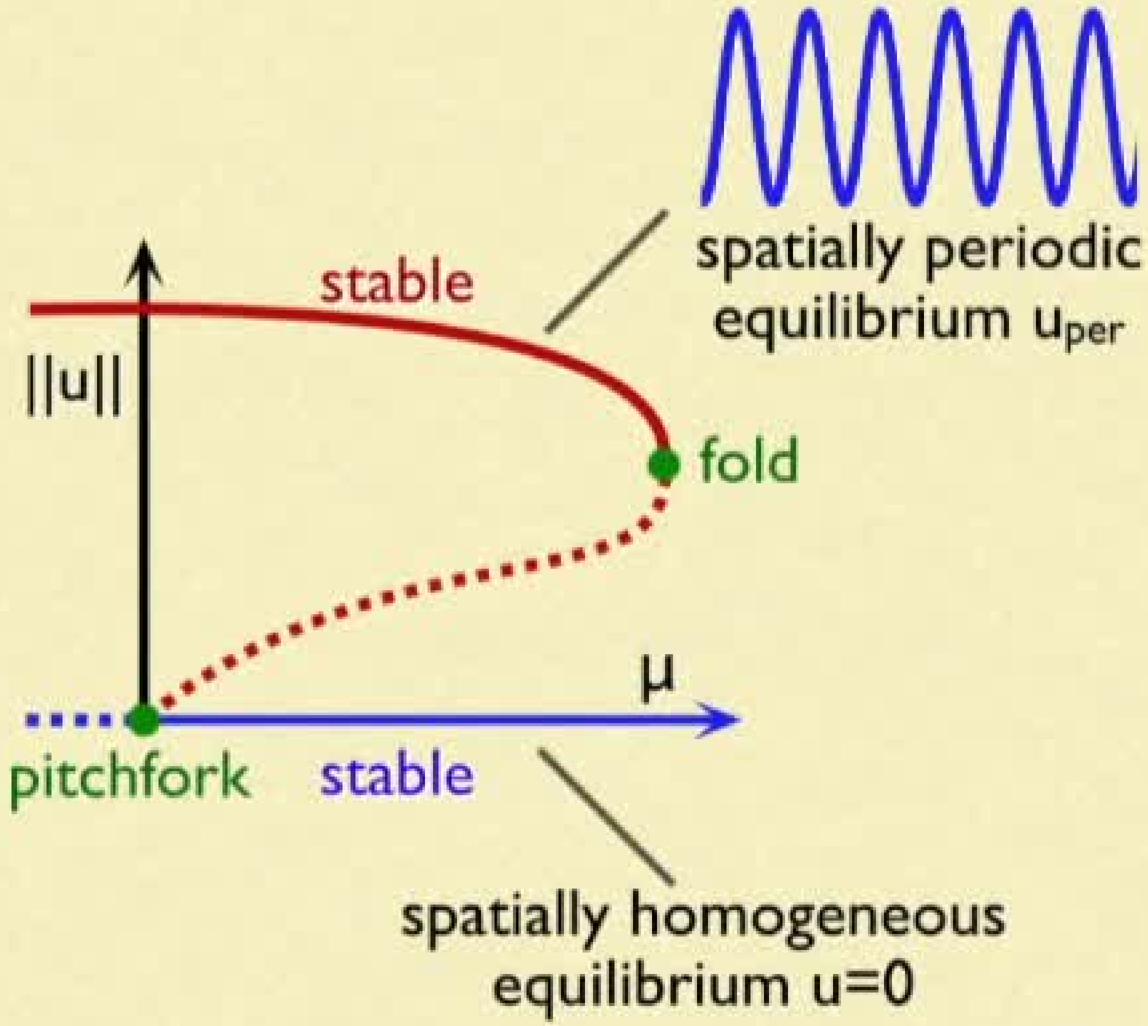

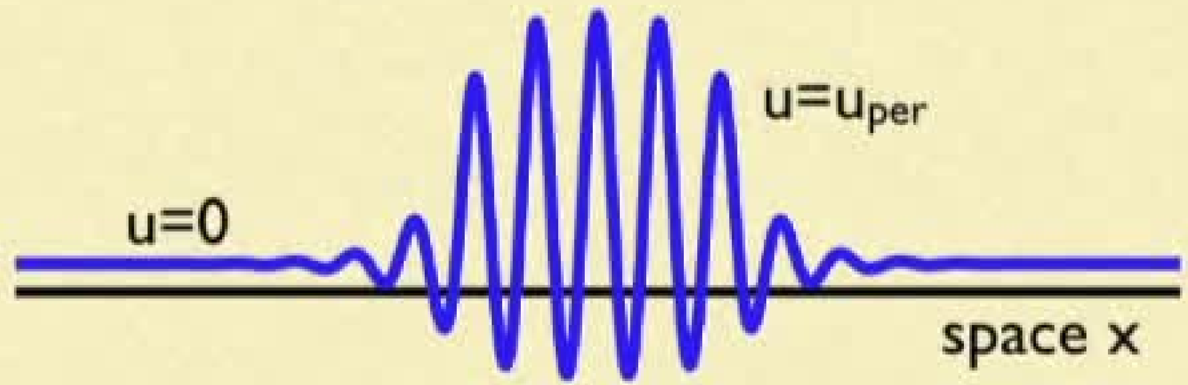

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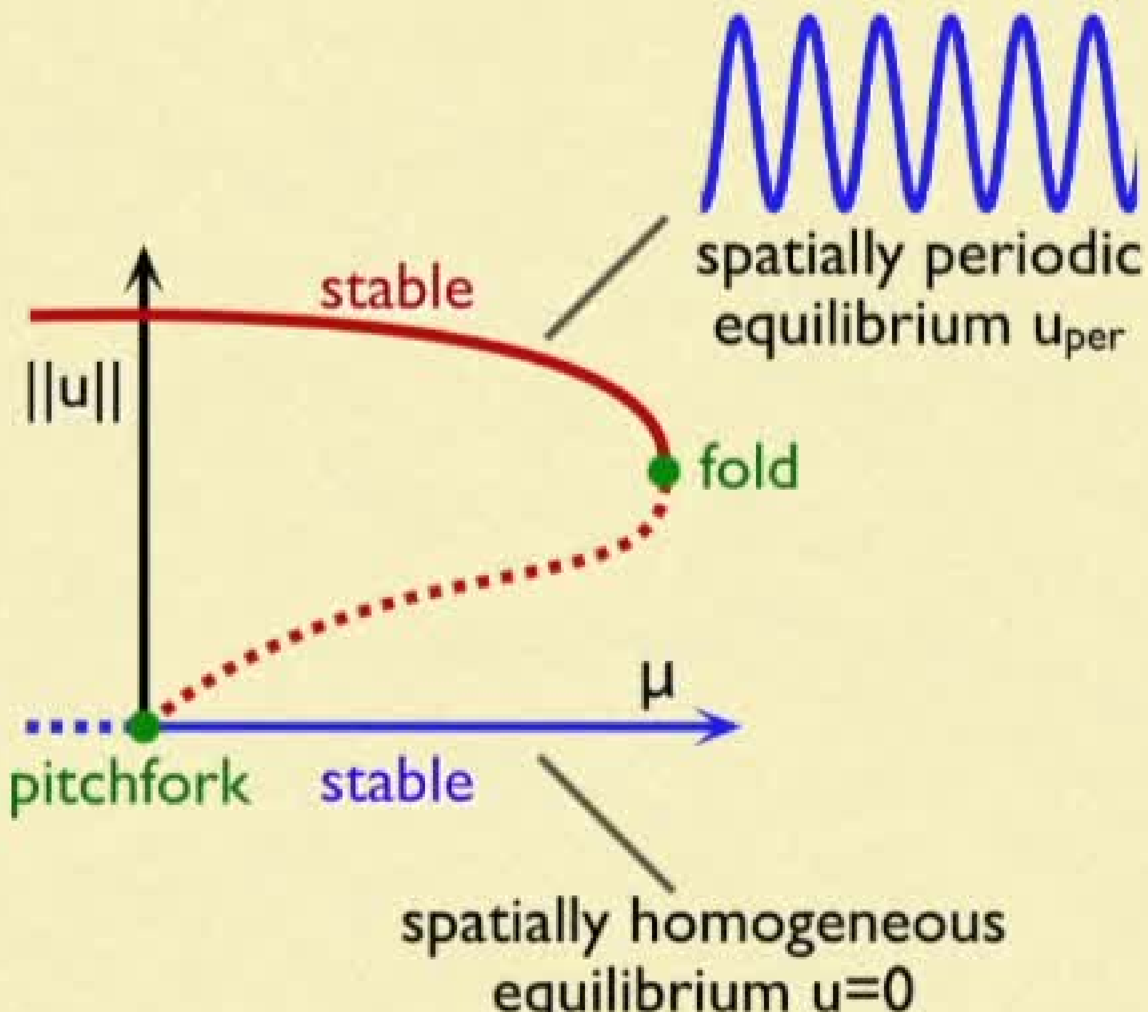

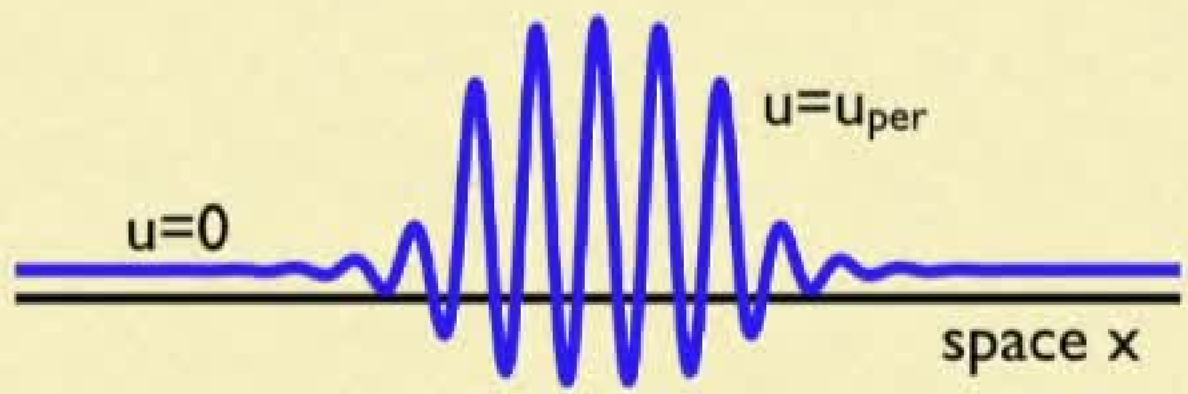

Examples

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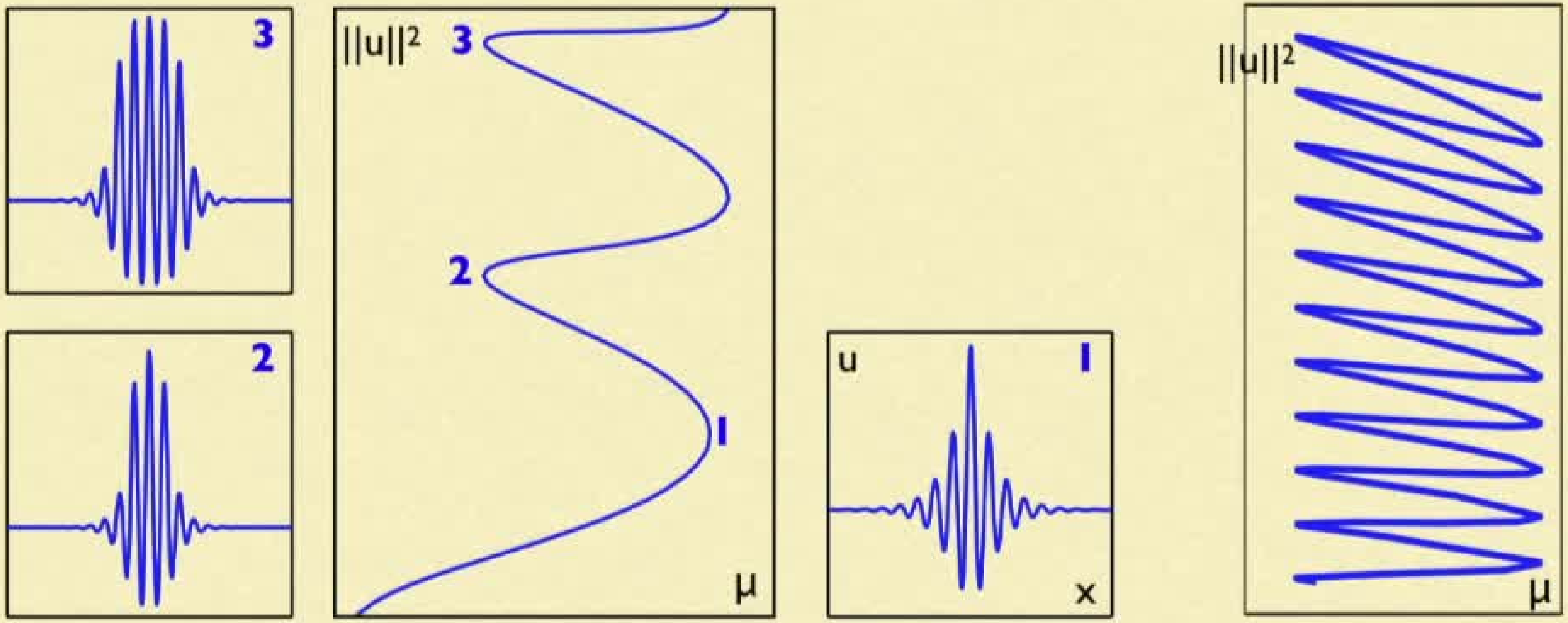
Bistability: snaking diagrams

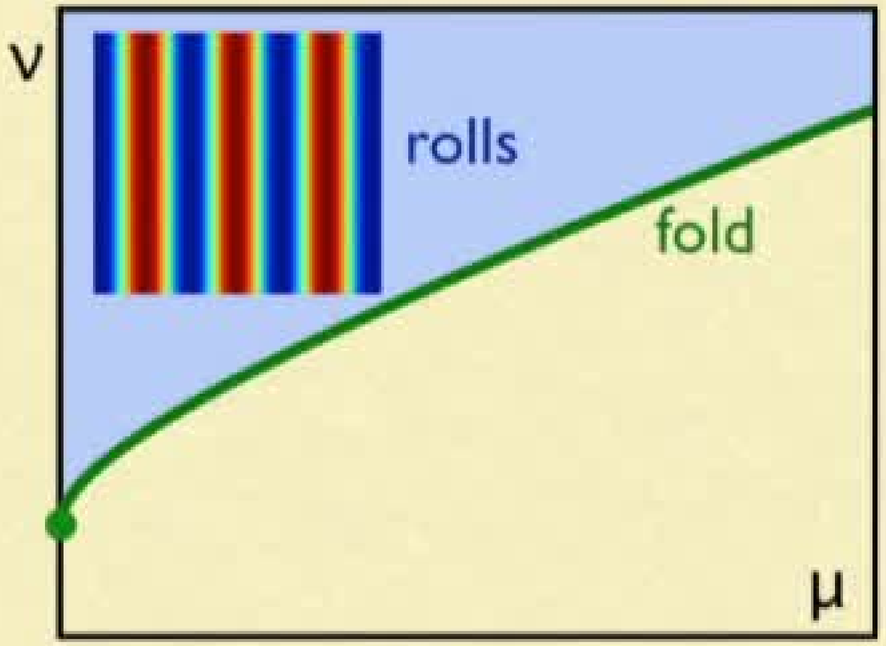


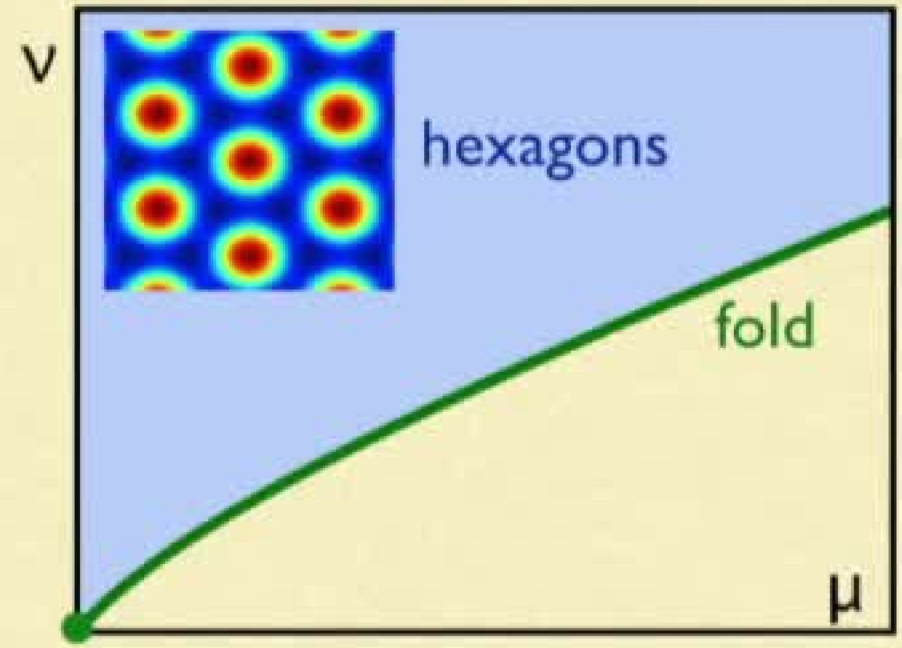
Stationary localized roll states exist in an open parameter interval!

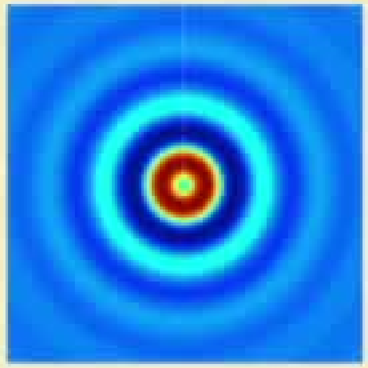
[Pomeau], [Woods & Champneys], [Coullet, Riera & Tresser], [Burke & E Knobloch]
[Chapman & Kozyreff], [Beck, J Knobloch, Lloyd, S., Wagenknecht]

Bistability in planar systems

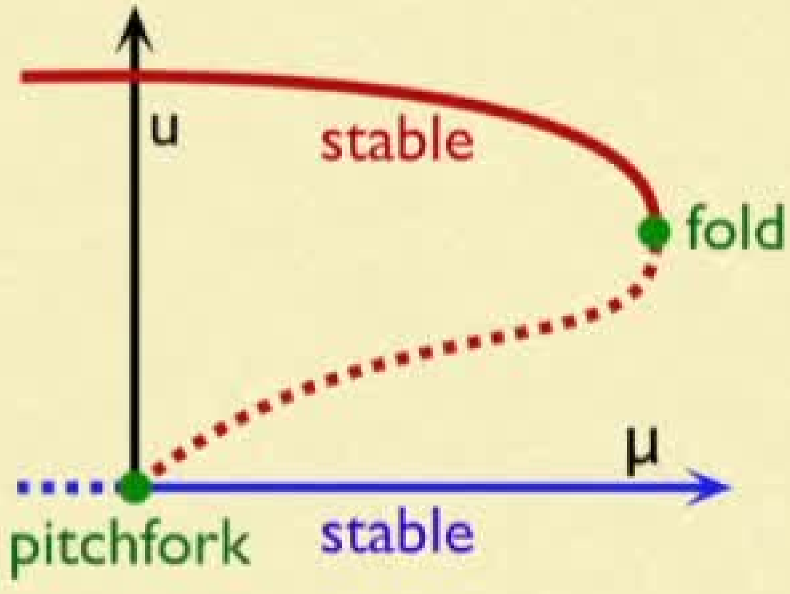
$$u_t = -[1+\Delta]^2 u - \mu u + \nu u^2 - u^3, \qquad x \in \mathbb{R}^2$$

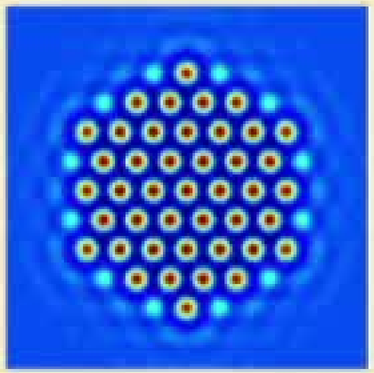






stationary radial roll pattern





stationary hexagon patch

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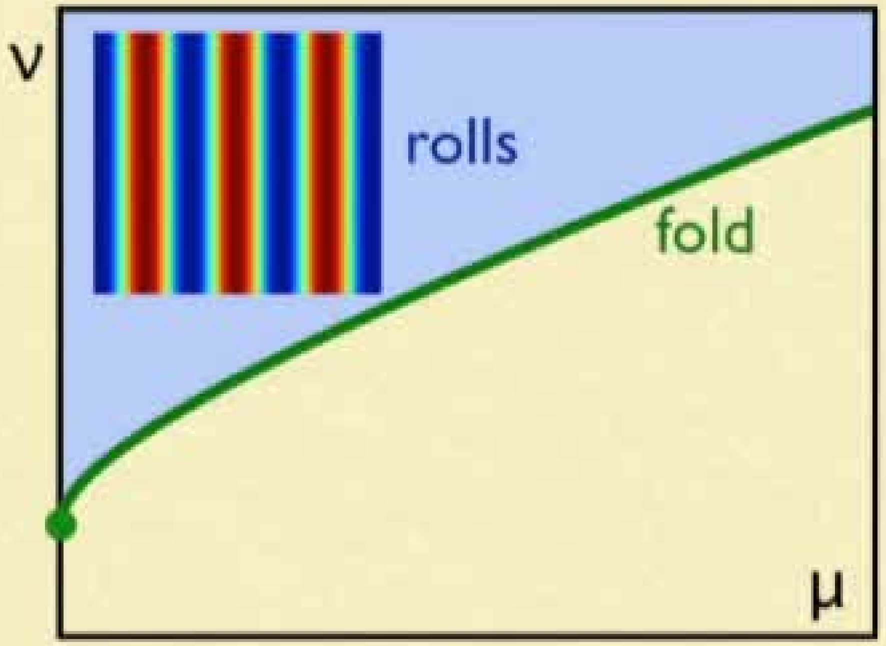

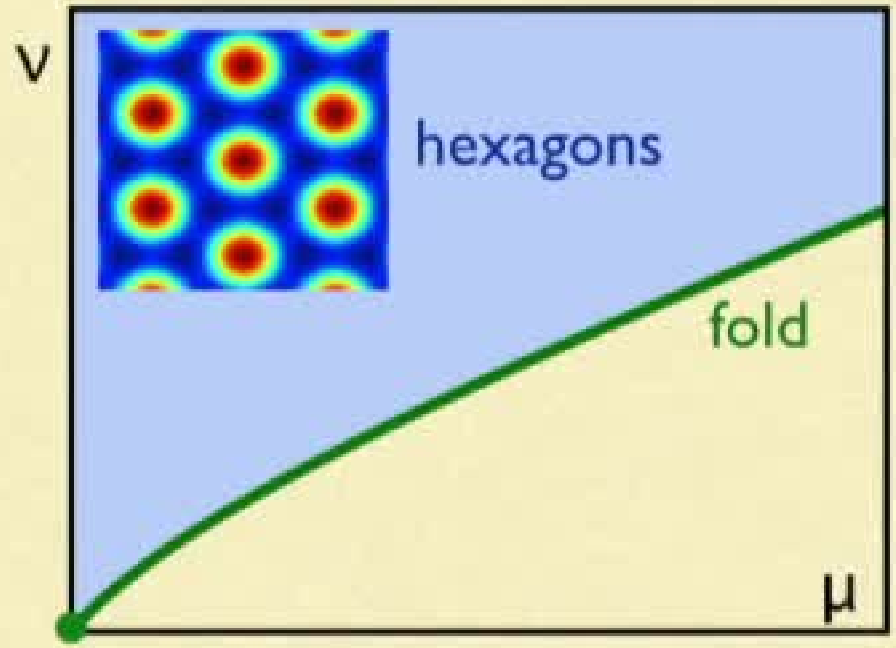

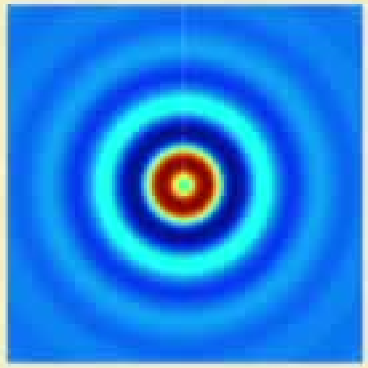

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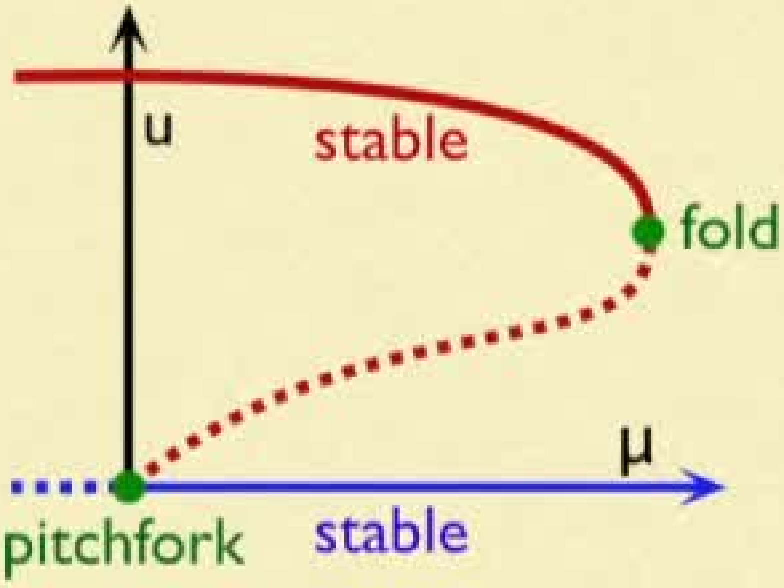

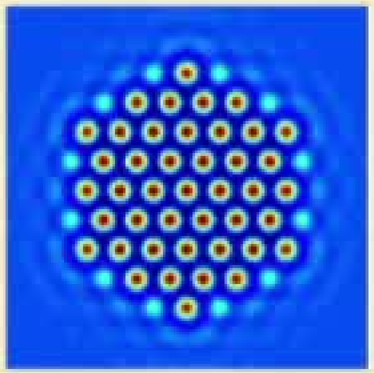

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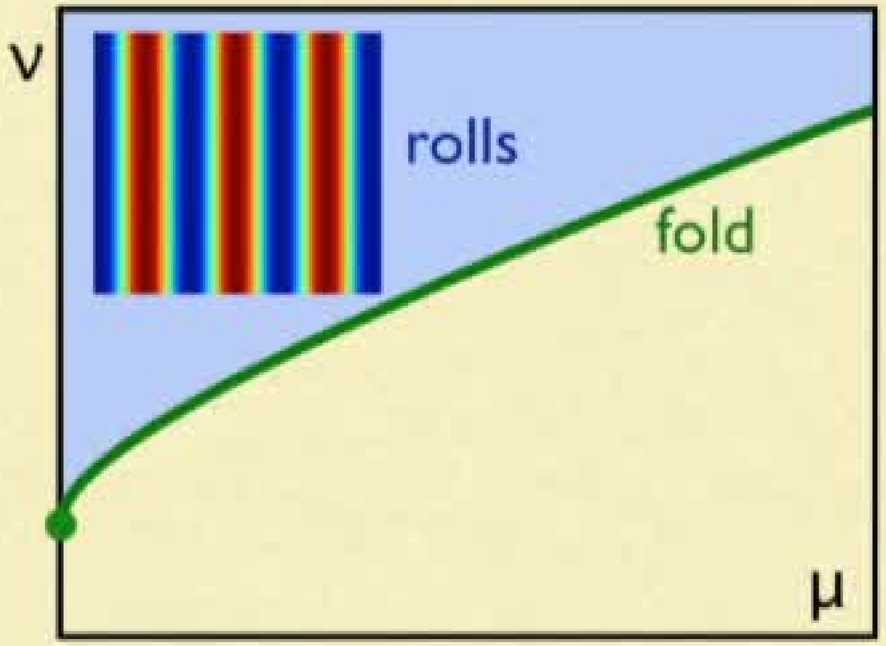

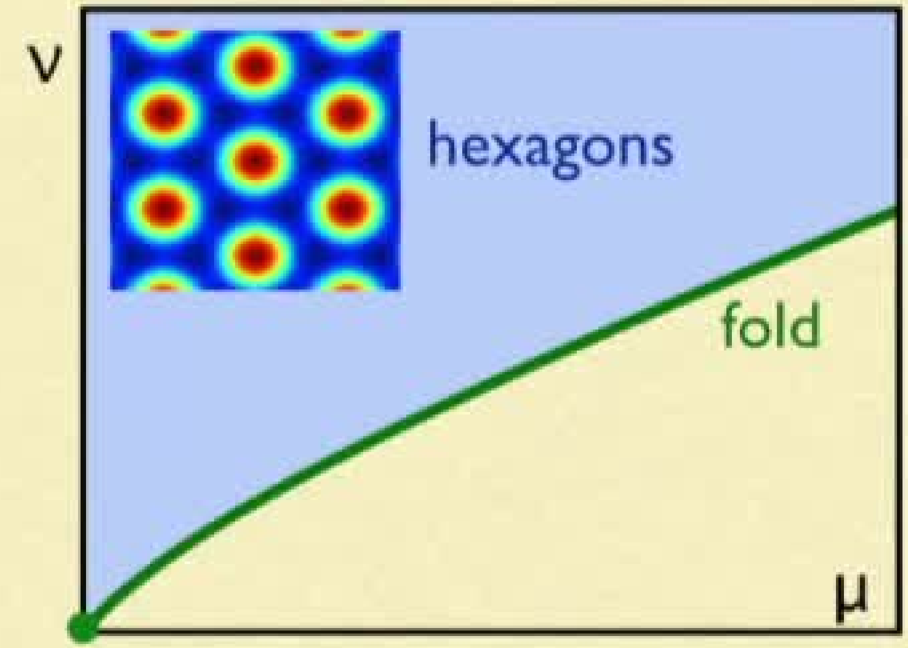

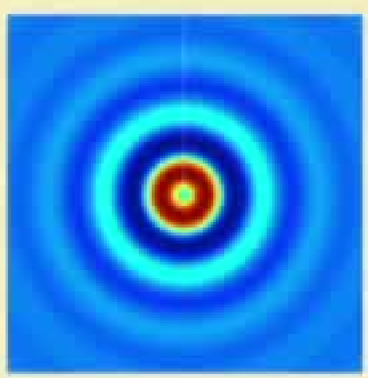

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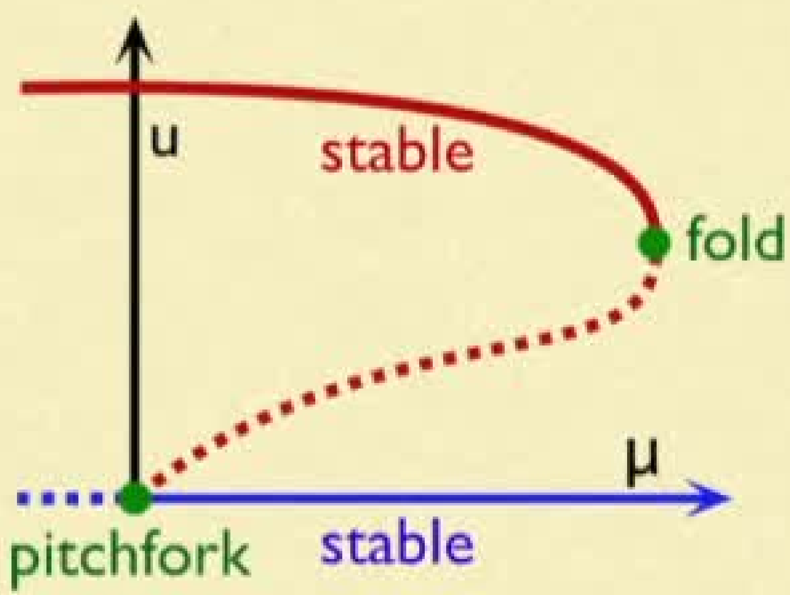
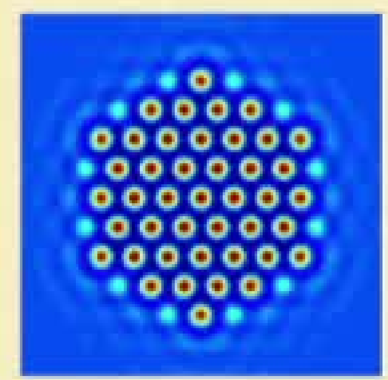
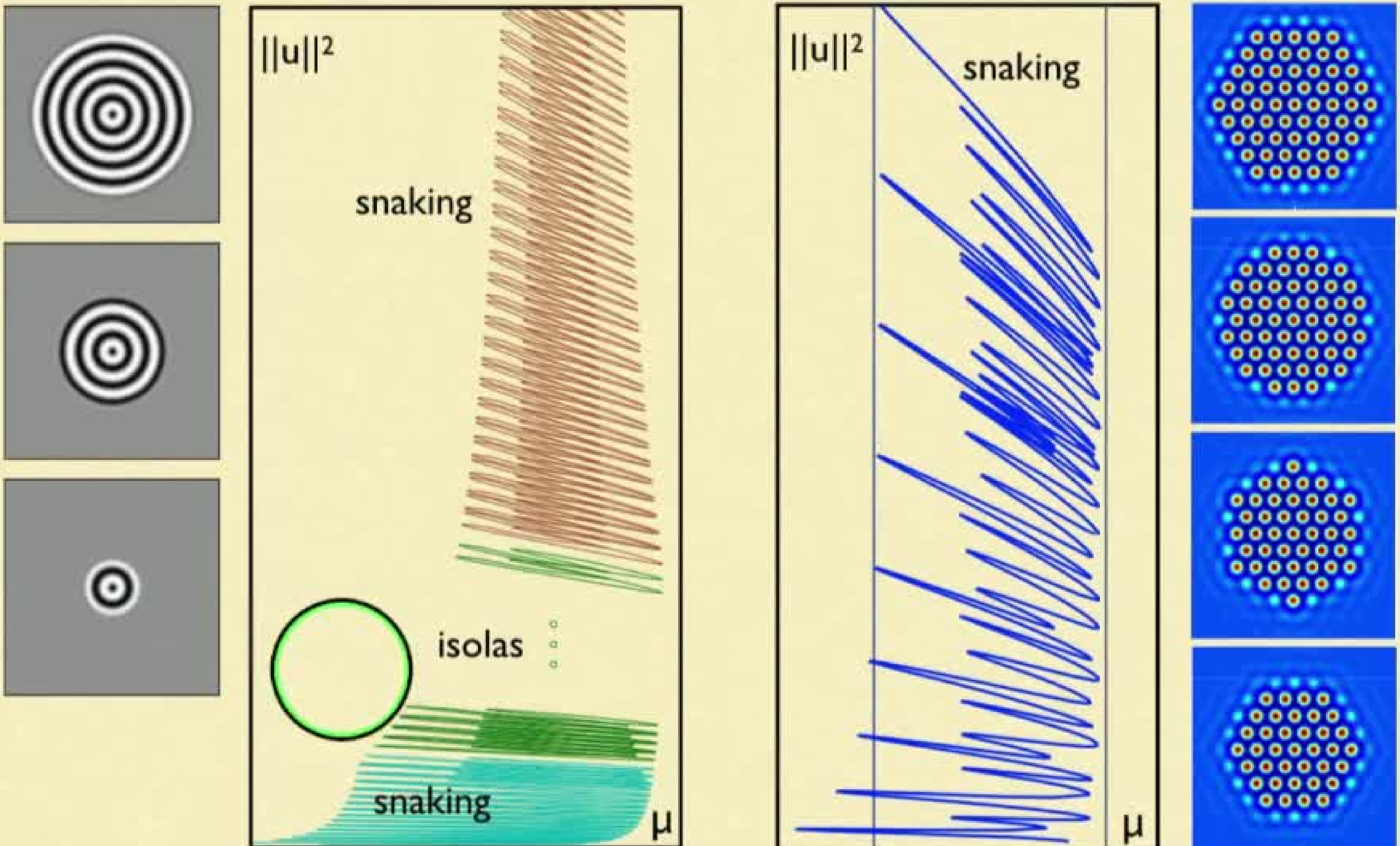




stationary hexagon patch

Snaking diagrams for localized planar patterns



Radial rolls

Hexagon patches

[McCalla & S.], [Lloyd, S., Avitabile, Champneys]

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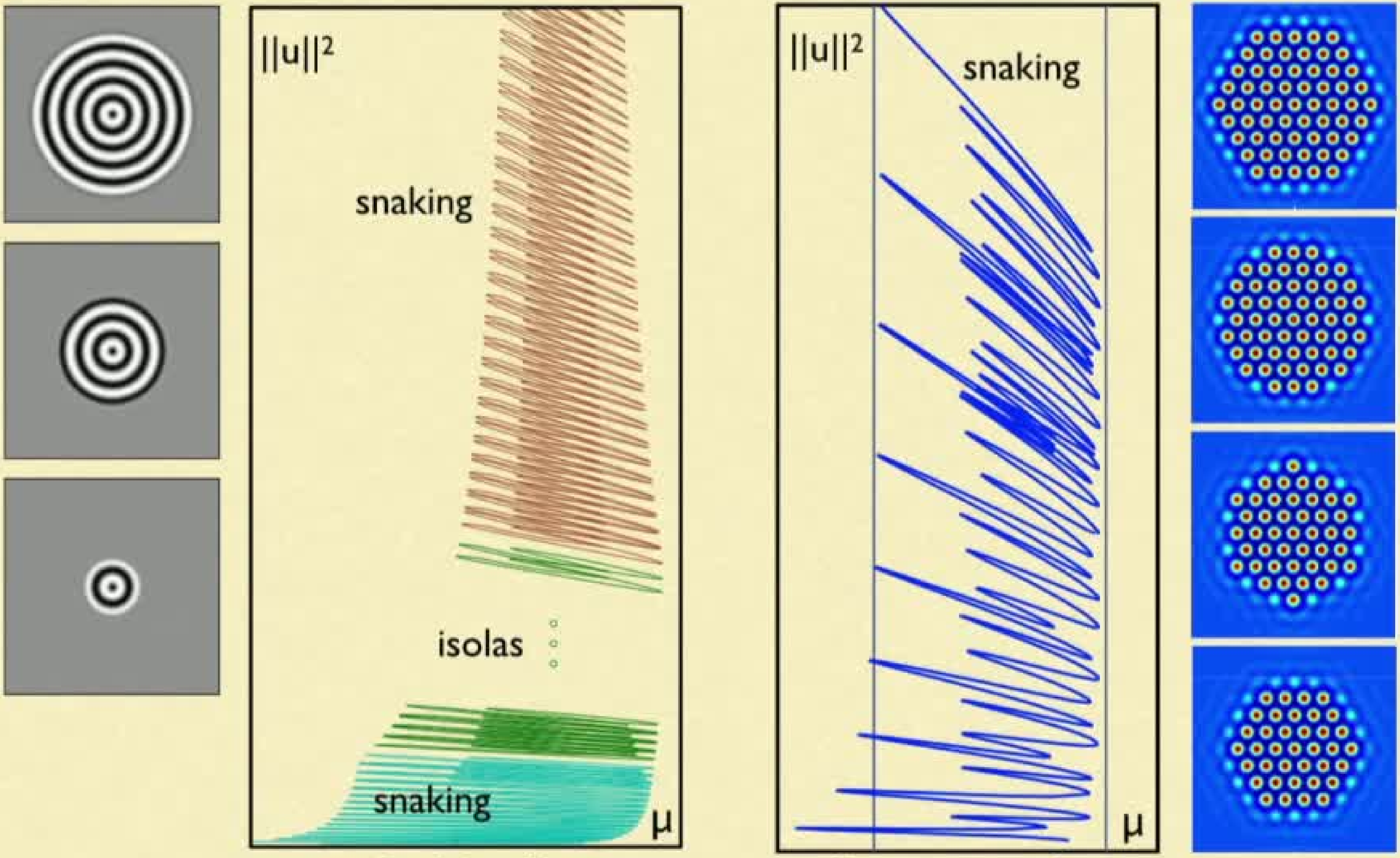

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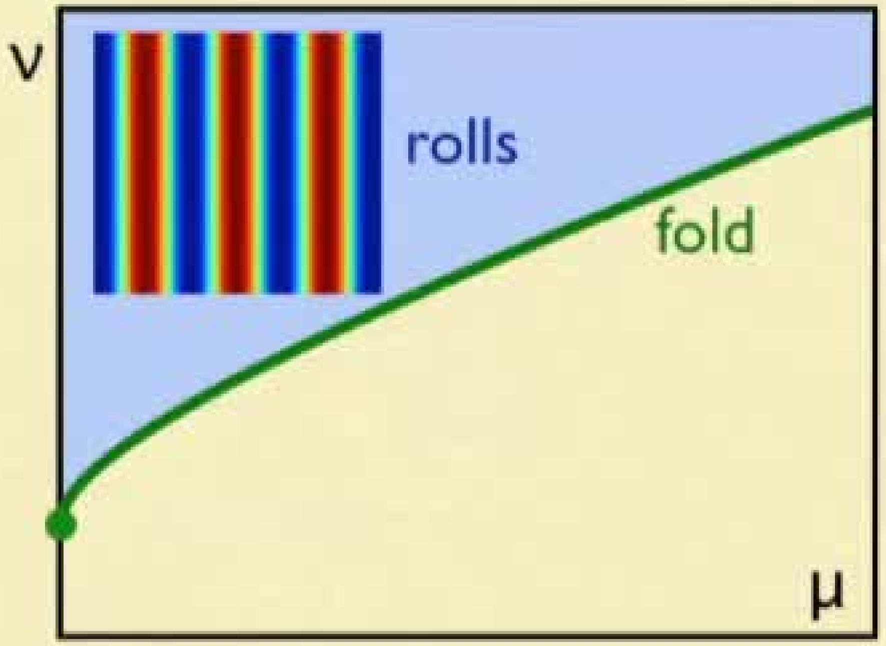

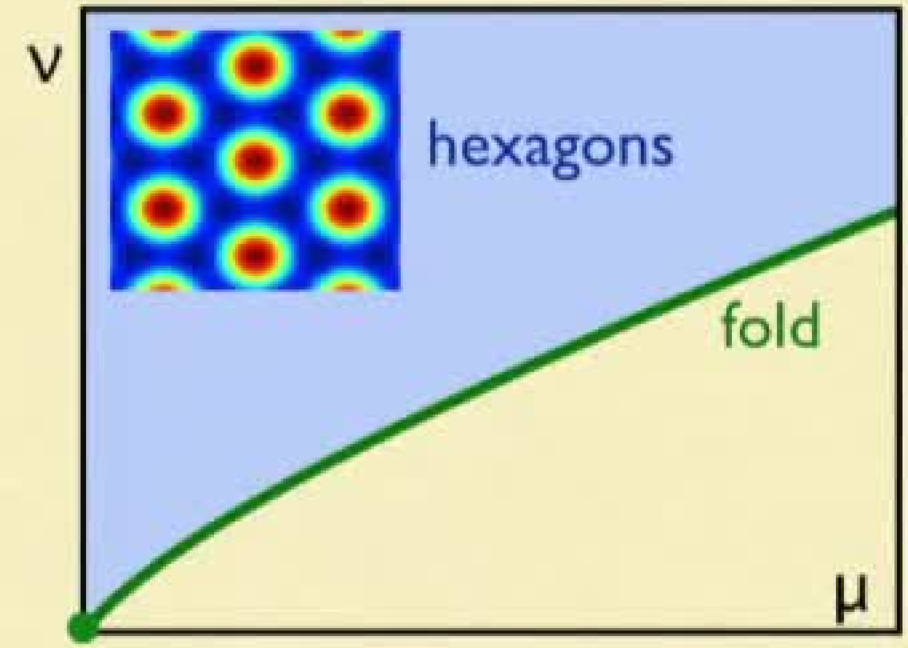

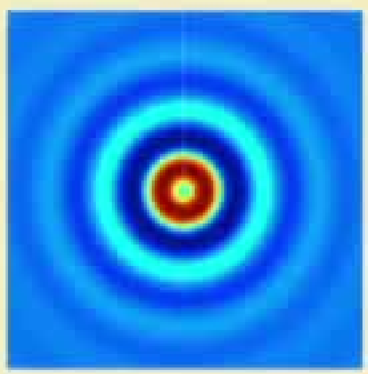

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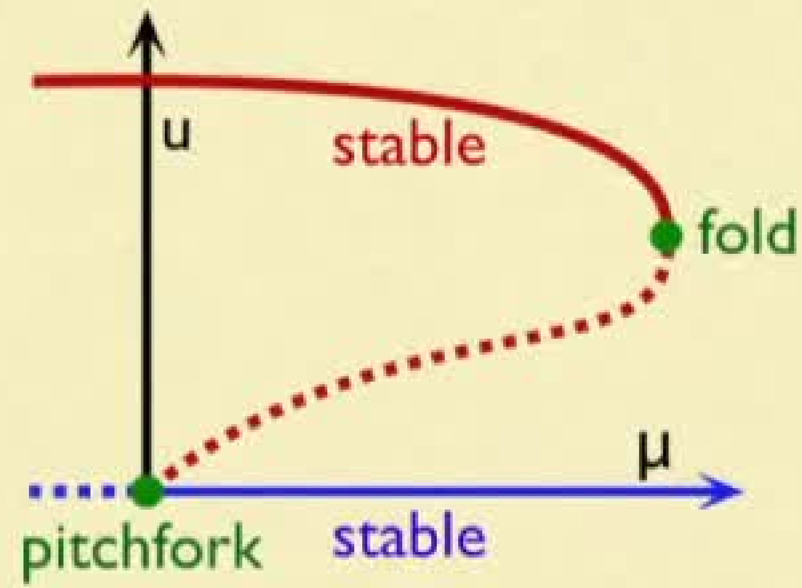
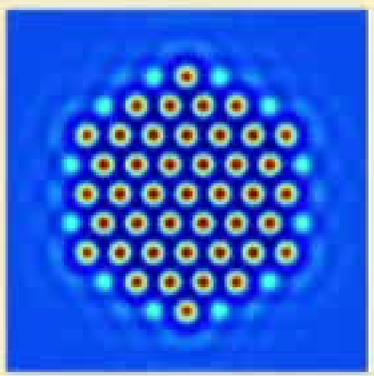
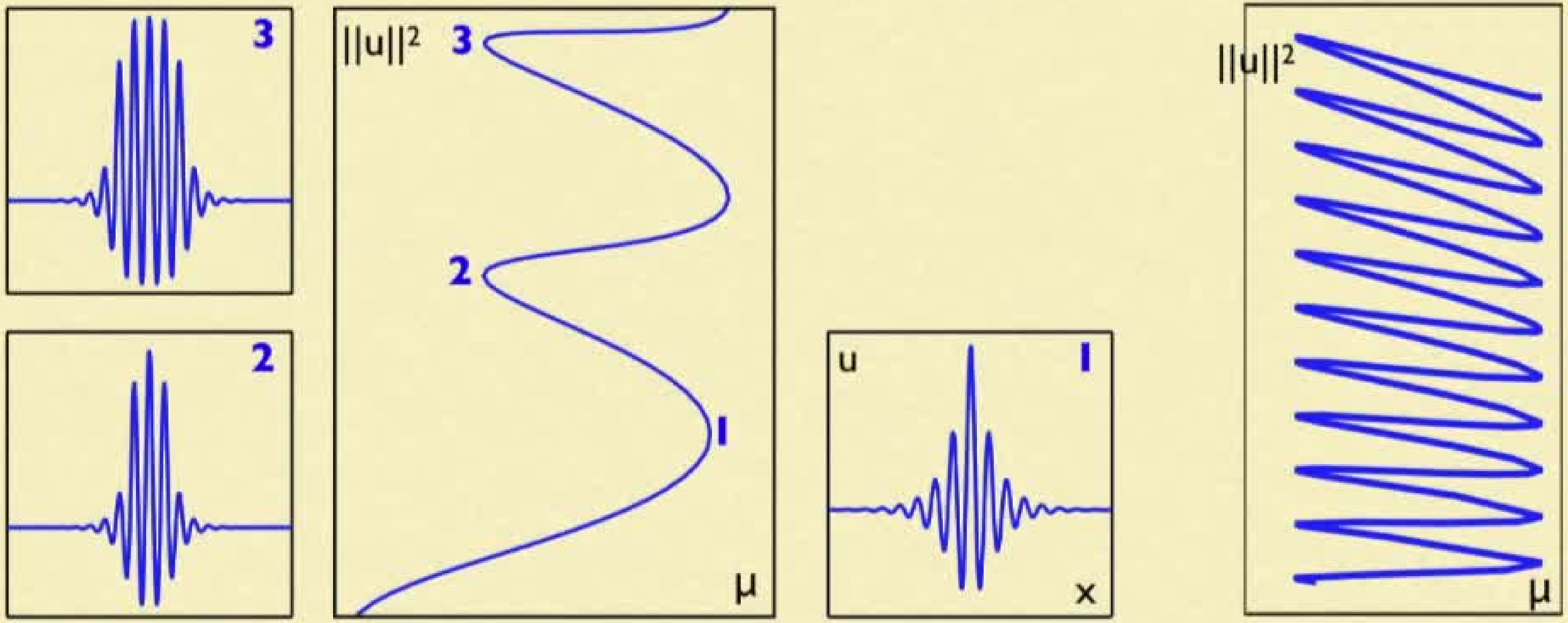




stationary hexagon patch

Bistability: snaking diagrams



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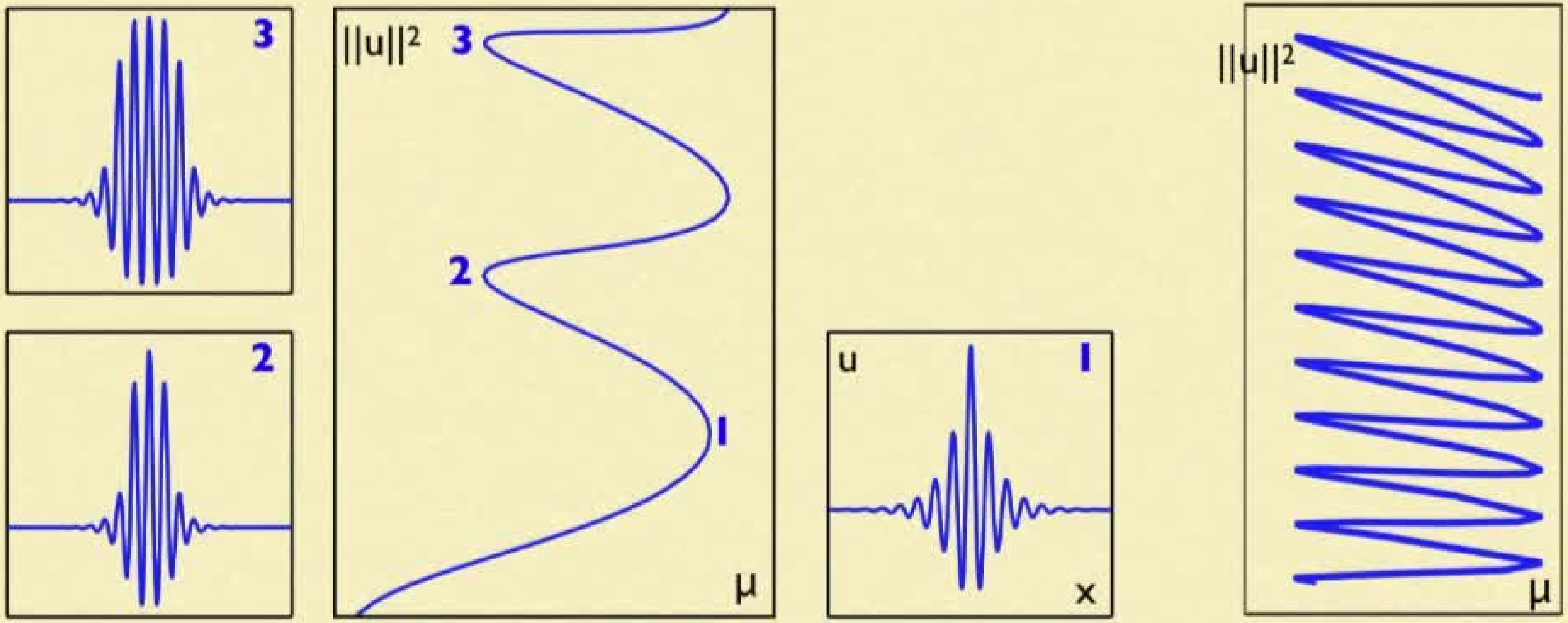

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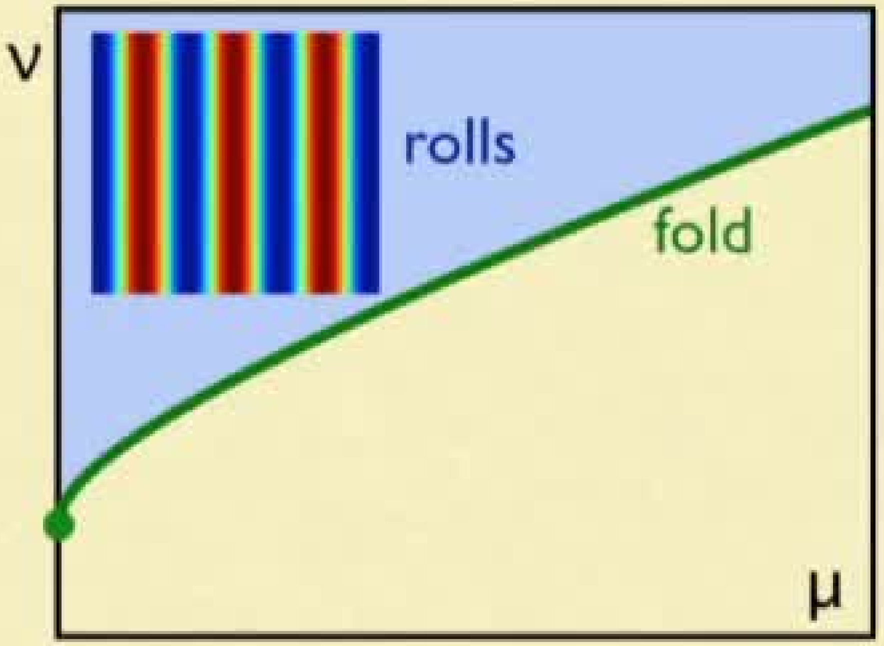
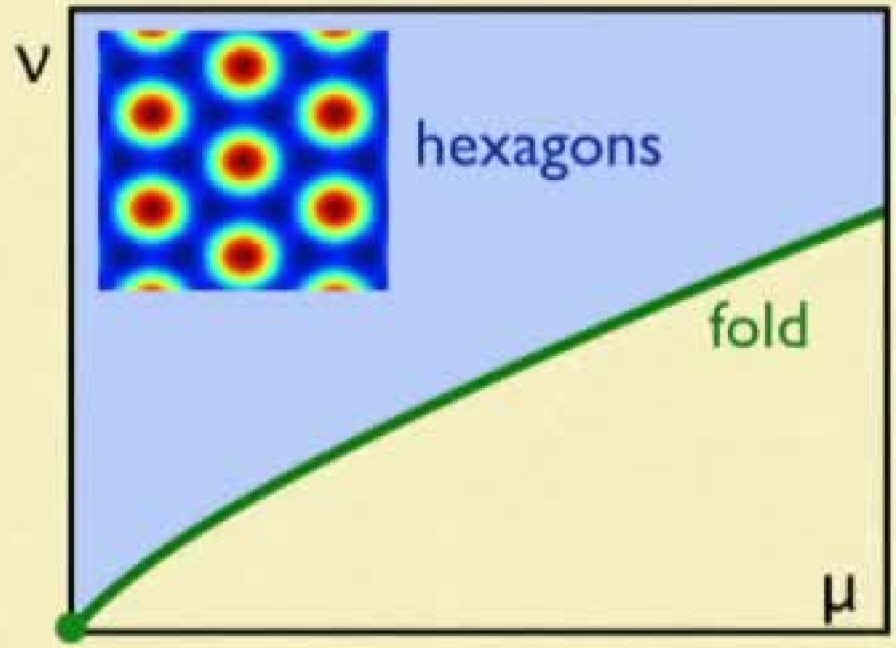
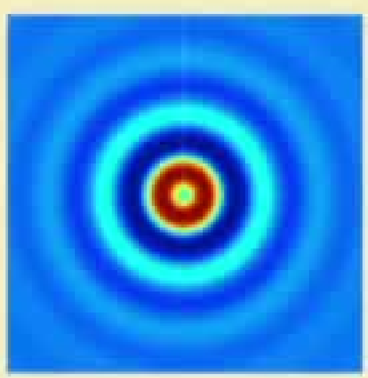
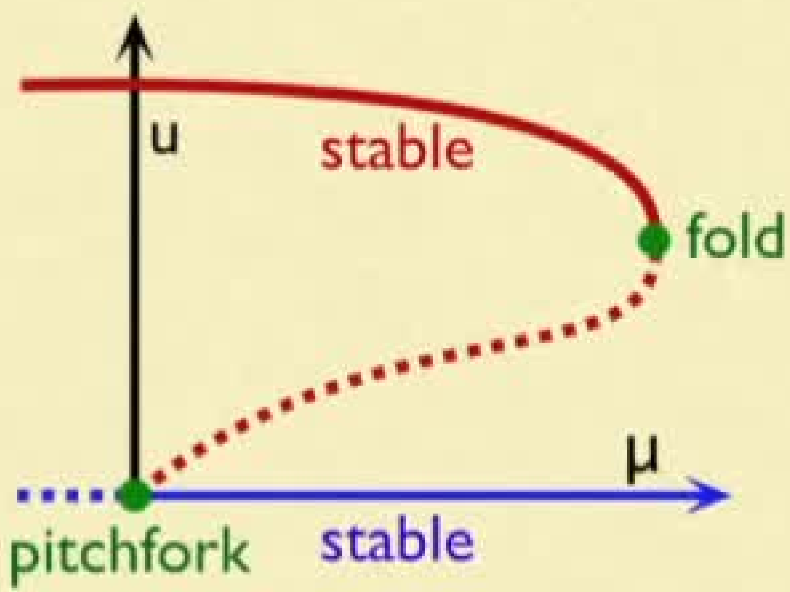
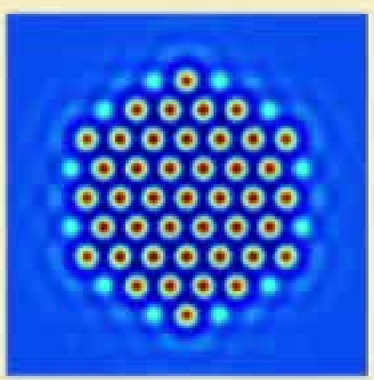






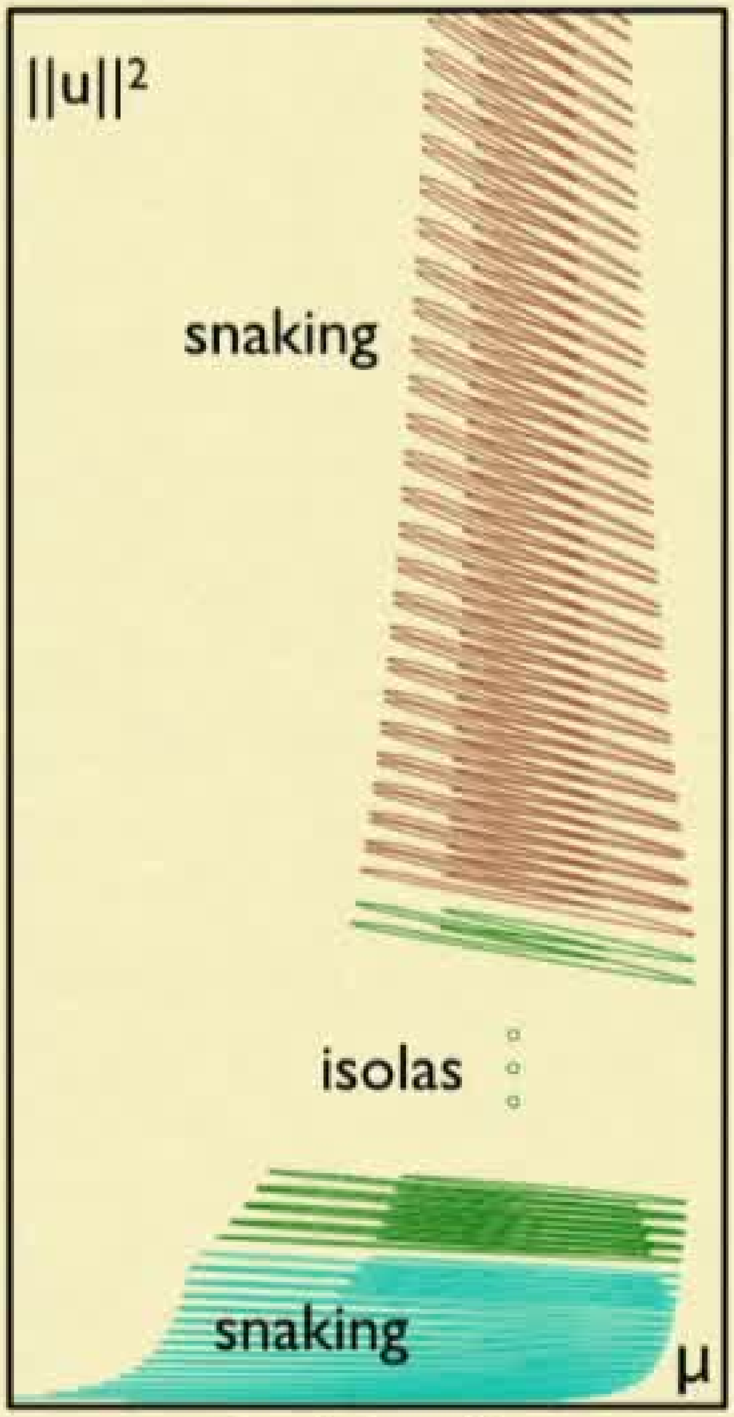
stationary radial roll pattern



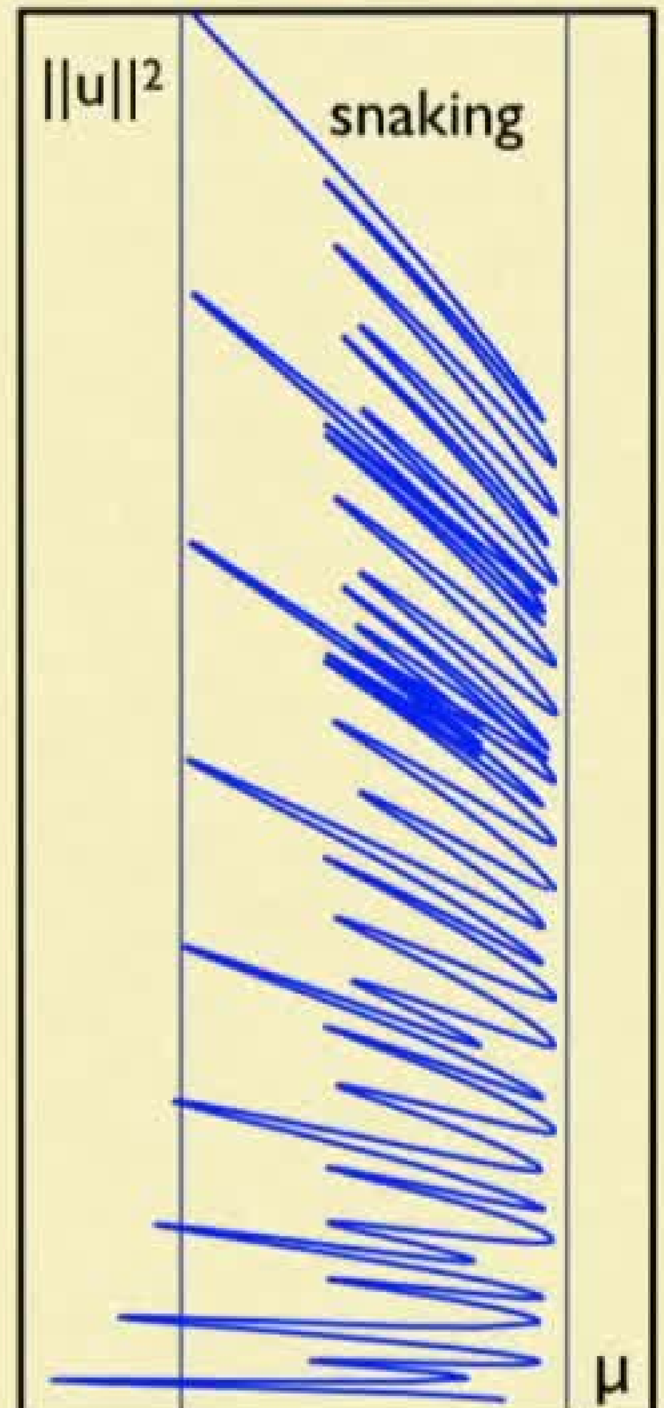


stationary hexagon patch

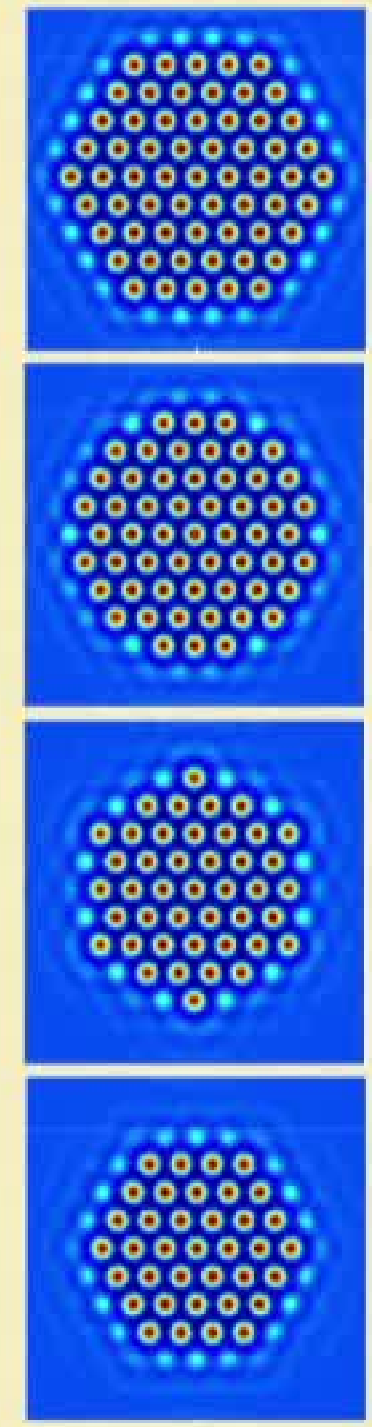
Snaking diagrams for localized planar patterns



Radial rolls



Hexagon patches



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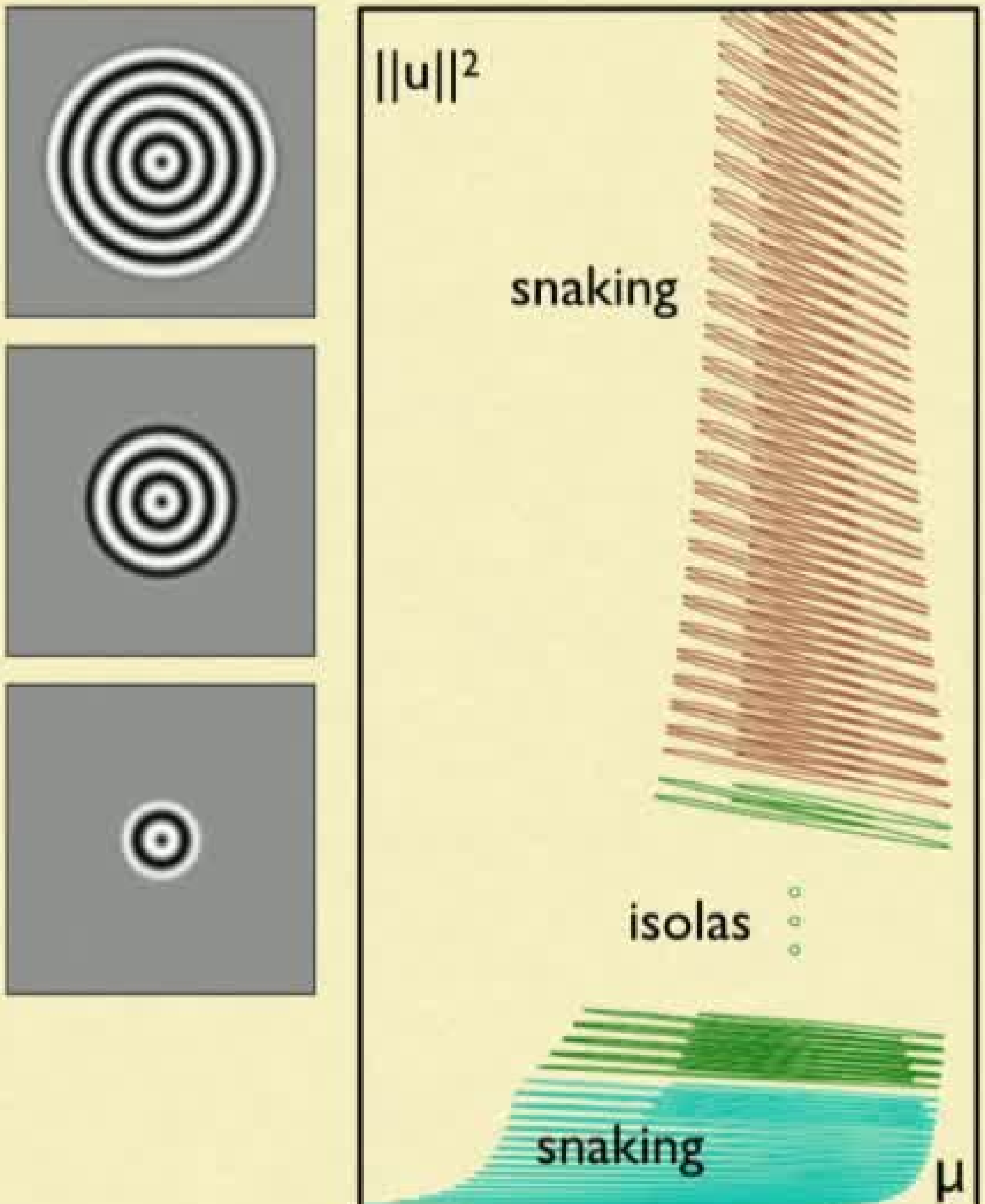

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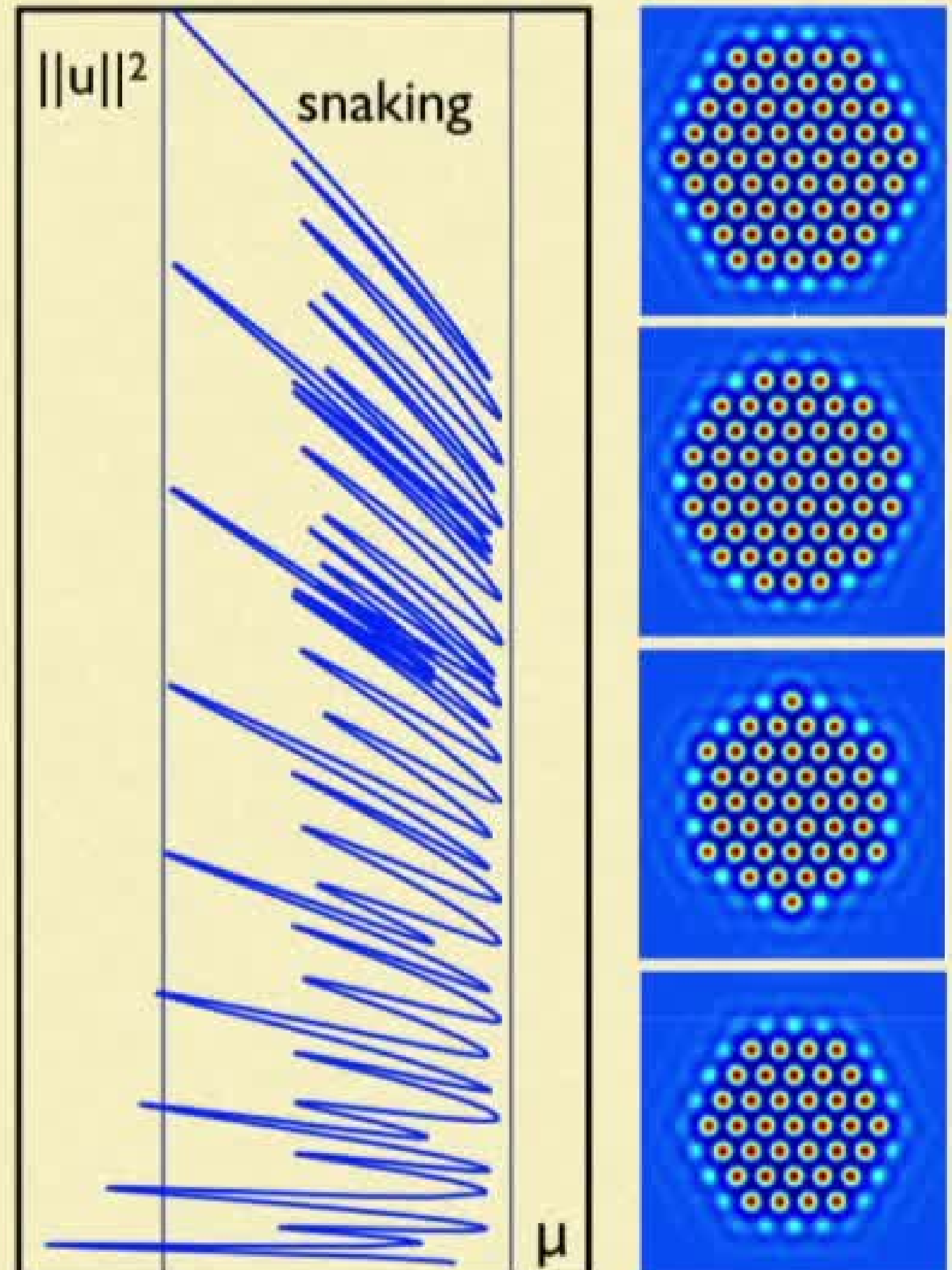

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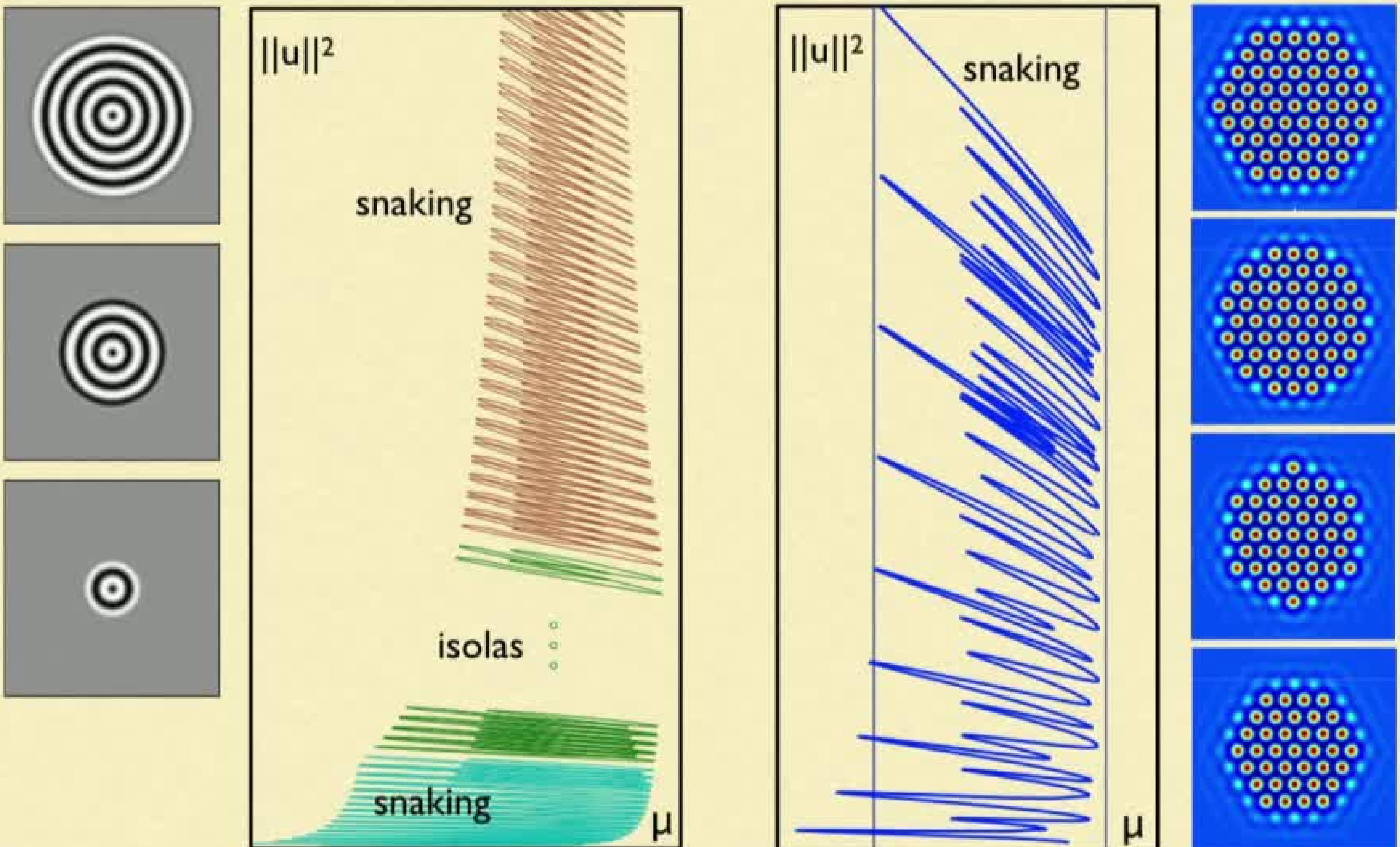

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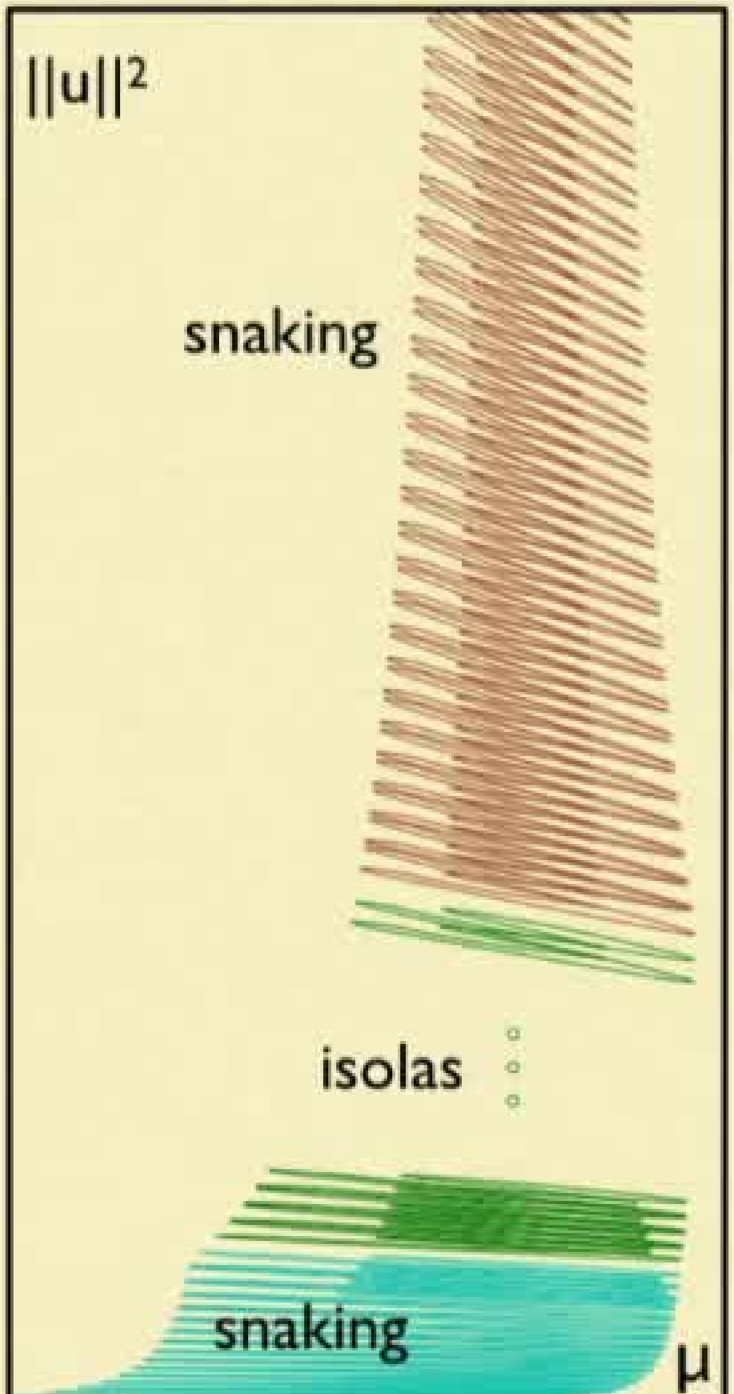

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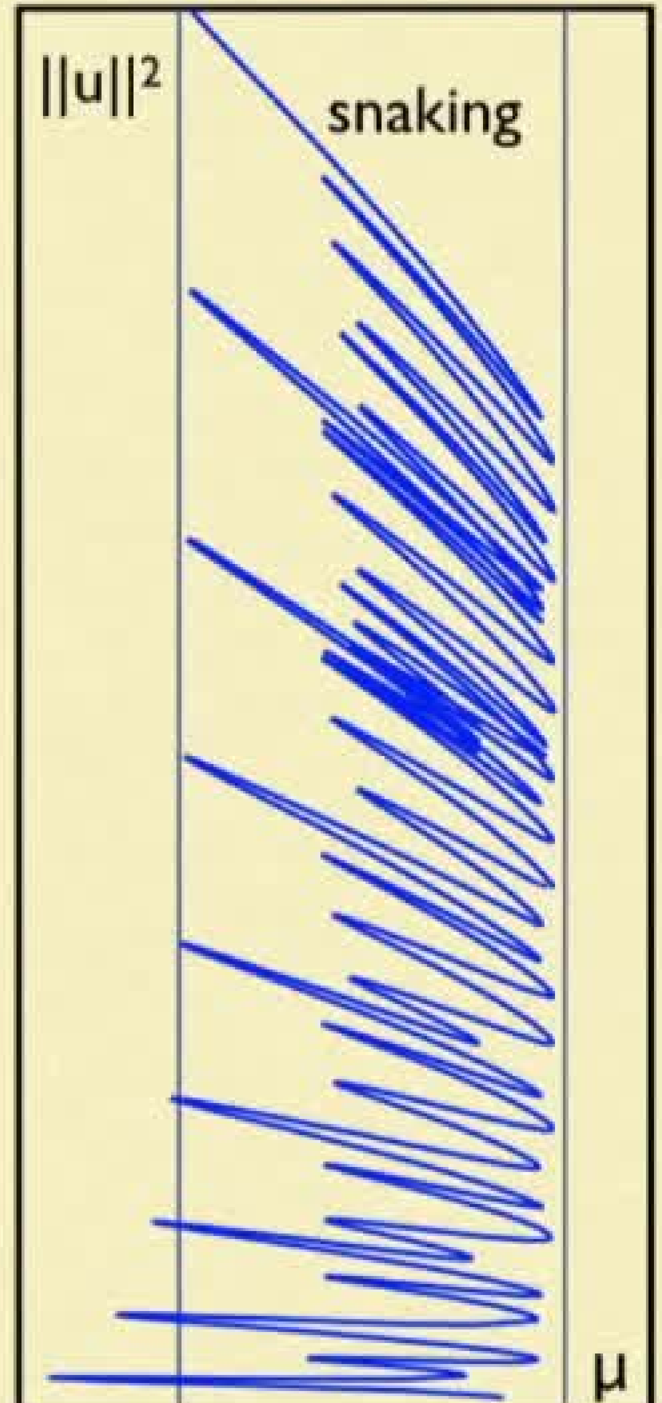

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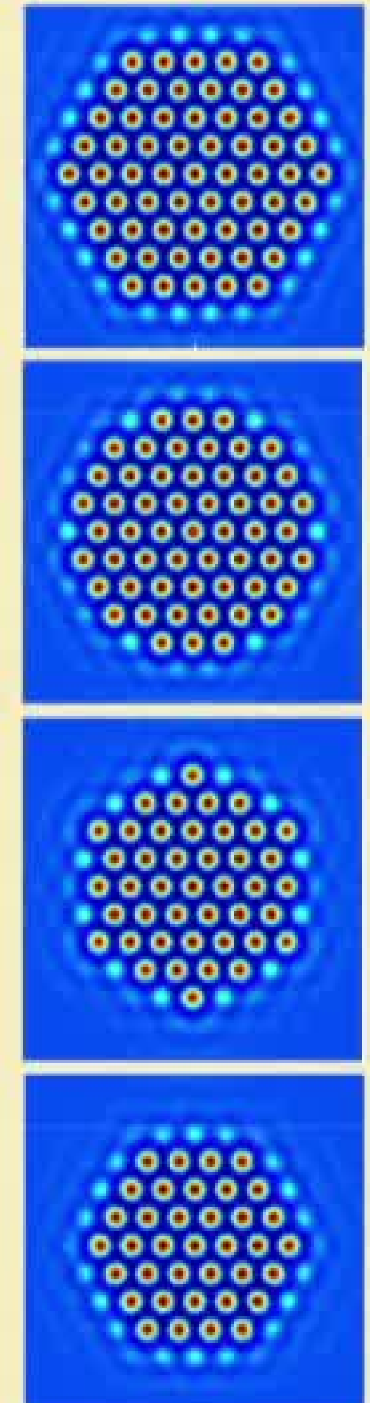

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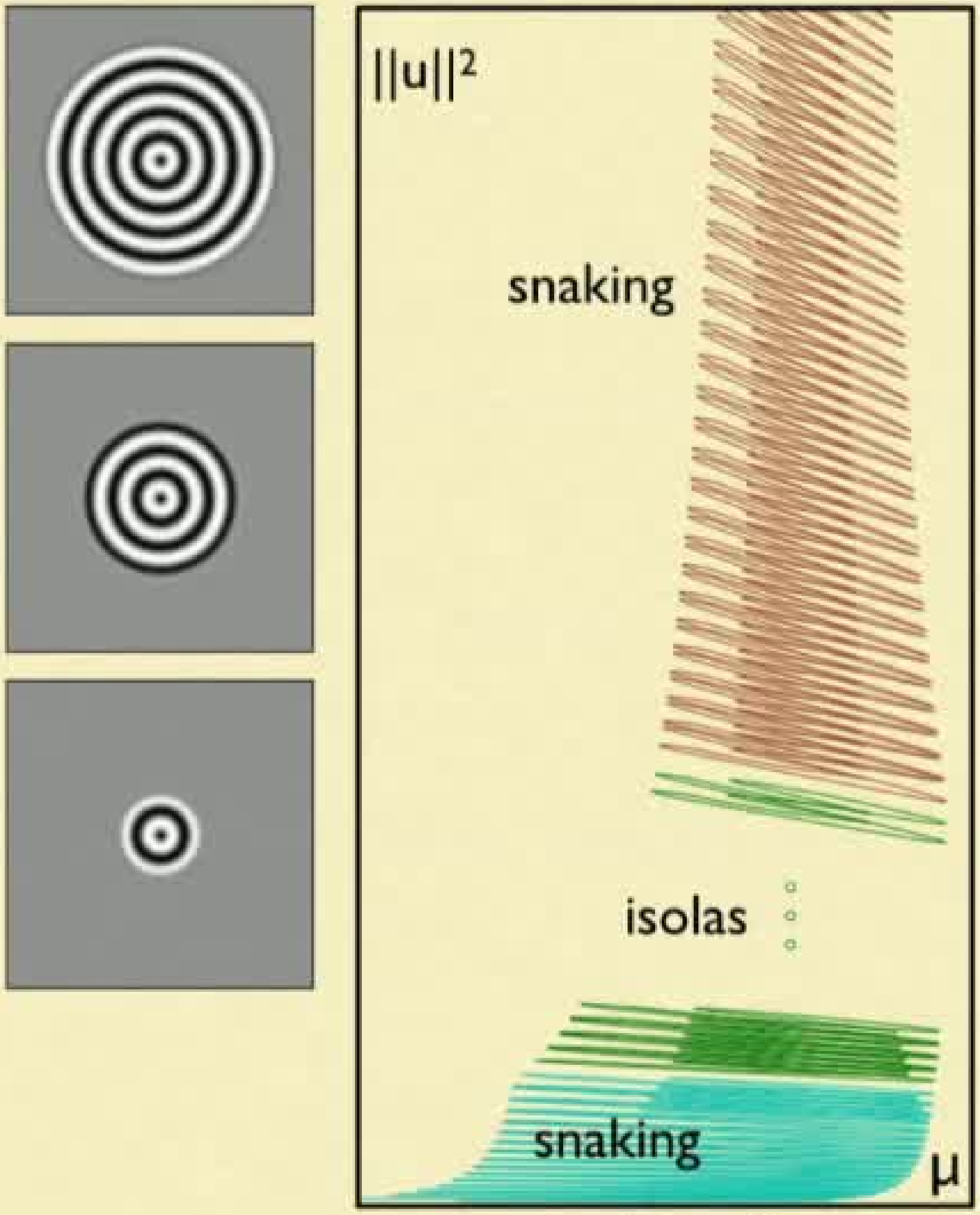

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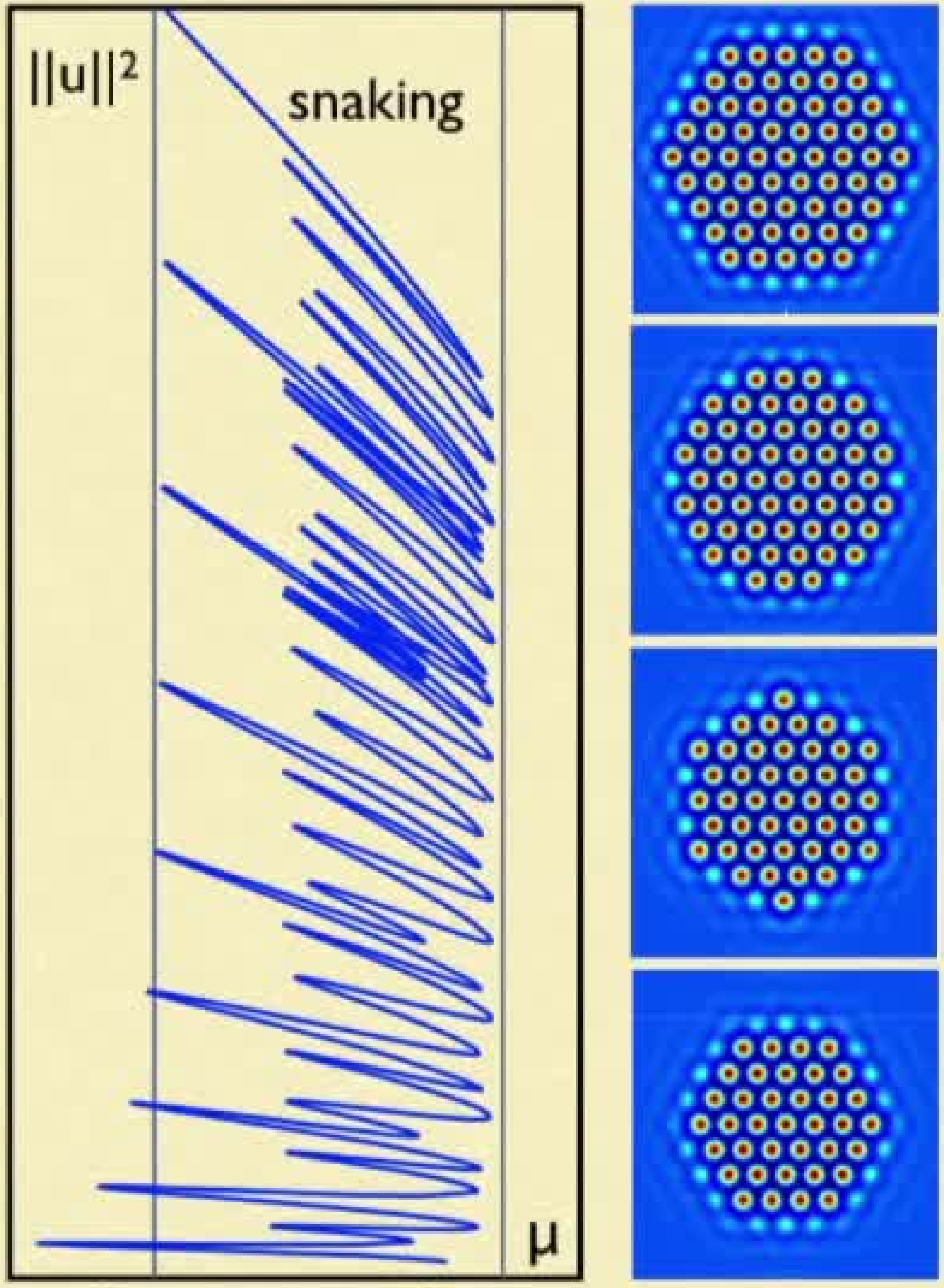

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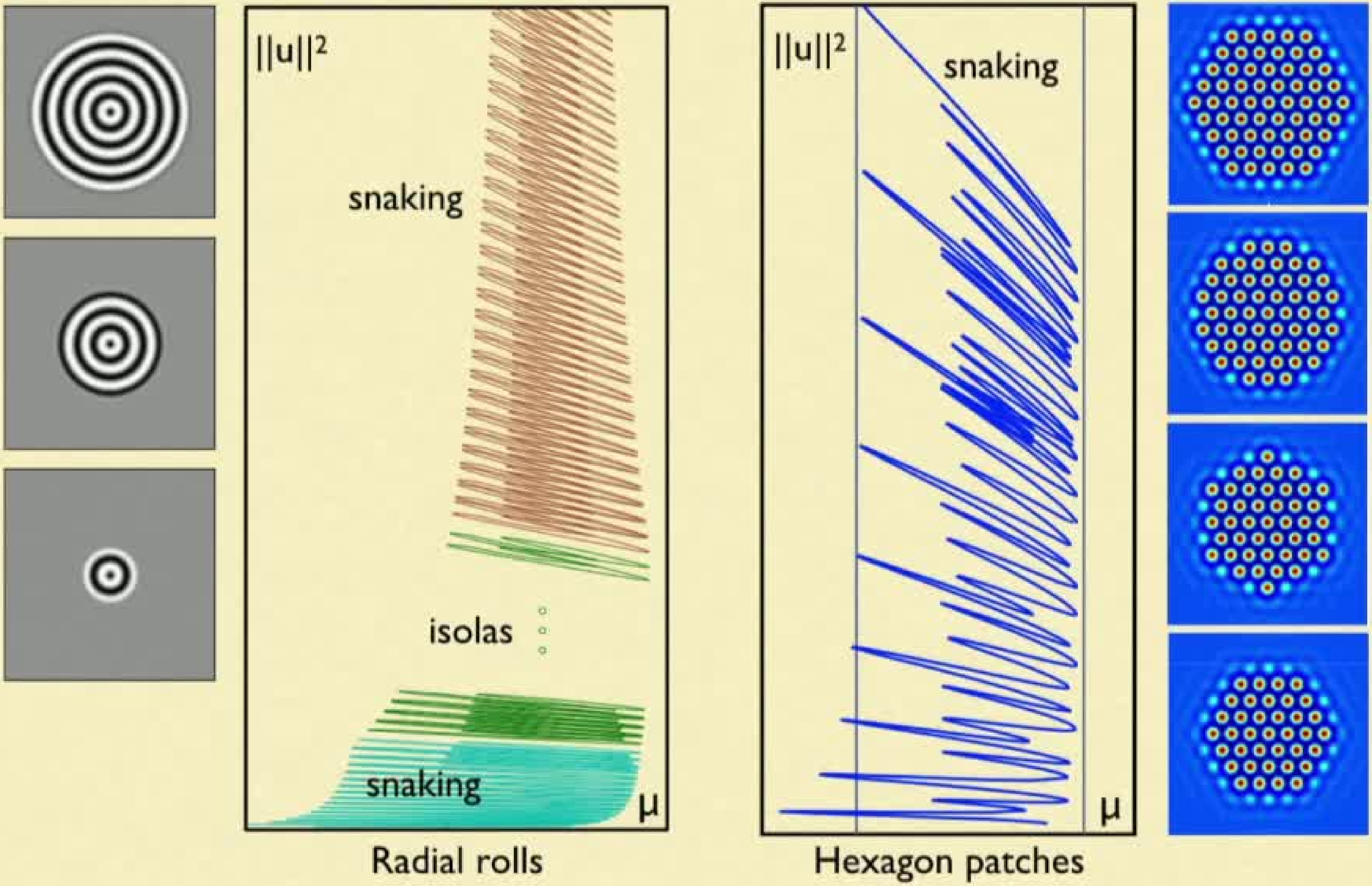

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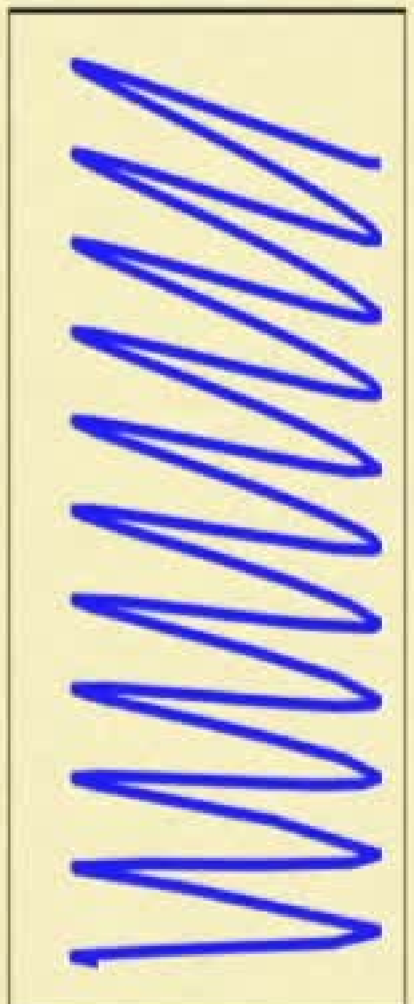
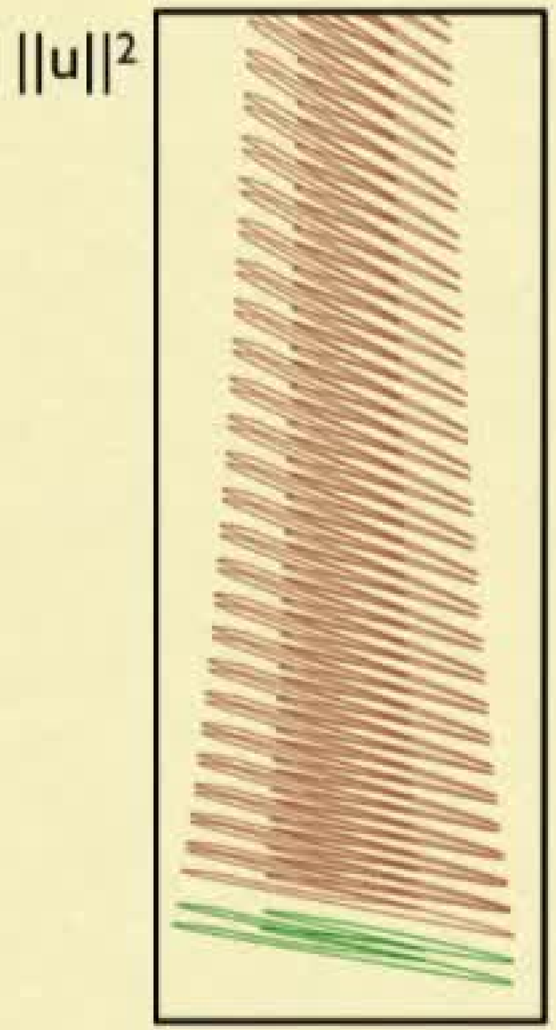
Questions

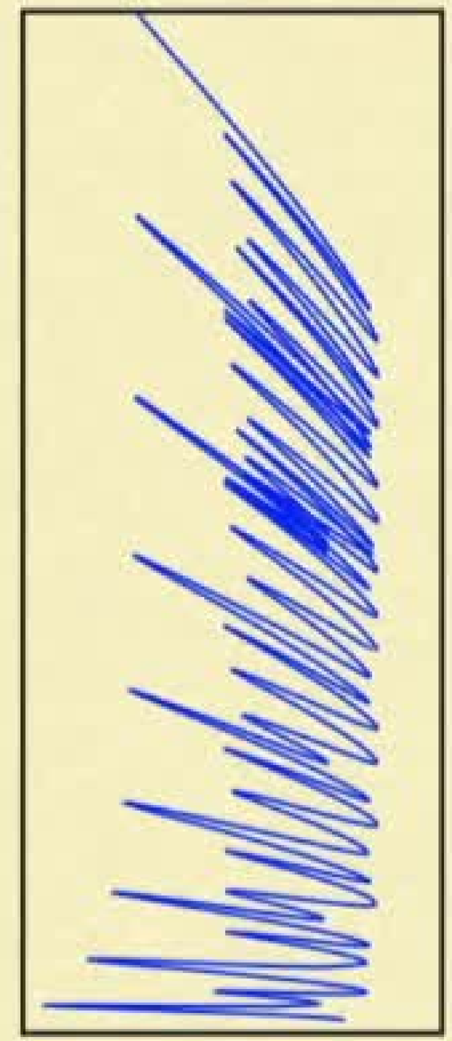
- Localized rolls in 1D: Why do we see snaking?
- Radial rolls in 2D: Why do we see collapsed snaking?
- Hexagon patches in 2D: Can we analyse these patterns?



Rolls 1D



Radial 2D



Hexagons 2D

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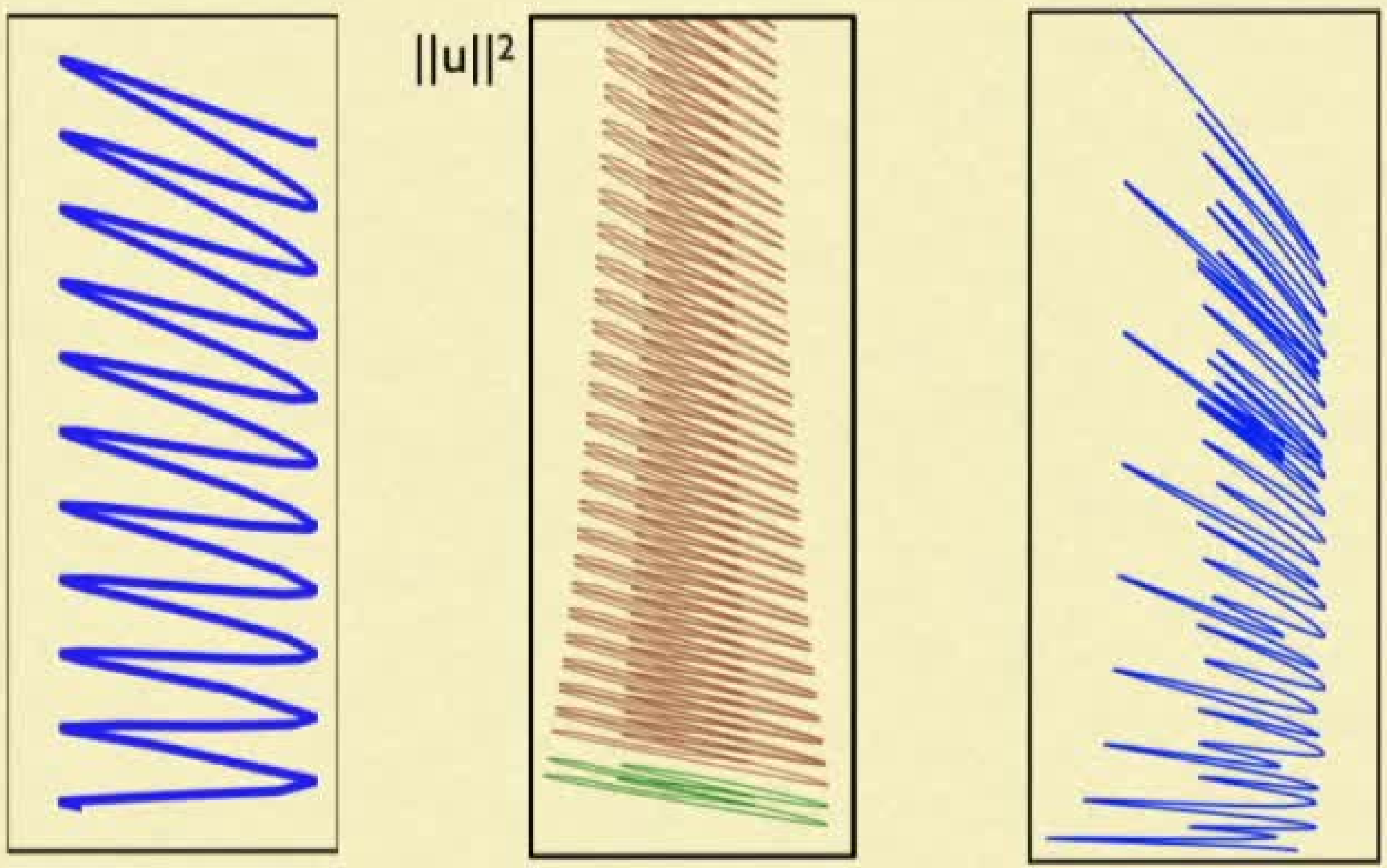
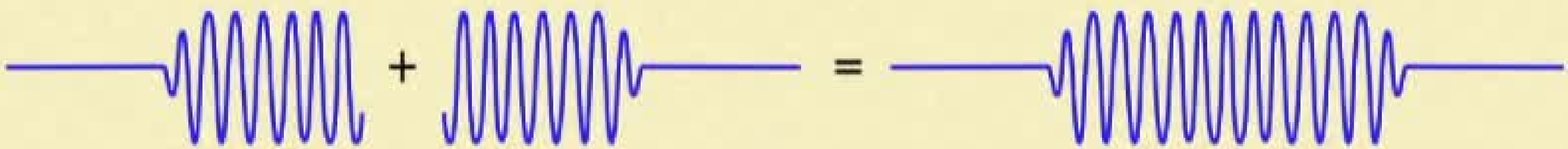


Rolls 1D   Radial 2D   Hexagons 2D

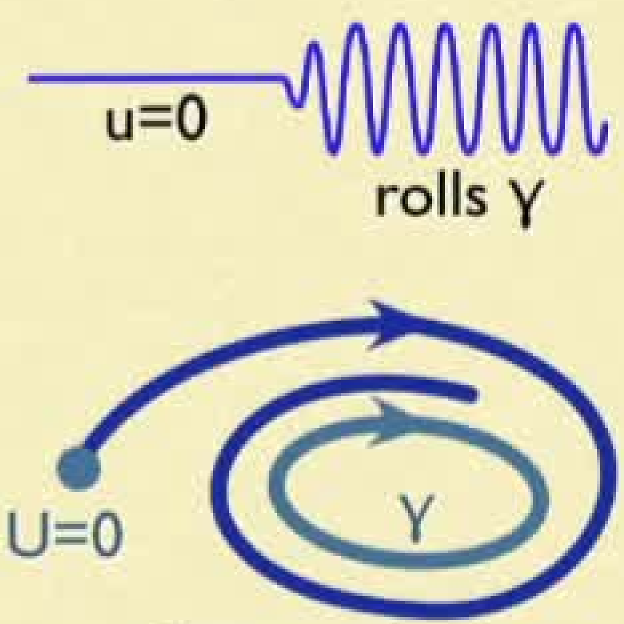
Geometry and analysis

$$0 = -[1 + \partial_x^2]^2 u - \mu u + \nu u^2 - u^3$$

- Symmetry: $x \longmapsto -x$
- Gradient-like structure $u_t = -\nabla E(u,\mu)$



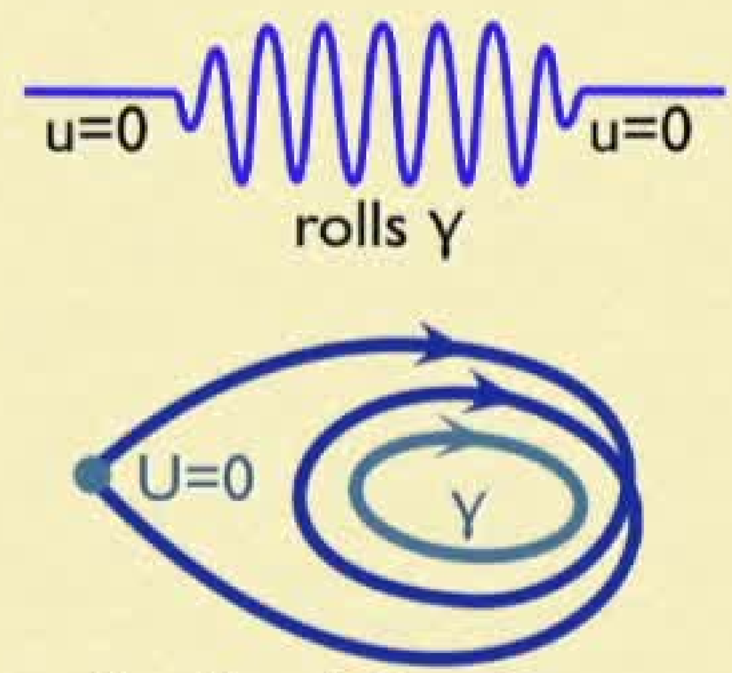
Glue fronts and backs together to create localized structures



fronts + backs

Dynamical system:
$U_x = f(U,\mu)$

- Reversibility: $x \longmapsto -x$
- Hamiltonian $H(U,\mu)$

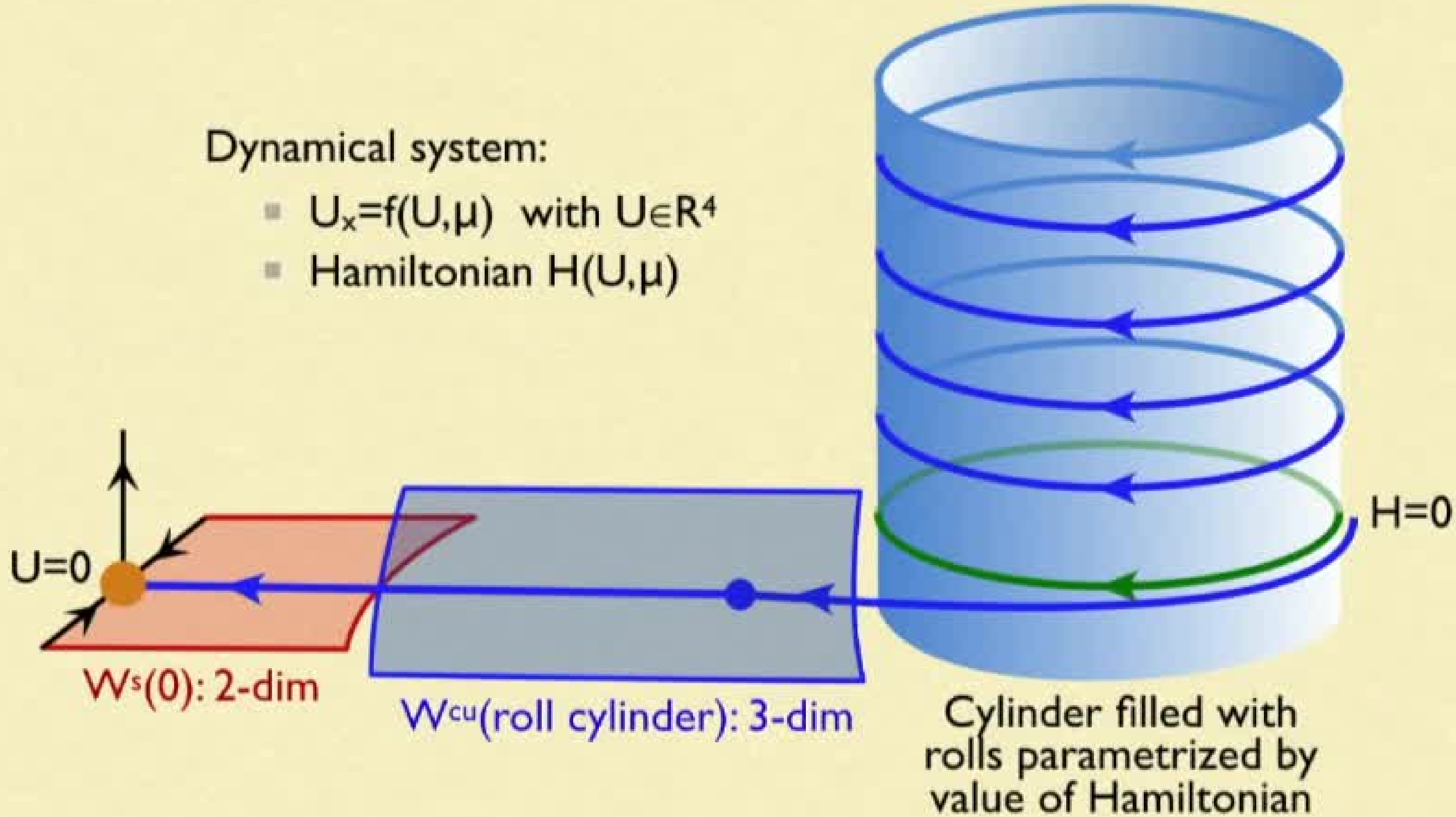


localized structures

Geometry and analysis

Dynamical system:

- $U_x = f(U, \mu)$ with $U \in R^4$
- Hamiltonian $H(U, \mu)$



- Robust intersection of $W^s(0)$ and W^{cu}(rolls) in R^4!
- Can glue fronts and backs together to construct localized rolls and establish snaking diagram
- Intersections disappear at tangencies of these manifolds

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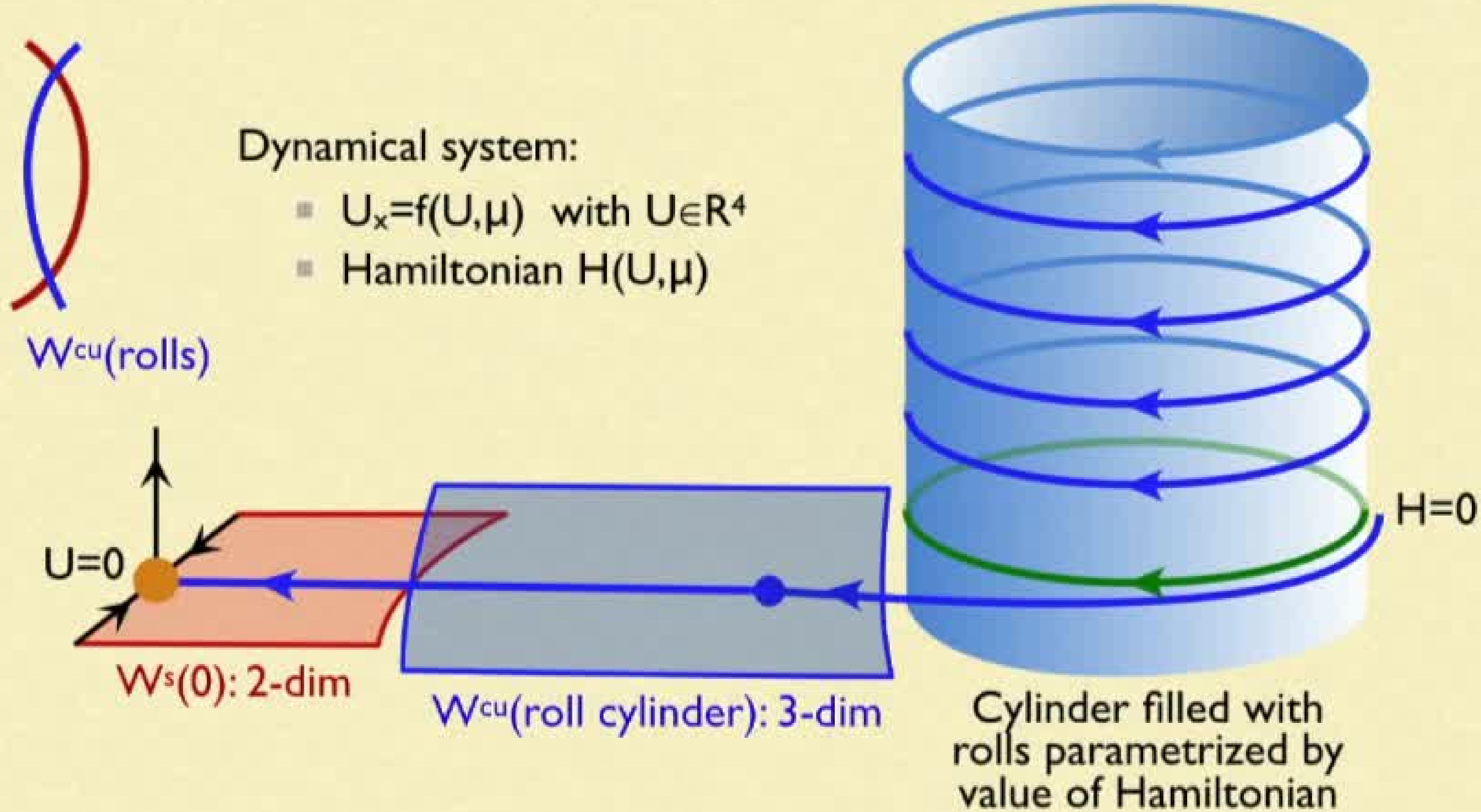

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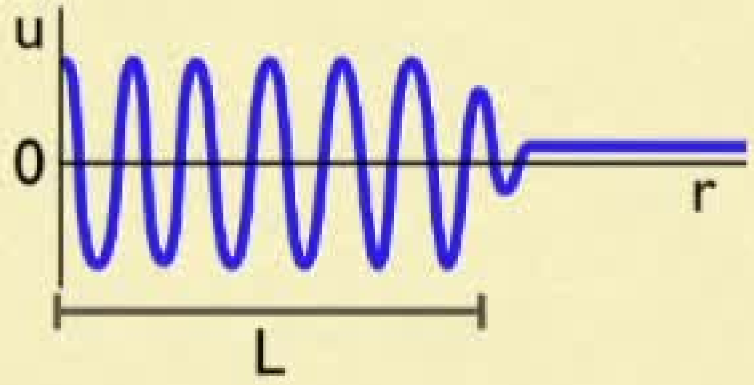
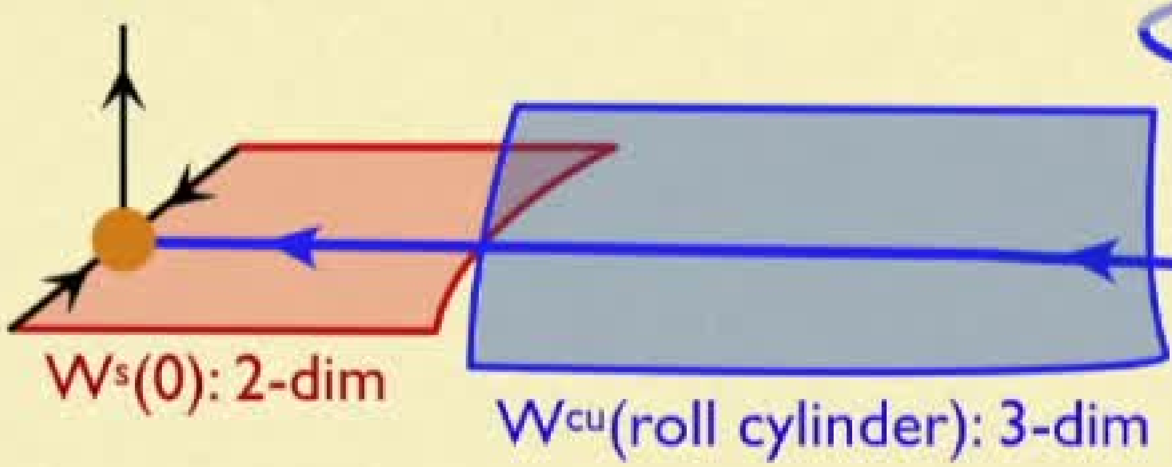
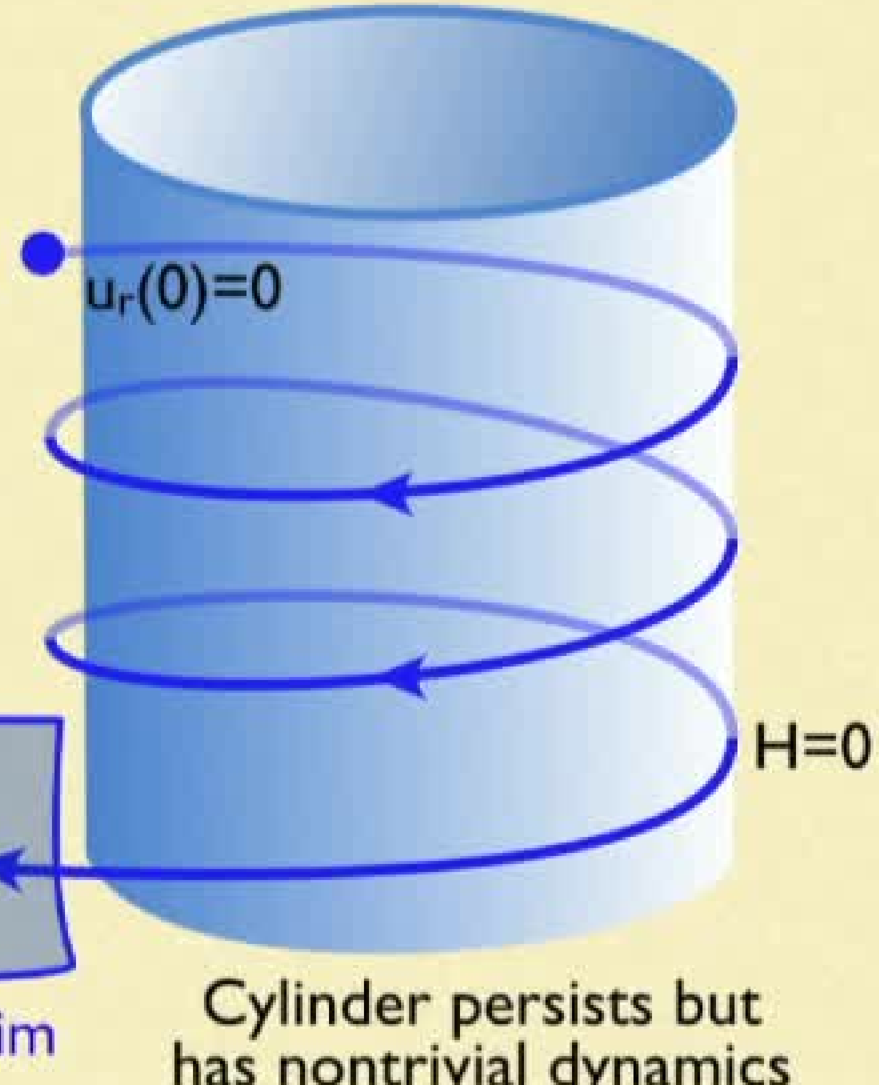
Radial planar rolls



$$0 = -\left(1 + \frac{1}{r}\partial_r + \partial_r^2\right)^2 u - \mu u + \nu u^2 - u^3, \; r \geq 0$$

- Radial u_r/r term is not a small perturbation
- Focus on solutions with $L \gg 1$
- $1/r$ dynamics on persisting cylinder allows solutions to leave cylinder after "time" $\log L$





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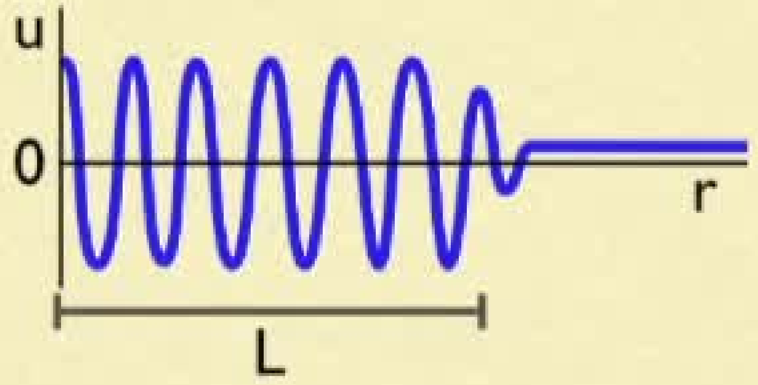

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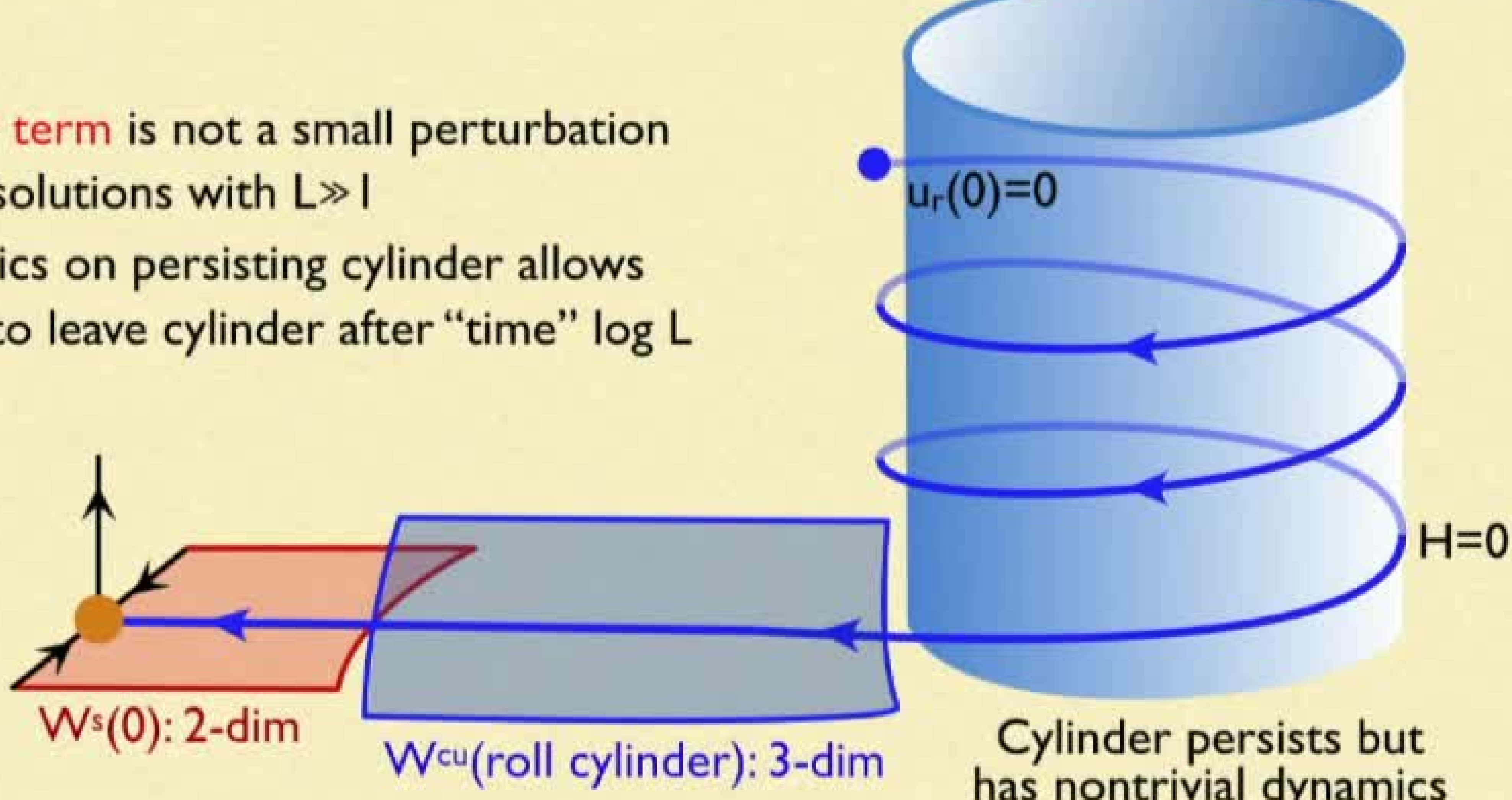
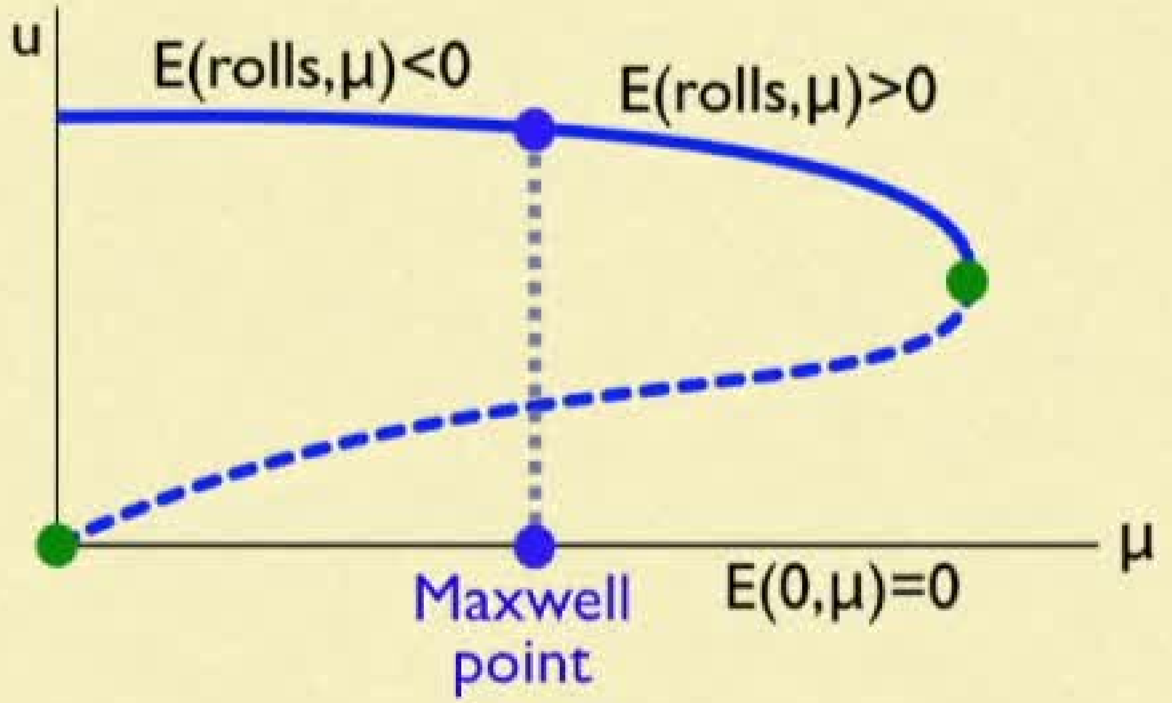


PDE energy and Maxwell point

$$u_t = -[1 + \partial_x^2]^2 u - \mu u + \nu u^2 - u^3 = -\nabla E(u, \mu)$$

Energy E(u,μ):	Decreases strictly in time t along non-stationary solutions
Hamiltonian H(U,μ):	Conserved pointwise in x along stationary solutions
Maxwell point μ_{Max}:	Energy of rolls with H=0 vanishes



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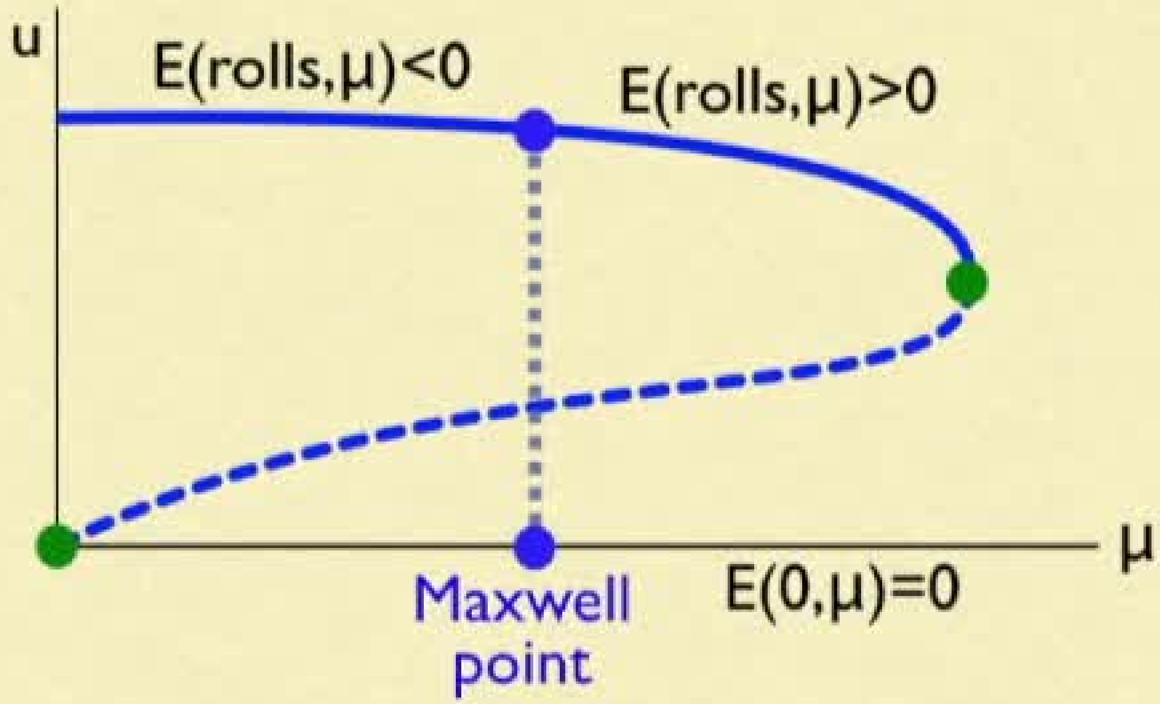

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- Radial pulses with large plateaus ($L \gg 1$) can exist only near the Maxwell point
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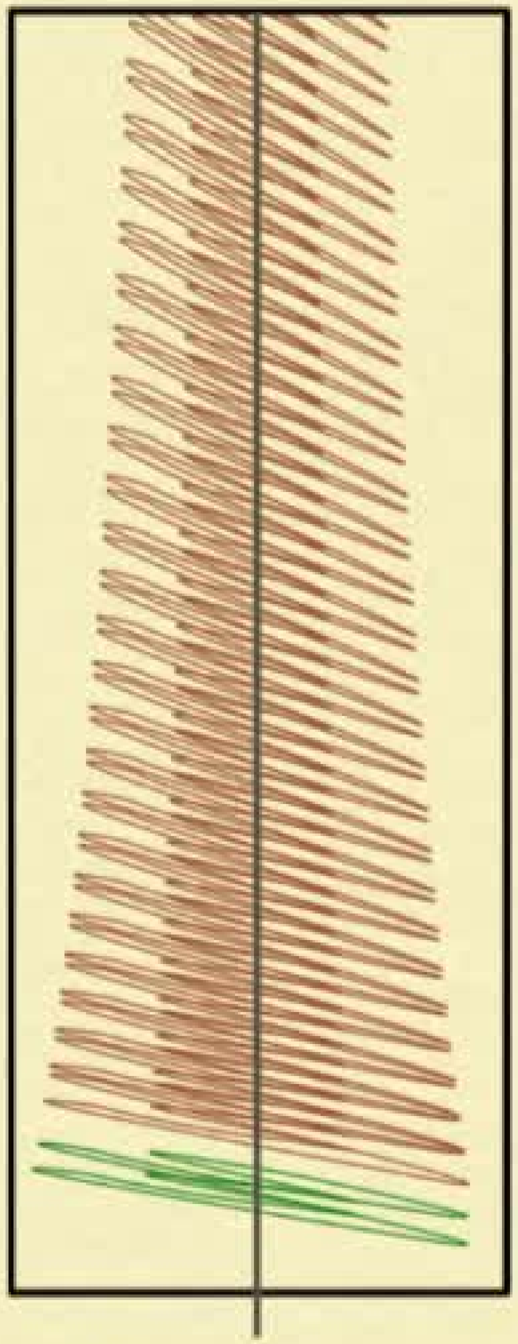

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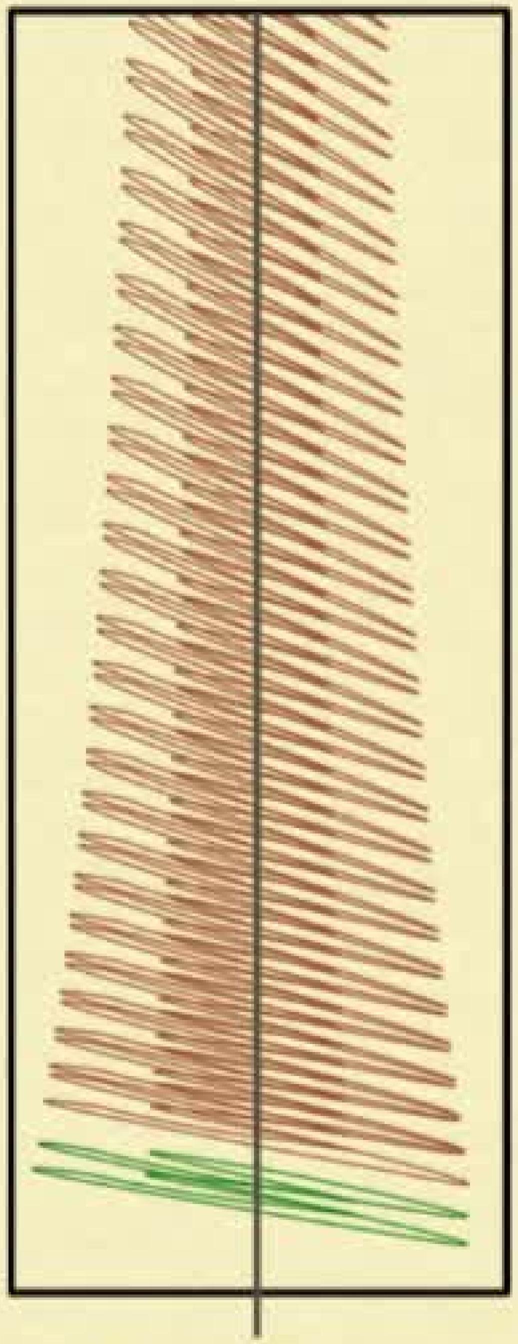

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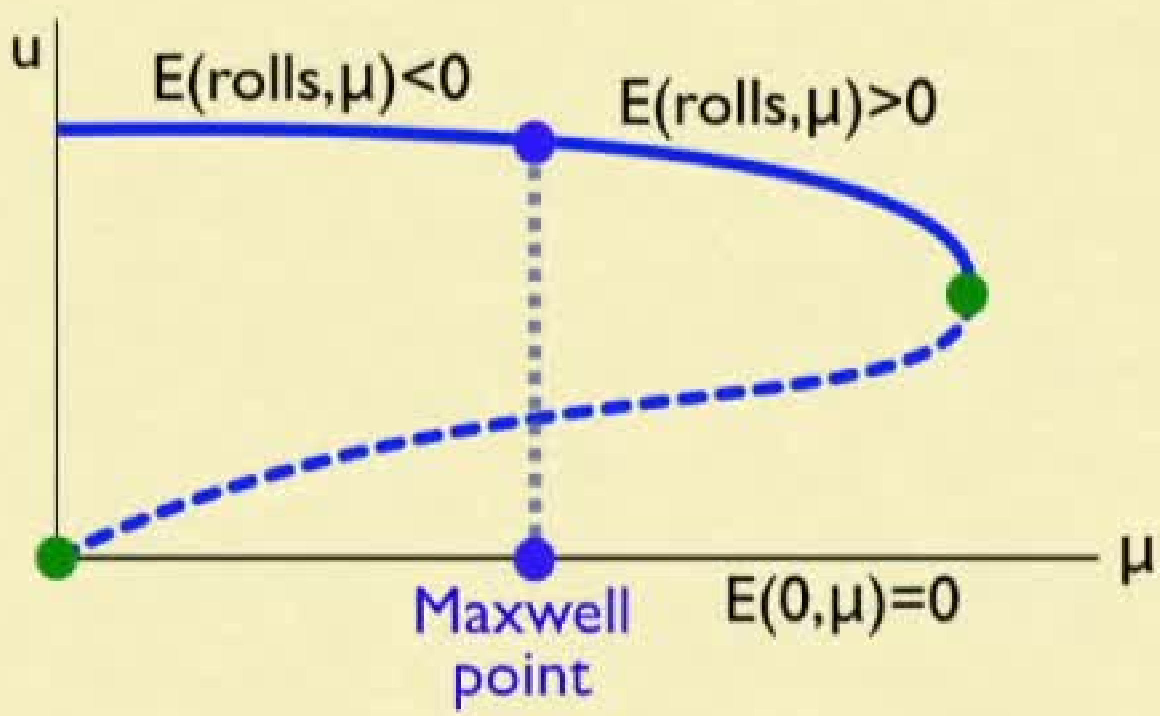

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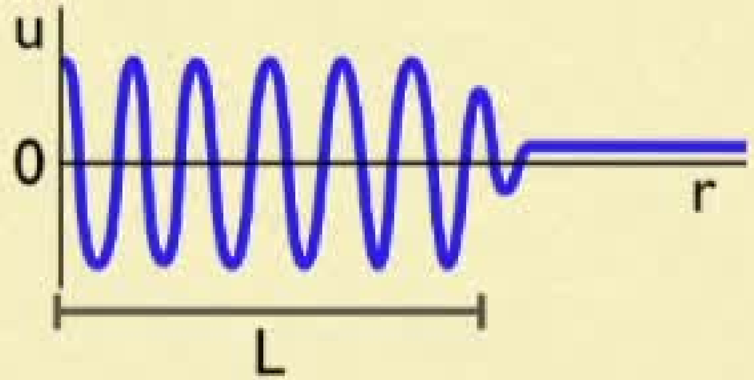

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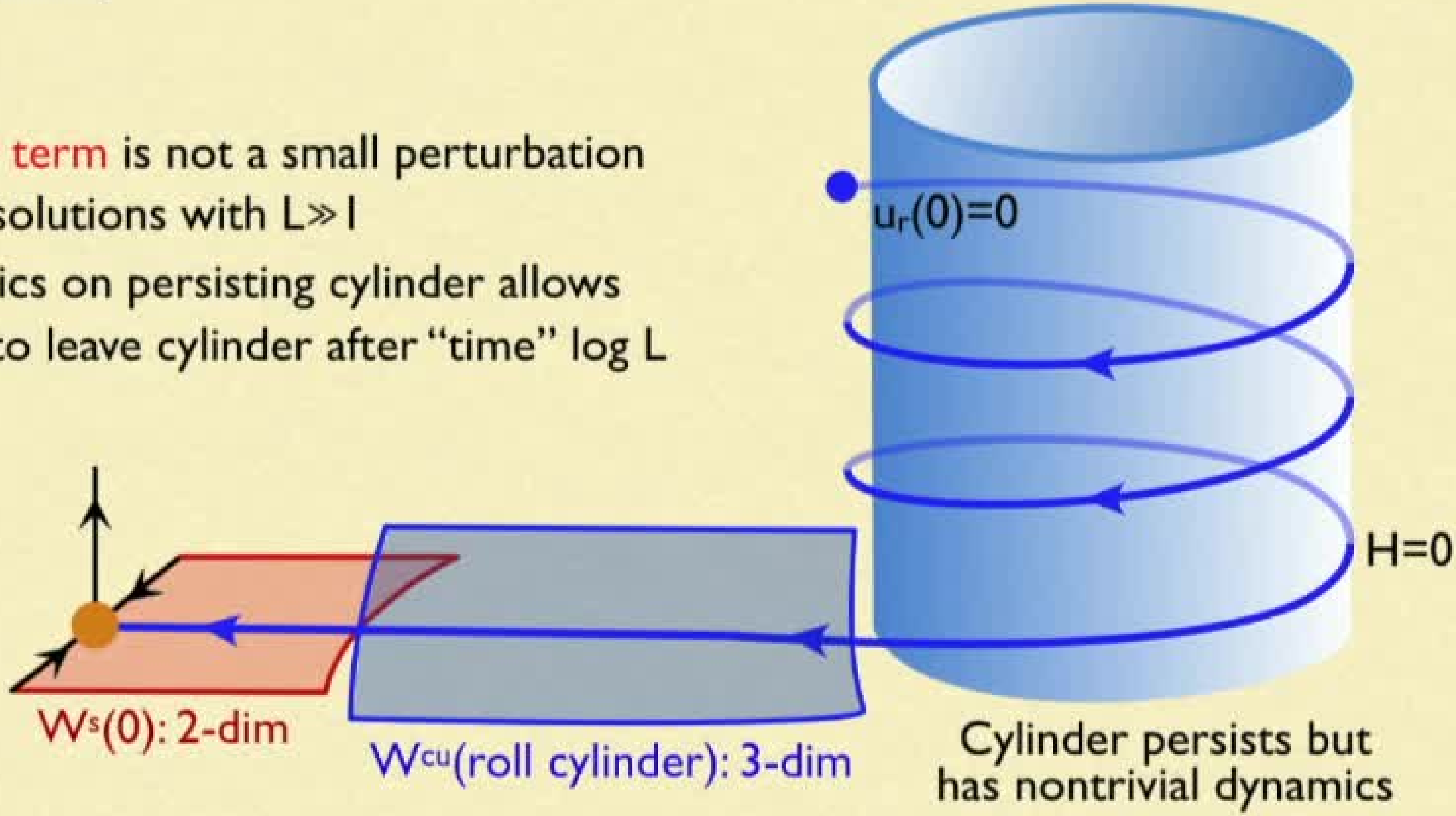

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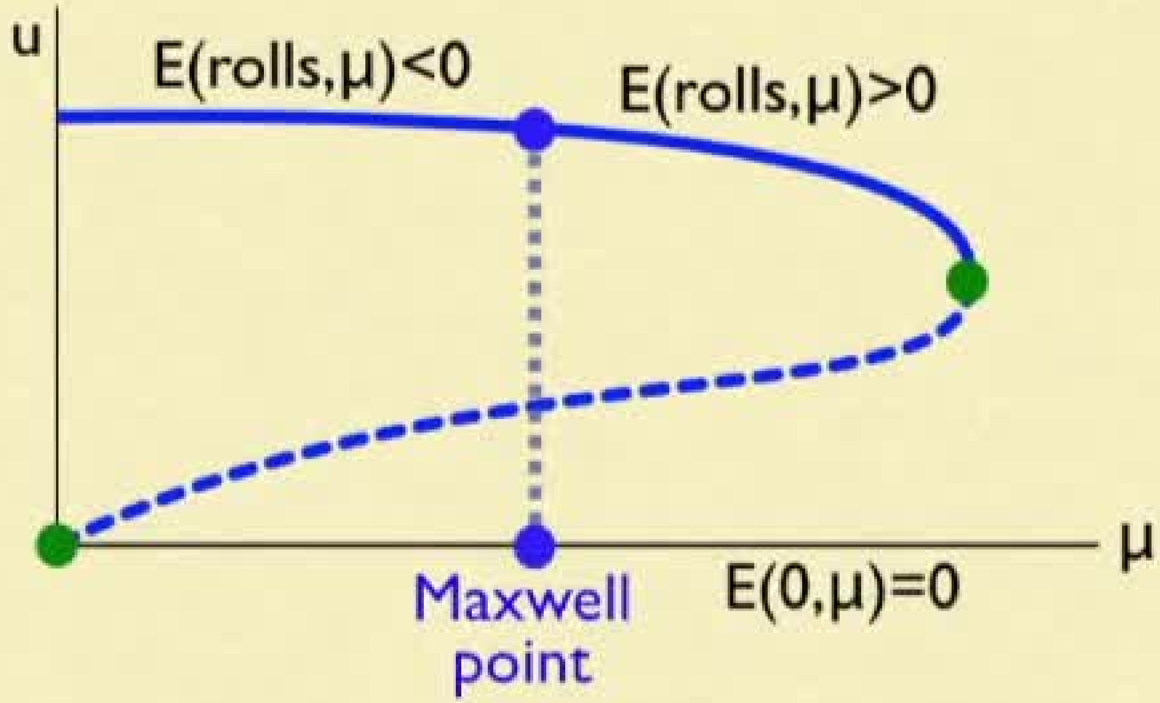

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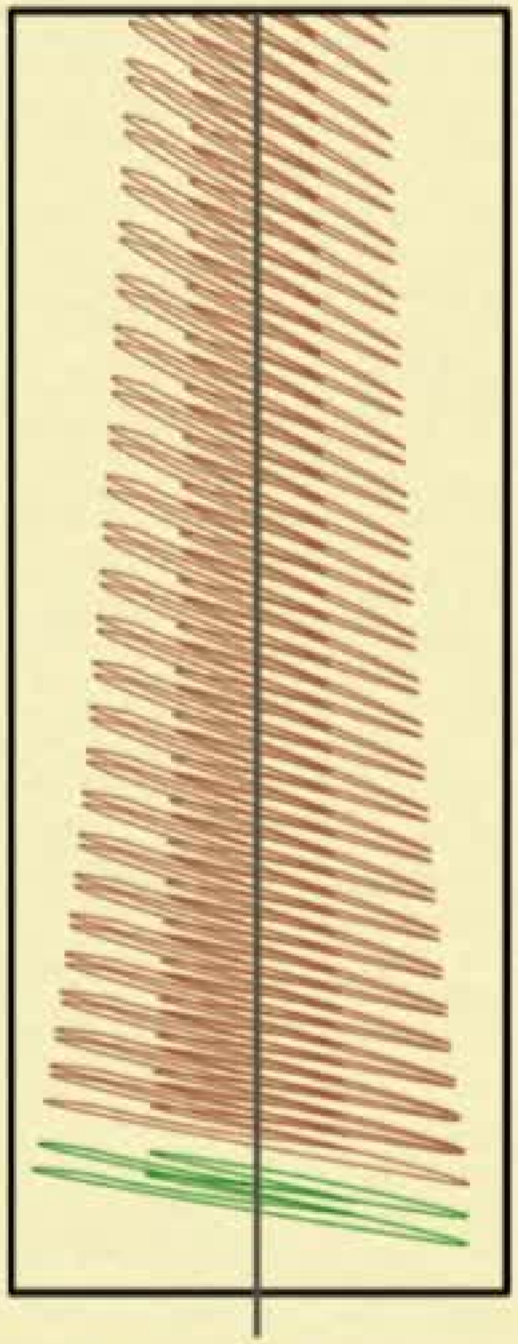
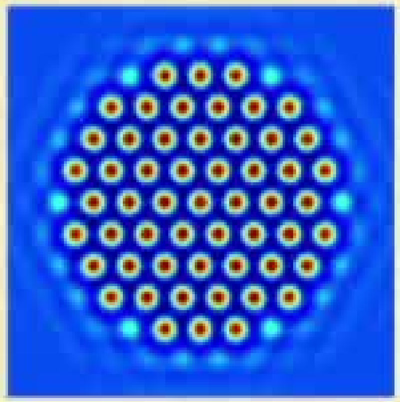


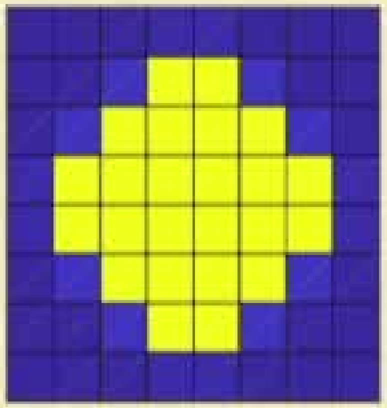
Planar hexagon patches

$$u_t = -[1+\Delta]^2 u - \mu u + \nu u^2 - u^3,\ x \in \mathbb{R}^2$$

Hexagon patches are difficult to analyse!



Pose system on square (or hexagonal) lattice:



$$\frac{du_{mn}}{dt} = d(\Delta u)_{mn} + f(u_{mn}), \quad m,n \in \mathbb{Z}^2$$

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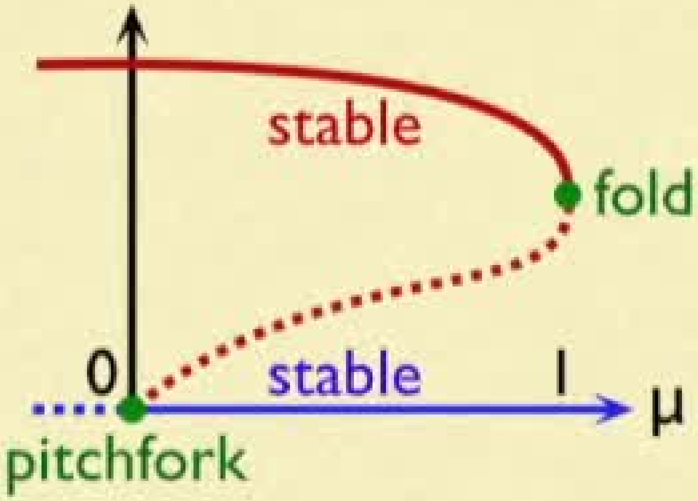

Consider anti-continuum limit $0<d\ll 1$

Motivated by numerical work by [Taylor, Dawes]

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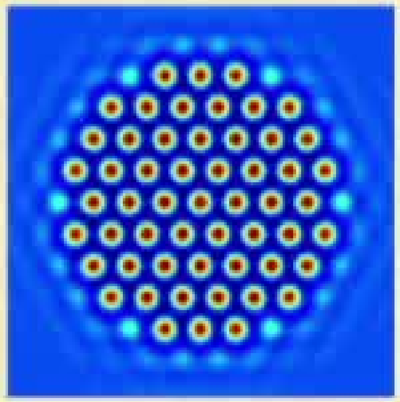

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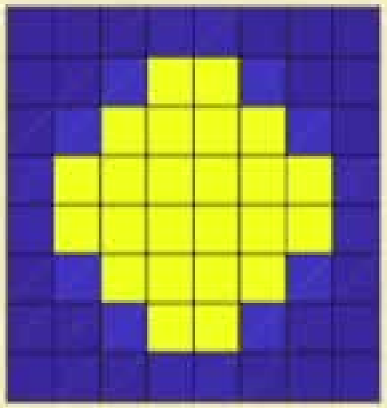

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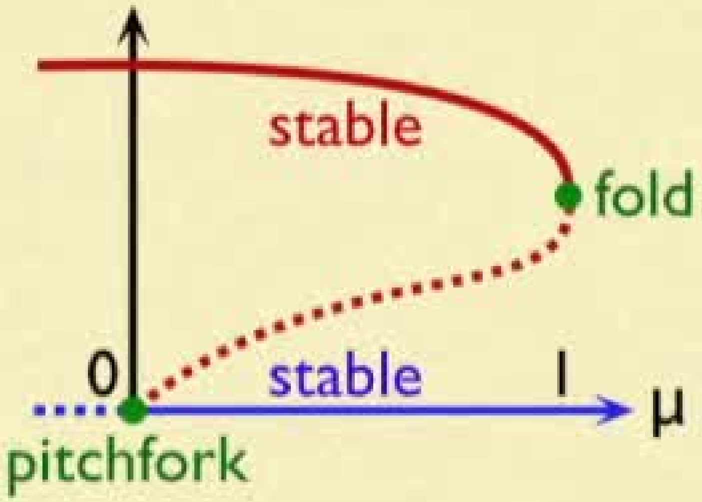

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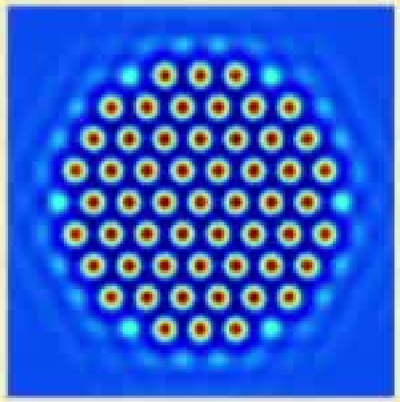

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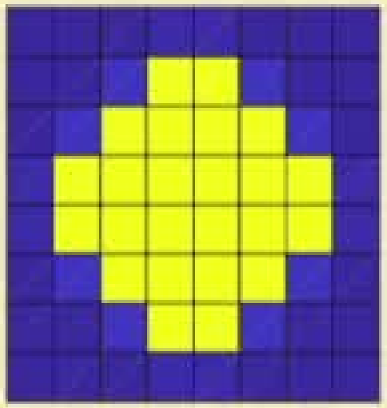

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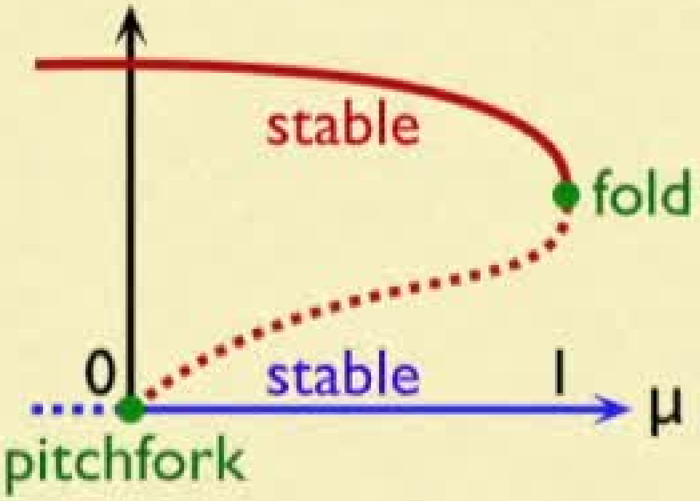

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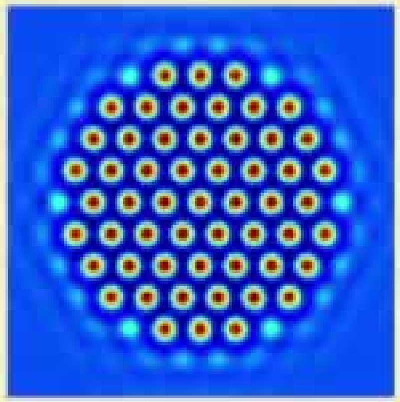

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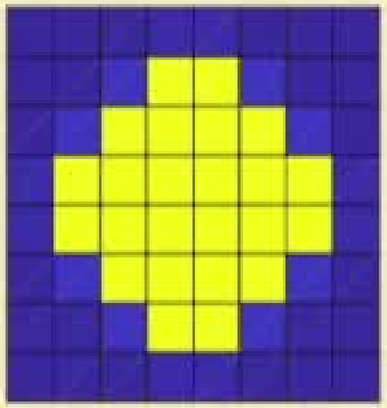

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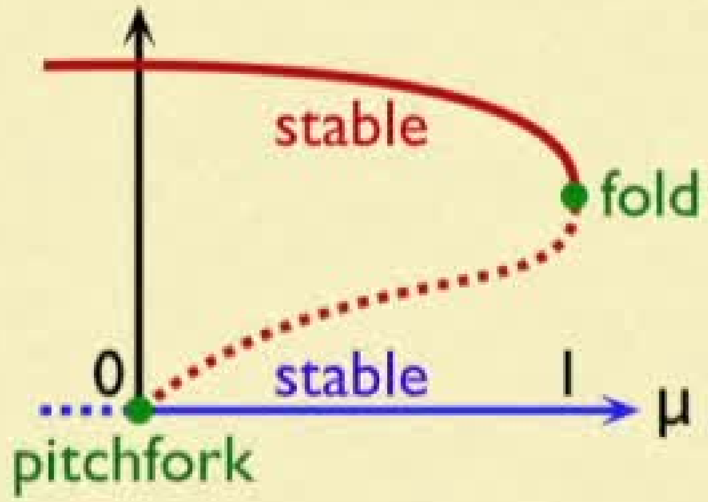

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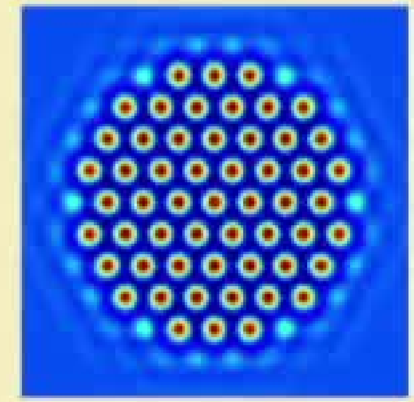

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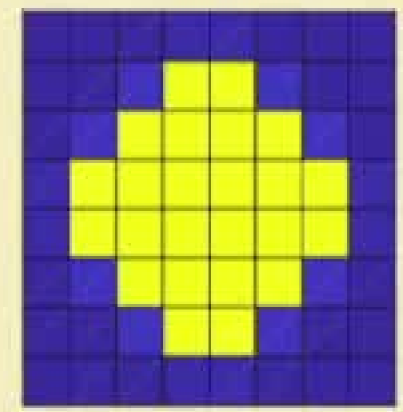

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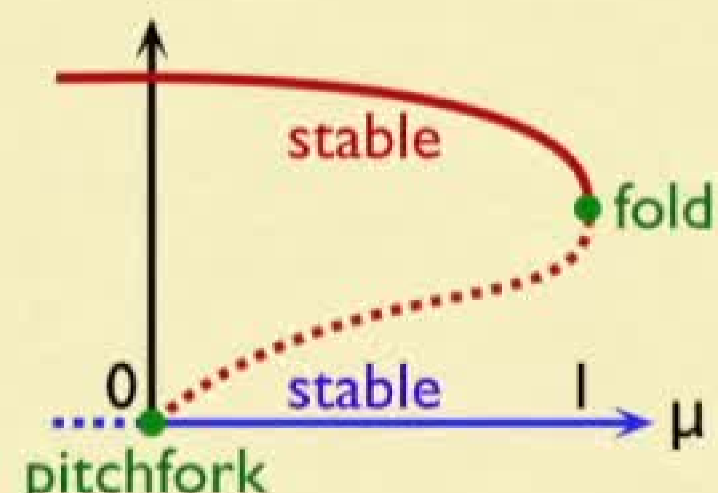

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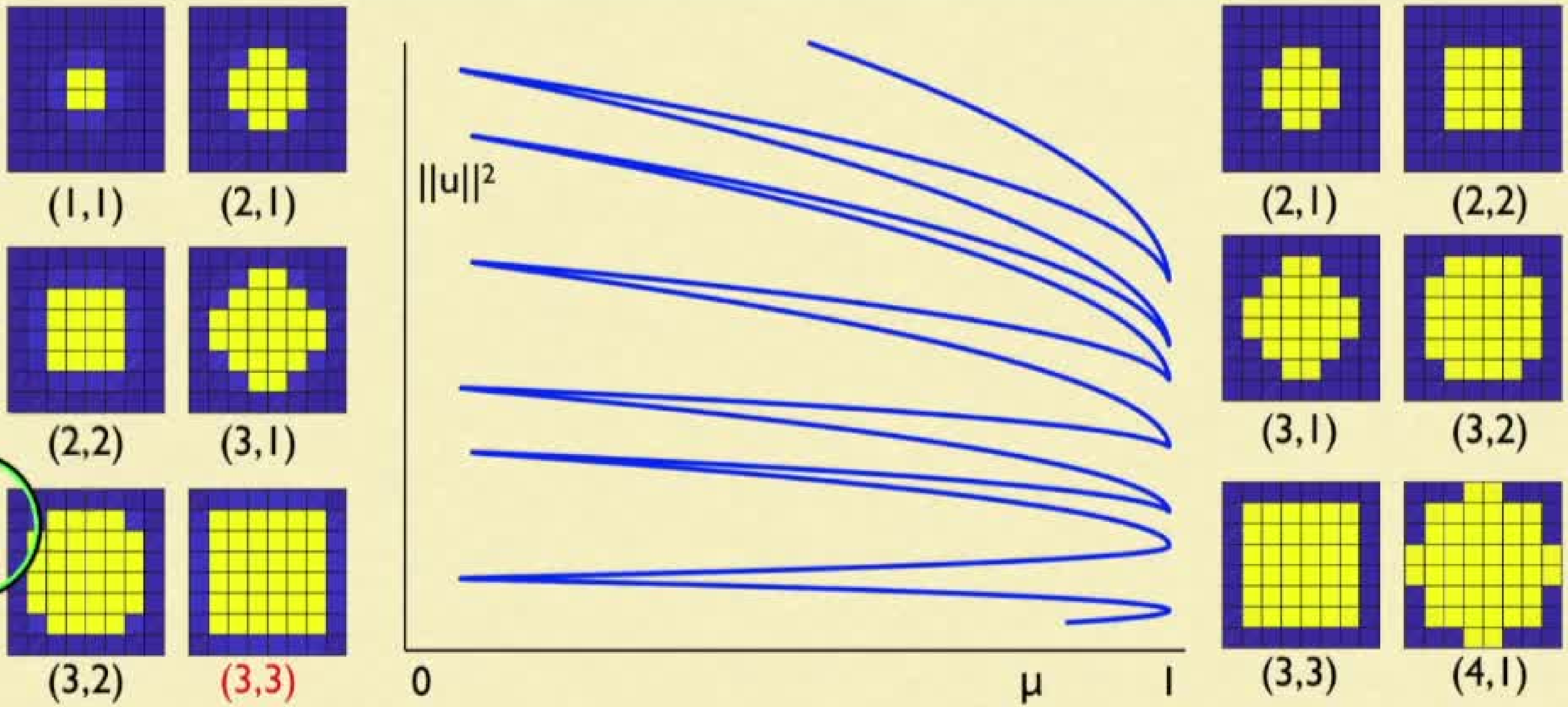
Patches on square lattice



Theorem [Bramburger, S.]

For $0<d\ll 1$, we have

- existence for $0<\mu<1$
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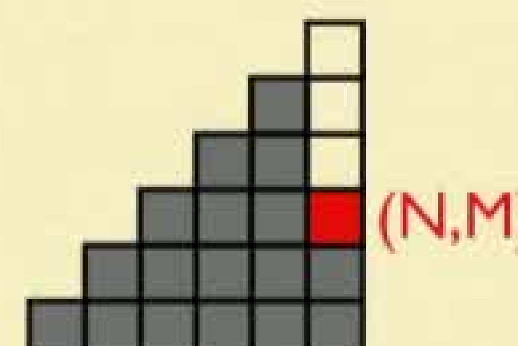

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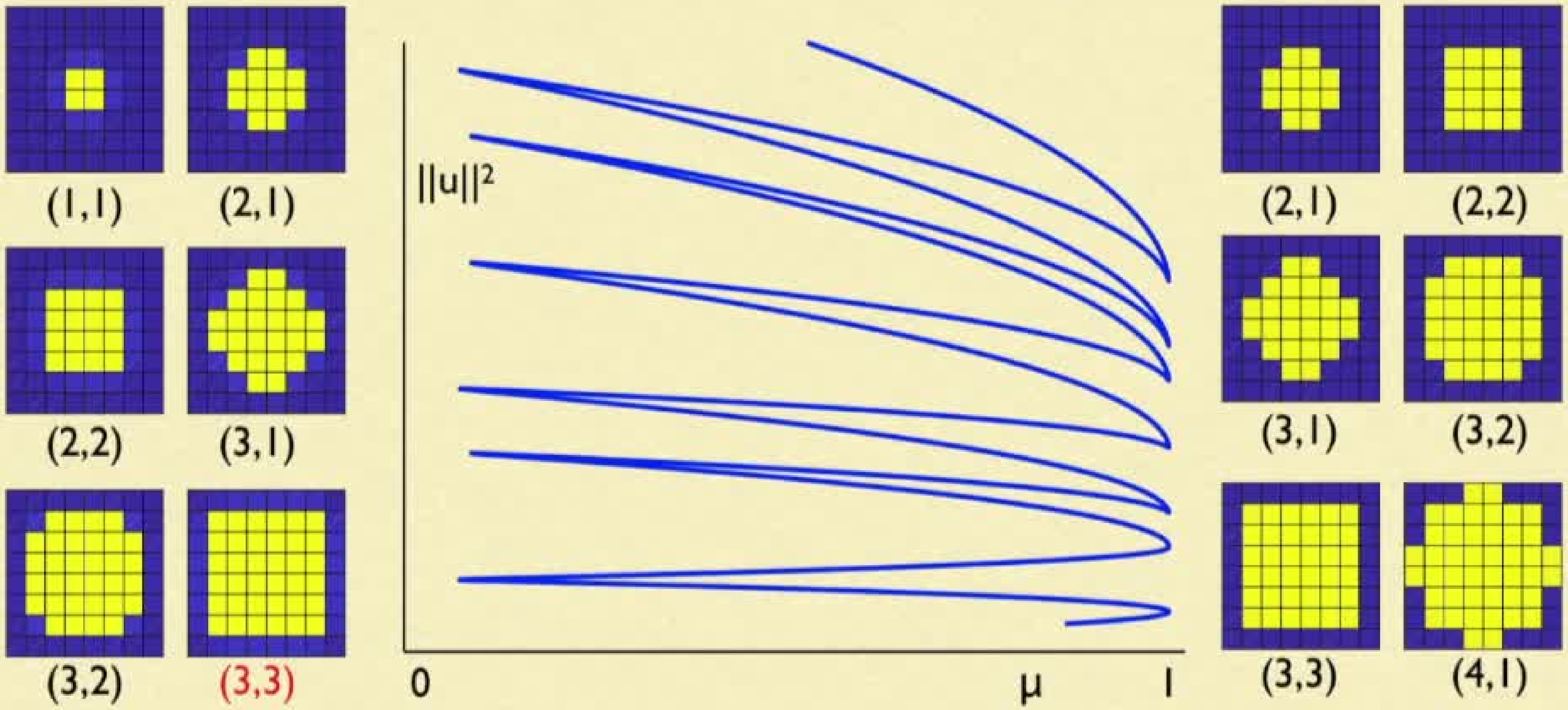

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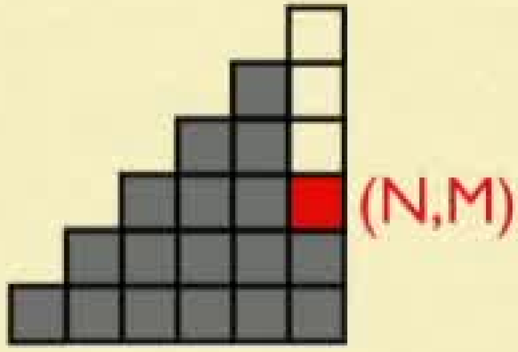

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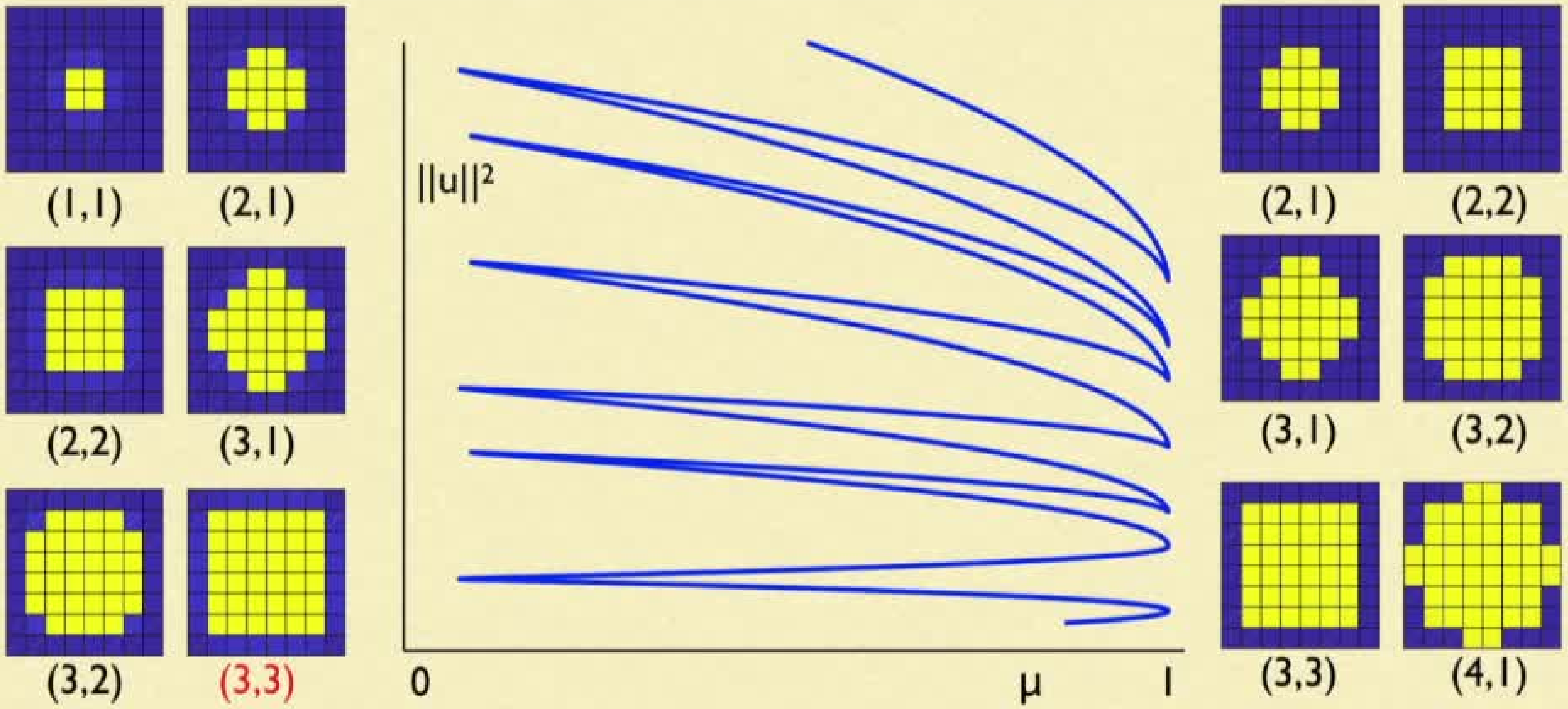

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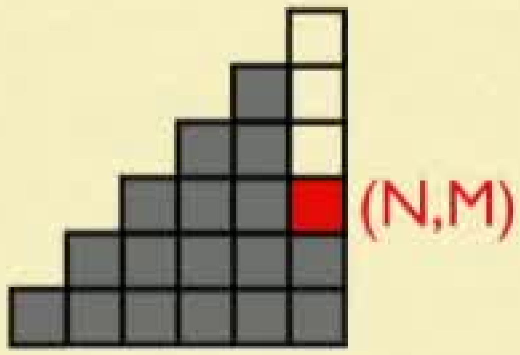

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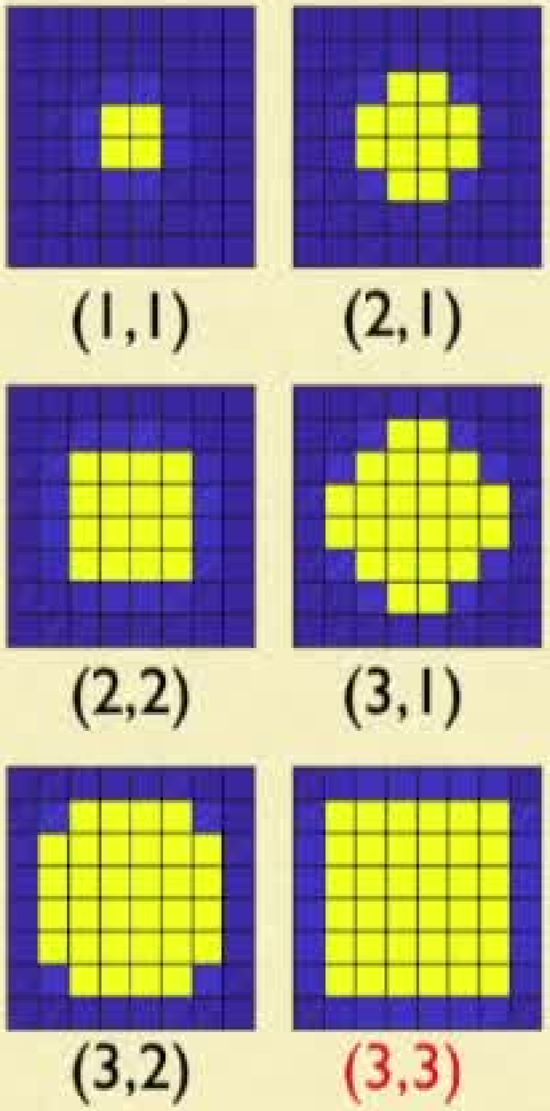

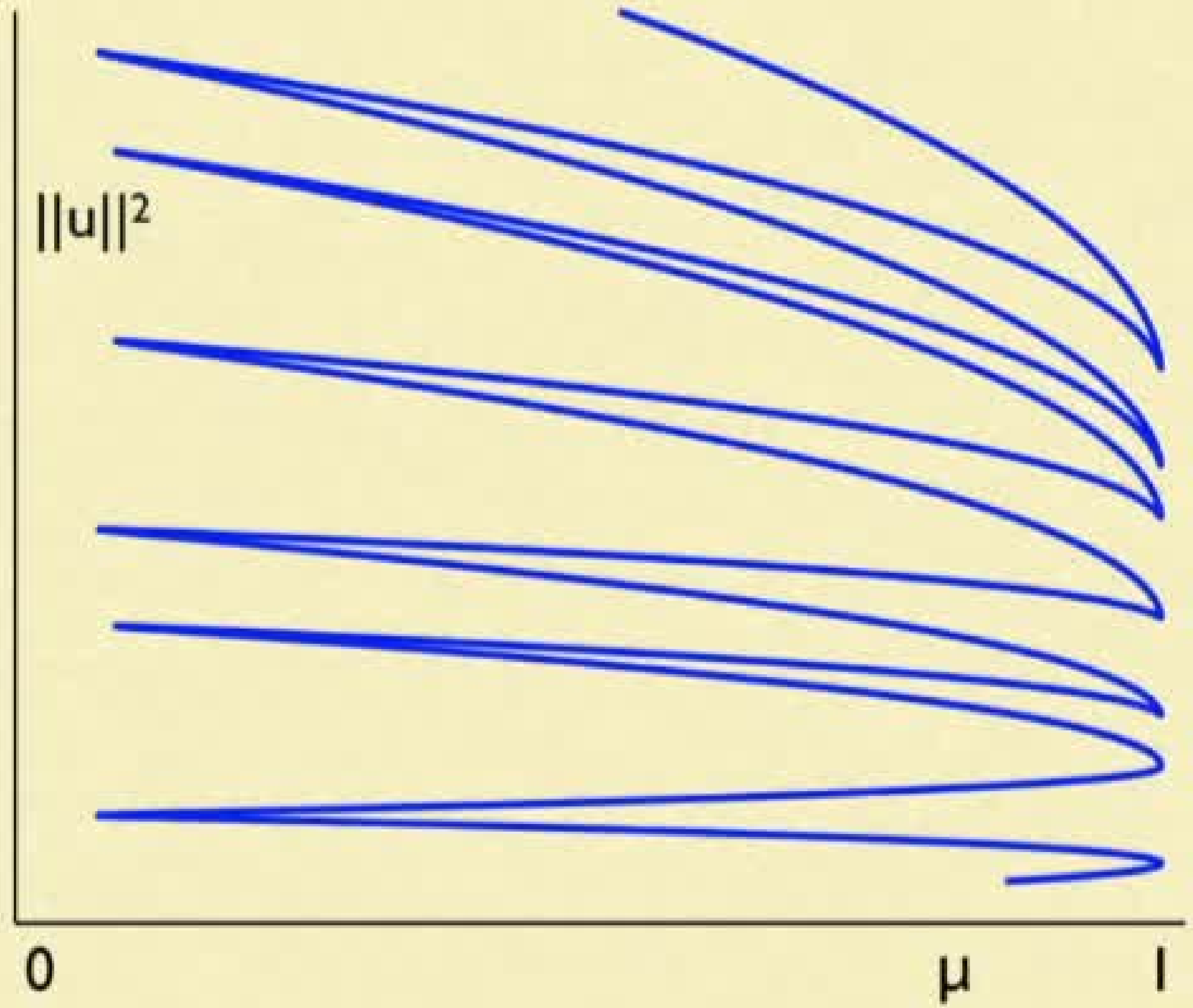

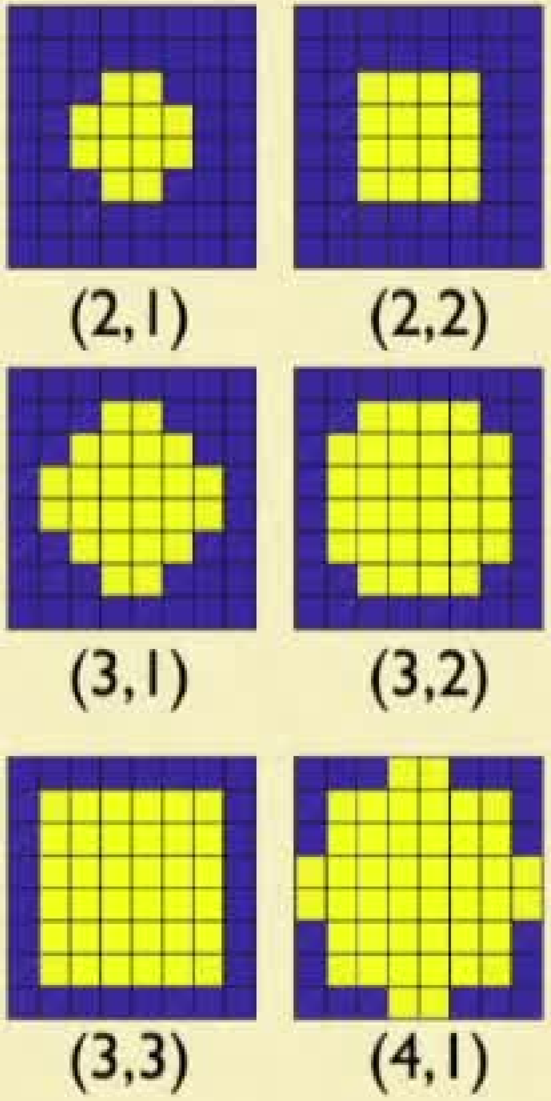

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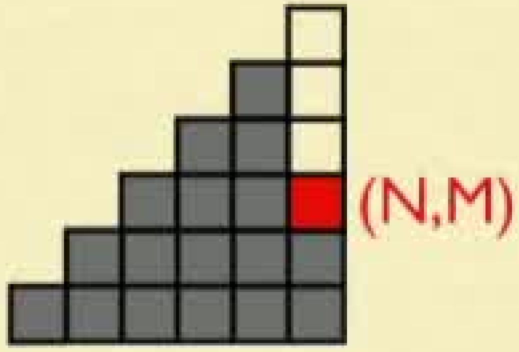

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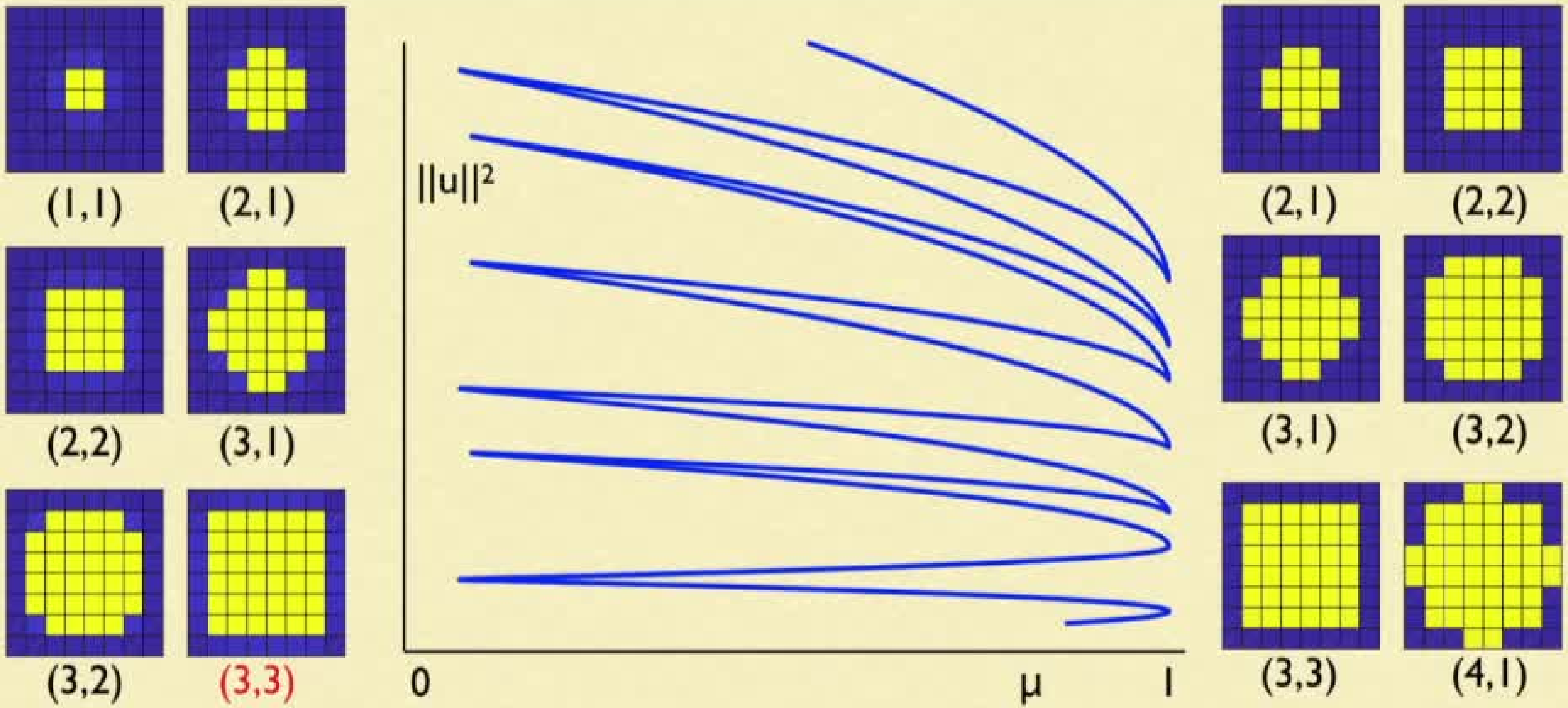

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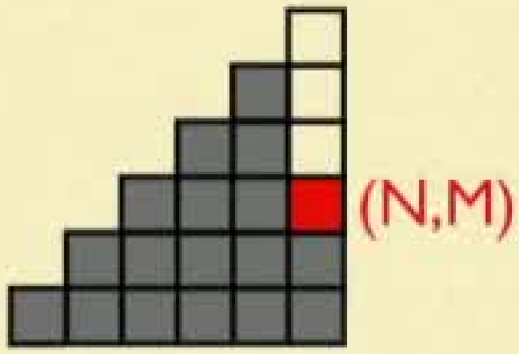

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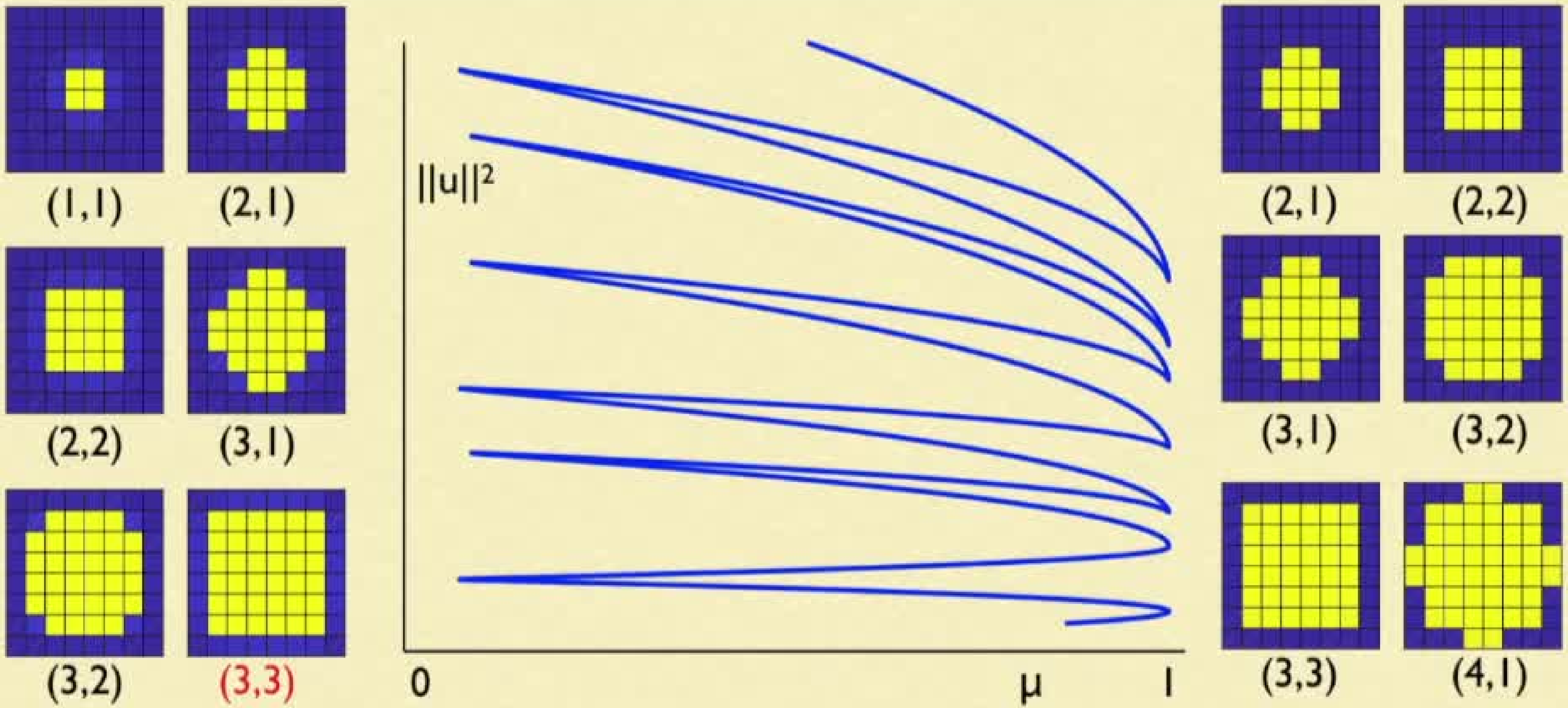

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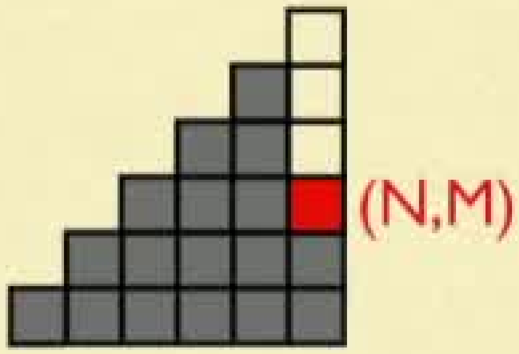

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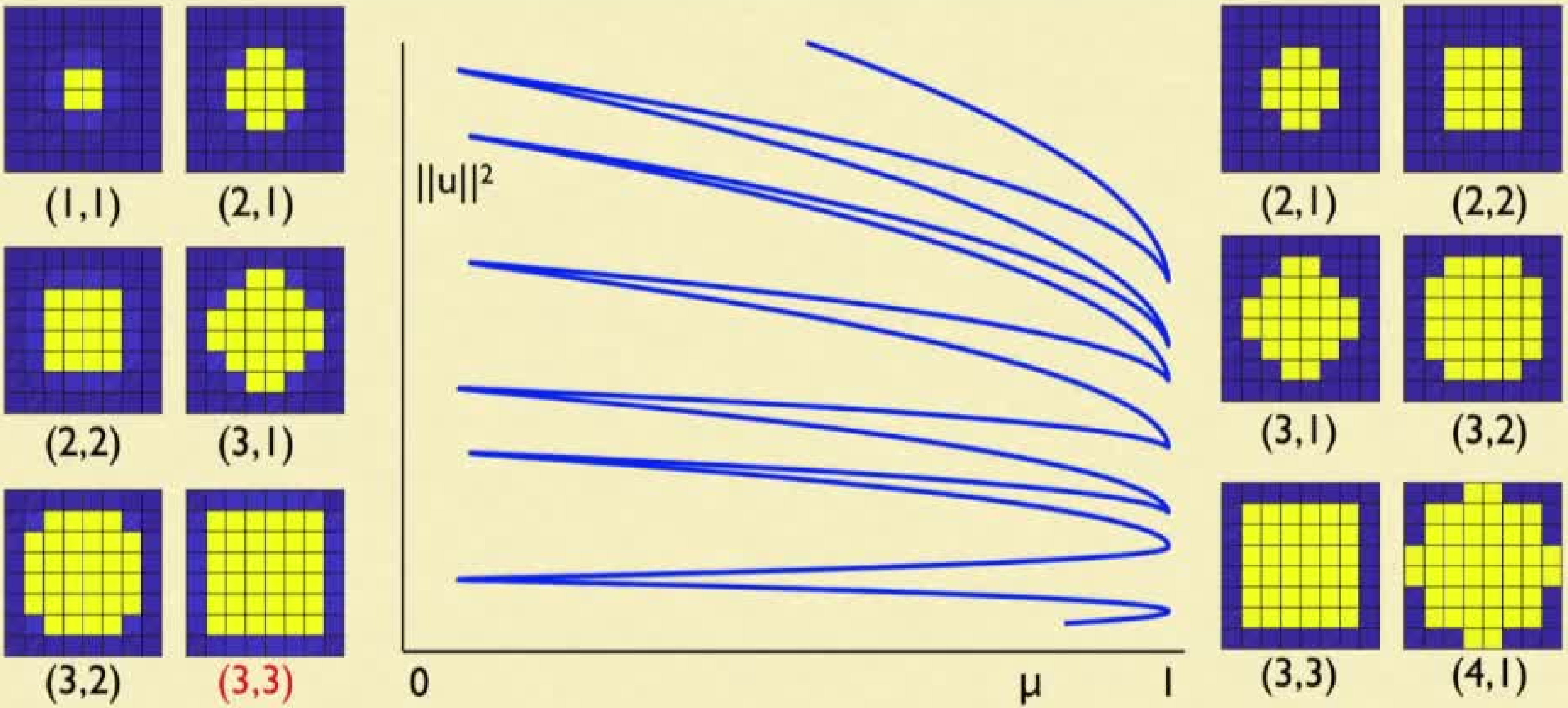

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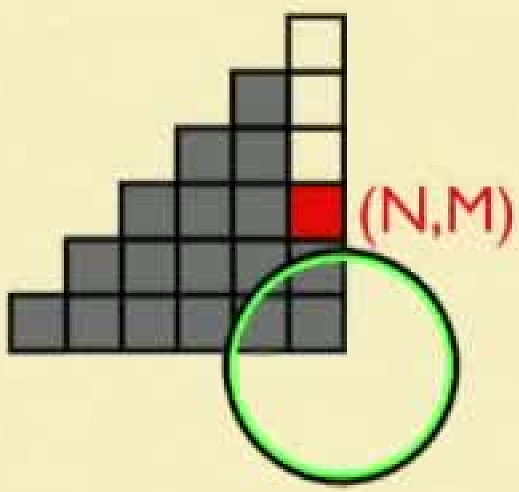
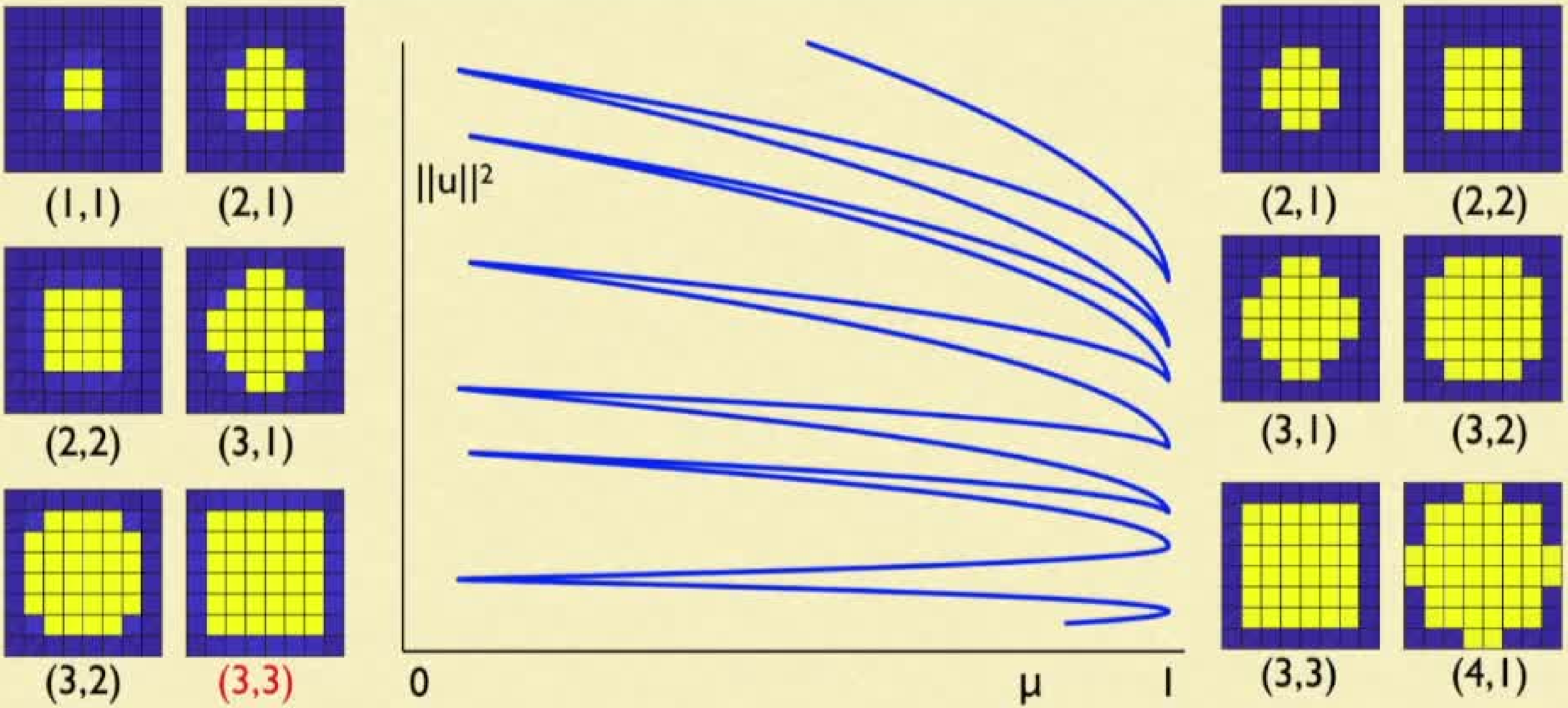
For 0<d≪1, we have

- existence for 0<μ<1
- continuation near μ=0 except for (N,N) with N≥3
- continuation near μ=1 except for (N,M) with 2≤M≤N-2



Patches on square lattice



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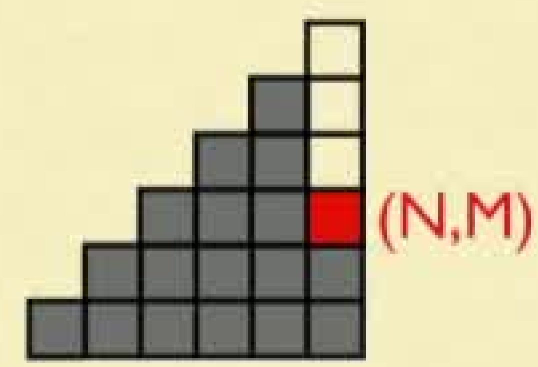
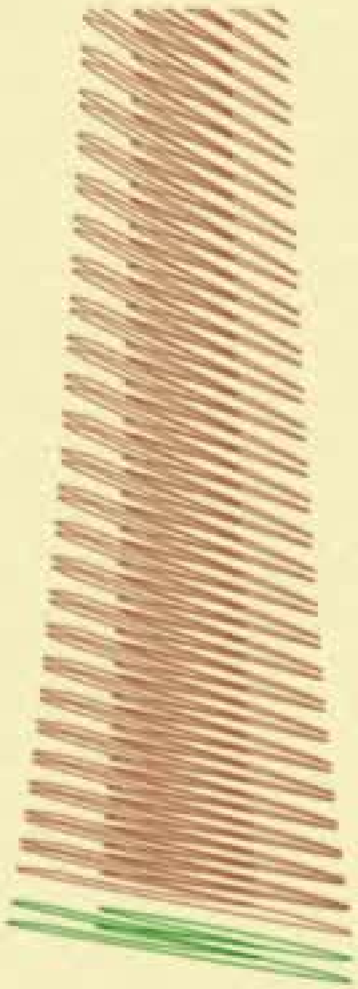


Conclusions



Radial 2D

- Proved that snaking has to collapse for radial rolls
- Proved partial snaking diagram for patches on square lattice near anti-continuum limit

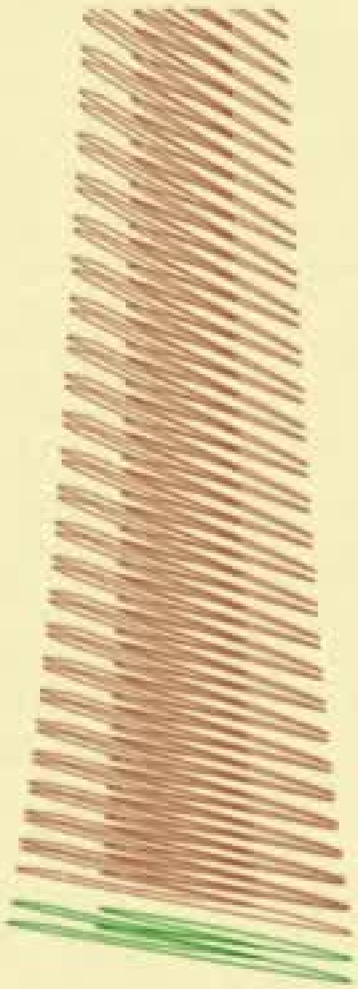
Open Problems:

- Prove continuation of patterns through all folds
- Extend to hexagon lattices

Square Lattice

Thank you for listening!

Conclusions



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