Signature of Optimal Solutions in Turbulent Rayleigh-Bénard Convection

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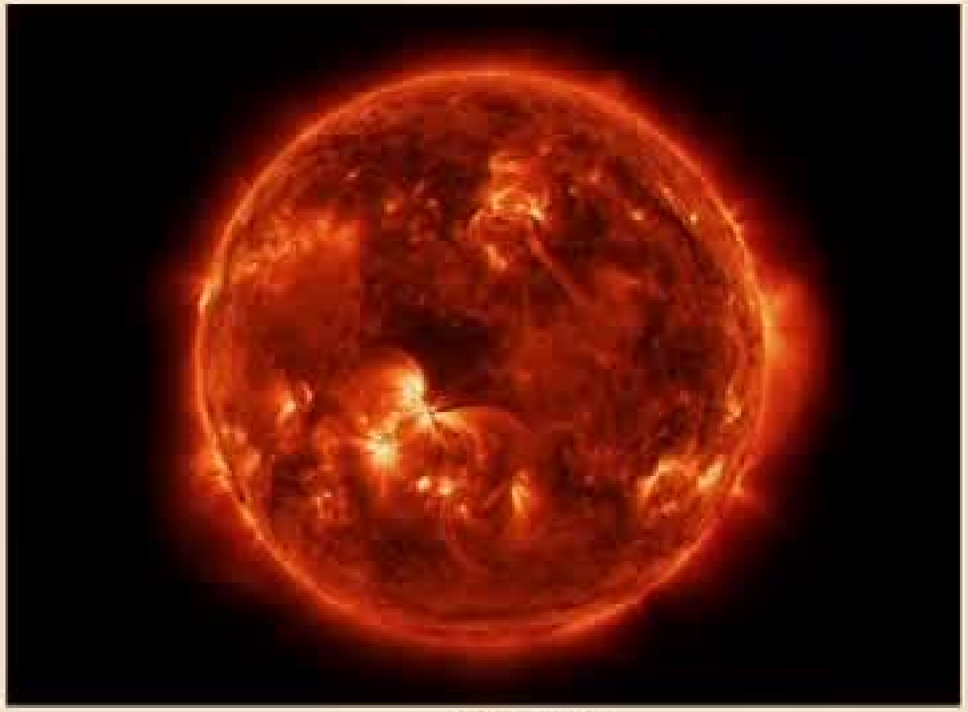
Outline

Motivation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Governing Equations I

We consider the Boussinesq equations:

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \alpha_V T g \widehat{\mathbf{y}}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T$$

$\mathbf{u}(\mathbf{x}, t) = (u,\ v,\ w)$: velocity field $\quad$ $T(\mathbf{x}, t)$: temperature field

ν: kinematic viscosity $\quad$ κ: thermal diffusivity

α_V: Coefficient of volume expansion

Governing Equations II

From the Boussinesq equations we:

1. Eliminate pressure with $\nabla \times \nabla \times$ of momentum equation;
2. Project momentum equation onto $\widehat{\mathbf{y}}$;
3. Consider flows such that $w = 0$ and $\partial_z(\cdot) = 0$.

$$\frac{\partial \nabla^2 v}{\partial t} + \partial_x \left(u\nabla^2 v - v\nabla^2 u\right) = \nu\nabla^2\nabla^2 v + g\alpha_V \partial_x^2 T$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa\nabla^2 T$$

$$\partial_x u + \partial_y v = 0$$

Nondimensionalization I

- Introduce the non-dimensionalizations,

$$T^* = T/(\Delta T/2), \quad \mathbf{x}^* = \mathbf{x}/h, \quad \mathbf{u}^* = \mathbf{u}/(h/t_f), \quad t^* = t/t_f$$

$$t_f = h/g', \quad g' = g\alpha_V \Delta T/2$$

- The resulting nondimensional equations are

$$\frac{\partial \nabla^2 v}{\partial t} + \partial_x \left(u\nabla^2 v - v\nabla^2 u\right) = \nu_\star \nabla^2\nabla^2 v + \partial_x^2 T$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa_\star \nabla^2 T$$

$$\partial_x u + \partial_y v = 0$$

- Dimensionless diffusivities are given by

$$\nu_\star = \frac{\nu}{g'\sqrt{h^3}}, \quad \kappa_\star = \frac{\kappa}{g'\sqrt{h^3}}.$$

Non-exhaustive Review of Results

- Variational approaches and rigorous bounds
  - Howard (1963), $Nu \sim \left(\frac{Ra}{248}\right)^{3/8}$; Busse (1969), $Nu \sim Ra^{1/2}$
  - Constantin and Doering (1990s), $Nu \leqslant 0.167 Ra^{1/2} - 1$; Whitehead and Doering (2011), $Nu \leqslant 0.2295 Ra^{5/12}$ [**free slip**], ...
- Statistical turbulence theory
  - Kraichnan (1962), $Nu \sim Pr^{\gamma} Ra^{1/2} \times$ log corrections [**ultimate regime**]; Shraiman and Siggia (1990); $Nu \sim 0.27 Ra^{2/7} Pr^{-1/7}$; Grossman-Lohse theory (2000s); ...
- Computational studies
  - Amati (2005), $Nu \sim Ra^{1/3}$; Verzicco (2003), $Nu \sim Ra^{0.309}$; Zhu (2018), transition to ultimate regime in 2D RBC; ...
- Experimental studies
  - Niemela (2000), $Nu \sim Ra^{0.34}$; He (2012), transition to ultimate regime; ...

Exact Coherent Structures

- Scaling laws have generally been derived through variational principles:
  - The goal is to maximize *Nu*;
  - *Nu* has a functional dependence on integral quantities derived from the Boussinesq equations;
  - Constraints include boundary conditions and incompressibility.
- We wish to maximize *Nu* subject to the *full* Boussinesq equations in addition to the usual constraints:
  - Solve the full Boussinesq equations numerically;
  - Find steady (possibly unstable) states;
  - Determine $Nu(Ra, Pr)$ for these states.

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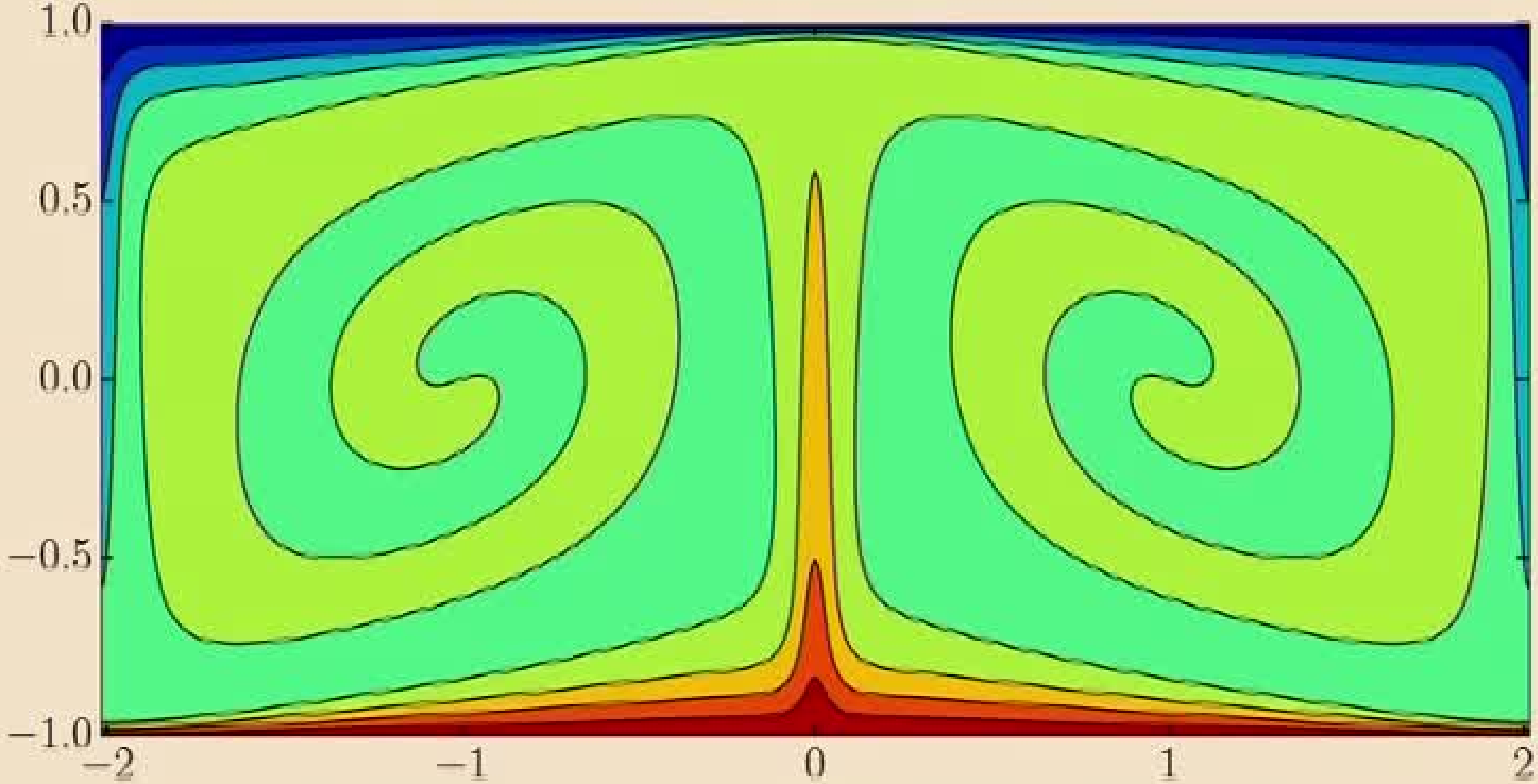
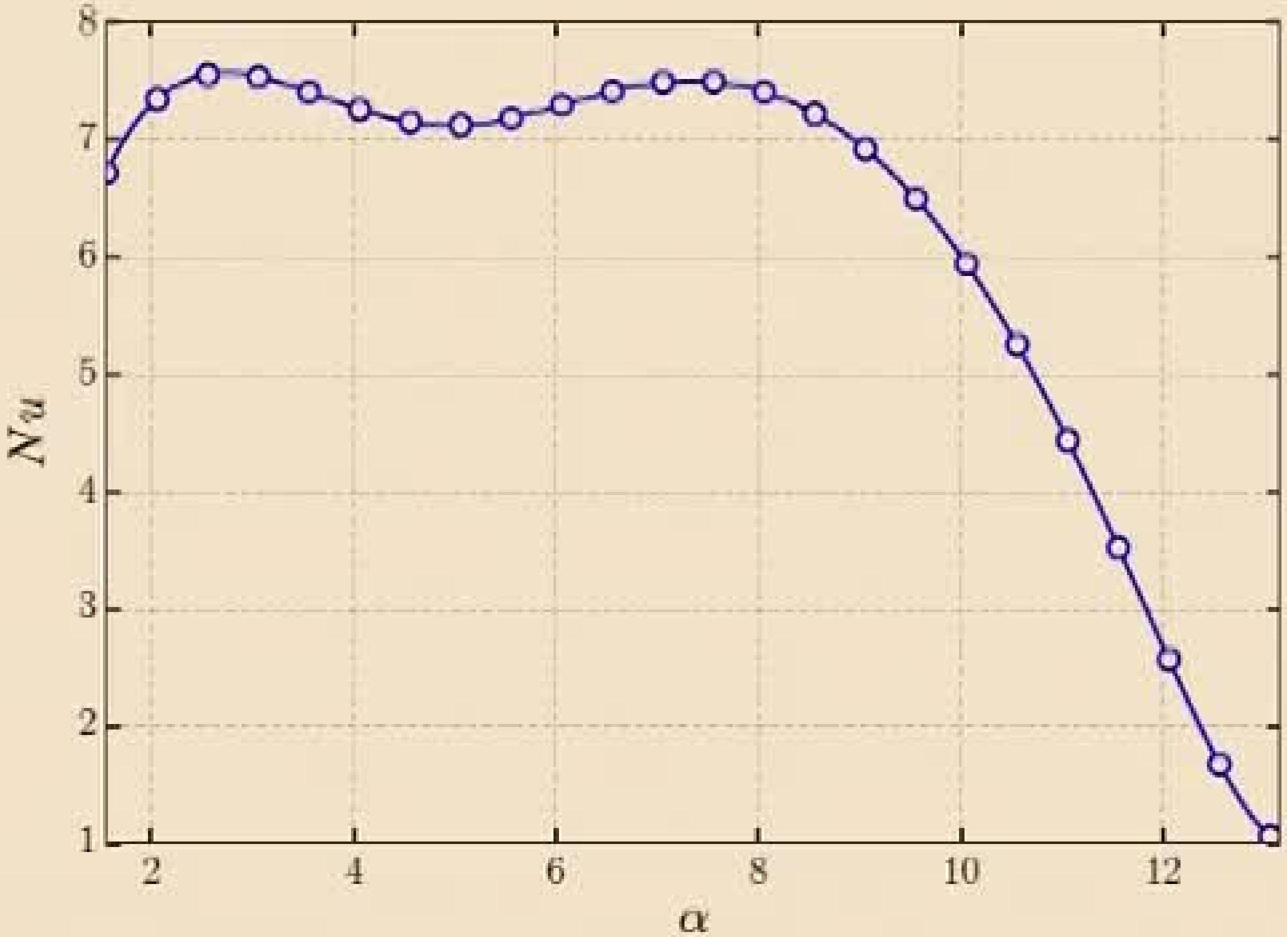
- Example of a non-optimal structure:



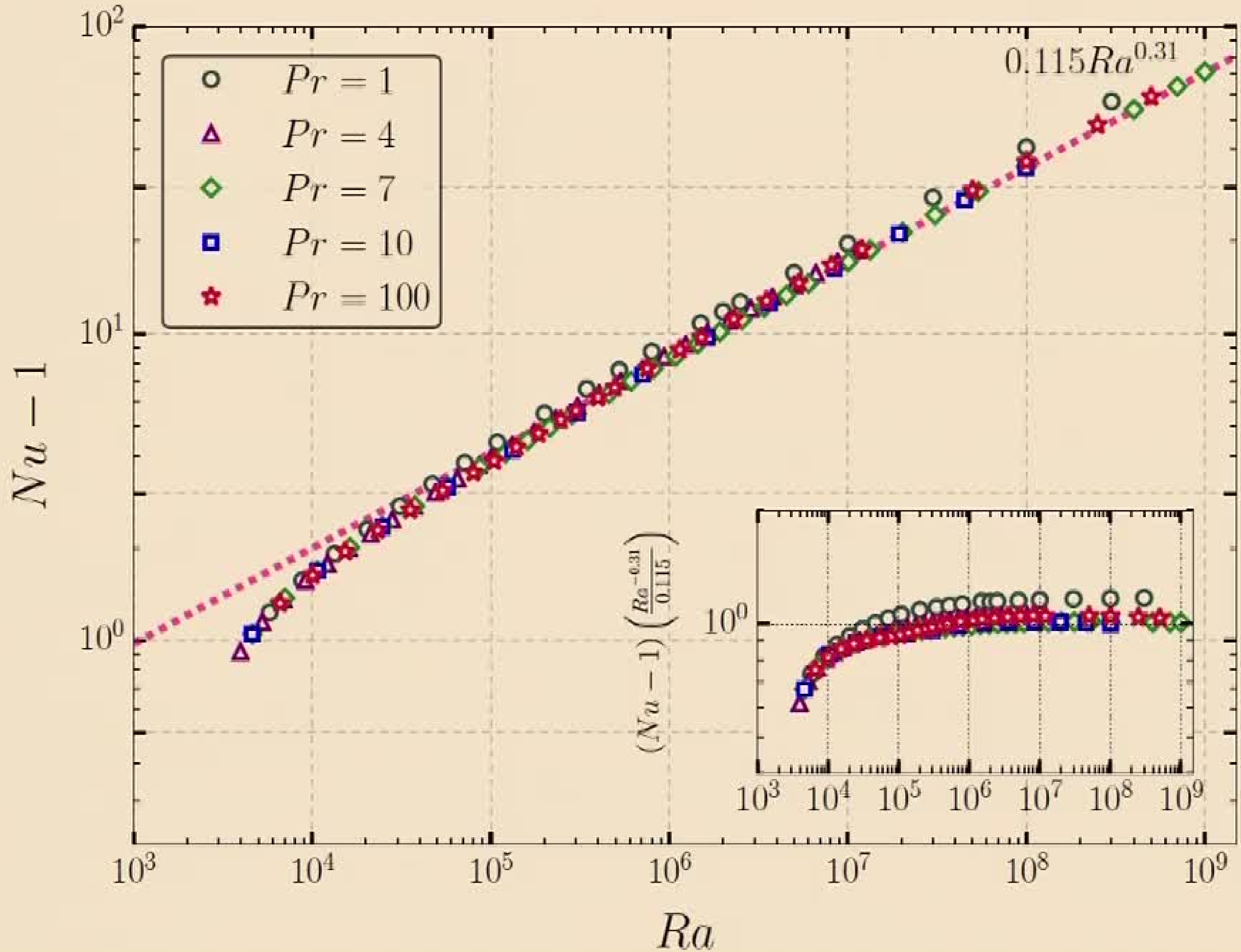
Optimal Exact Coherent Structures

- Vary the horizontal wavenumber (α) of solutions
- The domain size is $L = \dfrac{2\pi}{\alpha}$
- This selects the horizontal size of the structures
- Optimize Nu over α



- Small $\alpha \Rightarrow$ Larger scales
- Large $\alpha \Rightarrow$ smaller scales

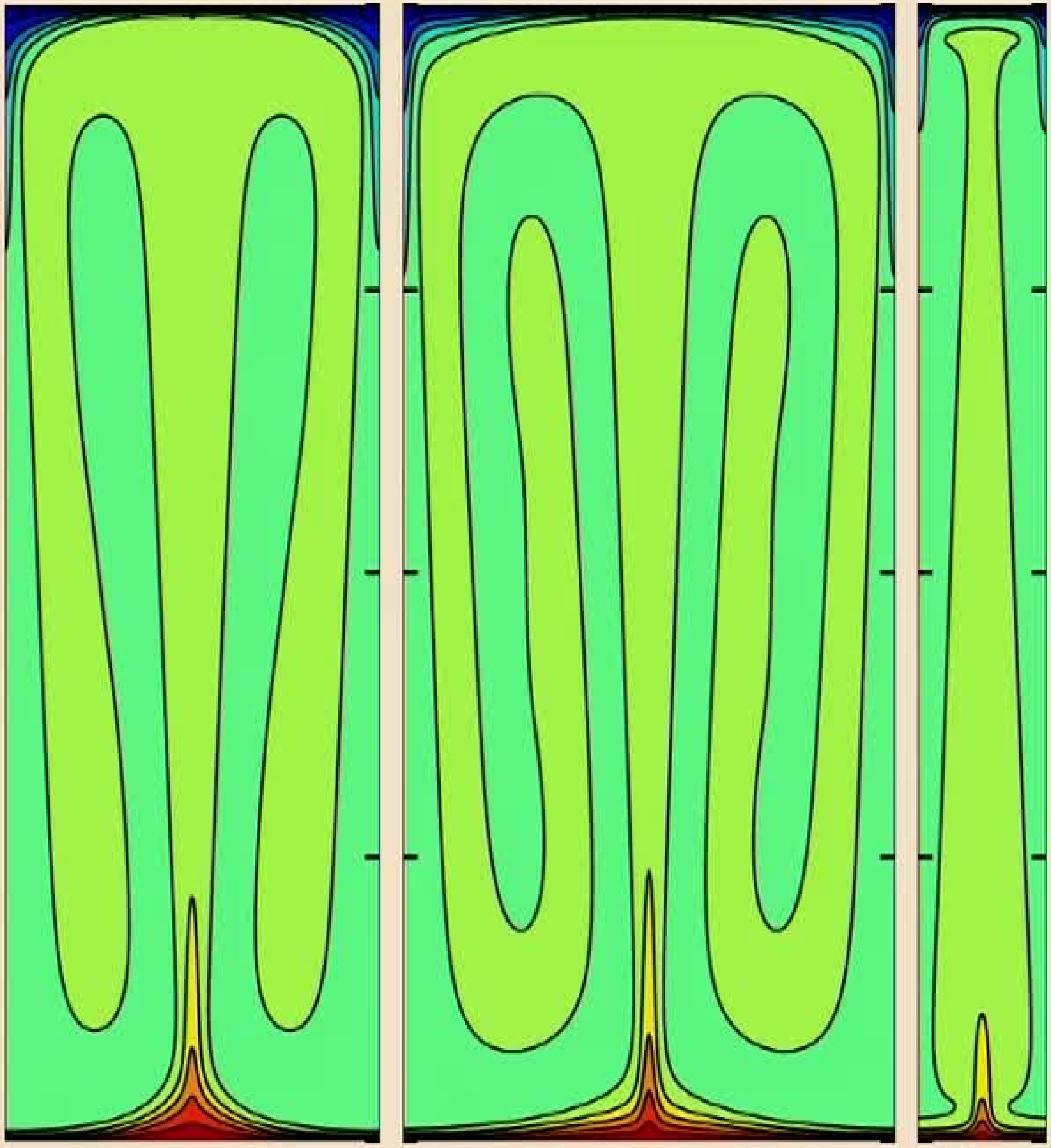
Nusselt Scaling with *Ra* and *Pr*



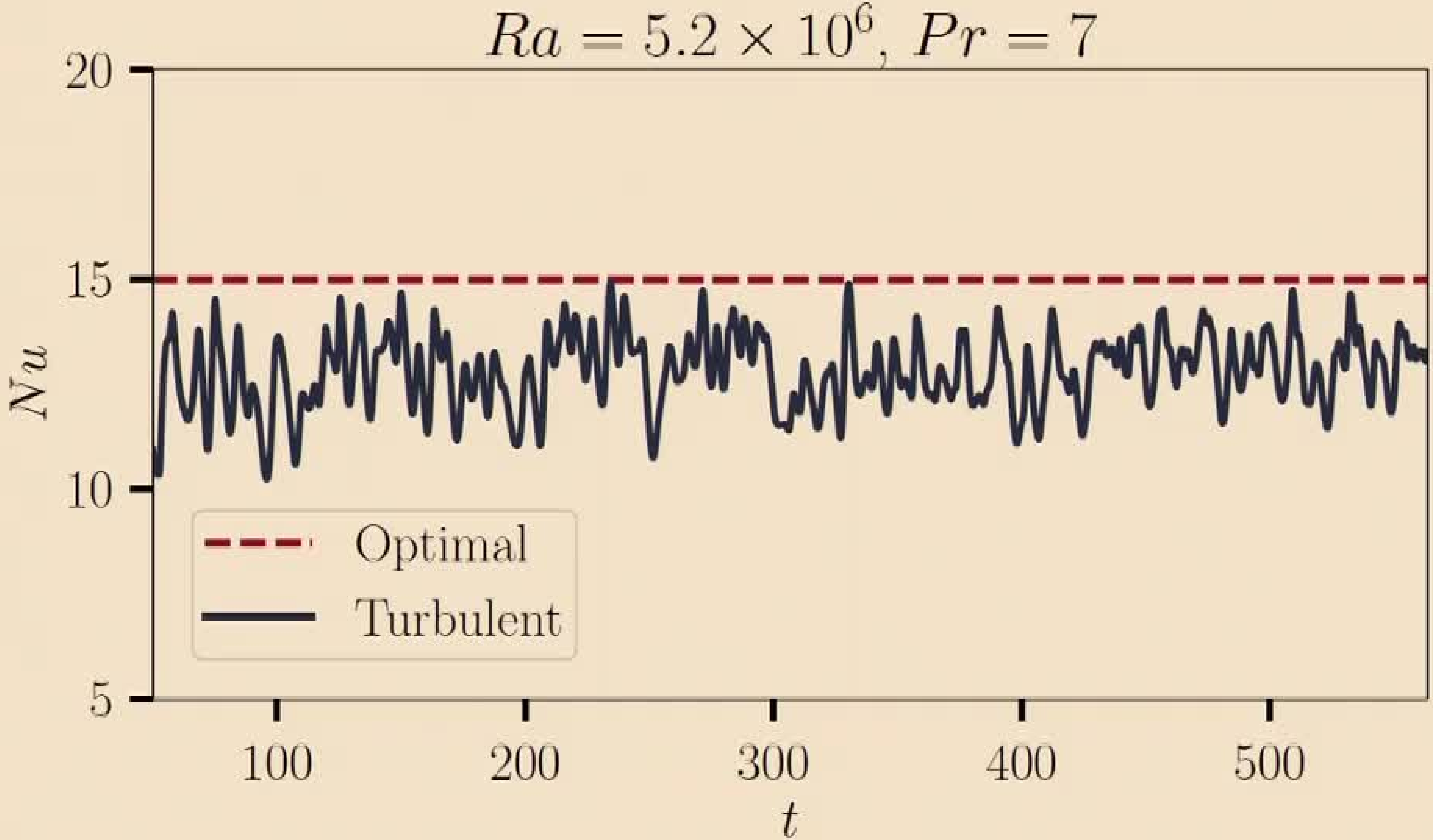
Summary of Scaling Results

1. Numerically solved the full 2D Boussinesq equations
2. Optimal solutions to the Boussinesq equations follow $Nu \sim Ra^{0.31}$ up to $Ra = 10^9$
3. $Pr-$ dependence of $Nu - Ra$ scaling is very weak for $Pr > 1$
4. Significant $Pr-$ dependence in lengthscales giving rise to optimal solutions
5. Fluids with $Pr \leqslant 7$ require larger horizontal lengthscales to optimize vertical heat transport
6. Fluids with $Pr > 7$ require smaller horizontal lengthscales to optimize vertical heat transport

$Ra = 10^8$: $Pr = 1$, $Pr = 7$, $Pr = 10$



Searching for Signatures



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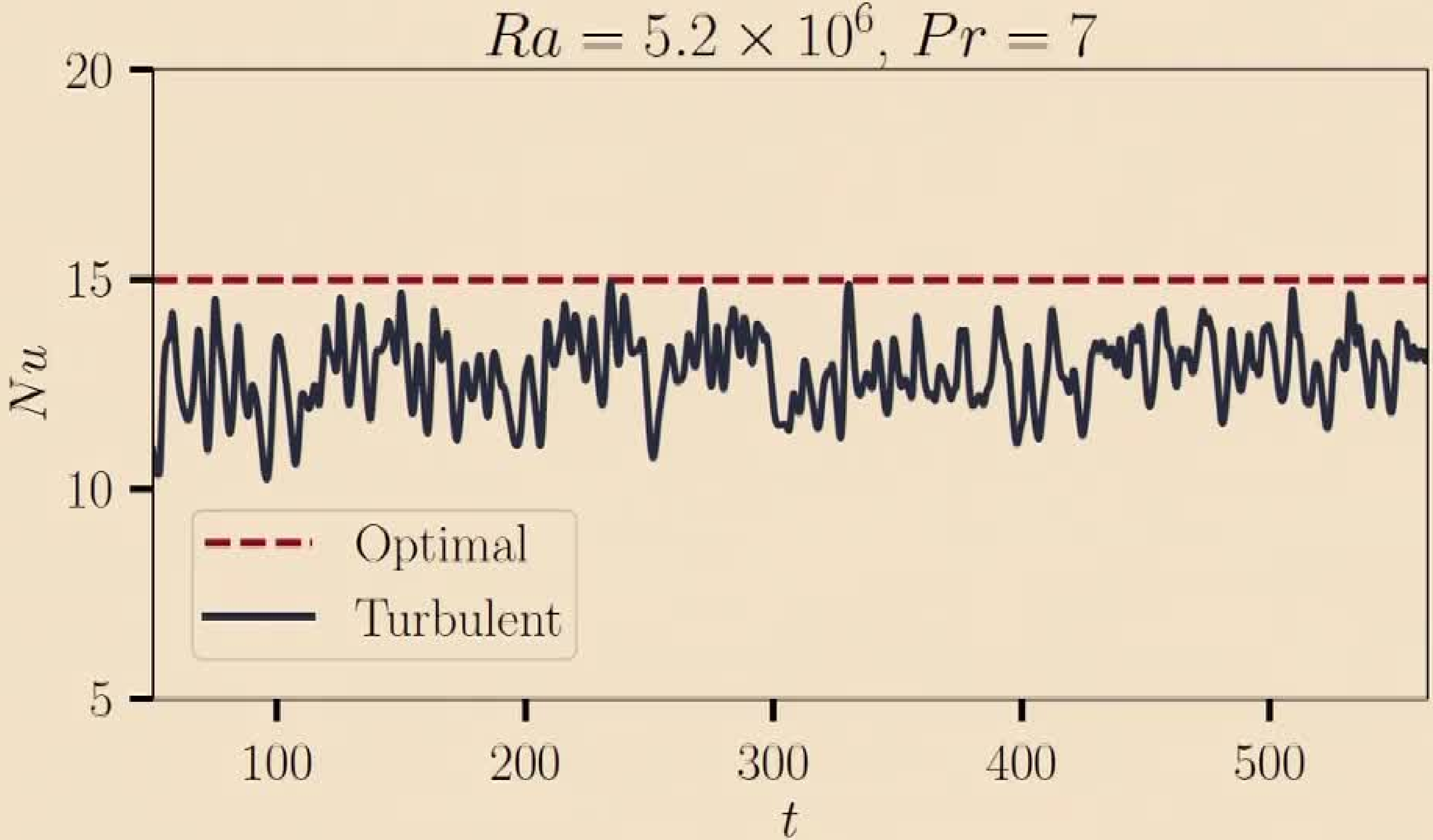
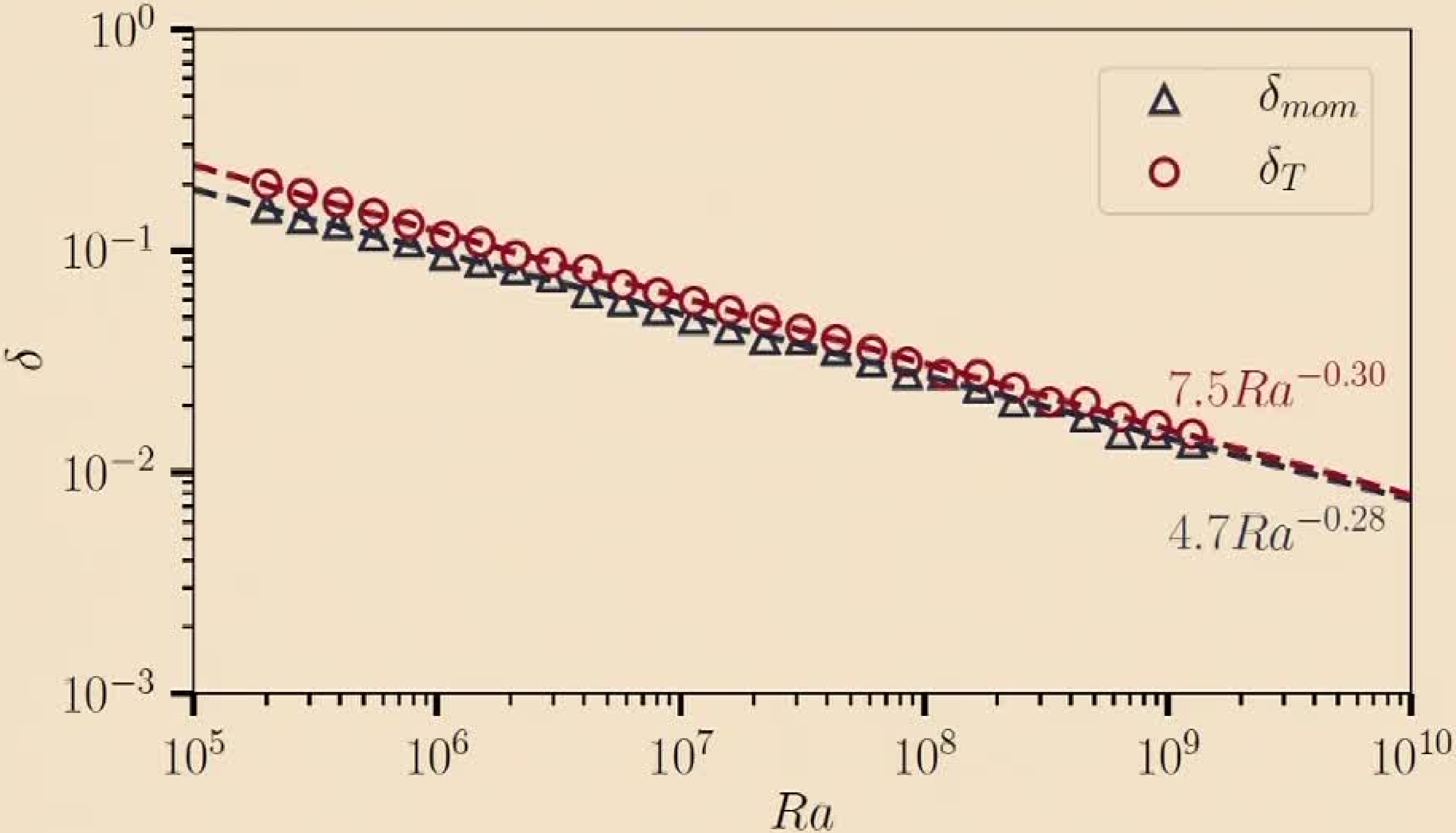
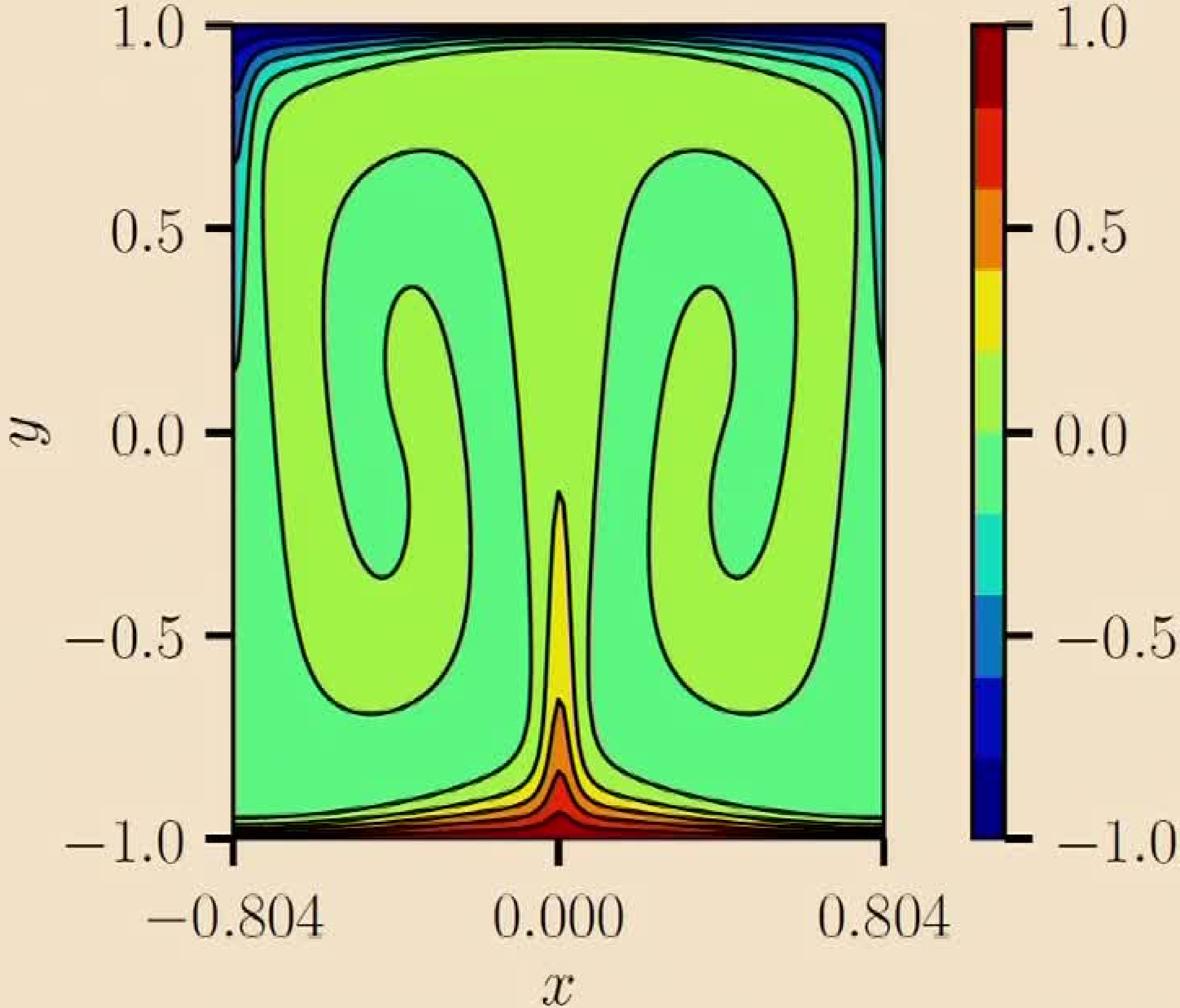


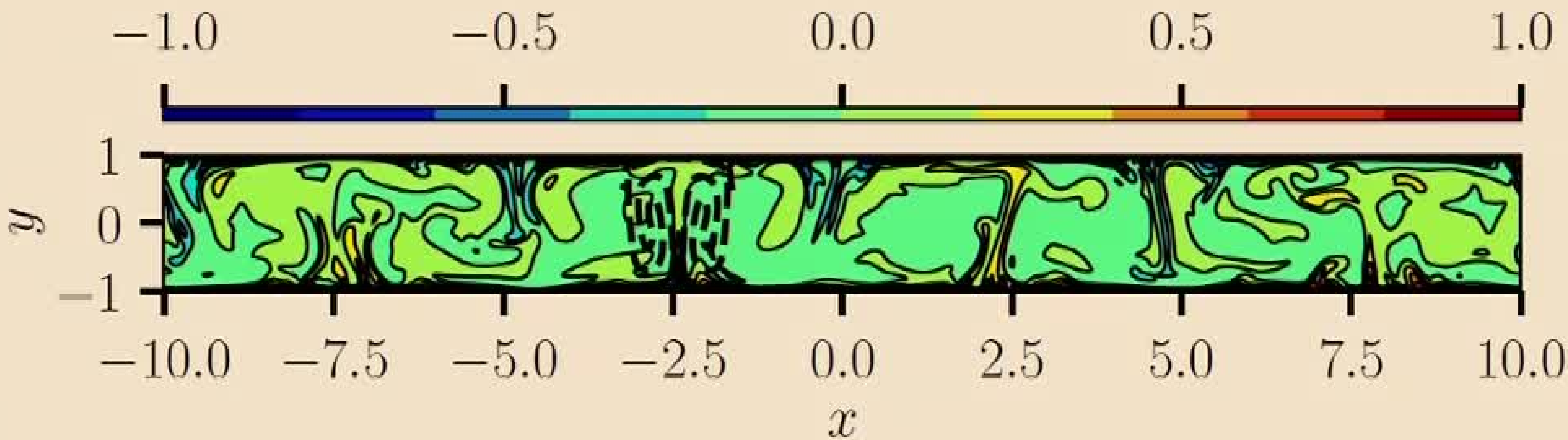
Boundary Layer Scaling: $Pr = 100$



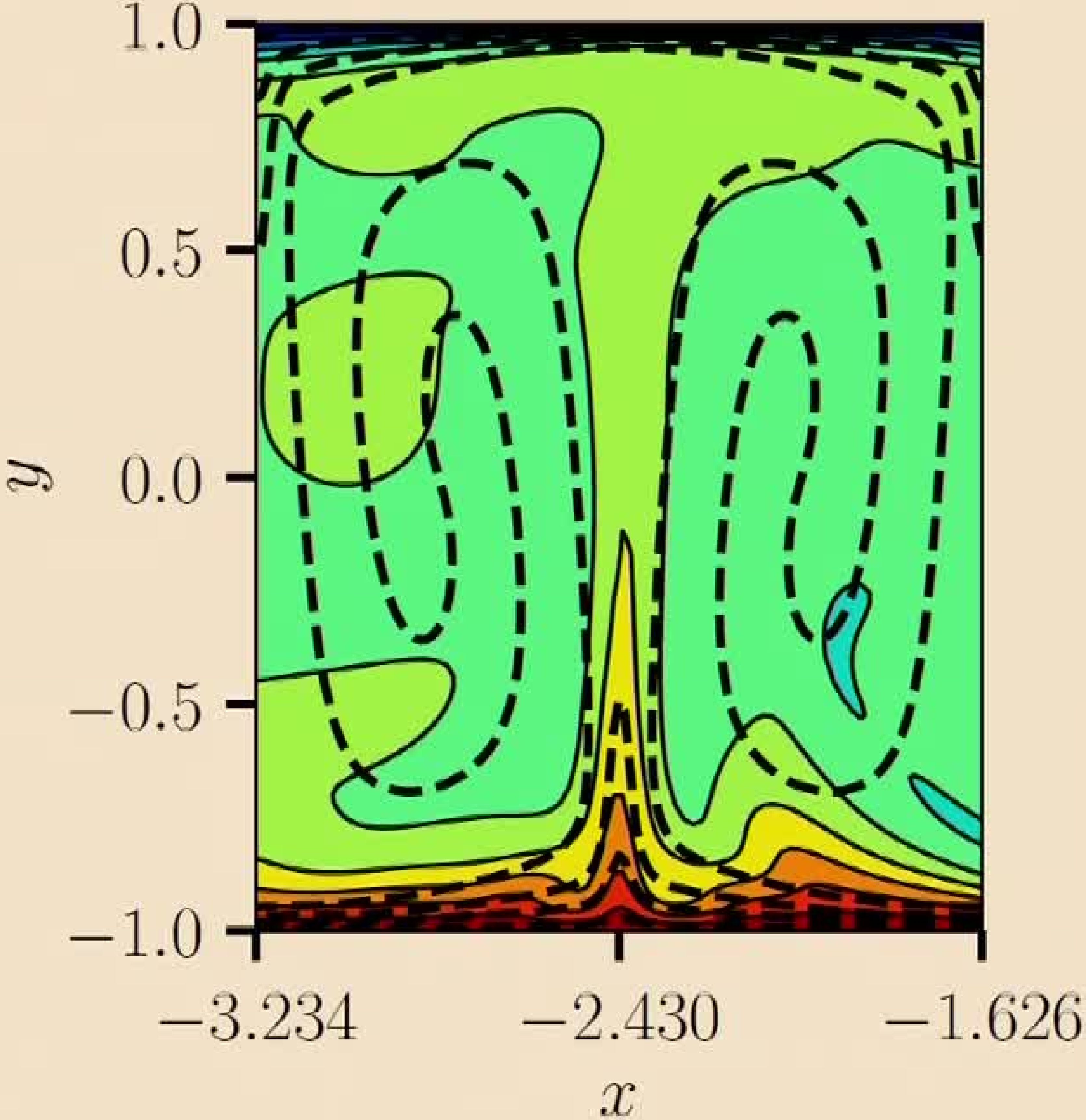
Optimal Solution at $Ra = 5 \times 10^6$, $Pr = 7$



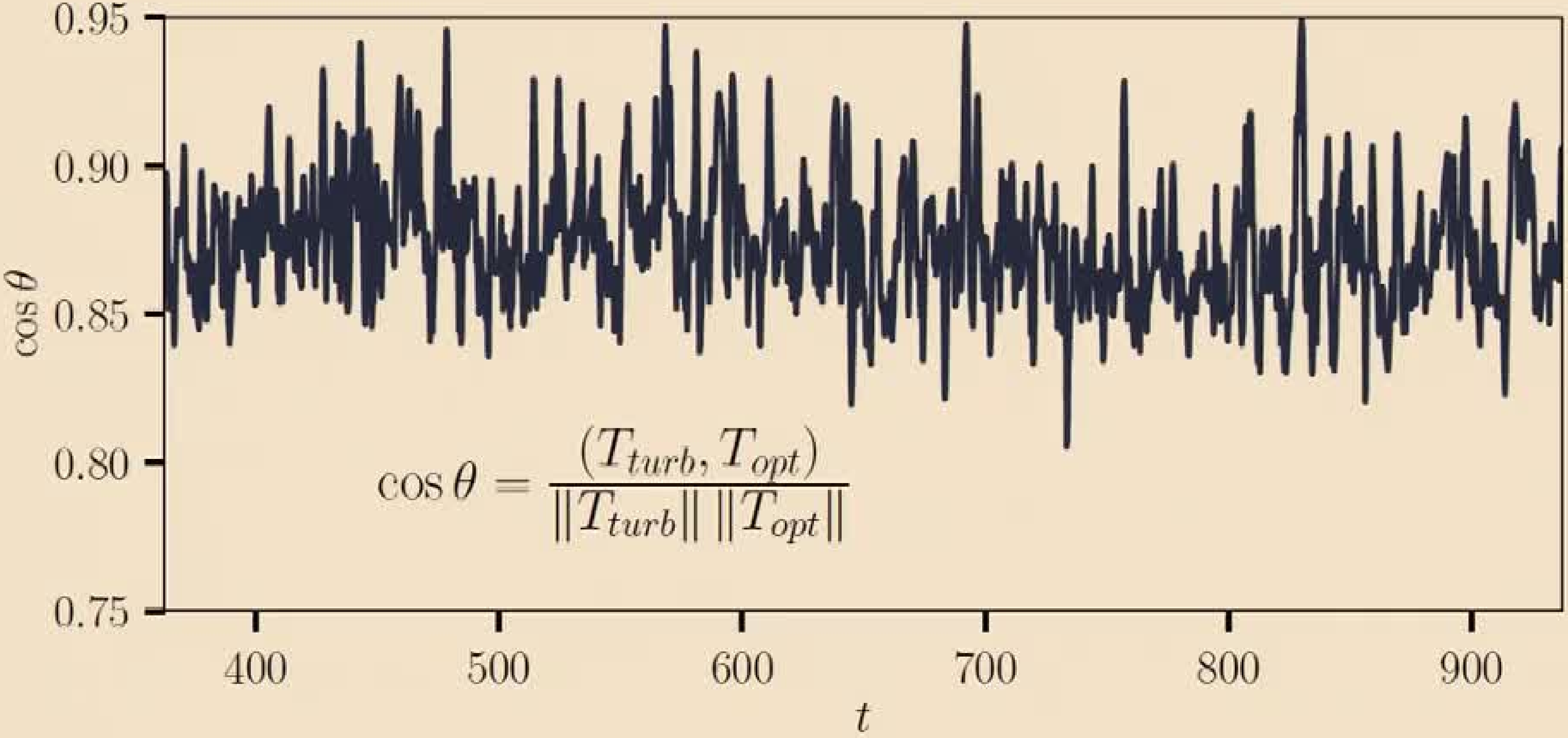
Superposition of Optimal and Turbulent Solutions



Superposition of Optimal and Turbulent Solutions: Zoomed



Alignment of Optimal and Turbulent Solutions



Summary

- Computed optimal solutions that optimize heat transport in 2D Rayleigh-Bénard convection
- 2D optimal solutions are consistent with rigorous theory
- 2D optimal solutions are in good agreement with results from 3D turbulence theory and computation
- The exact optimal solutions don't appear in the turbulent flow, but some key characteristics are observed
- A maximally aligned optimal solution and turbulent solution have correlation of ≈ 0.95.

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Current and Future Work

- Extension to 3D
  - 3D optimal solutions
  - Signature of 2D optimal solutions in 3D data
- Incorporate rotation
- Stability analysis
- Software development