ALORA: Affine low-rank approximation

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Classical low-rank algorithms can generate large errors of approximation.
Introduction and motivation

- Classical low-rank algorithms can generate large errors of approximation.

- The SVD approximation can be constructed iteratively as (affine) subspace fitting of a set of columns.
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Matrix (hierarchical) structure must be exploited to increase precision with small cost.
Introduction and motivation

- Classical low-rank algorithms can generate large errors of approximation.

- The SVD approximation can be constructed iteratively as (affine) subspace fitting of a set of columns.

- Matrix (hierarchical) structure must be exploited to increase precision with small cost.

- Black-box fast solvers can efficiently replace classical solvers for PDE’s and integral equations.
Truncated SVD

Given $A \in \mathbb{R}^{m \times n}$, $m \geq n$, there exists orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that

$$A = U\Sigma V^T = [u_1 \cdots u_m] \begin{bmatrix} \sigma_1 & \cdots & \sigma_k \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{bmatrix} [v_1 \cdots v_n]^T.$$  

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values and $u_j$ and $v_j$ are the left and right singular vectors associated to $\sigma_j$. 

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where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values and $u_j$ and $v_j$ are the left and right singular vectors associated to $\sigma_j$. Cost: $O(mn^2)$. 

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The truncated SVD decomposition is defined as

$$\mathcal{T}_k(A) := U_k \Sigma_k V_k^T,$$

where $U_k := [u_1 \cdots u_k]$, $\Sigma_k := \text{diag}(\sigma_1, \ldots, \sigma_k)$ and $V_k := [v_1 \cdots v_k]$. 
Error of TSVD approximation

For the spectral and Frobenius norms it holds

\[ \| T_k(A) - A \|_2 = \sigma_{k+1}, \quad \| T_k(A) - A \|_F = \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_n^2}. \]
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Theorem (Eckart and Young)

Let \( A \in \mathbb{R}^{m \times n} \), then

\[ \| \mathcal{T}_k(A) - A \| = \min \{ \| A - B \| : B \in \mathbb{R}^{m \times n} \text{ has at most rank } k \} \] (2)

holds for any unitarily invariant norm.
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Remark

- Problem (2) has a unique solution when the Frobenius norm is used, provided all \( \sigma_j \) are different.
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Remark

- Problem (2) has a unique solution when the Frobenius norm is used, provided all \( \sigma_j \) are different.
- If the spectral norm is used, the solutions are not unique since, e.g. for any \( 0 \leq \theta \leq 1 \), \( B = T_k(A) - \theta \sigma_{k+1} U_k V_k^T \) is a solution, [Gu, M., 2014].
Householder reflections

Definition (Householder reflector)

It is a linear transformation that describes a reflection about an hyperplane containing the origin and orthogonal to \( u \),

\[
\mathcal{H}_u := I - \frac{2}{\|v\|^2} vv^T,
\]

where \( v = u - \|u\|e \) is the Householder vector and \( e = (1, 0, \cdots, 0)^T \).
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Since \( \mathcal{H}_u(u) = \|u\|e \), a complete pivoted QR factorization can be constructed via Householder reflections, this is

\[
A\Pi = \underbrace{Q_1 \cdots Q_n}_Q R = QR,
\]

(4)

where \( \Pi \) is a permutation, \( Q_1 = \mathcal{H}_1 \) and for \( j = \{2 \cdots n\} \)

\[
Q_j = \begin{bmatrix}
I_j & 0 \\
0 & \mathcal{H}_j
\end{bmatrix}
\]

\( I_j \): Identity matrix of size \((j - 1) \times (j - 1)\).
Error of QR approximation

For a rank-$k$ QR approximation only consider the first $k$ reflections as follows

$$A = QR \Pi^T = m \begin{bmatrix} k & r - k \\ Q_{11} & Q_{12} \end{bmatrix} \begin{bmatrix} k \\ r - k \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \Pi^T$$

$$= Q_{11} \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} \Pi^T + Q_{12} \begin{bmatrix} 0 & R_{22} \end{bmatrix} \Pi^T.$$

where $Q = Q_1 \cdots Q_k$, and

$$\|A - A_k\| = \|Q_{12}[0 \ R_{22}]\Pi^T\| = \|[0 \ R_{22}]\| = \|R_{22}\|. \quad (5)$$
Error of QR approximation

For a rank-\( k \) QR approximation only consider the first \( k \) reflections as follows

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\]

\[
= \underbrace{Q_{11} \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} \Pi^T}_{=: A_k} + \underbrace{Q_{12} \begin{bmatrix} 0 & R_{22} \end{bmatrix} \Pi^T}_{"\text{residual"}}.
\]

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- Computing \(A_k\) is typically faster than computing the TSVD.
- The choice of \(\Pi\) is of great importance to control the error.
- Note that \(\sigma_k(A) = \sigma_k(R)\).
Choosing the pivot using the maximal volume criteria

Theorem (Goreinov and Tyrtyshnikov, 2001)

Let us consider

\[
R = \begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix}
\]

where \( R_{11} \in \mathbb{R}^{k \times k} \) has maximal volume (i.e., maximum determinant in absolute value) among all \( k \times k \) submatrices of \( R \). Then

\[
\| R_{22} - R_{21} R_{11}^{-1} R_{12} \|_{\text{max}} \leq (k + 1) \sigma_{k+1}(R).
\]

where \( \| M \|_{\text{max}} := \max_{i,j} | M(i, j) | \).
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Good news: Since for a low-rank QR factorization we have \( R_{21} = 0 \), then

\[ \| R_{22} \|_{\text{max}} \leq (k + 1) \sigma_{k+1}(A). \]
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Good news: Since for a low-rank QR factorization we have $R_{21} = 0$, then

$$\| R_{22} \|_{\text{max}} \leq (k + 1)\sigma_{k+1}(A).$$

Bad news: Finding a submatrix of maximum volume has been proven to be NP-hard, Civril and Magdon-Ismail (2011).
Choosing the pivot using the classical column pivoting QRCP

QRCP takes as pivot the column of largest norm at each step, the error is bounded as

\[ \| R_{22} \|_2 \leq 2^k \sqrt{n - k} \sigma_{k+1}(A). \] (6)

In general, \( \| R_{22} \|_2 \leq g(k, n) \sigma_{k+1}(A) \).
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<table>
<thead>
<tr>
<th>Method</th>
<th>Reference</th>
<th>( g(k,n) )</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pivoted QR</td>
<td>[Golub, 1965]</td>
<td>( \sqrt{(n - k)2^k} )</td>
<td>( O(mnk) )</td>
</tr>
<tr>
<td>High RRQR</td>
<td>[Foster, 1986]</td>
<td>( \sqrt{n(n - k)2^{n-k}} )</td>
<td>( O(mn^2) )</td>
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<tr>
<td>High RRQR</td>
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</tr>
<tr>
<td>RRQR</td>
<td>[Hong and Pan, 1992]</td>
<td>( \sqrt{k(n - k) + k} )</td>
<td>-</td>
</tr>
<tr>
<td>Low RRQR</td>
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<td>( \sqrt{(k + 1)n2^{k+1}} )</td>
<td>( O(mn^2) )</td>
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<tr>
<td>Hybrid-I RRQR</td>
<td>[Chandr. and Ipsen, 1994]</td>
<td>( \sqrt{(k + 1)(n - k)} )</td>
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</tr>
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<tr>
<td>Algorithm 3</td>
<td>[Gu and Eisenstat, 1996]</td>
<td>( \sqrt{k(n - k) + 1} )</td>
<td>-</td>
</tr>
<tr>
<td>Algorithm 4</td>
<td></td>
<td>( \sqrt{f^2k(n - k) + 1} )</td>
<td>( O(kmn \log_f(n)) )</td>
</tr>
<tr>
<td>DGEQPY</td>
<td>[Bischof and Orti, 1998]</td>
<td>( O((k + 1)^2(n - k)) )</td>
<td>-</td>
</tr>
<tr>
<td>DGEQPX</td>
<td></td>
<td>( O((k+1)(n - k)) )</td>
<td>-</td>
</tr>
<tr>
<td>SPQR</td>
<td>[Stewart, 1999]</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>PT Algorithm 1</td>
<td>[Pan and Tang, 1999]</td>
<td>( O((k + 1)(n - k)) )</td>
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<tr>
<td>PT Algorithm 2</td>
<td></td>
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<tr>
<td>Pan Algorithm 2</td>
<td>[Pan, 2000]</td>
<td>( O(\sqrt{k(n - k) + 1}) )</td>
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**Figure:** Different algorithms for low-rank QR approximation, Mahoney et al. (2010).
Low rank approximation using subspace iteration

The following algorithm is the basic subspace iteration method,

\textbf{Algorithm 1} \hspace{1em} [A_k] = \text{SubspaceIter}(A,\Omega,k,q)

- \textbf{Requires:} $\Omega \in \mathbb{R}^{n \times l}$, with $l \geq k$.
- \textbf{Returns:} rank-$k$ approximation of $A$.

1: Perform $Y = (A A^T)^q A \Omega$.
2: Compute (economic) QR decomposition $Y = QR$.
3: Form $B = Q^T A$.
4: Set $A_k := Q T_k(B)$.

Note that setting $k = l = 1$ then Algorithm 1 is the classical power method.

If $\Omega$ is a random Gaussian matrix, then setting $l = 2k$ and $q = 0$, we get the expected error \cite{Halko, N. et al, 2014} $\|A - A_k\|_2 \leq (2 + 4\sqrt{2} \min\{m,n\} k^{-1}) \sigma_{k+1}$. 

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Low rank approximation using subspace iteration

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Algorithm 2 \([A_k] = \text{SubspaceIter}(A, \Omega, k, q)\)

**Requires:** \(\Omega \in \mathbb{R}^{n \times l}\), with \(l \geq k\).

**Returns:** rank-\(k\) approximation of \(A\).

1. Perform \(Y = (AA^T)^q A\Omega\).
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- If \(\Omega\) is a random Gaussian matrix, then setting \(l = 2k\) and \(q = 0\), we get the expected error [Halko, N. et al, 2014]

\[
\mathbb{E}\|A - A_k\|_2 \leq \left(2 + 4\sqrt{\frac{2\min\{m, n\}}{k - 1}}\right)\sigma_{k+1}.
\]
Error of subspace iteration approximation
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To find the error of approximation, consider the SVD of $A = U \Sigma V^T$ and the partition

$$\hat{\Omega} := V^T \Omega = \begin{bmatrix} \hat{\Omega}_1 \\ \hat{\Omega}_2 \end{bmatrix}, \quad 0 \leq p \leq l - k.$$ 

If $\hat{\Omega}_1$ is full row rank, then the error is bounded as ([Gu, M., 2014])

$$\|A - A_k\|_2 \leq \sqrt{\sigma_{k+1}^2 + \omega^2 \|\hat{\Omega}_2\|_2^2 \|\hat{\Omega}_1^\dagger\|_2^2},$$

where $\omega = \sqrt{k} \sigma_{l-p+1} \left( \frac{\sigma_{l-p+1}}{\sigma_k} \right)^2$ and $\hat{\Omega}_1 \hat{\Omega}_1^\dagger = I$. 
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Remark

*If $G$ is a $(l - p) \times l$ is a Gaussian matrix, then $\text{rank}(G) = l - p$ with probability 1.*
How do the singular vectors converge?

We need to investigate the rate at which we are approaching to a best fitting subspace.
How do the singular vectors converge?

1. We need to investigate the rate at which we are approaching to a best fitting subspace.

2. How do we measure the distance between subspaces?
   - Consider $W_1, W_2 \in \mathbb{R}^{m \times k}$ with orthogonal columns.
   - Let let $S_1 := \text{ran}(W_1)$ and $S_2 := \text{ran}(W_2)$, then
     
     \[
     \text{dist}(S_1, S_2) := \|W_1 W_1^T - W_2 W_2^T\|_2.
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Theorem (Ayala et al., 2017)

Using the notation from Algorithm 1, Let $S_u = \text{ran}([u_1 \cdots u_l])$ and $S_q = \text{ran}(Q)$, considering $\hat{\Omega}_1$ nonsingular and $p = 0$, then

\[
\text{dist}(S_u, S_q) \leq \left(\frac{\sigma_{l+1}}{\sigma_l}\right)^{2q+1} \|\hat{\Omega}_2\|_2 \|\hat{\Omega}_1^{-1}\|_2,
\]

provided $\sigma_{l+1} > \sigma_l$. 

Finding a good Householder vector

When choosing the pivot as one of the columns of $A$ the questions that arise are: How close the error is with respect to the truncated SVD?, Which choice of pivot is the optimal?.

Figure: Householder reflection: $p_j$ and $d_j$ denote the projections of $a_j$ along and orthogonal to $u$ respectively.
Finding a good Householder vector

When choosing the pivot as one of the columns of \( A \) the questions that arise are: How close the error is with respect to the truncated SVD?, Which choice of pivot is the optimal?.

Given \( A = [a_1 \ a_2 \ \cdots \ a_n] \), let \( u \in \mathbb{R}^m \) be any unitary vector, then

\[
H_u A = [h_{a_1} \ h_{a_2} \ \cdots \ h_{a_n}].
\]

**Figure:** Householder reflection: \( p_j \) and \( d_j \) denote the projections of \( a_j \) along and orthogonal to \( u \) respectively.
Error for a rank-one approximation with arbitrary Householder vector.

\[ H_u A = \begin{bmatrix} \|a_1\|_2 \cos(\varphi_1) & \|a_2\|_2 \cos(\varphi_2) & \cdots & \|a_n\|_2 \cos(\varphi_n) \\ r_1 & r_2 & \cdots & r_n \end{bmatrix}, \] (8)

where \( r_j \in \mathbb{R}^{m-1} \).
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where \( r_j \in \mathbb{R}^{m-1} \). The rank-one matrix

\[
A_1 = \frac{u}{\|u\|_2} (\|a_1\|_2 \cos(\varphi_1), \cdots, \|a_n\|_2 \cos(\varphi_n))
\]

(9)

approximates \( A \) with an error given by the norm of the residual matrix \( E := [r_1 \cdots r_n] \). By the Pythagorean theorem \( \|r_j\|_2 = \|a_j\|_2 \sin(\varphi_j) \), then

\[
\|A - A_1\|_F^2 = \|E\|_F^2 = \sum_{j=1}^{n} \|r_j\|_2^2 = \sum_{j=1}^{n} \|a_j\|_2^2 \sin^2(\varphi_j).
\]

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Error for a rank-one approximation with arbitrary Householder vector.

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\[ \|A - A_1\|_F^2 = \|E\|_F^2 = \sum_{j=1}^n \|r_j\|_2^2 = \sum_{j=1}^n \|a_j\|_2^2 \sin^2(\varphi_j). \]  

(10)

Since \( d_j = a_j - p_j \), then

\[ \|E\|_F^2 = \sum_{j=1}^n \|d_j\|_2^2. \]  

(11)
Error for a rank-one approximation with arbitrary Householder vector.

\[ \mathcal{H}_u A = \begin{bmatrix} \|a_1\|_2 \cos(\varphi_1) & \|a_2\|_2 \cos(\varphi_2) & \cdots & \|a_n\|_2 \cos(\varphi_n) \\ r_1 & r_2 & \cdots & r_n \end{bmatrix} , \]  

(8)

where \( r_j \in \mathbb{R}^{m-1} \). The rank-one matrix

\[ A_1 = \frac{u}{\|u\|_2} (\|a_1\|_2 \cos(\varphi_1), \cdots, \|a_n\|_2 \cos(\varphi_n)) \]

(9)

approximates \( A \) with an error given by the norm of the residual matrix \( E := [r_1 \cdots r_n] \). By the Pythagorean theorem \( \|r_j\|_2 = \|a_j\|_2 \sin(\varphi_j) \), then

\[ \|A - A_1\|_F^2 = \|E\|_F^2 = \sum_{j=1}^{n} \|r_j\|_2^2 = \sum_{j=1}^{n} \|a_j\|_2^2 \sin^2(\varphi_j). \]

(10)

Since \( d_j = a_j - p_j \), then

\[ \|E\|_F^2 = \sum_{j=1}^{n} \|d_j\|_2^2. \]  

(11)

Which choice of \( u \) minimizes this error?
Solving the optimization problem

We seek the hyperline in the $m$ dimensional space that minimizes the sum of squared orthogonal distances from the points $a_j$’s to itself. This is the *total least-square problem*. 

Define the matrix $Y := \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$. 

(12) 

The best fitting line of the points \{a_j’s\} is given by 

$L := \{ g + u\tau | \tau \in \mathbb{R} \}$. 

(13) 

where $g := (1/n) \sum_{j=1}^{n} a_j$ and $u = u_1(Y)$. [Schneider et al., 2003].

If we impose the condition that the line passes through the origin, then the solution would be 

$\tilde{L} := \{ \tilde{u}\tau | \tau \in \mathbb{R} \}$. 

(14) 

where $\tilde{u} = u_1(A)$. 

Solving the optimization problem

We seek the hyperline in the $m$ dimensional space that minimizes the sum of squared orthogonal distances from the points $a_j$’s to itself. This is the total least-square problem.

Define the matrix

$$Y := [a_1 - g \cdots a_n - g].$$ (12)

- The best fitting line of the points $\{a_j\}$ is given by

$$\mathcal{L} := \{ g + u\tau \mid \tau \in \mathbb{R} \}. \quad (13)$$

where $g := (1/n) \sum_{j=1}^{n} a_j$ and $u = u_1(Y)$, [Schneider et al., 2003].
Solving the optimization problem

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- If we impose the condition that the line passes through the origin, then the solution would be

$$\tilde{\mathcal{L}} := \{ \tilde{u} \tau \mid \tau \in \mathbb{R} \}.$$ 

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where $\tilde{u} = u_1(A)$. 
Geometry analysis of pivoting

Setting the optimization problem

Best fitting (affine) subspace.

Figure: Solution of the total least-square problem (left) and its solution by imposing the condition to pass through the origin $O$ (right).
Best fitting (affine) subspace.

\[ \mathcal{L}(\tau) = g + u\tau \]

\[ \tilde{\mathcal{L}}(\tau) = \tilde{u}\tau \]

**Figure:** Solution of the total least-square problem (left) and its solution by imposing the condition to pass through the origin \( O \) (right).

- To approximate \( u_1(Y) \) we can use the fact that it is the principal component of \( C = YY^T \), the covariance matrix.
- There exists work on PCA on trimming around affine subspaces [Croux et al., 2014].
Error approximation for ALORA

Consider \( c = [1, \cdots, 1]^T \in \mathbb{R}^m \). Let \( u = u_1(Y) = u_1(A - gc) \) and define

\[
B = A - T, \quad T = (g - \alpha u)c,
\]

where \( \alpha \in \mathbb{R} \).

- Considering \( g_B = (1/n) \sum_{j=1}^{n} b_j \), then clearly \( g_B = u \).
Error approximation for ALORA

Consider $c = [1, \cdots, 1]^T \in \mathbb{R}^m$. Let $u = u_1(Y) = u_1(A - gc)$ and define

$$B = A - T, \quad T = (g - \alpha u)c,$$

where $\alpha \in \mathbb{R}$.

- Considering $g_B = (1/n) \sum_{j=1}^{n} b_j$, then clearly $g_B = u$.
- Next, we prove that $u_1(B) = \frac{g_B}{\|g_B\|}$ and then the best fitting line of $B$ is

$$\mathcal{L}^{(B)} := \left\{ \frac{g_B}{\|g_B\|} \tau \mid \tau \in \mathbb{R} \right\}.$$
Error approximation for ALORA

Consider \( c = [1, \cdots, 1]^T \in \mathbb{R}^m \). Let \( u = u_1(Y) = u_1(A - gc) \) and define

\[
B = A - T, \quad T = (g - \alpha u)c, \tag{15}
\]

where \( \alpha \in \mathbb{R} \).

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- Next, we prove that \( u_1(B) = \frac{g_B}{\|g_B\|} \) and then the best fitting line of \( B \) is

\[
\mathcal{L}^{(B)} := \left\{ \frac{g_B}{\|g_B\|} \tau \mid \tau \in \mathbb{R} \right\}.
\]

Lemma

Let \( r = \text{rank}(Y) \) \( \alpha \in \mathbb{R} \), then \( \text{rank}(B) = r \) and

\[
u_j(B) = u_j(Y) \quad \forall j \in \{1 \cdots r\}.
\]

\[
\sigma_1(B) = \sqrt{\sigma_1(Y)^2 + n\alpha^2} \quad \text{and} \quad v_1(B) = (\alpha c + \sigma_1(Y) v_1(Y)) / \sigma_1(B).
\]

\[
\sigma_j(B) = \sigma_j(Y) \quad \text{and} \quad v_j(B) = v_j(Y) \quad \forall j \in \{2 \cdots r\}.
\]
Lemma

Let $B_k$ be a rank-$k$ approximation of $B$ such that

$$
\|B - B_k\|_2 \leq g(k, n)\sigma_{k+1}(B),
$$

where $g$ is a function of $k$ and $n$. Define $A_{k+1} = B_k + T$, then

$$
\|A - A_{k+1}\|_2 \leq g(k, n)\sigma_{k+1}(A).
$$
Lemma

Let $B_k$ be a rank-$k$ approximation of $B$ such that

$$
\| B - B_k \|_2 \leq g(k, n)\sigma_{k+1}(B),
$$

where $g$ is a function of $k$ and $n$. Define $A_{k+1} = B_k + T$, then

$$
\| A - A_{k+1} \|_2 \leq g(k, n)\sigma_{k+1}(A).
$$

Corollary

$$
\sigma_{k+1}(B) \leq \sigma_{k+1}(A) \leq \sigma_k(B). \quad (16)
$$
Error approximation for ALORA

Lemma

Consider $A_l = B_{l-1} + T$, where $B_{l-1}$ is a rank $l - 1$ approximation of $B$, then

$$\|A - A_l\|_2 \leq g(l, n, C) \sigma_{l+1}(A),$$

where $C = (A - g)(A - g)^T$ is the covariance matrix and

$$g(l, n, C) = \sqrt{\frac{r + s \sqrt{\frac{n-l}{l}}}{r - s \sqrt{\frac{l}{n-l}}}},$$

with $r = \frac{\text{tr}(C)}{n}$ and $s = \sqrt{\frac{\text{tr}(C^2)}{n} - r^2}$. 

Proof.

Use Theorem 3.1 from [Merikosky et al., 1983] on matrix $C$. 

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Lemma

Consider $A_l = B_{l-1} + T$, where $B_{l-1}$ is a rank $l-1$ approximation of $B$, then

$$\|A - A_l\|_2 \leq g(l, n, C) \sigma_{l+1}(A),$$

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$$g(l, n, C) = \sqrt{\frac{r + s\sqrt{n-l}}{r - s\sqrt{n-l}}}$$

with $r = \frac{\text{tr}(C)}{n}$ and $s = \sqrt{\frac{\text{tr}(C^2)}{n} - r^2}$.

Proof.

Use Theorem 3.1 from [Merikosky et al., 1983] on matrix $C$.\hfill \square
Affine low rank approximation (ALORA)

**Algorithm 4** \( [A_{k+1}] = \text{ALORA}(A,k) \)

**Require:** \( A = [a_1 \ a_2 \ \cdots \ a_n] \in \mathbb{R}^{m \times n} \).

**Returns:** rank \( k + 1 \) approximation of \( A \).

1: \( g = (1/n) \sum_{j=1}^{n} a_j, \ c = [1 \ \cdots \ 1] \in \mathbb{R}^{1 \times n} \).

2: \( u := \text{first singular vector of } Y \).

3: \( \alpha = g(1)/u(1) \).

4: \( T = (g - \alpha u)c \).

5: Compute \( B_k \): a rank-\( k \) approximation of \( B = Y + \alpha uc \).

6: \( A_{k+1} = T + B_k \)

**Ensure:** \( \|A - A_{k+1}\|_2 \leq \sigma_k(A) \)

Note that if the directions of the fitting lines are computed using a rank-revealing QR algorithm, then ALORA will produce a translated QR factorization.
Approximation error using ALORA with QRCP

- Using QRCP to approximate the direction of the best fitting line, then ALORA yields a QRCP factorization plus a rank-one translation matrix.

**Figure:** Low-rank approximation of a random matrix with slowly decreasing singular values (left), and the Kahan matrix (right), size $m = n = 256$. 
ALORA with QRCP

- For matrices with slowly decreasing singular values, typically the first part of the spectrum is better approximated by ALORA.

Figure: Low-rank approximation of matrices GKS (left), and Baart1 (right), size $m = n = 256$. 
Approximation error using ALORA with Subspace Iteration

- Using Subspace iteration (Alg. 1 with $p = 2$, $q = 1$), to approximate the direction of the best fitting line, then ALORA improves the convergence error.
- The error get smaller while increasing $p$ or $q$ in Alg. 1.

Figure: Low-rank approximation of matrices GKS (left), and Baart1 (right), size $m = n = 256$. 
Approximation of singular values

- For QRCP (top) we plot $\frac{|R(i,i)|}{\sigma_i}$.
- For ALORA (bottom) we plot $\frac{|R^B(i,i)|}{\sigma_i}$. 
Reduction with tournament pivoting

- Tournament pivoting scheme (CARRQR, [Demmel et al., 2015]) on a $m$-by-10 matrix using 3 processors.
Reduction with tournament pivoting

- Tournament pivoting scheme (CARRQR, [Demmel et al., 2015]) on a $m$-by-10 matrix using 3 processors.
- The number of messages (two) is independent of the number of columns and it is obviously optimal.
Numerical Experiments
Parallel implementation

Reduction with tournament pivoting

- Tournament pivoting scheme (CARRQR, [Demmel et al., 2015]) on a $m$-by-10 matrix using 3 processors.
- The number of messages (two) is independent of the number of columns and it is obviously optimal.
- We use this reduction to in general select approximative directions instead of pivot columns.

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- PALORA: Parallel ALORA using QRCP.
- CALRQR: Low-rank version of CARRQR.
- PDGEKQP: A low-rank version of the ScaLapack routine PDGEQP.

**Figure:** Low-rank approximation of matrices GKS (left), and Phillips (right), size $m = n = 512$. 
ALORA_IE: modified ALORA for integral equations

- We create a (hierarchical) partition of the domain.
- In such a way that the matrix corresponding to each subdomain has a best fitting line which direction can be approximated with its gravity center.
- Take advantage of the rapidly decreasing singular values.
- Construct a **linear cost** Householder reflection.

**Example:** Consider the inner Dirichlet problem $Au = f$

$$Au(x) = \frac{1}{4\pi} \int_{\Gamma} \frac{u(y)}{|x-y|} ds_y.$$ 

Defined over a 3D domain $\Gamma$.

1. Discretize the equation by the classical Boundary element method and get the linear system $Ax = b$.
Numerical Experiments

Modified ALORA for integral equations.
Numerical Experiments

Modified ALORA for integral equations.

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Numerical Experiments

Modified ALORA for integral equations.
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