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University of Bristol
Michael J. Ward
University of British Columbia

On How Turing Triggers Biochemical Spot Dynamics in a Plant Root Hair Initiation Model

MS77 Computes Shoots and Leaves: Alan Turing, Phyllotaxis and Beyond - Part I of II
SIAM Conference on Applications of Dynamical Systems
May 23, 2017

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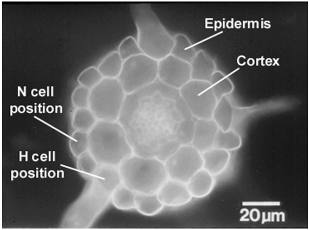
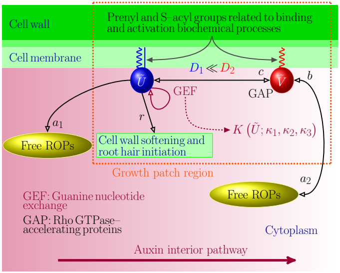
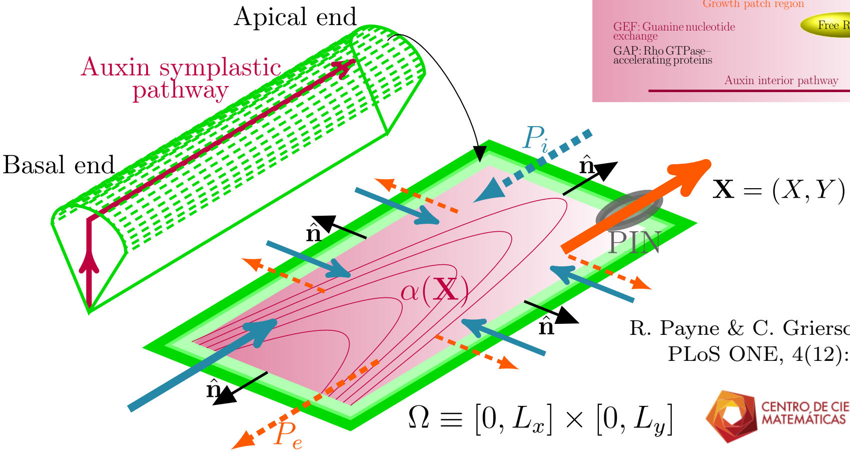


C. Grierson & J. Schiefelbein, 2002

Generalised Schnakenberg system

$$\begin{cases} U_t = D_1 \Delta U + k_{20}\alpha(\mathbf{X}) U^2 V - (c+r)U + k_1 V\,, \\ V_t = D_2 \Delta V - k_{20}\alpha(\mathbf{X}) U^2 V + cU - k_1 V + b\,. \end{cases}$$





R. Payne & C. Grierson (2009). PLoS ONE, 4(12):e8337

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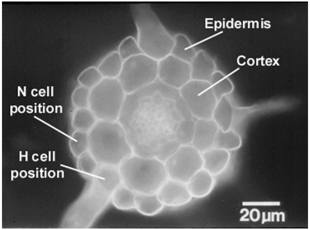
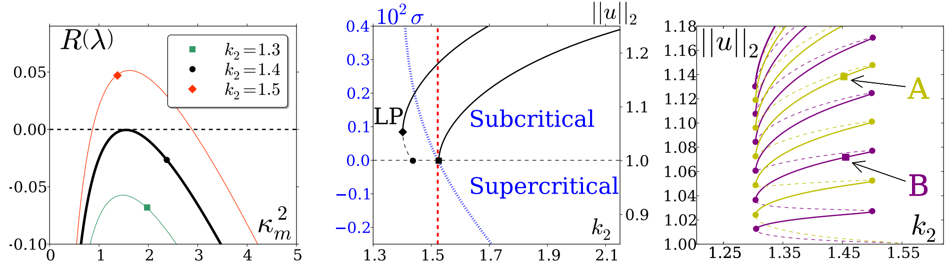


C. Grierson & J. Schiefelbein, 2002



Dispersion relation | Criticality transition | Homoclinic snaking

$$\color{blue}{q_1} \underbrace{\left(k_2 - k_{2_c}\right)}_{\color{red}{\mu}} z + \color{blue}{q_3} z^3 + q_5 z^5 + \text{h.o.t.} = 0\,, \qquad \color{blue}{\sigma \equiv q_1 q_3}\,.$$

- Root of multiplicity two.
- Long domain.
- Super- to subcritical transition.
- Spatial reversibility for steady-states.
- Turing bifurcation ↔ Hamilton–Hopf bifurcation.

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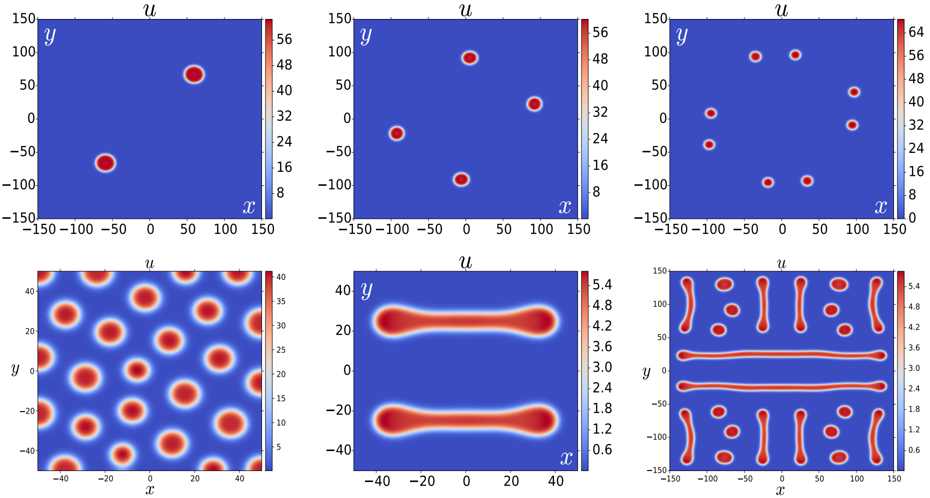

... joint work with A. Champneys
Phys. Rev. E **90**, 032923 (2014).

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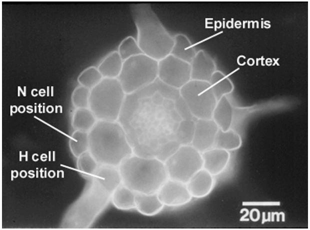
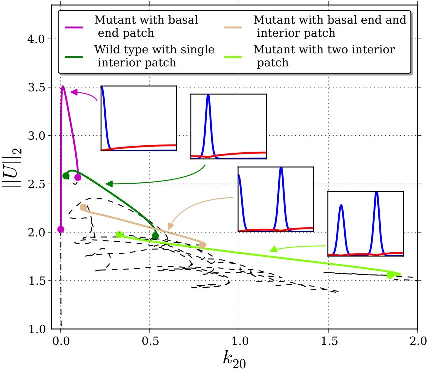
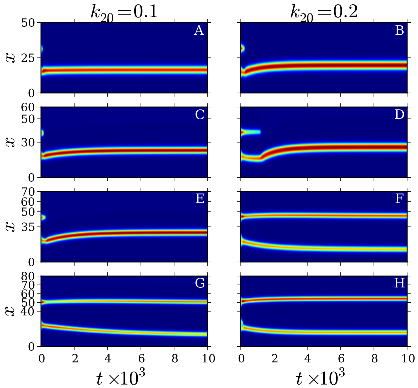


C. Grierson & J. Schiefelbein, 2002

Hysteretical structure





- Slow drifting spikes.
- Gradient controls location.
- Asymmetric amplitudes.
- Over-crowding instability.

Perturbed subcritical Turing bifurcation.

... joint work with A. Champneys, C. Grierson & M. Ward (2014). SIAM J. Appl. Dyn. Syst. **13**(1), pp. 210–248.

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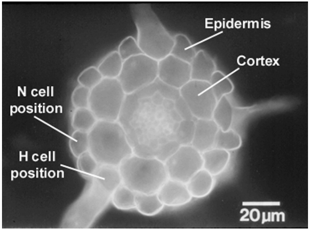
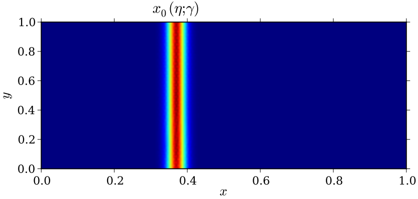
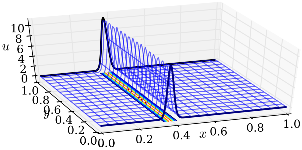
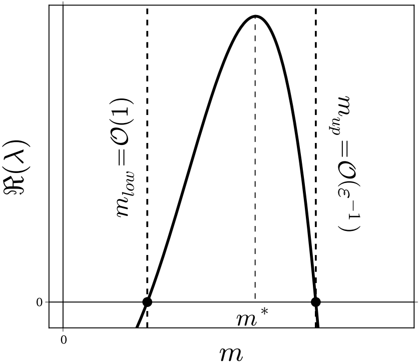


C. Grierson & J. Schiefelbein, 2002

Breakup instability







$$\mathcal{L}_0\Phi_0 - \theta_h(\lambda; m)w^2 \frac{\int_{-\infty}^{\infty} w\Phi_0 \, d\xi}{\int_{-\infty}^{\infty} w^2 \, d\xi} = \left(\lambda + s\varepsilon^2 m^2\right) \Phi_0 \,, \qquad s = \left(\frac{L_x}{L_y}\right)^2 .$$

... joint work with D. Avitabile, A. Champneys, & M. Ward (2015). SIAM J. Appl. Math. **75**(3), pp. 1090–1119.

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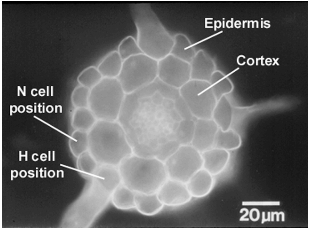

C. Grierson & J. Schiefelbein, 2002

Conservation principle

$$\int_\Omega U_0 \, \mathrm{d}\Omega = \frac{d_y}{\beta\gamma} \,, \quad d_y \equiv \frac{L_y}{L_x} \,, \quad \beta\gamma = \frac{(c+r) r k_1}{k_{20} b^2} \,,$$

$$\text{then} \quad U = \varepsilon^{-2} U_j \quad \Rightarrow \quad \sum_{j=1}^{N} \int_{\mathbb{R}^2} U_j \, \mathrm{d}\boldsymbol{\xi} \sim \frac{d_y}{\beta\gamma} \,, \quad \boldsymbol{\xi} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j) \,.$$

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Re-scaling

$$U = \varepsilon^{-2} u\,, \quad V = \varepsilon^{2} v\,, \quad D = \varepsilon^{-2} D_0\,,$$

$$\text{where} \quad \varepsilon^2 \equiv \frac{D_1}{L_x^2(c+r)} \text{ and } D \equiv \frac{D_2}{L_x^2 k_1}\,.$$

Asymptotic expansion

$$u = \underbrace{u_{0j}(\rho)}_{\text{profile}} + \varepsilon \underbrace{u_{1j}(\rho)}_{\text{location}} \ldots\,, \quad v = \underbrace{v_{0j}(\rho)}_{\text{profile}} + \varepsilon \underbrace{v_{1j}(\rho)}_{\text{location}} \ldots\,.$$

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Conservation principle

$$\int_\Omega U_0 \,\mathrm{d}\Omega = \frac{d_y}{\beta\gamma}\,, \quad d_y \equiv \frac{L_y}{L_x}\,, \quad \beta\gamma = \frac{(c+r)rk_1}{k_{20}b^2}\,,$$

then $\quad U = \varepsilon^{-2}U_j \quad \Rightarrow \quad \sum_{j=1}^{N}\int_{\mathbb{R}^2} U_j \,\mathrm{d}\boldsymbol{\xi} \sim \frac{d_y}{\beta\gamma}\,, \quad \boldsymbol{\xi} = \varepsilon^{-1}(\mathbf{x}-\mathbf{x}_j)\,.$

Re-scaling

$$U = \varepsilon^{-2}u\,, \quad V = \varepsilon^{2}v\,, \quad D = \varepsilon^{-2}D_0\,,$$

where $\quad \varepsilon^2 \equiv \dfrac{D_1}{L_x^2(c+r)}$ and $D \equiv \dfrac{D_2}{L_x^2 k_1}\,.$

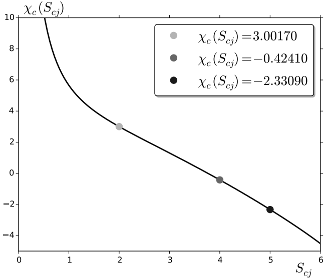
Canonical core problem

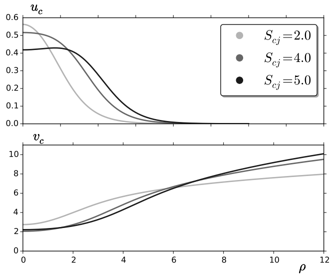
$$\Delta_\rho u_c + u_c^2 v_c - u_c = 0\,, \quad \Delta_\rho v_c - \frac{\tau}{\beta}\left(u_c^2 v_c - u_c\right) - u_c = 0\,, \quad \rho = |\boldsymbol{\xi}|\,,$$

$$u_c \to 0\,, \quad v_c \sim S_{cj}\log\rho + \chi_c(S_{cj})\,,$$

$$\rho \to \infty\,.$$

Source parameter and auxin





$$u_{0j} \equiv \sqrt{\frac{D_0}{\beta\gamma\alpha(\mathbf{x}_j)}}\, u_c\,, \quad v_{0j} \equiv \sqrt{\frac{\beta\gamma}{D_0\alpha(\mathbf{x}_j)}}\, v_c\,, \quad S_j \equiv \sqrt{\frac{\beta\gamma}{D_0\alpha(\mathbf{x}_j)}}\, S_{cj}\,.$$

$$S_j = \frac{\beta\gamma}{D_0}\int_0^\infty \left[\frac{\tau}{\beta}\left(\alpha\left(\mathbf{x}_j\right) u_{0j}^2 v_{0j} - u_{0j}\right) + u_{0j}\right]\rho\, d\rho\,.$$

$$\frac{\tau}{\beta} = \frac{c+r}{r}\,.$$

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Proposition

Let be $\varepsilon \ll 1$, $U = \mathcal{O}(\varepsilon^{-2})$, $V = \mathcal{O}(\varepsilon)$, $D = \mathcal{O}(\varepsilon^{-2})$ and stable N-spot quasi steady-state solution on an $\mathcal{O}(1)$ time-scale, the slow dynamics on the long time-scale $\eta = \varepsilon^2 t$ of this quasi steady-state spot pattern consists of the ODEs

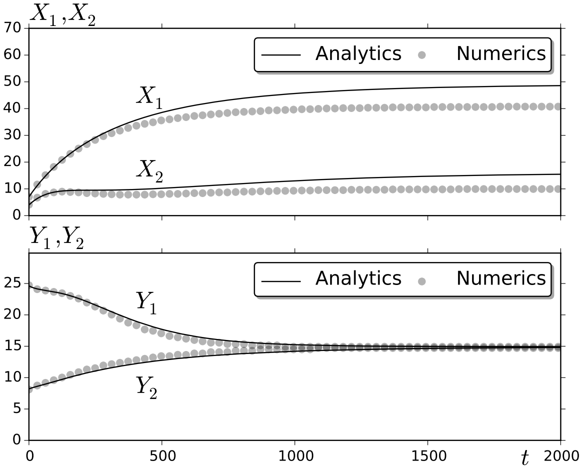
$$\frac{d\mathbf{x}_j}{d\eta} = n_1 \mathbf{\Psi}_j + n_2 \frac{\nabla \alpha(\mathbf{x}_j)}{\alpha(\mathbf{x}_j)} , \qquad j = 1, \ldots, N ,$$

where n_1 y n_2 satisfy a nonlinear algebraic system and depend on S_{cj} and the ratio τ/β, and the interaction vector is defined by

$$\mathbf{\Psi}_j = -2\pi \left(S_{cj} \nabla_{\mathbf{x}} R_j + \sum_{i \neq j}^{N} S_{ci} \nabla_{\mathbf{x}} G_{ji} \right) ,$$

where $R_j = R(\mathbf{x}_j, \mathbf{x}_j)$ and $G_{ji} = G(\mathbf{x}_j, \mathbf{x}_i)$.

Auxin gradient and shape



... joint work with D. Avitabile & M. Ward (in review);
`arXiv:1703.02608` and `bioRxiv:114876`.

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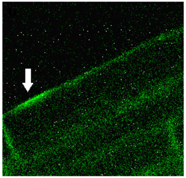
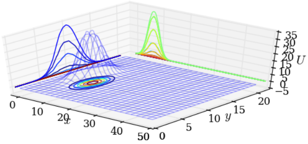
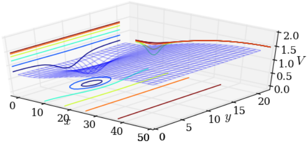
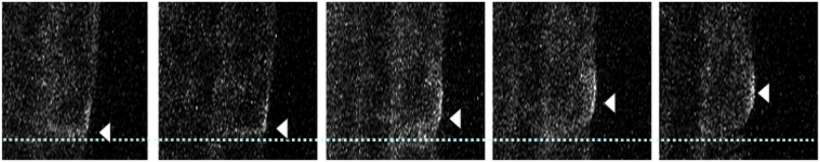
Key ROP features



R. Payne & C. Grierson, 2009







C. Gierson

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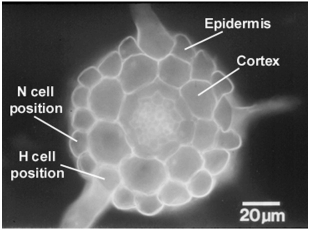

C. Grierson & J. Schiefelbein, 2002

- **Turing bifurcation determines the onset of localised steady-states.**
- Inverse relationship between the auxin and longitudinal length controls over-crowding instability and give rise to **hysteresis**.
- ROPs tend to form a maximal entropy configuration.
- Activated ROPs location is directly controlled by auxin gradient and physical features.
- Under-crowding (peanut-splitting) instability is induced as a threshold is overpassed by source parameter $S_c = S_c\left(k_{20}\alpha\left(\mathbf{X}_j\right), L_x\right)$; not shown.
- More realistic transport processes (hyperbolic diffusion or anomalous diffusion) for auxin or/and ROPs; dynamics upon considering torsion and transversal curvature; multi-scale interactions between a set of RH cells, non-RH cells and auxin…(R. Plaza, D. Hernández, M. Ward).

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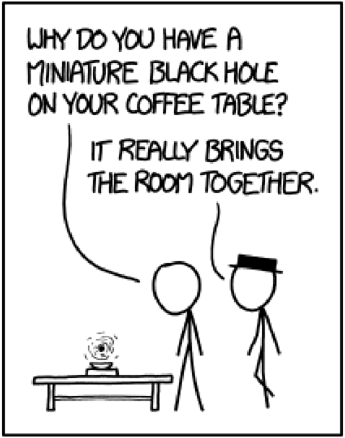

"It also brings all the boys, and everything else, to the yard."
Randall Munroe

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