Stability of Wavefronts in a Diffusive Model for Porous Media Combustion

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Porous media combustion

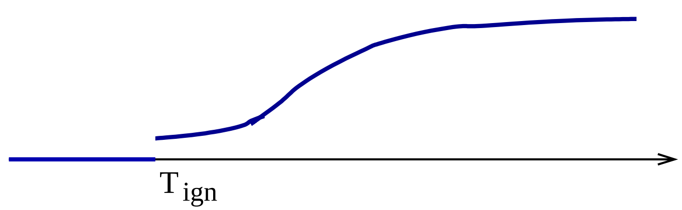
$$
\begin{aligned}
T_t - (1-\gamma^{-1})P_t &= \epsilon T_{xx} + YF(T) \qquad \text{[Sivashinsky 2002]}\\
P_t - T_t &= P_{xx}\\
Y_t &= \epsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y F(T)
\end{aligned}
$$

$\epsilon \ll 1$ - thermal diffusivity / pressure diffusivity $(\sim 10^{-4} - 10^{-7})$

$\gamma > 1$ - specific heat ratio, Le - Lewis number

$F(T)$ of Arrhenius type

with an ignition cut-off at $T = T_{ign}$



Combustible gas or gas mixture: oxygen, methane-air, hydrogen-air, propane-air...

Porous medium: coal, ceramic fiber felt, polyurethane foam...

Applications: chemical technology, ecology, fire and explosion safety

Close relatives: convective burning of granular explosives, combustion in thin rough tubes

Before Simplifications

Energy $c_p\rho(\Theta_\tau + u\Theta_\xi) - (\Pi_\tau + u\Pi_\xi) = qW + (c_p\rho D_{th}\Theta_\xi)_\xi$

Concentration $\rho(C_\tau + uC_\xi) = -W + (\Theta^{-1}D_{mol}(\rho\Theta C)_\xi)_\xi$

Chemical kinetics $W = Z_\rho C exp(-E/R\Theta)$

Continuity $\rho_\tau + (\rho u)_\xi = 0$

Momentum $\rho u = -K\nu^{-1}\Pi_\xi$

State $\rho = P/(c_p - c_\nu)\Theta$

u - gas velocity, C - concentration of the deficient reactant, ρ, Π, Θ - density, pressure, temperature of the gas-solid system, W - chemical reaction rate, ν - kinematic viscosity, Z -frequency factor, E - activation energy, R - universal gas constant, q - heat release, c_p /c_v/ - specific heat at constant pressure /volume/, D_{th} /D_{mol}/ - thermal /molecular/ diffusivity

Derivation of Simplified Model

- Small heat release approximation: variation of pressure, temperature, density and gas velocity assumed small

  nonlinear effects are ignored everywhere except in the reaction term

- Scaling:

$$T = \frac{\Theta - \Theta_0}{\Theta_\infty - \Theta_0}, \quad P = \frac{\Pi - \Pi_0}{\Pi_\infty - \Pi_0}, \quad Y = \frac{C}{C_0}$$

  Θ_0, Π_0, C_0 -temperature, pressure, concentration at $\tau = 0$,

  $\Theta_\infty, \Pi_\infty$ at $\tau \to \infty$ in case of homogeneous explosion

- $t = \frac{\tau}{\bar{\tau}}$, $x = \frac{\xi}{\bar{\xi}}$, where $\bar{\tau}, \bar{\xi} = const$

$$T_t - (1 - \gamma^{-1}) P_t = \epsilon T_{xx} + Y F(T)$$ partially lin. eqn for conservation of energy

$$P_t - T_t = P_{xx}$$ lin. continuity eqn with the eqn of state and Darcy law

$$Y_t = \epsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y F(T)$$ partially lin. eqn for conservation of reactant

The standard reduction of the system: $\varepsilon = 0$, $\mathrm{Le} = \mathrm{O}(1)$

$\epsilon = 0,\ \mathrm{Le} = \mathrm{O}(1)$

$$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= YF(T) \\ P_t - T_t &= P_{xx} \\ Y_t &= -\gamma Y \Omega(T) \end{aligned}$$

Time-conserved quantity: $T_t - (1-\gamma^{-1})P_t + \gamma^{-1}Y_t = 0$

Initial conditions: $T(x,0) = T_0(x), \quad P(0,x) = 0, \quad Y(0,x) = 1$

A unique front exists that connects the cold state $(0,0,1)$ with the burnt state $(1,1,0)$.

This front persists in the full system with $0 < \epsilon \ll 1$

Equivalent system

For $$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= \epsilon T_{xx} + Y\Omega(T) \\ P_t - T_t &= P_{xx} \\ Y_t &= \epsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y \Omega(T) \end{aligned}$$

transformation $$\begin{aligned} T &= hu + (1-h)v \\ P &= (1-\varepsilon)^{-1}u - \varepsilon(1-\varepsilon)^{-1}v, \quad Y = y \\ \varepsilon &= \epsilon\gamma(1-\mu)^2, \;\; h = \mu/(1-\varepsilon) = \mu/(1-\epsilon\gamma(1-\mu)^2) \\ \mu &= \frac{\sqrt{\gamma^2(\epsilon+1)^2 - 4\gamma\epsilon} + \gamma(\epsilon-1)}{2\gamma\epsilon} \\ \tau &= \gamma\, t, \quad z = \sqrt{\gamma(1-\mu)}\, x \end{aligned}$$

leads to $$\begin{aligned} u_\tau &= u_{zz} + yF(hu + (1-h)v) \\ v_\tau &= \varepsilon v_{zz} + yF(hu + (1-h)v) \\ y_\tau &= \varepsilon(\gamma(1-\mu)\mathrm{Le})^{-1} y_{zz} - yF(hu + (1-h)v) \end{aligned}$$

Reduction, special initial conditions, $\mathrm{Le} = \mathrm{Le}^*$

When $\mathrm{Le}^{-1} = \gamma(1-\mu)$

$$\begin{aligned} u_\tau &= u_{zz} + yF(hu + (1-h)v) \\ v_\tau &= \varepsilon v_{zz} + yF(hu + (1-h)v) \\ y_\tau &= \varepsilon(\gamma(1-\mu)\mathrm{Le})^{-1} y_{zz} - yF(hu + (1-h)v) \end{aligned}$$

reads

$$u_t = u_{zz} + yF(hu + (1-h)v)$$

$$v_t = \varepsilon v_{zz} + yF(hu + (1-h)v)$$

$$y_t = \varepsilon y_{zz} - yF(hu + (1-h)v)$$

If initially $y(0,x) = 1 - v(0,x)$, then $y(t,x) = 1 - v(t,x)$ for $t > 0$, $x \in \mathbb{R}$.

Therefore the system reduces [Gordon, 2007] to

$$\begin{aligned} u_t &= u_{xx} + yF(hu + (1-h)(1-y)) \\ y_t &= \varepsilon y_{xx} - yF(hu + (1-h)(1-y)) \end{aligned}$$

Reduction, no restrictions on initial conditions, $\mathrm{Le} = \mathrm{Le}^*$

In

$$u_t = u_{zz} + yF(hu + (1-h)v)$$

$$v_t = \varepsilon v_{zz} + yF(hu + (1-h)(1-y))$$

$$y_t = \varepsilon y_{zz} - yF(hu + (1-h)(1-y))$$

take $g = v + y$, **to obtain**

$$u_t = u_{zz} + yF(hu + (1-h)(1-y))$$

$$y_t = \varepsilon y_{zz} - yF(hu + (1-h)(1-y))$$

$$g_t = \varepsilon g_{zz}$$

Plan:

1) Study stability of fronts in the system obtained $y(0,x) = 1 - v(0,x)$

2) Extend the stability result for fronts in that system to the general case.

Traveling fronts

In the moving with the front frame $\xi = x - ct$

$$u_t = u_{xx} + yF(hu + (1-h)(1-y))$$

$$y_t = \varepsilon y_{xx} - yF(hu + (1-h)(1-y))$$

where $h \in (0,1)$,

$$u_t = u_{\xi\xi} + cu_\xi + yF(hu + (1-h)(1-y))$$

$$y_t = \varepsilon y_{\xi\xi} + cy_\xi + yF(hu + (1-h)(1-y))$$

For small ε, there exists a unique c such that there is $(\widehat{u}(\xi), \widehat{y}(\xi))$ that solves ([Gordon, Kamin, Sivashinksi, 2002], [G., Gordon, Jones, 2008] for the full system)

$$\begin{aligned} u_{\xi\xi} + cu_\xi + yF(hu + (1-h)(1-y)) &= 0 \\ \varepsilon y_{\xi\xi} + cy_\xi - yF(hu + (1-h)(1-y)) &= 0 \end{aligned} \tag{1}$$

and $(u,y) \to (1,0)$ as $\xi \to -\infty$, $(u,y) \to (0,1)$ as $\xi \to +\infty$.

Reaction rate

$$F(w) = F_{jump}(w) = \begin{cases} \exp\left(Z\left\{\frac{w-h}{\sigma+(1-\sigma)w}\right\}\right), & w \geq w_{ign}, \\ 0, & w < w_{ign}. \end{cases}$$

However, for the stability analysis, we consider a smooth $\tilde{F}$

$$\tilde{F}(w) = \begin{cases} \exp\left(Z\left\{\frac{w-h}{\sigma+(1-\sigma)w}\right\}\right), & w \geq w_{ign} + 2\delta, \\ F_{jump}(w)\, H_\delta(w - w_{ign} - \delta), & w_{ign} \leq w < w_{ign} + 2\delta, \\ 0, & w < w_{ign}, \end{cases}$$

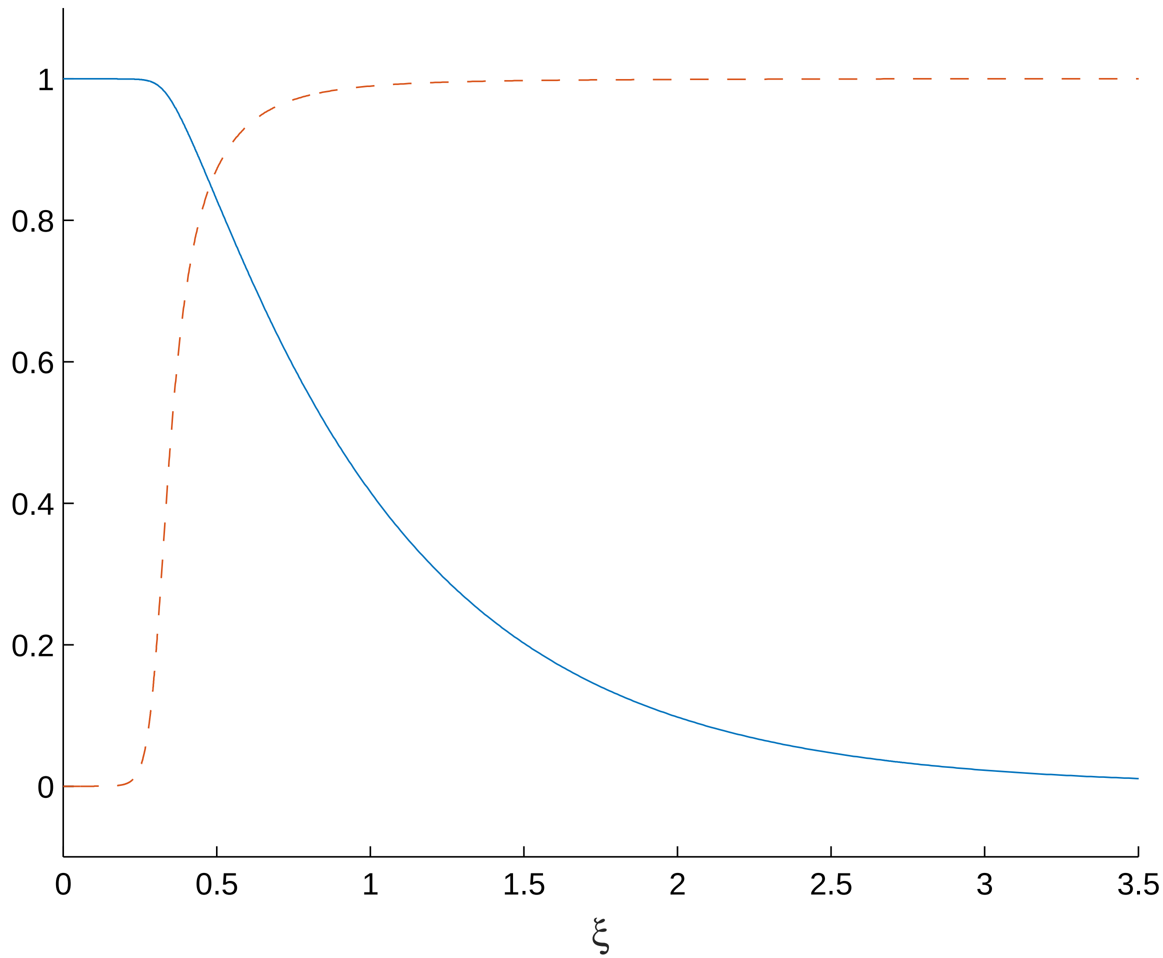
where H_δ is a smooth approximation of the Heaviside function H such that

$H_\delta(x) = \frac{1}{1+e^{\frac{4x\delta}{x^2-\delta^2}}}$, for $|x| < \delta$

A front $(\hat{u}, \hat{v})$ exists ($c = O(1)$) that connects $(1, 1)$ at $-\infty$ to $(0, 0)$ at ∞

Traveling front: numerics

$\varepsilon = 0.1$, $h = 0.3$, $\sigma = 0.25$, $\delta = 0.0005$, $T_{ign} = 0.01$, and $Z = 6$, which results in $c = 1.8588$. δ is chosen so that the speed c is close to the speed in the discontinuous system.



Linearization about $(\widehat{u}, \widehat{y})$

$$\lambda p = p_{\xi\xi} + cp_\xi + F_w(\widehat{w})\,\widehat{y}\,(hp - (1-h)q) + F(\widehat{w})q,$$

$$\lambda q = \varepsilon q_{\xi\xi} + cq_\xi - F_w(\widehat{w})\,\widehat{y}\,(hp - (1-h)q) - F(\widehat{w})q,$$

where $w = hu + (1-h)(1-y)$ and $\widehat{w} = w(\widehat{u}, \widehat{y})$.

Essential Spectrum of the linearization L about the front:

- Essential spectrum in $L^2(\mathbb{R})$ is bounded by a parabola touching the imaginary axis at the origin from the right
- Consider $L^2_\alpha(\mathbb{R})$: $\|f\|^2_\alpha = \int_{-\infty}^{\infty} |\rho_\alpha(\xi) f(\xi)|^2 \, \mathrm{d}\xi$, with weight $\rho_\alpha = e^{\alpha\xi}$. Essential spectrum in $L^2(\mathbb{R}) \cup L^2_\alpha(\mathbb{R})$ has the right most boundary to the left of the imaginary axis and it is a parabola

Energy-like estimates allow to obtain a bound on the unstable point spectrum.

Point spectrum: there are parameter regimes, where

- 0 is a simple eigenvalue
- the rest of the point spectrum $\subset \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda < 0\}$

Evans function

$$\lambda p = p_{\xi\xi} + c p_{\xi} + \tilde{\Omega}'(\hat{w})\,\hat{y}\,(hp - (1-h)q) + \tilde{\Omega}(\hat{w})q \iff X' = A(\xi,\lambda)\,X$$

$$\lambda q = c q_{\xi} - \tilde{\Omega}'(\hat{w})\,\hat{y}\,(hp - (1-h)q) - \tilde{\Omega}(\hat{w})q$$

where

$$A(\xi,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda - h\,F_w(\widehat{w})\,\widehat{y} & -c & (1-h)\,F_w(\widehat{w})\,\widehat{y} - F(\widehat{w}) & 0 \\ 0 & 0 & 0 & 1 \\ h\,F_w(\widehat{w})\,\widehat{y}/\varepsilon & 0 & \left(\lambda + (1-h)\,F_w(\widehat{w})\widehat{y} + F(\widehat{w})\right)/\varepsilon & -c/\varepsilon \end{pmatrix}$$

Asymptotic matrix, $+\infty$

$$\mathcal{A}^\infty(\lambda) = \lim_{\xi\to\infty} A(\xi,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda & -c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda/\varepsilon & -c/\varepsilon \end{pmatrix}$$

For $\mathrm{Re}\,\lambda > 0$, $\mathcal{A}^\infty$ has two eigenvalues with negative real part:

$$\mu_{1+} = -\frac{1}{2\varepsilon}(c + \sqrt{c^2 + 4\varepsilon\lambda}), \quad \mu_{2+} = -\frac{1}{2}(c + \sqrt{c^2 + 4\lambda}),$$

and their corresponding eigenvectors are

$$v_{1+} = \left(0, 0, 1, \mu_{1+}\right)^T, \quad v_{2+} = \left(1, \mu_{2+}, 0, 0\right)^T.$$

Asymptotic matrix, $-\infty$

$$\mathcal{A}^{-\infty}(\lambda) = \lim_{\xi\to-\infty} A(\xi,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda & -c & e^{(1-h)Z} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (\lambda + e^{(1-h)Z})/\varepsilon & -c/\varepsilon \end{pmatrix}$$

For $\operatorname{Re}\lambda > 0$, $\mathcal{A}^{-\infty}$ has two eigenvalues with positive real part:

$$\mu_{1-} = -\frac{1}{2\varepsilon}\left(c - \sqrt{c^2 + 4\varepsilon\lambda + 4\varepsilon e^{Z(1-h)}}\right), \quad \mu_{2-} = -\frac{1}{2}\left(c - \sqrt{c^2 + 4\lambda}\right)$$

and the corresponding eigenvectors are

$$v_{1-} =$$

$$\left(1, \mu_{1-}, \frac{(1-\varepsilon)(\lambda - c\,\mu_{1-})}{\varepsilon\, e^{Z(1-h)}} + \frac{1}{\varepsilon}, \frac{\left(\left(c^2+\lambda\,\varepsilon\right)(1-\varepsilon)+\varepsilon\, e^{Z(1-h)}\right)\mu_{1-} - c\left(\lambda + e^{Z(1-h)}\right)(1-\varepsilon)}{\varepsilon^2 e^{Z(1-h)}}\right)^T$$

$$v_{2-} = \left(1,\ \mu_{2-}, 0, 0\right)^T$$

Definition of Evans Function

So $X' = A(\xi, \lambda)\, X$ has two linearly independent solutions X_{1+} and X_{2+} converging to zero as $\xi \to \infty$ and two solutions X_{1-} and X_{2-} converging to zero as $\xi \to -\infty$, satisfying

$$\lim_{\xi \to \pm\infty} X_{i\pm} e^{-\mu_{i\pm} \xi} = v_{i\pm}, \quad i = 1, 2.$$

λ is an eigenvalue if and only if the space of solutions spanned by $\{X_{1+},\ X_{2+}\}$, and the space of solutions spanned by $\{X_{1-},\ X_{2-}\}$, have a non-empty intersection.

Those values of λ can be located with the help of the Evans function. The Evans function is a function of the spectral parameter λ; it is analytic, real for λ real, and it vanishes on the point spectrum.

We define the Evans function using exterior algebra.

Evans function using exterior algebra

The dimension of the eigen-value system is $n = 4$ and the dimensions of the stable and unstable manifolds are $n_s = n_u = 2$.

We consider the wedge-space $\bigwedge^2(\mathbb{C}^4)$, the space of all two forms on $\mathbb{C}^4$. The induced dynamics of $X' = A(\xi, \lambda)\, X$ on $\bigwedge^2(\mathbb{C}^4)$ can be written as

$$U' = \mathrm{A}^{(2)}(\xi, \lambda) U.$$

Here the matrix $A^{(2)}$ is matrix generated by $A = \{a_{ij}\}$ on $\bigwedge^2(\mathbb{C}^4)$. Using the standard basis of $\bigwedge^2(\mathbb{C}^4)$: $\omega_1 = \mathrm{e}_1 \wedge \mathrm{e}_2$, $\omega_2 = \mathrm{e}_1 \wedge \mathrm{e}_3$, $\omega_3 = \mathrm{e}_1 \wedge \mathrm{e}_4$, $\omega_4 = \mathrm{e}_2 \wedge \mathrm{e}_3$, $\omega_5 = \mathrm{e}_2 \wedge \mathrm{e}_4$, $\omega_6 = \mathrm{e}_3 \wedge \mathrm{e}_4$, ($\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4\}$ is the standard basis of $\mathbb{C}^4$) the matrix $A^{(2)}$ is given by

$$A^{(2)} = \begin{pmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{pmatrix}$$

Evans function

In our case, the asymptotic matrices are given by

$$\lim_{\xi \to \pm\infty} A^{(2)}(\xi, \lambda) = \left(\mathcal{A}^{\pm\infty}\right)^{(2)}$$

The eigenvalue of $(\mathcal{A}^{\infty})^{(2)}$ with the smallest real part is $\mu_{1+} + \mu_{2+}$ with eigenvector $v_{1+} \wedge v_{2+}$. The solution of $U' = \mathrm{A}^{(2)}(\xi, \lambda)U$ given by $U_{+} = X_{1+} \wedge X_{2+}$ then behaves as

$$\lim_{\xi \to \infty} U_{+} e^{-(\mu_{1+} + \mu_{1+})\xi} = w_{+} \equiv v_{1+} \wedge v_{2+}.$$

Similarly, the solution $U_{-} = X_{1-} \wedge X_{2-}$ behaves as

$$\lim_{\xi \to -\infty} U_{-} e^{-(\mu_{1-} + \mu_{-})\xi} = w_{-} \equiv v_{1-} \wedge v_{2-}.$$

We define the Evans function as

$$E(\lambda) \equiv U_{-} \wedge U_{+},$$

where $U_{\pm}$ are evaluated at, say, $\xi = 0$.

Evans function

The Evans function is

$$E(\lambda) = U_-^T \, \Sigma \, U_+,$$

where Σ is the matrix

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The function $E(\lambda)$ will be analytic in the any region of the complex plane where the eigenvalues $\mu_{1+} + \mu_{2+}$ and $\mu_{1-} + \mu_{2-}$ are, respectively, the eigenvalues with smallest and largest real part of $(\mathcal{A}^{\infty})^{(2)}$ and $\left(\mathcal{A}^{-\infty}\right)^{(2)}$. To define such a region, it suffices to implement the condition $\mathrm{Re}\,\lambda > -\frac{1}{4}\min\left(1,\ \frac{1}{\varepsilon}\right)$.

Numerical calculation of Evans function

To find zeroes of $E(\lambda)$, compute the integral of the logarithmic derivative of $E(\lambda)$ on a closed curve and obtain the winding number of $E(\lambda)$ along that curve.

In our case, the contour of integration is chosen so that it lies in the region defined by energy-like estimates.

Numerical winding number computation then that the Evans function has no zeroes other than the one at the origin.

There is a regime in which the front is spectrally stable, with the exception of essential spectrum touching the imaginary axis.

Nonlinear Stability

[G, Latushkin, Schecter, 2011] $\Longrightarrow$ convective nature of the instability due to the marginal essential spectrum

1. If the initial perturbations are small in the regular and the weighted norms, then
   - y component decays exponentially in the regular norm (therefore Y does)
   - u component stays bounded, so T and P do too
   - in the weighted norm all components decay exponentially

2. If the initial perturbation in addition are small in L^1-norm, then the perturbation to the u-component decays diffusively in L_∞-norm, so T and P do so too.

Full system, $\mathrm{Le}^{-1} = \gamma(1-\mu)$, no restriction on initial conditions.

$$u_t = u_{zz} + yF(hu + (1-h)v), \qquad y_t = \varepsilon y_{zz} - yF(hu + (1-h)v)$$

$$g_t = \varepsilon g_{zz}$$

Spectral stability: $g_t = \varepsilon g_{zz}$ **does not produce point spectrum. The lin operator then has only marginally unstable essential spectrum** $\Longrightarrow$ **The spectral results extend to the full system.**

Nonlinear stability: [G, Latushkin, Schecter, 2011] $\Longrightarrow$ **The time evolution of perturbations to** u **and** y **are the same as in reduced system. If initial perturbations to the front are small in both regular and weighted norm,** g **stays bounded in the norm without the weight and decays exponentially in the weighted norm. Since perturbation to** y **decays exponentially in all norms, perturbations to the** v **behave the same way as perturbations to** $g = y + v$**.**

If, in addition, initial perturbations are also small in L^1**-norm, then the perturbations to** g**, and therefore** v **not only stay bounded but decay algebraically in** L_∞**-norm.**

Nonlinear convective stability: assumptions

$$Y_t = DY_{xx} + R(Y)$$

Traveling wave: $Y_*(\xi), \xi = x - ct, c > 0 \; Y_- = 0$

Assume that $Y_*(\xi)$ **is spectrally stable in** $\mathcal{E}_\alpha$ **and** $Y_*' \in \mathcal{E}_\alpha$.

Assume that in appropriate variables $Y = (U, V)$ **and** $R(U, 0) = 0$ **such that**

$$U_t = D_1 U_{xx} + cU_x + R_1(U, V)$$

$$V_t = D_2 V_{xx} + cV_x + R_2(U, V)$$

D_1 **and** D_2 **nonneg. diag. matrices, and** $R_i(U, 0) = 0$ **and at** $Y_- = (0, 0)$

$$U_t = D_1 U_{\xi\xi} + cU_\xi + D_2 R_1(0, 0)V = L^{(1)}U + D_2 R_1(0, 0)V$$

$$V_t = D_2 V_{\xi\xi} + cV_\xi + D_2 R_2(0, 0)V = L^{(2)}V$$

Assume that in $\mathcal{E}_0$, $\sigma(L^{(2)}) \in \{\mathrm{Re}\,\lambda \leq -\rho, \; \rho > 0\}$, **and** $\mathrm{sp}(L^{(1)})$ **touches or crosses the imaginary axis, but** $L^{(1)}$ **generates a bounded semigroup**

Nonlinear convective stability: theorems

Theorem. With the given assumptions, perturbations of the traveling wave that are initially small in $\mathcal{E}_0 \cap \mathcal{E}_\alpha$ decay exponentially in $\mathcal{E}_\alpha$ to some shift of the wave. These solutions can be written

$$(U, V)(\xi, t) = U_*(\xi + \tilde{c}(t)) + \tilde{U}(\xi, t), V_*(\xi + \tilde{c}(t)) + \tilde{V}(\xi, t))$$

with, for each t, $(\tilde{U}(\xi, t), \tilde{V}(\xi, t))$ in a fixed subspace of $\mathcal{E}_0 \cap \mathcal{E}_\alpha$ complementary to Y_*'. $\tilde{U}(\xi, t)$ stays small in $\mathcal{E}_0$, and $\tilde{V}(\xi, t)$ decays exponentially $\mathcal{E}_0$.
So in the unweighted norm, the U-component of a perturbation stays small, and the V-component decays.

Theorem. In addition to the given assumptions, suppose the linear equation $\tilde{U}_t = L^{(1)}\tilde{U}$ is parabolic, i .e., the diagonal entries of D_1 are all positive. If the perturbation of the traveling wave is also small in L^1, then $\tilde{U}(\xi, t)$ stays small in the L^1-norm and decays like $t^{-\frac{1}{2}}$ in the L^∞-norm.

[G, Latushkin, Schecter, 2011]