The Moment Problem for Rational Measures: Convexity in the Spirit of Krein

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Special recognition

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C. I. Byrnes and A. Lindquist, The moment problem for rational measures: convexity in the spirit of Krein, in *Modern Analysis and Applications: To the Centenary of Mark Krein*, Vol I, Birkhäuser 2009.

Outline of talk

- The classical theory of moments in the style of Krein
- The moment problem for rational measures
- A Dirichlet principle for the rational moment problem

The generalized moment problem in the spirit of Krein

$\mathfrak{P}$ finite-dimensional subspace of $C[a,b]$

$(u_0, u_1, \ldots, u_n)$ basis in $\mathfrak{P}$

$p \in \mathfrak{P} \Rightarrow$

$p(t) = \sum_{k=0}^{n} p_k u_k(t)$

Given $c := (c_0, c_1, \ldots, c_n) \in \mathbb{C}^{n+1}$,

find positive measure $d\mu$ such that

$$\int_a^b u_k(t) d\mu = c_k, \quad k = 0, 1, \ldots, n$$

Dual cones

$p \in \mathfrak{P} \Rightarrow$

$p(t) = \sum_{k=0}^{n} p_k u_k(t)$

$$\mathfrak{P}_+ := \{p \in \mathfrak{P} \mid P(t) := \mathrm{Re}(p) \geq 0 \quad \forall t \in [a,b]\}$$

closed convex cone

$$\langle c, p \rangle := \mathrm{Re}\left\{ \sum_{k=0}^{n} c_k p_k \right\}$$ where $c := (c_0, c_1, \ldots, c_n) \in \mathbb{C}^{n+1}$

$$\mathfrak{C}_+ := \{c \in \mathbb{C}^{n+1} \mid \langle c, p \rangle \geq 0 \quad \forall p \in \mathfrak{P}_+\}$$

dual cone

closed convex

$$\mathfrak{C}_+ = \mathfrak{P}_+^{\top}$$

$c \in \mathfrak{C}_+$ positive sequence

Ex 1: Power moment problem

$$u_k(t) = t^k, \quad k = 0, 1, \ldots, n$$

Every $p \in \mathfrak{P}$ is a polynomial.

$$c \in \mathfrak{C}_+ \; (n \text{ even}) \iff \begin{cases} \left[\; c_{j+k} \;\right]_{j,k=0}^{n/2} \geq 0 \\ \left[\; (a+b)c_{j+k+1} - abc_{j+k} - c_{j+k+2} \;\right]_{j,k=0}^{n/2-1} \geq 0 \end{cases}$$

Ex 2: Trigonometric moment problem

$$u_k(t) := e^{ikt}, \quad k = 0, 1, \ldots, n \qquad [a, b] = [-\pi, \pi]$$

$$c \in \mathfrak{C}_+ \quad \Longleftrightarrow \quad T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ \bar{c}_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}_n & \bar{c}_{n-1} & \cdots & c_0 \end{bmatrix} \geq 0$$

Toeplitz matrix

Ex 3: Nevanlinna-Pick interpolation

$$u_k(t) = \frac{e^{it} + z_k}{e^{it} - z_k}, \quad k = 0, 1, \ldots, n$$

For $d\mu = F dt$, where $F = \mathrm{Re}(f)$ with f analytic in $\mathbb{D}$,

$$c_k = \int_{-\pi}^{\pi} u_k(t) F(t) dt = f(z_k) \quad k = 0, 1, \ldots, n$$

$$c \in \mathfrak{C}_+ \quad \Longleftrightarrow \quad P_n = \left[\frac{c_j + \bar{c}_k}{1 - z_j \bar{z}_k} \right]_{j,k=0}^{n} \geq 0$$

Pick matrix

The moment map

$$\mathfrak{M} : C[a,b]^* \to \mathbb{C}^{n+1}, \qquad d\mu \mapsto c = \int_a^b u(t)d\mu$$

$\mathcal{M}_+ \subset C[a,b]^*$ space of positive measures

$c \in \mathfrak{M}(\mathcal{M}_+)$ ⇒

$$P(t) = \mathrm{Re}\{p(t)\}$$

$$\langle c, p \rangle := \mathrm{Re}\left\{ \sum_{k=0}^{n} c_k p_k \right\} = \int_a^b P(t)d\mu \geq 0 \quad \forall p \in \mathfrak{P}_+$$

⇒ $c \in \mathfrak{C}_+$

$$\mathfrak{M}(\mathcal{M}_+) \subset \mathfrak{C}_+$$

HYPOTHESIS 1. $\mathring{\mathfrak{P}}_+ \neq \emptyset$, where $\mathring{\mathfrak{P}}_+$ is the interior of $\mathfrak{P}_+$.

THEOREM(Krein-Nudelman). Suppose that Hypothesis 1 holds. Then

$$\mathfrak{M}(\mathcal{M}_+) = \mathfrak{C}_+.$$

In other words, the moment problem is solvable if and only if c is positive.

We have already shown that $\mathfrak{M}(\mathcal{M}_+) \subset \mathfrak{C}_+$

It remains to prove that $\mathfrak{M}(\mathcal{M}_+) \supset \mathfrak{C}_+$

Proof. Consider the curve $U = \{u(t);\ t \in [a,b]\} \subset \mathbb{C}^{n+1}$,
where $u(t) = (u_0(t), u_1(t), \ldots, u_n(t)) \quad a \le t \le b.$

For $p \in \mathfrak{P}$,

$$\langle u(t), p\rangle := \mathrm{Re}\left\{\sum_{k=0}^{n} p_k u_k(t)\right\} = \mathrm{Re}\{p(t)\} = P(t)$$

$K(U)$ convex conic hull of U

$$K(U)^{\mathsf{T}} = \{p \in \mathfrak{P} \mid \langle \phi, p\rangle \ge 0,\ \forall \phi \in K(U)\} = \mathfrak{P}_+$$

$$\Rightarrow \quad \boxed{K(U) = \mathfrak{C}_+}$$

$\mathfrak{M}(\mathcal{M}_+) \subset \mathfrak{C}_+$ is closed, by the Helly selection theorem.

Show that $\mathfrak{C}_+ \subset \mathfrak{M}(\mathcal{M}_+)$: $\quad \mathfrak{M}(\delta_t) = u(t), \quad a \le t \le b$

$$\Rightarrow \quad U \subset \mathfrak{M}(\mathcal{M}_+) \quad \Rightarrow \quad \underbrace{K(U)}_{\mathfrak{C}_+} \subset \mathfrak{M}(\mathcal{M}_+)$$

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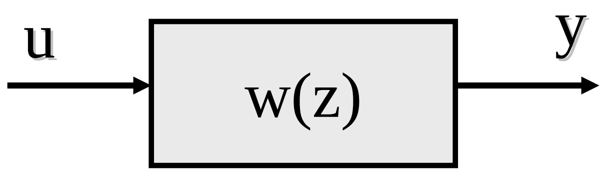
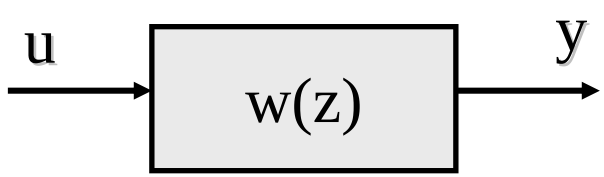
A motivating example



$$y(t) = \sum_{k=-\infty}^{t} w_{t-k} u(k)$$

System is finite-dimensional iff $w(z) := \sum_{k=0}^{\infty} w_k z^{-k}$ is rational

white noise



stationary process
with spectral density
$\Phi(e^{i\theta}) = |w(e^{i\theta})|^2$

$$\int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) \frac{d\theta}{2\pi} = c_k := E\{y(t+k)y(t)\}$$

$$\Phi(z) = \frac{P(z)}{Q(z)}$$

$\mathfrak{P}$ consists of trigonometric polynomials

$$P(e^{i\theta}) = \mathrm{Re}\{p(e^{i\theta})\}, \quad Q(e^{i\theta}) = \mathrm{Re}\{q(e^{i\theta})\}, \quad \text{where } p, q \in \mathfrak{P}_+$$

The moment problem for rational measures

DEF. $p \in \mathfrak{P}, \quad p = \sum_{k=0}^{n} p_k u_k$ polynomial in $\mathfrak{P}$

$P = \mathrm{Re}(p)$

P/Q, where $p, q \in \mathfrak{P}$ real rational function for $\mathfrak{P}$

$$\mathcal{R}_+ = \left\{ d\mu \mid d\mu = \frac{P(t)}{Q(t)} dt, \ p, q \in \mathring{\mathfrak{P}}_+ \right\} \subset \mathcal{M}_+$$

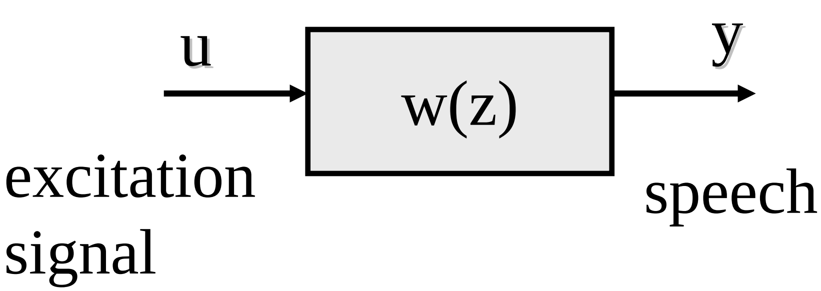
rational positive measure

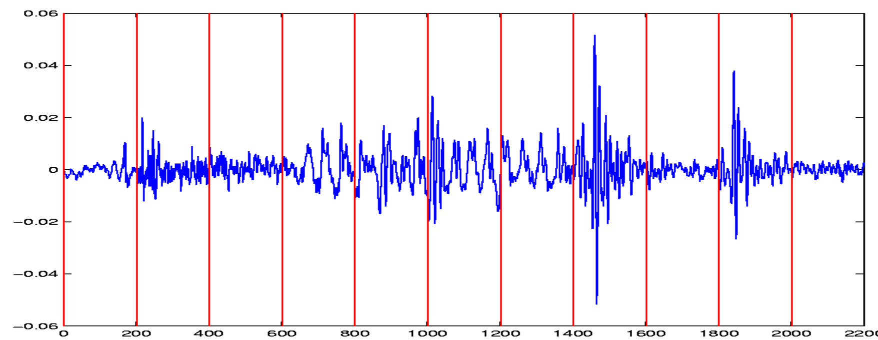
$$\int_a^b u_k d\mu = c_k, \quad k = 0, 1, \ldots, n \quad (\dagger)$$

Find $d\mu \in \mathcal{M}_+$ satisfying $(\dagger)$ linear problem

Find $d\mu \in \mathcal{R}_+$ satisfying $(\dagger)$ nonlinear problem

Ex: Modeling speech





on each (30 ms) subinterval
w(z) constant, y stationary

observation: $y_0, y_1, \ldots, y_N$

$N \approx 250$

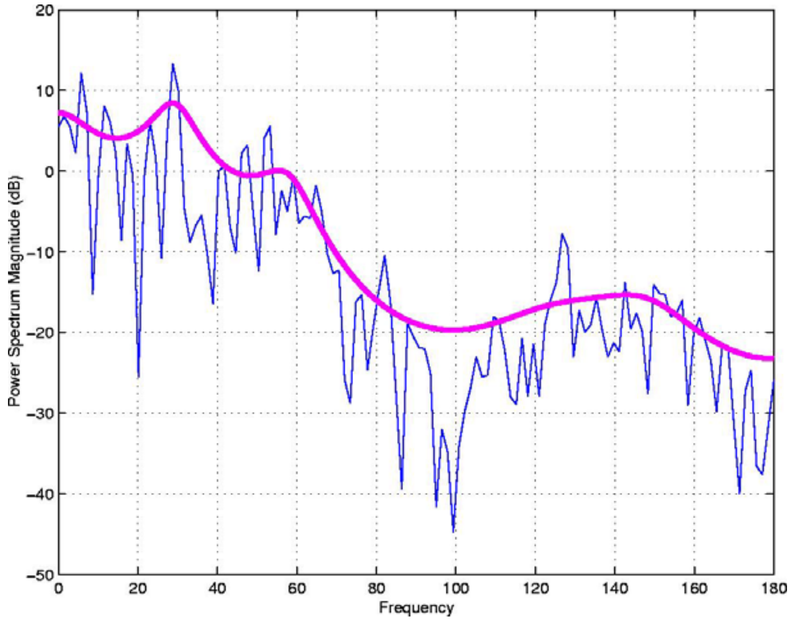
$$\int_{-\pi}^{\pi} e^{ikt} d\mu = c_k := \frac{1}{N+1} \sum_{t=0}^{N-k} y_{t+k} y_t, \quad k = 0, 1, \ldots, n \quad n = 10$$

$$\mathfrak{P} = \operatorname{span}\{1, e^{it}, e^{2it}, \ldots, e^{int}\}$$

$$d\mu = \left| w(e^{it}) \right|^2 dt \in \mathcal{R}_+$$

Cellular telephone:

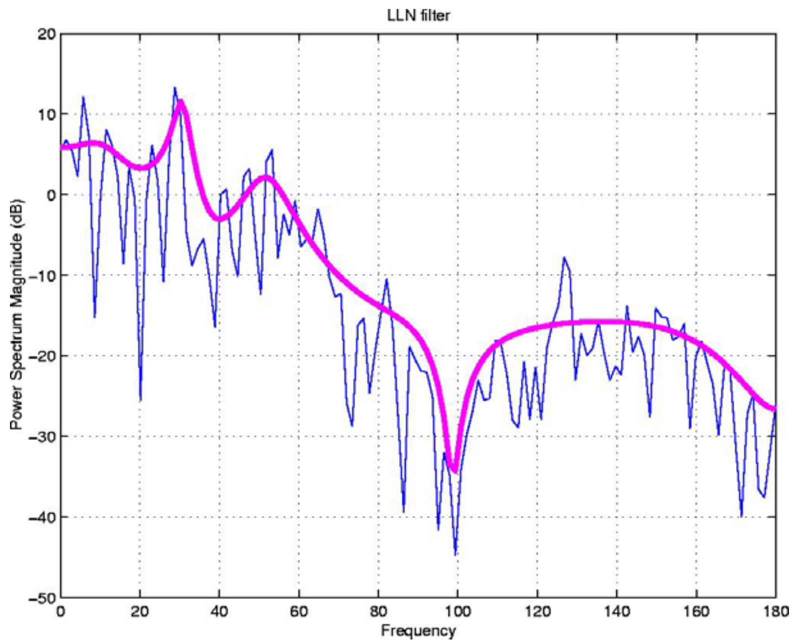
$$d\mu = \frac{\rho_n}{|\varphi_n(e^{it})|^2} dt$$



$\varphi_n(z)$ n:th Szegö polynomial orthogonal on the unit circle

FFT in blue

Another rational positive measure of the same degree



$\mathring{\mathfrak{C}}_+$ interior of $\mathfrak{C}_+$

$c \in \mathring{\mathfrak{C}}_+ \quad \Longleftrightarrow \quad \langle c, p \rangle > 0 \quad \forall p \in \mathfrak{P}_+ \setminus \{0\}$ strictly positive sequence

HYPOTHESIS 2. The vector space $\mathfrak{P}$ consists of Lipschitz continuous functions.

THEOREM. Suppose that Hypotheses 1 and 2 hold. Then

$$\mathfrak{M}(\mathcal{R}_+) = \mathring{\mathfrak{C}}_+.$$

In other words, the moment problem for rational measures is solvable if and only if c is strictly positive.

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Proof. As in the classical case:

$$\langle c, p \rangle = \int_a^b P d\mu > 0 \quad \forall p \in \mathfrak{P}_+ \setminus \{0\} \quad \Rightarrow \quad \mathfrak{M}(\mathcal{R}_+) \subset \mathring{\mathfrak{C}}_+$$

To show that $\mathfrak{M}(\mathcal{R}_+) = \mathring{\mathfrak{C}}_+$ it suffices to show:

PROPOSITION. Suppose that Hypotheses 1 and 2 hold. Then there is a nonempty subset $\mathcal{P}_+ \subset \mathcal{R}_+$ such that $\mathfrak{M}(\mathcal{P}_+)$ is both open and closed in the convex set $\mathring{\mathfrak{C}}_+$.

$$\Rightarrow \quad \mathfrak{M}(\mathcal{P}_+) = \mathring{\mathfrak{C}}_+ \quad \Rightarrow \quad \mathfrak{M}(\mathcal{R}_+) = \mathring{\mathfrak{C}}_+$$

PROPOSITION. Suppose that Hypotheses 1 and 2 hold. Then there is a nonempty subset $\mathcal{P}_+ \subset \mathcal{R}_+$ such that $\mathfrak{M}(\mathcal{P}_+)$ is both open and closed in the convex set $\mathring{\mathfrak{C}}_+$.

For a fixed $p \in \mathring{\mathfrak{P}}_+$, consider the set

$$\mathcal{P}_+ = \left\{ d\mu \in \mathcal{R}_+ \mid d\mu = \frac{P}{Q} dt,\ q \in \mathring{\mathfrak{P}}_+ \right\}$$

and the map $\mathfrak{M}_{|\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+$.

- Jac $\mathfrak{M}_{|\mathcal{P}_+}$ full rank $\Longrightarrow$ $\mathfrak{M}(\mathcal{P}_+) \subset \mathring{\mathfrak{C}}_+$ open
- Q Lipschitz $\Longrightarrow$ $\mathfrak{M}_{|\mathcal{P}_+}$ proper $\Longrightarrow$ $\mathfrak{M}(\mathcal{P}_+) \subset \mathring{\mathfrak{C}}_+$ closed

COROLLARY. Suppose that Hypotheses 1 and 2 hold. Then the moment map $\mathfrak{M}_{|\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+$ is surjective. In other words, for each $c \in \mathring{\mathfrak{C}}_+$, the moment problem

$$\mathfrak{M}(d\mu) = c \quad \text{for } d\mu \in \mathcal{P}_+$$

has a solution.

We want to show that $\mathfrak{M}_{|\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+$ is also injective. In other words, for each $c \in \mathring{\mathfrak{C}}_+$, the moment problem

$$\mathfrak{M}(d\mu) = c \quad \text{for } d\mu \in \mathcal{P}_+$$

has a unique solution.

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A Dirichlet principle

Solving the moment problem

$$\mathfrak{M}(d\mu) = c \quad \text{for } d\mu \in \mathcal{P}_+ \qquad\qquad c \in \mathring{\mathfrak{C}}_+$$

is equivalent to solving the equations

$$f_k^p(q) := c_k - \int_a^b u_k \frac{P}{Q} dt = 0, \quad k = 0, 1, \ldots, n,$$

where $p \in \mathring{\mathfrak{P}}_+$ is fixed.

In this case, the Dirichlet Principle would say that $f_k^p(q) = 0$ should be the critical point equations for some smooth function $\mathbb{J}_p : \mathring{\mathfrak{P}}_+ \to \mathbb{R}$. In fact, a Dirichlet Principle would assert that $\mathbb{J}_p$ should have a unique minimum and no other critical points.

Define a 1-form on $\mathring{\mathfrak{P}}_+$:

$$\omega = \mathrm{Re}\left\{\sum_{k=0}^{n} f_k^p(q)\,dq_k\right\}$$

$$dQ = \mathrm{Re}\sum_{k=0}^{n} u_k\,dq_k$$

$$\begin{aligned}\omega &= \mathrm{Re}\left\{\sum_{k=0}^{n} c_k\,dq_k - \int_a^b \sum_{k=0}^{n} u_k\,dq_k \frac{P}{Q}\,dt\right\}\\ &= \mathrm{Re}\sum_{k=0}^{n} c_k\,dq_k - \int_a^b \frac{P}{Q}\,dQ\,dt\end{aligned}$$

$$d\omega = \int_a^b \frac{P}{Q^2}\,dQ \wedge dQ\,dt = 0$$ $\Rightarrow$ ω closed

$\mathring{\mathfrak{P}}_+$ convex $\Rightarrow$ ω exact (The Poincaré Lemma)

$$\omega = \mathrm{Re} \sum_{k=0}^{n} c_k dq_k - \int_a^b \frac{P}{Q} dQ dt$$

By the Poincaré Lemma, we can integrate along any curve:

$$\mathbb{J}_p(q_1) := \int_{q_0}^{q_1} \left(\mathrm{Re} \sum_{k=0}^{n} c_k dq_k - \int_a^b \frac{P}{Q} dQ dt \right)$$

$$\boxed{\mathbb{J}_p(q) = \langle c, q \rangle - \int_a^b P \log Q \, dt}$$

(modulo a constant of integration)

- This is a strictly convex functional

$$\boxed{\mathbb{J}_p(q) = \langle c, q\rangle - \int_a^b P \log Q \, dt}$$

strictly convex function $\mathbb{J}_p : \mathring{\mathfrak{P}}_+ \to \mathbb{R}$ satisfying

$$\frac{\partial \mathbb{J}_p}{\partial q_k} = c_k - \int_a^b u_k \frac{P}{Q} dt, \quad k = 0, 1, \ldots, n \quad (\dagger)$$

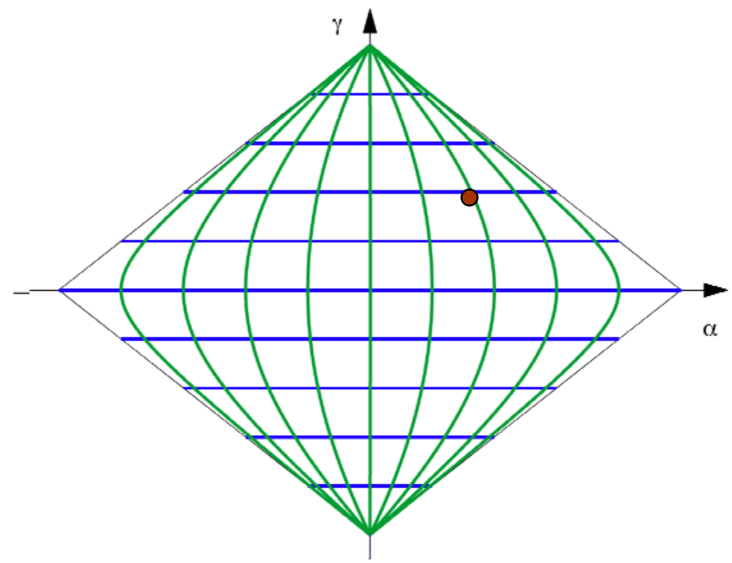
1. We have already shown that $\mathfrak{M}_{|\mathcal{P}_+}$ is surjective so that the moment equations $(\dagger)$ have a solution $\hat{q} \in \mathring{\mathfrak{P}}_+$.
2. Therefore, since $\mathbb{J}_p$ is strictly convex, $\mathbb{J}_p$ has a unique minimum.
3. Hence $\mathfrak{M}_{|\mathcal{P}_+}$ is also injective. In fact, $\mathfrak{M}_{|\mathcal{P}_+}$ is a diffeomorphism.
4. Fix $c \in \mathring{\mathfrak{C}}_+$. The map $g^c : \mathring{\mathfrak{P}}_+ \to \mathring{\mathfrak{P}}_+$ that sends p to $\hat{q}$ is a diffeomorphism onto its image. In other words, the solutions to the moment problem with rational positive measures are completely parameterized by $p \in \mathring{\mathfrak{P}}_+$; i.e., the spectral zeros.

EXAMPLE. $\mathfrak{P} = \text{span}\{1, e^{it}, \ldots, e^{int}\}$

The solutions $d\mu \in \mathcal{R}_+$ form a manifold of dimension $2n$.

A foliation with one leave for each choice of $p \in \mathring{\mathfrak{P}}_+$ (Kalman filtering)

A foliation with one leave for each choice of $c \in \mathring{\mathfrak{C}}_+$



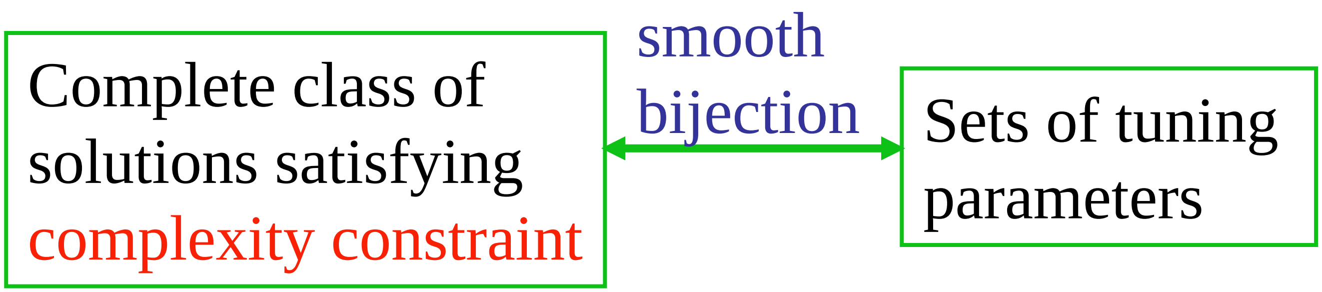
THEOREM. The two foliations intersect transversely so that each leaf in one meets each leaf in the other in exactly one point.

$$\min_{q \in \mathring{\mathfrak{P}}_+} \mathbb{J}(q)$$

unique solution $d\mu = \frac{P}{Q} dt$

Basic paradigm

- Find complete parameterization



- For any choice of tuning parameters, determine the corresponding solution by convex optimization

- Choose a solution that best satisfies additional design specifications (without increasing the complexity)

Conclusions

- The classical moment problem has a solution for each positive sequence.
- Natural constraints in applications, e.g., in systems and control, motivate the moment problems for rational measures.
- The moment problem for rational measures
  - has a solution for each strictly positive sequence
  - is completely parameterized by spectral zeros
  - can be solved by convex optimization