Multiparameter Full-Waveform Inversion with Near-Interface Sources using Staggered-grid Finite Differences

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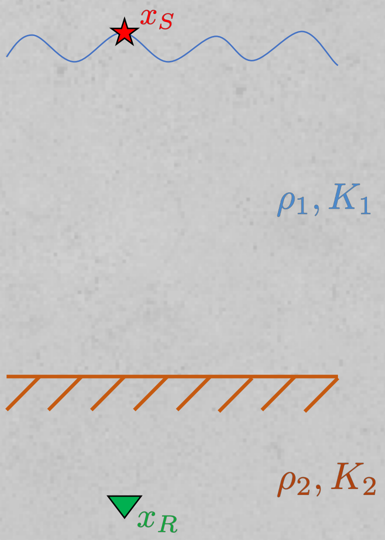
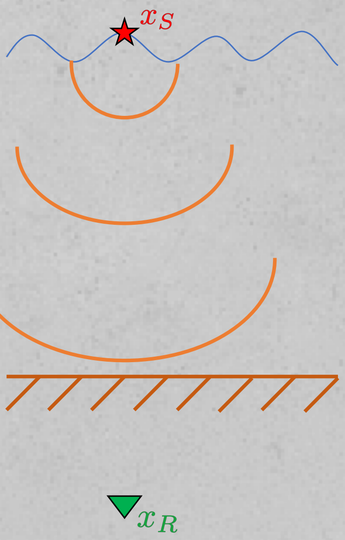
Problem setup

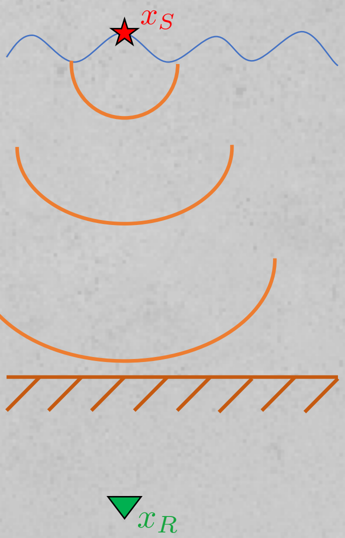


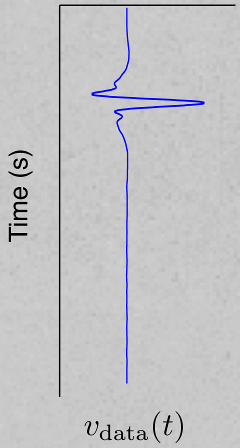
Forward problem



$$\rho\dot{v} + p_x = 0$$
$$\frac{1}{K}\dot{p} + v_x = \delta(x - x_S) f(t)$$

Recorded data





$p_{\mathrm{data}}(t)$ can also exist

Full-Waveform Inversion

$$\underset{\rho(x),K(x),f(t)}{\text{minimize}} \quad F(v,p), \quad \text{where}$$

$$F(v,p) = \frac{w_1}{2}\int_0^T \left(v(x_R,t) - v_{\text{data}}(t)\right)^2 dt$$

$$+ \frac{w_2}{2}\int_0^T \left(p(x_R,t) - p_{\text{data}}(t)\right)^2 dt$$

subject to $\rho\dot{v} + p_x = 0,$

$$\frac{1}{K}\dot{p} + v_x - f(t)\delta(x - x_S) = 0.$$

Full-Waveform Inversion

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Loss function (least squares)

subject to

$$\rho\dot{v} + p_x = 0,$$

$$\frac{1}{K}\dot{p} + v_x - f(t)\delta(x - x_S) = 0.$$

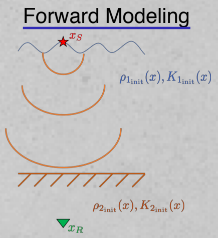
PDE constraint

Optimizing the functional

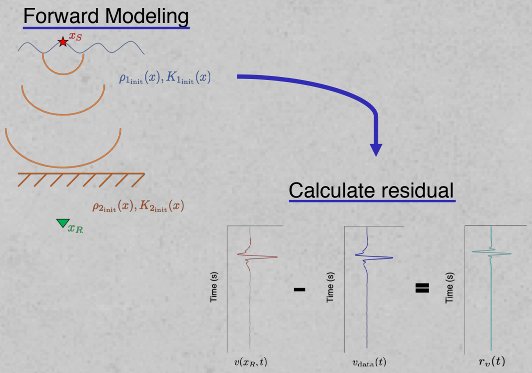
- Global optimization $\rightarrow$ too many PDE solves

- Local/gradient-based optimization is preferred
  - **Requires use of the adjoint method**

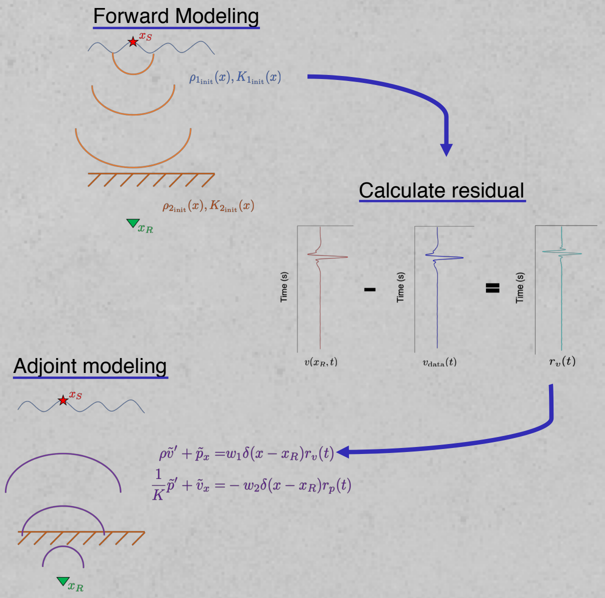
Adjoint method: Forward modeling



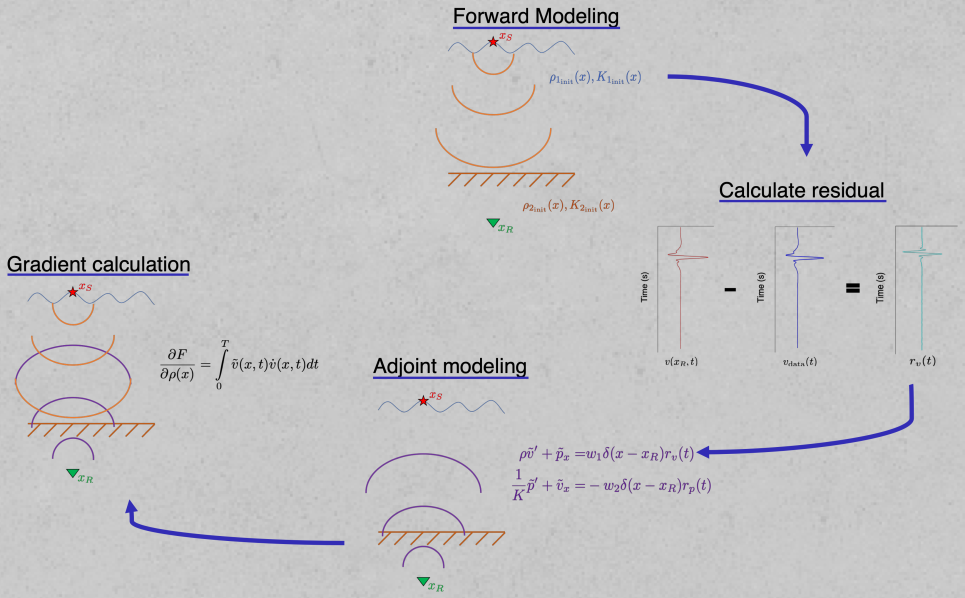
Adjoint method: Calculate residual



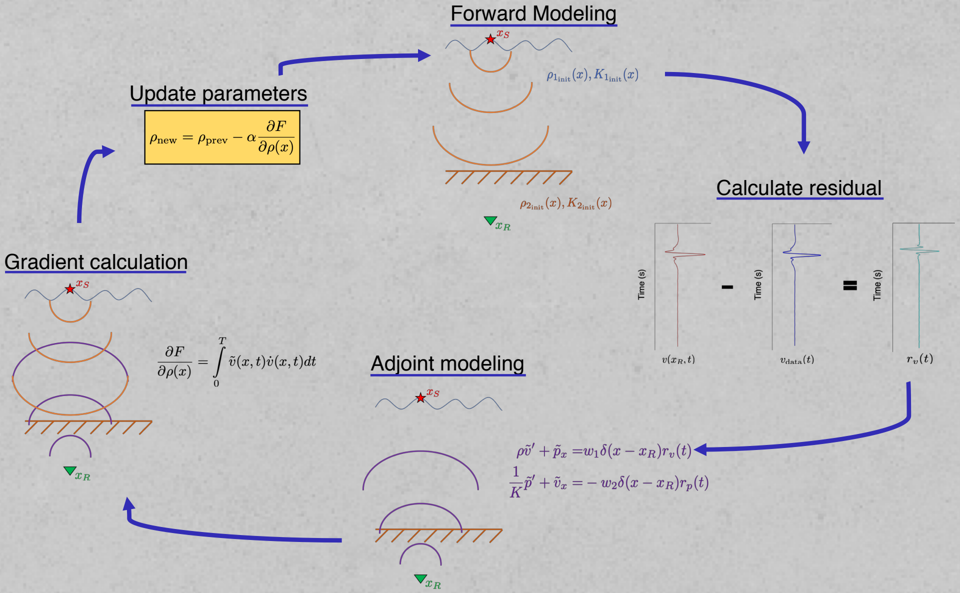
Adjoint method: Adjoint modeling



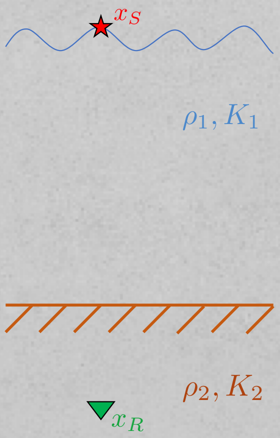
Adjoint method: Calculate gradient



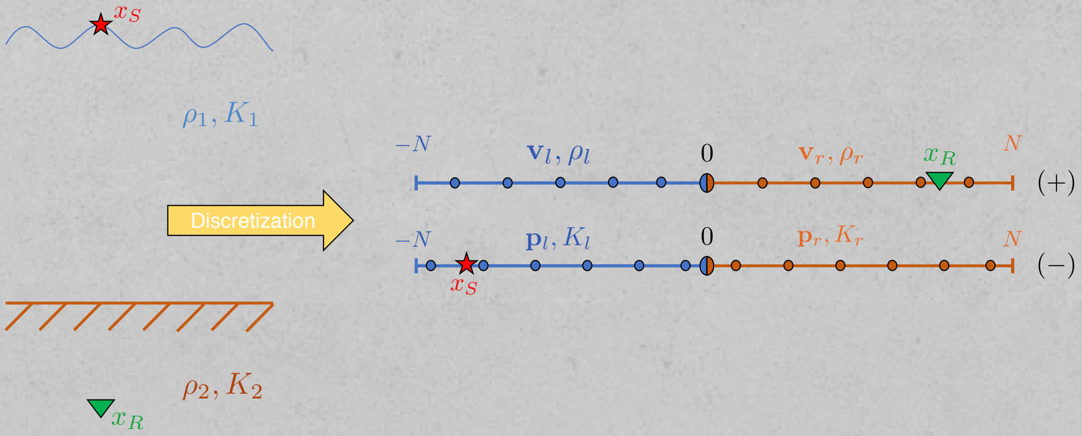
Adjoint method: Update parameters



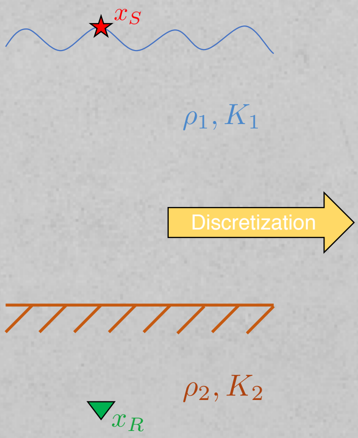
Problem setup

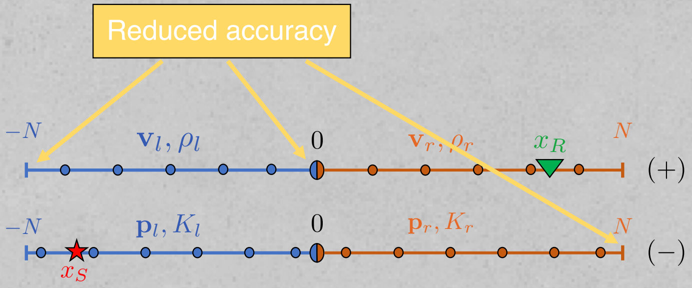


Finite difference discretization



Reduced accuracy at boundaries

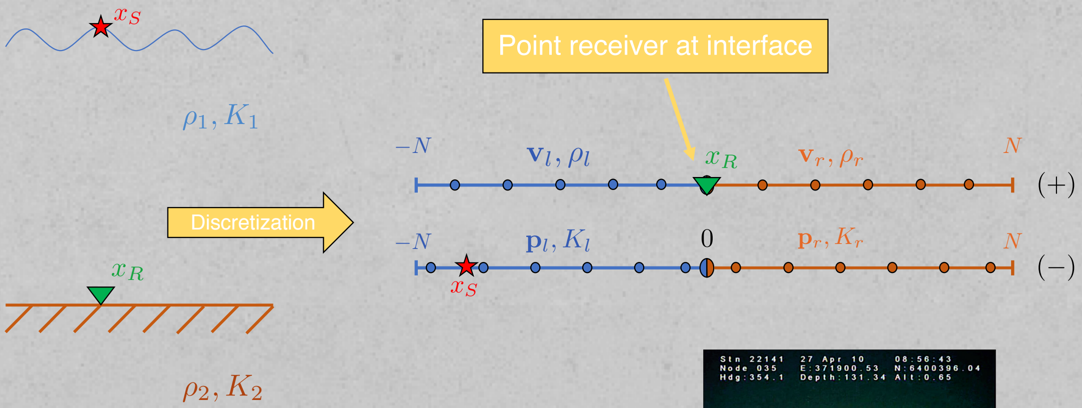




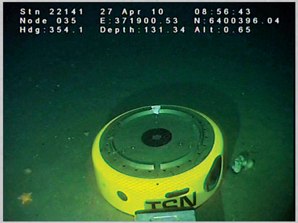
Issues with finite differences

1. Reduced accuracy at interfaces and boundaries

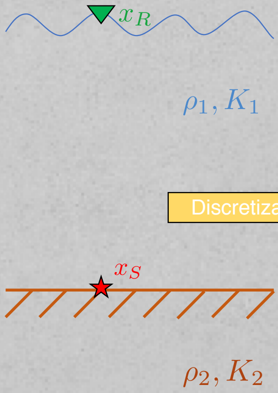
Ocean-bottom node acquisition (OBN)

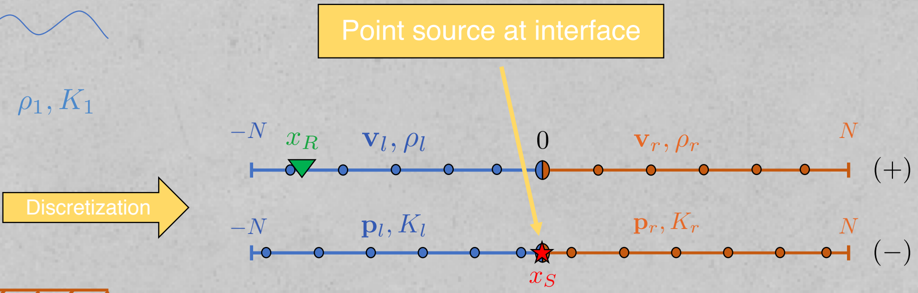


Ocean bottom node
(Ronen et al., 2012)



OBN: Adjoint problem

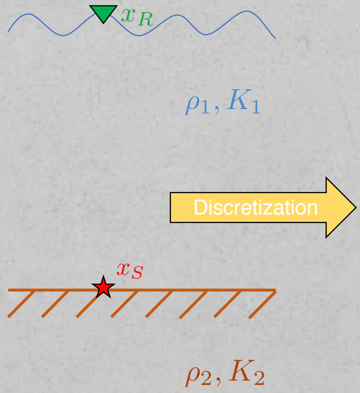


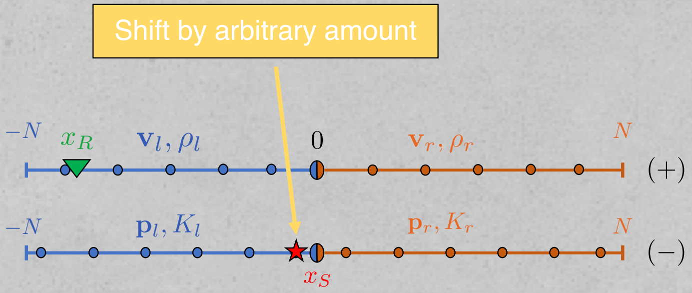


Issues with finite differences

1. Reduced accuracy at interfaces and boundaries
2. Sources can be on/near interfaces

Shifting leads to gradient errors





Issues with finite differences

1. Reduced accuracy at interfaces and boundaries
2. Sources can be on/near interfaces
   - Shifting of sources leads to **gradient errors**

Presentation takeways

1. **High-order accurate modeling of the forward problem**

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   - Accurate finite difference (FD) stencils at boundaries and interfaces
   - Modeling of point sources at interfaces

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2. **Immediate availability of the discrete adjoint**

Presentation takeways

1. **High-order accurate modeling of the forward problem**
   - Accurate finite difference (FD) stencils at boundaries and interfaces
   - Modeling of point sources at interfaces

2. **Immediate availability of the discrete adjoint**
   - Semi-discrete adjoint equations are the same as the forward with different source

Dual consistency (Berg and Nordstrom, 2012)

Continuous

- Forward:

$$
\begin{aligned}
\rho \dot{v} + p_x &= 0 \\
\frac{1}{K}\dot{p} + v_x &= \delta(x - x_S) f(t)
\end{aligned}
$$

- Adjoint:

Discrete

- Forward:

- Adjoint:

Dual consistency (Berg and Nordstrom, 2012)

Continuous

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$$\rho\dot{v} + p_x = 0$$
$$\frac{1}{K}\dot{p} + v_x = \delta(x - x_S)f(t)$$

Discretize

Discrete

- Forward:

$$\rho\dot{\mathbf{v}} + \mathbf{D}_+\mathbf{p} = \mathbf{0}$$
$$K^{-1}\dot{\mathbf{p}} + \mathbf{D}_-\mathbf{v} = f(t)\mathbf{d}_{S-}$$

- Adjoint:

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Dual consistency (Berg and Nordstrom, 2012)

Continuous

- Forward:

$$\begin{aligned}\rho\dot{v} + p_x &= 0\\ \frac{1}{K}\dot{p} + v_x &= \delta(x - x_S)f(t)\end{aligned}$$

Discretize

Compute adjoint

- Adjoint:

$$\begin{aligned}\rho\tilde{v}' + \tilde{p}_x &= w_1\delta(x - x_R)r_v(t)\\ \frac{1}{K}\tilde{p}' + \tilde{v}_x &= -w_2\delta(x - x_R)r_p(t)\end{aligned}$$

Discrete

- Forward:

$$\begin{aligned}\rho\dot{\mathbf{v}} + \mathbf{D}_+\mathbf{p} &= \mathbf{0}\\ K^{-1}\dot{\mathbf{p}} + \mathbf{D}_-\mathbf{v} &= f(t)\mathbf{d}_{S-}\end{aligned}$$

- Adjoint:

Dual consistency (Berg and Nordstrom, 2012)

Continuous

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$$\rho \dot{v} + p_x = 0$$
$$\frac{1}{K}\dot{p} + v_x = \delta(x - x_S) f(t)$$

Discretize →

Discrete

- Forward:

$$\rho \dot{\mathbf{v}} + \mathbf{D}_+ \mathbf{p} = \mathbf{0}$$
$$K^{-1}\dot{\mathbf{p}} + \mathbf{D}_- \mathbf{v} = f(t)\mathbf{d}_{S-}$$

↓ Compute adjoint (continuous)

↓ Compute adjoint (discrete)

Continuous

- Adjoint:

$$\rho \tilde{v}' + \tilde{p}_x = w_1 \delta(x - x_R) r_v(t)$$
$$\frac{1}{K}\tilde{p}' + \tilde{v}_x = -w_2 \delta(x - x_R) r_p(t)$$

Discretize →

Discrete

- Adjoint:

$$\rho \tilde{\mathbf{v}}' + \mathbf{D}_+ \tilde{\mathbf{p}} = w_1 \mathbf{d}_{R-} r_v(t)$$
$$K^{-1}\tilde{\mathbf{p}} + \mathbf{D}_- \tilde{\mathbf{v}} = -w_2 \mathbf{d}_{R+} r_p(t)$$

Outline for presentation

1. Summation-by-parts with simultaneous approximation term method (SBP-SAT)

2. Adjoint optimization with SBP-SAT

3. Numerical examples

Summation-by-parts (SBP) operator

$$\frac{\partial p}{\partial x} \approx \mathbf{D}\mathbf{p},$$

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$$\frac{\partial p}{\partial x} \approx \mathbf{Dp},$$

$$\mathbf{D} = \frac{1}{h}\begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & & & \\ \theta_{21} & 0 & \theta_{23} & \theta_{24} & & & \\ \theta_{31} & \theta_{32} & 0 & \theta_{33} & \theta_{34} & & \\ \theta_{41} & \theta_{42} & \theta_{43} & 0 & \theta_{45} & \theta_{46} & \\ 0 & 0 & \frac{1}{12} & -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

(Del Rey Fernández et al., 2014)

Summation-by-parts (SBP) operator

$$\frac{\partial p}{\partial x} \approx \mathbf{D}\mathbf{p},$$

Interior: 4th-order accuracy

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(Del Rey Fernández et al., 2014)

Summation-by-parts (SBP) operator

$$\frac{\partial p}{\partial x} \approx \mathbf{D}\mathbf{p},$$

Interior: 4th-order accuracy

Boundary: 2nd-order accuracy

$$\mathbf{D} = \frac{1}{h}\begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & & & \\ \theta_{21} & 0 & \theta_{23} & \theta_{24} & & & \\ \theta_{31} & \theta_{32} & 0 & \theta_{33} & \theta_{34} & & \\ \theta_{41} & \theta_{42} & \theta_{43} & 0 & \theta_{45} & \theta_{46} & \\ 0 & 0 & \frac{1}{12} & -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

(Del Rey Fernández et al., 2014)

Summation-by-parts (SBP) operator

$$\frac{\partial p}{\partial x} \approx \mathbf{D}\mathbf{p},$$

Interior: 4th-order accuracy

Boundary: 2nd-order accuracy

Global: 3rd -order accuracy

$$\mathbf{D} = \frac{1}{h}\begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & & & \\ \theta_{21} & 0 & \theta_{23} & \theta_{24} & & & \\ \theta_{31} & \theta_{32} & 0 & \theta_{33} & \theta_{34} & & \\ \theta_{41} & \theta_{42} & \theta_{43} & 0 & \theta_{45} & \theta_{46} & \\ 0 & 0 & \frac{1}{12} & -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

(Del Rey Fernández et al., 2014)

Discrete form of integration-by-parts

1. Inner-product
   - $(u, v) = \int_a^b u(x)v(x)dx$
   - $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T\mathbf{H}\mathbf{v}$, $(\mathbf{H}$: discrete quadrature)

Discrete form of integration-by-parts

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   - $(u, v) = \int_a^b u(x)v(x)dx$
   - $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T\mathbf{H}\mathbf{v}$, $(\mathbf{H}$: discrete quadrature)
2. Integration-by-parts and SBP
   - $\left(u, \frac{dv}{dx}\right) = -\left(\frac{du}{dx}, v\right) + u(x)v(x)\Big|_a^b$
   - $(\mathbf{u}, \mathbf{D}\mathbf{v}) = -(\mathbf{D}\mathbf{u}, \mathbf{v}) + u_N v_N - u_0 v_0$

Why is this useful?

1. Discrete energy balance $\Rightarrow$ provable stability

2. Consistent approximation of continuous adjoint problem

    - Dual consistency

Simultaneous approximation term method

Simultaneous approximation term (SAT) method with SBP (SBP-SAT):

$\rightarrow$ Weak imposition of boundary conditions (BC)

Simultaneous approximation term method

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$$
\begin{aligned}
\rho \frac{\partial \mathbf{v}}{\partial t} &= \cdots - c\mathbf{H}^{-1}\left[\mathbf{e}_0(v_0 - \hat{v}_0) + \mathbf{e}_n(v_N - \hat{v}_N)\right], \\
\frac{1}{K}\frac{\partial \mathbf{p}}{\partial t} &= \cdots - c\mathbf{H}^{-1}\left[\mathbf{e}_0(p_0 - \hat{p}_0) + \mathbf{e}_n(p_N - \hat{p}_N)\right].
\end{aligned}
$$

- $\hat{p}_0, \hat{v}_0$: “target” variables that satisfy BC

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3. Numerical examples

Continuous optimization problem

$$\underset{\rho(x),K(x),f(t)}{\text{minimize}} \quad F(v,p), \quad \text{where}$$

$$F(v,p) = \frac{w_1}{2}\int_0^T \left(v(x_R,t) - v_{\text{data}}(t)\right)^2 dt + \frac{w_2}{2}\int_0^T \left(p(x_R,t) - p_{\text{data}}(t)\right)^2 dt$$

Loss function (least squares)

subject to

$$\rho\dot{v} + p_x = 0,$$

$$\frac{1}{K}\dot{p} + v_x - f(t)\delta(x - x_S) = 0.$$

PDE constraint

Semi-discrete (SD) optimization problem

$$\underset{\boldsymbol{\rho},\mathbf{K},f(t)}{\text{minimize}} \quad F(\mathbf{v},\mathbf{p}), \quad \text{where}$$

$$F(\mathbf{v},\mathbf{p}) = \frac{w_1}{2}\int_0^T \left((\mathbf{H}_+\mathbf{d}_{R+})^T\mathbf{v} - v_{\text{data}}(t)\right)^2 dt + \frac{w_2}{2}\int_0^T ((\mathbf{H}_-\mathbf{d}_{R-})^T\mathbf{p} - p_{\text{data}}(t))^2 dt$$

subject to $\rho\dot{\mathbf{v}} + \mathbf{D}_+\mathbf{p} = \mathbf{0},$

$$K^{-1}\dot{\mathbf{p}} + \mathbf{D}_-\mathbf{v} - f(t)\mathbf{d}_{S-} = \mathbf{0}.$$

Point source/receiver operators

$$\underset{\boldsymbol{\rho},\mathbf{K},f(t)}{\text{minimize}} \quad F(\mathbf{v},\mathbf{p}), \quad \text{where}$$

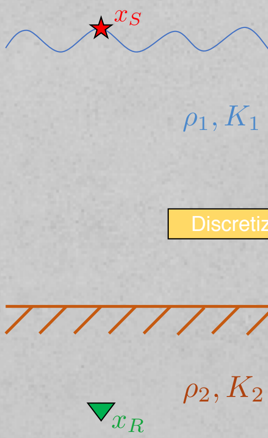
$$F(\mathbf{v},\mathbf{p}) = \frac{w_1}{2}\int_0^T \left((\mathbf{H}_+\mathbf{d}_{R+})^T\mathbf{v} - v_{\text{data}}(t)\right)^2 dt$$

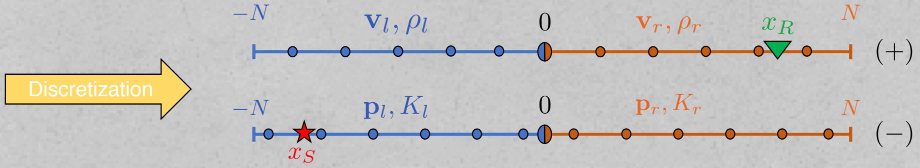
$$+ \frac{w_2}{2}\int_0^T ((\mathbf{H}_-\mathbf{d}_{R-})^T\mathbf{p} - p_{\text{data}}(t))^2 \, dt$$

$$\text{subject to} \quad \rho\dot{\mathbf{v}} + \mathbf{D}_+\mathbf{p} = \mathbf{0},$$

$$K^{-1}\dot{\mathbf{p}} + \mathbf{D}_-\mathbf{v} - f(t)\mathbf{d}_{S-} = \mathbf{0}.$$

Finite difference discretization





Continuous

$$
\begin{aligned}
\rho \dot{v} + p_x &= 0 \\
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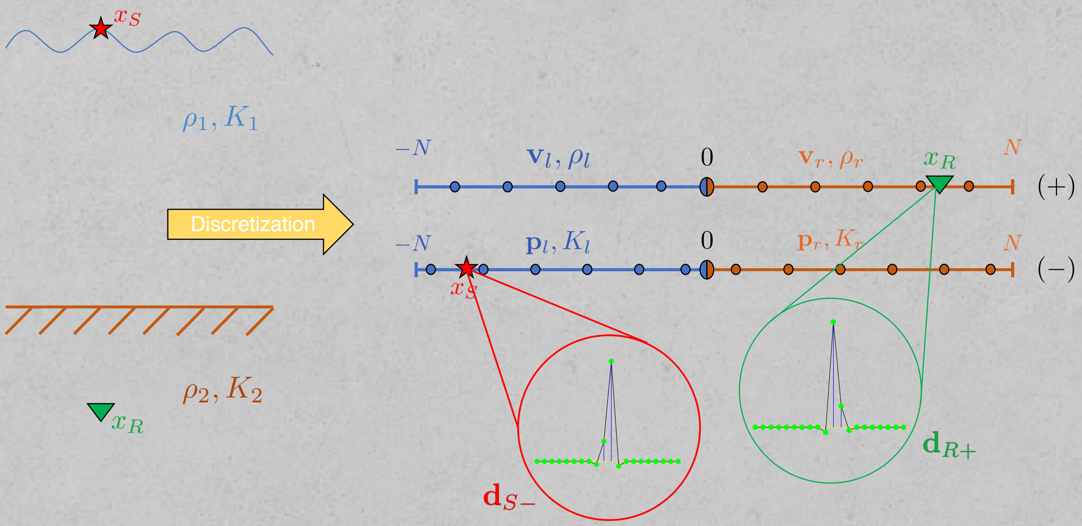
\end{aligned}
$$

Semi-Discrete

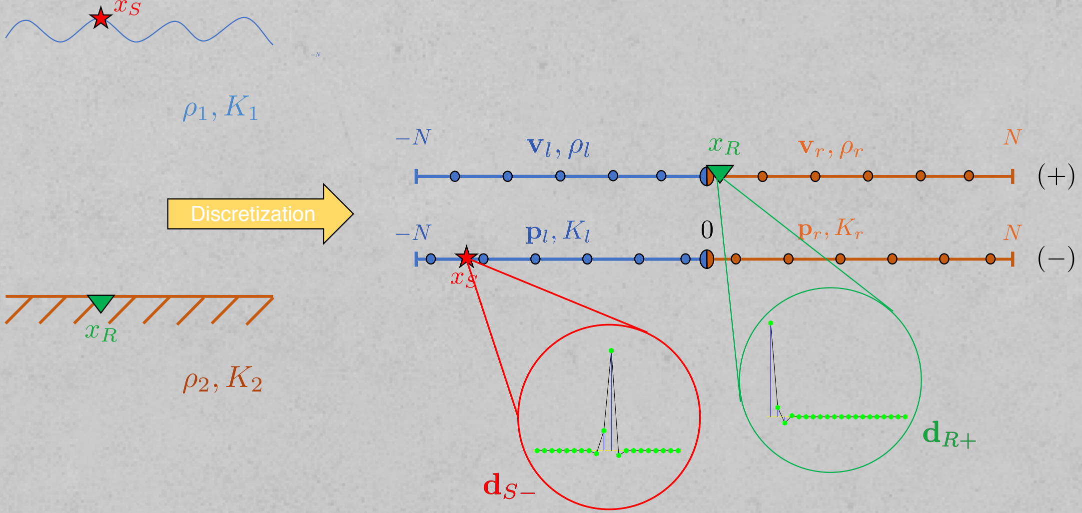
$$
\begin{aligned}
\rho \dot{\mathbf{v}} + \mathbf{D}_+ \mathbf{p} &= \mathbf{0} \\
K^{-1}\dot{\mathbf{p}} + \mathbf{D}_- \mathbf{v} &= f(t)\mathbf{d}_{S-}
\end{aligned}
$$

Point source/receiver discretization



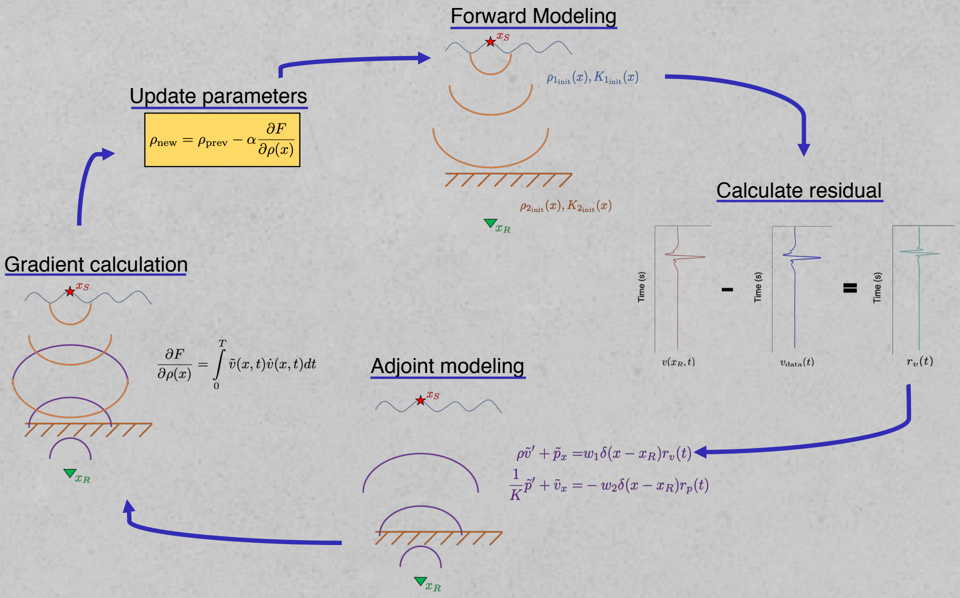
(Petersson et al., 2016)

Point receiver near interface



(Petersson et al., 2016)

Residual becomes adjoint source



Source-receiver dual consistency

- SD adjoint equations

$$\rho \tilde{\mathbf{v}}' + \mathbf{D}_+ \tilde{\mathbf{p}} = w_1 \mathbf{d}_{R+} \left(\mathbf{d}_{R+}^T \mathbf{H}_+ \mathbf{v} - v_{\text{data}}(t) \right),$$
$$K^{-1} \tilde{\mathbf{p}}' + \mathbf{D}_- \tilde{\mathbf{v}} = -w_2 \mathbf{d}_{R-} \left(\mathbf{d}_{R-}^T \mathbf{H}_- \mathbf{p} - p_{\text{data}}(t) \right).$$

- $\mathbf{d}_{R\pm}$: receiver restriction in functional
- $\mathbf{d}_{R\pm}$: delta function (point source) in adjoint

Easy to obtain adjoint equations

- SD adjoint equations

$$\rho\tilde{\mathbf{v}}' + \mathbf{D}_+\tilde{\mathbf{p}} = w_1\mathbf{d}_{R+}r_p(t),$$
$$K^{-1}\tilde{\mathbf{p}}' + \mathbf{D}_-\tilde{\mathbf{v}} = -w_2\mathbf{d}_{R-}r_v(t).$$

- SD forward equations

$$\rho\dot{\mathbf{v}} + \mathbf{D}_+\mathbf{p} = \mathbf{0},$$
$$K^{-1}\dot{\mathbf{p}} + \mathbf{D}_-\mathbf{v} = \mathbf{d}_{s-}f(t).$$

Easy to obtain adjoint equations

- SD adjoint equations

$$\rho\tilde{\mathbf{v}}' + \mathbf{D}_+\tilde{\mathbf{p}} = w_1\mathbf{d}_{R+}r_p(t),$$
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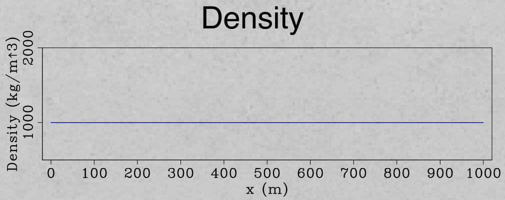
$$K^{-1}\dot{\mathbf{p}} + \mathbf{D}_-\mathbf{v} = \mathbf{d}_{s-}f(t).$$

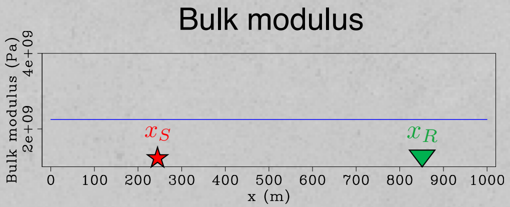
- We can use same code for forward and adjoint!

Outline for presentation

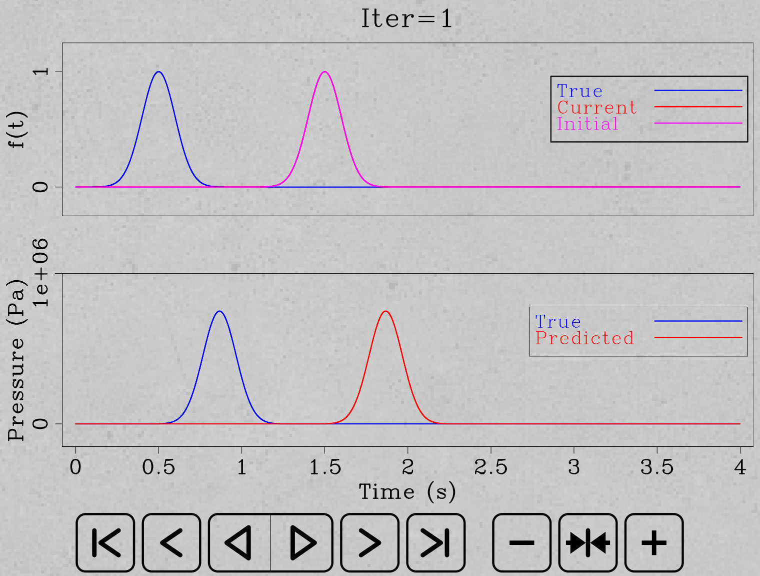
1. Summation-by-parts with simultaneous approximation term method (SBP-SAT)
    - High-order accurate, energy stable, dual consistency
2. Adjoint optimization with SBP-SAT
    - Source and receiver discretization
    - Source-receiver dual consistency
    - Same code for forward and adjoint
4. Numerical examples

Setup for source inversion

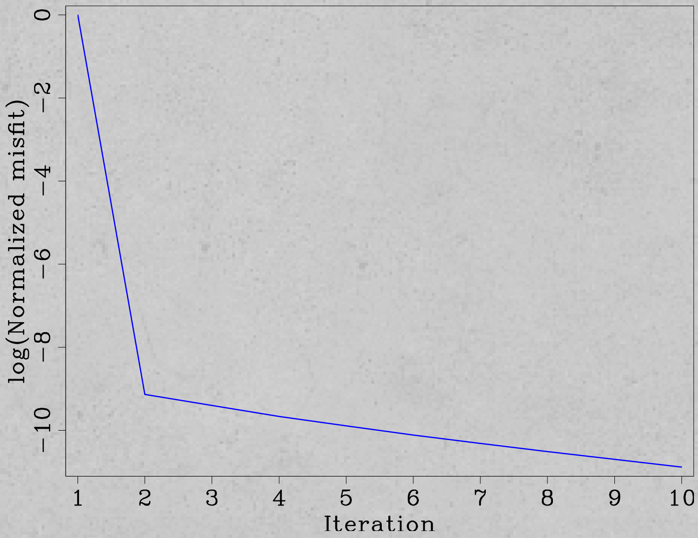




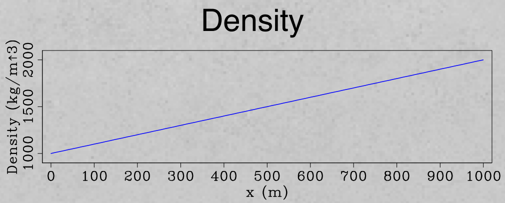
Inversion for $f(t)$: iteration movie

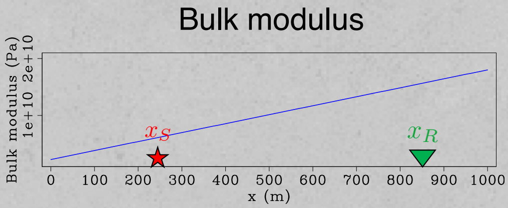


Inversion for $f(t)$: misfit reduction

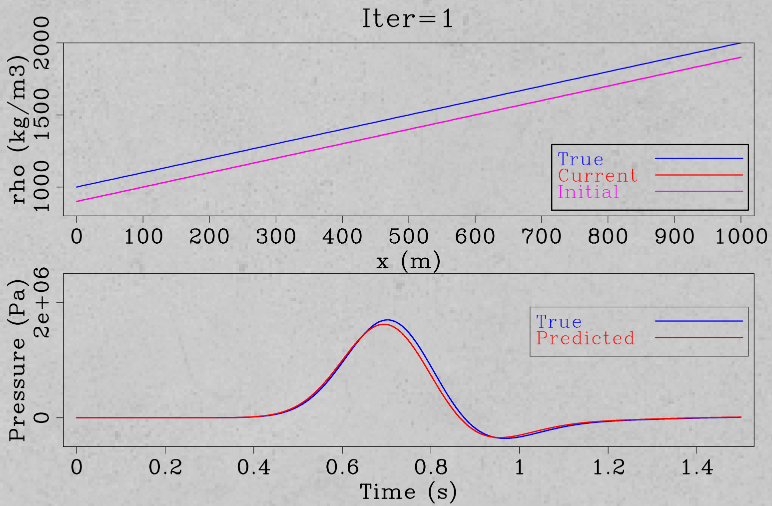


Setup for medium parameter inversion

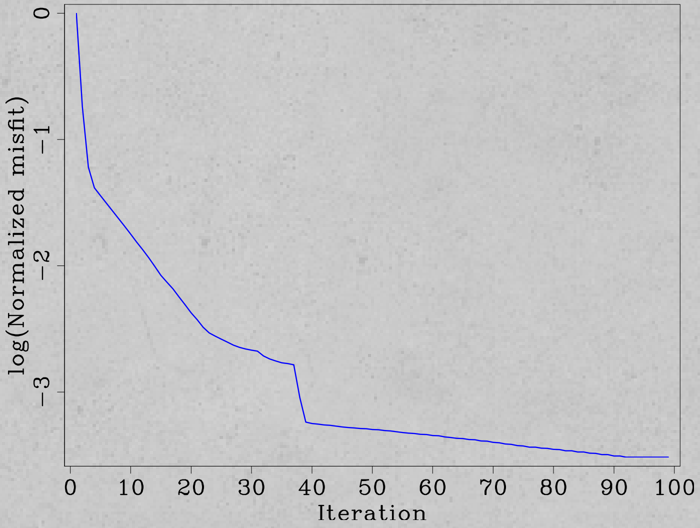




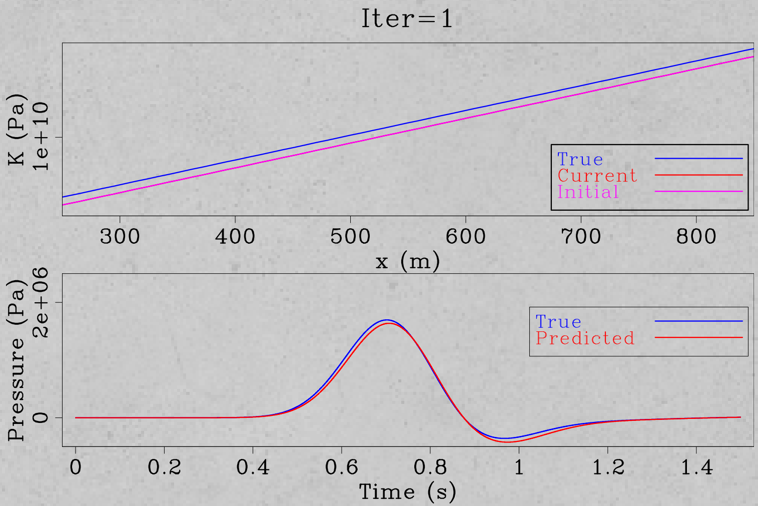
Inversion for $\rho(x)$: iteration movie



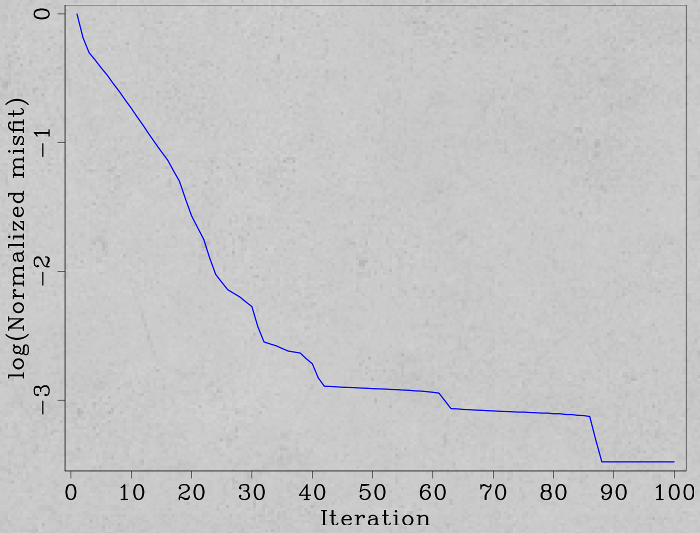
Inversion for $\rho(x)$: misfit reduction



Inversion for $K(x)$: results



Inversion for $K(x)$: misfit reduction



Conclusions and road ahead

1. SBP-SAT framework is ideal for adjoint optimization

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2. Apply to more complicated numerical examples

Questions?

Specifying SATs

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