Defining the “Phase” of a Stochastic Oscillator

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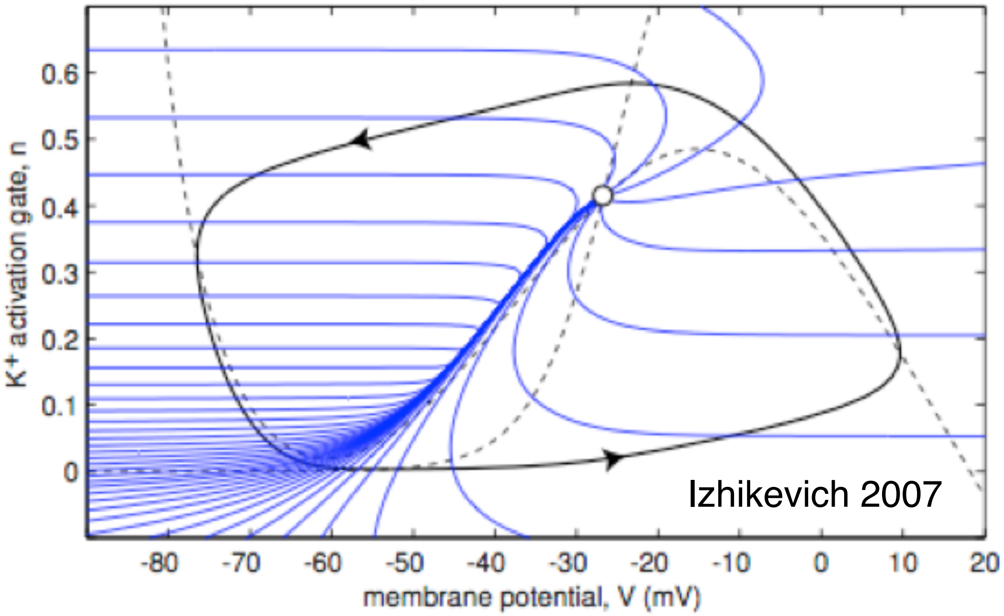
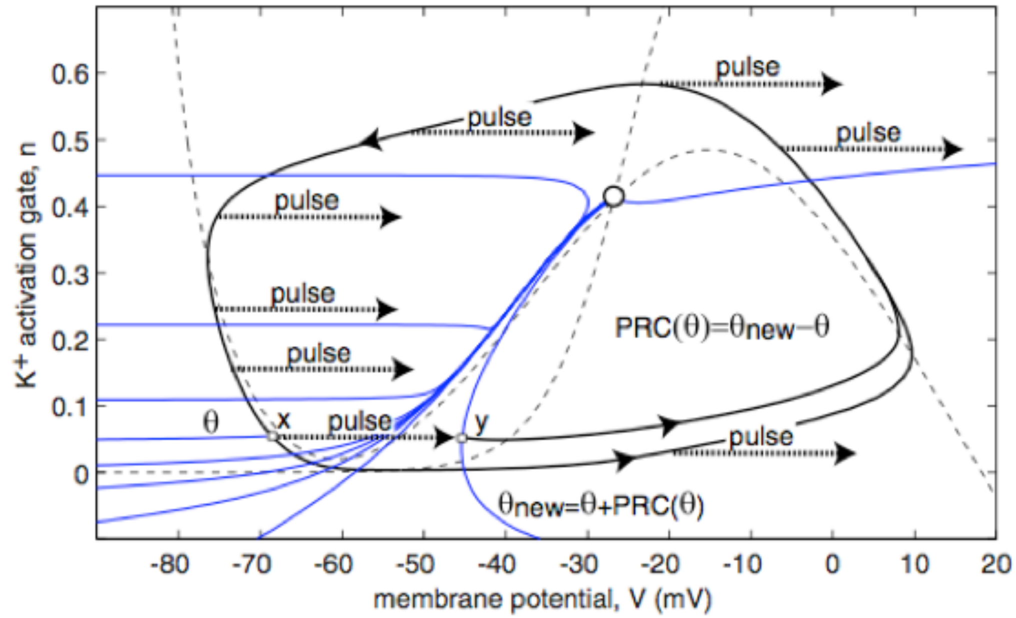
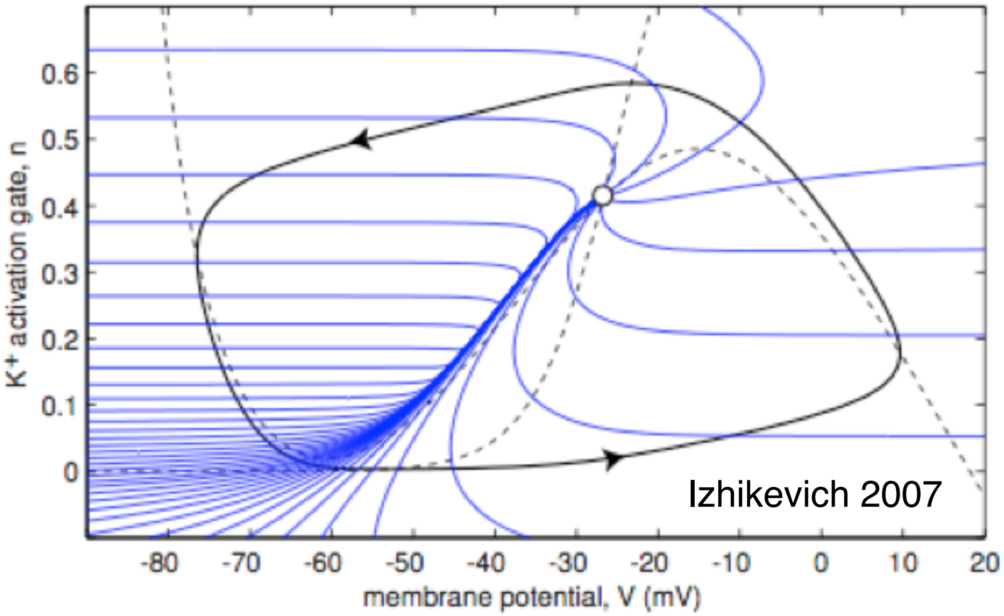


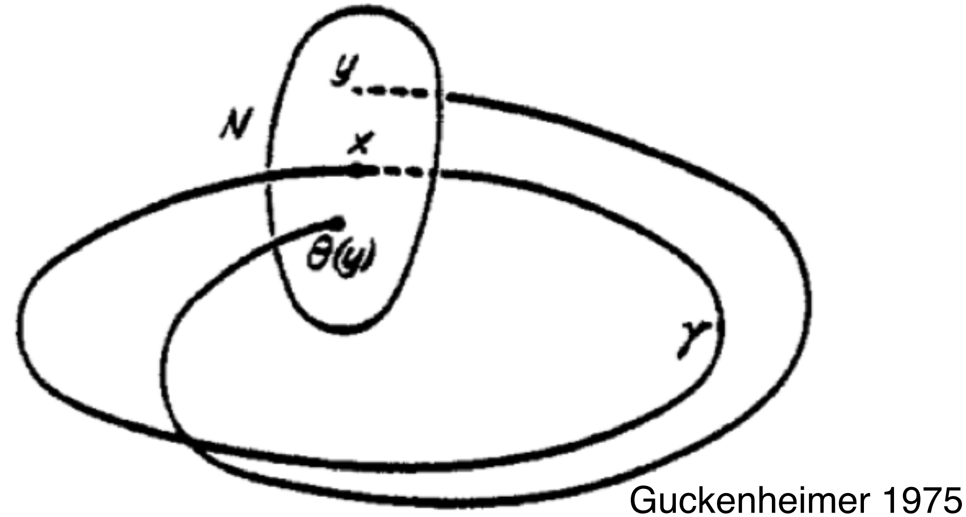


Definition (Asymptotic Phase)

Let $\mathcal{M}$ be a smooth n-dimensional manifold with distance metric d, let $\Phi : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$ be a flow satisfying the autonomous differential equation $\frac{\partial \Phi(x,t)}{\partial t} = F(x)$, and let $\gamma(t) = \Phi(x_0, t)$ be a hyperbolic, stable limit cycle with period T passing through the point $x_0 \in \mathcal{M}$. Let $\Gamma = \{\gamma(t), 0 \leq t < T\}$ be the point set comprising the limit cycle, and let $\mathcal{W}^s(\Gamma)$ be the stable manifold of Γ, the set of points that converge to Γ as $t \rightarrow \infty$. For any point $x \in \mathcal{W}^s(\Gamma)$, define the asymptotic phase function $\theta(x) \in [0, T)$ to be the unique value such that as $t \rightarrow \infty$,
$d(\Phi(t,x) - \gamma(t + \theta(x))) \rightarrow 0$.

[Guckenheimer 1975]

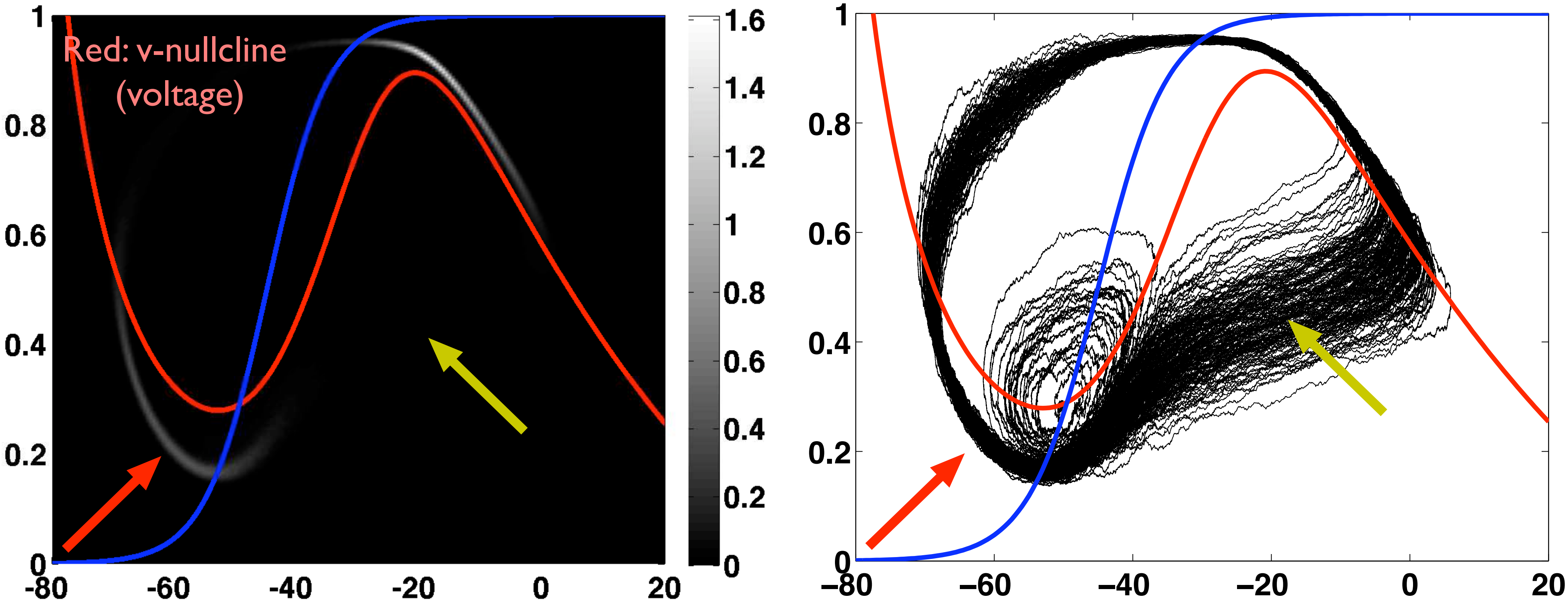




Definition (Isochron)

Let $x_0 \in \Gamma \subset \mathcal{M}$ and Φ be as above, and let Σ_0 be the largest section through x_0 such that the first return time, $\tau = \inf\{t > 0 | \Phi(t, x) \in \Sigma_0\}$ satisfies $\tau = T$ for all $x \in \Sigma_0$. Then Σ_0 is the isochron for Γ through x_0.

[Guckenheimer 1975]



- For a deterministic oscillator, the asymptotic phase disambiguates initial points.
- For a stochastic oscillator, all initial points (densities) converge to the same stationary density $P_0(\mathbf{x})$ as $t \to \infty$. Initial data is forgotten at arbitrarily long times.
- But at each point $\mathbf{x}$ the convergence to P_0 is oscillatory, with a well defined "phase".

Stochastic Limit Cycle

SDE: $dX = A(X)\,dt + \mathcal{B}(X)\,dW$ (Itô interpretation)

Define $B = \mathcal{B}\mathcal{B}^\intercal$. For $t > s$, density is:

$$\rho(y,t\,|\,x,s) = \frac{1}{dy}\Pr\{X(t) \in [y, y+dy)\,|\,X(s) = x\}$$

$$\frac{\partial}{\partial t}\rho(y,t\,|\,x,s) = \mathcal{L}_y[\rho] \text{ (forward operator)}$$

$$= -\sum_i \frac{\partial}{\partial y_i}\left(A_i(y)\rho(y,t\,|\,x,s)\right)\frac{1}{2}\sum_i\sum_j \frac{\partial^2}{\partial y_i \partial y_j}\left(B_{ij}(y)\rho(y,t\,|\,x,s)\right)$$

$$-\frac{\partial}{\partial s}\rho(y,t\,|\,x,s) = \mathcal{L}_x^\dagger[\rho] \text{ (backward operator)}$$

$$= \sum_i A_i(x)\frac{\partial}{\partial x_i}\rho(y,t\,|\,x,s) + \frac{1}{2}\sum_i\sum_j B_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\rho(y,t\,|\,x,s)$$

Assuming $\mathcal{L}$ can be diagonalized, we can expand the density into a biorthogonal system of eigenfunctions

$$\rho(\mathbf{y}, t|\mathbf{x}, s) = P_0(\mathbf{y}) + \sum_\lambda e^{\lambda(t-s)} P_\lambda(\mathbf{y}) Q^*_\lambda(\mathbf{x}), \tag{2}$$

where the eigentriples (λ, P, Q^*) satisfy

$$\mathcal{L}[P_\lambda] = \lambda P_\lambda, \;\; \mathcal{L}^\dagger[Q^*_\lambda] = \lambda Q^*_\lambda, \tag{3}$$

$$\langle Q_\lambda | P_{\lambda'} \rangle = \int d\mathbf{x}\, Q^*_\lambda(\mathbf{x}) P_{\lambda'}(\mathbf{x}) = \delta_{\lambda,\lambda'}. \tag{4}$$

All eigenvalues but one will have negative real parts, representing decaying modes.

Asymptotic Phase for Stochastic Oscillators. Thomas and Lindner. Phys. Rev. Lett. 2014

Assumptions

We say to the system is *robustly oscillatory* if

1. The nontrivial eigenvalue with least negative real part, $\lambda_1 = \mu + i\omega$, is complex (with $\omega > 0$).
2. The oscillation is faster than the decay: $|\omega/\mu| \gg 1$.
3. The slowest decaying mode is separated from the faster decaying modes, i.e. for all other eigenvalues λ', $\Re[\lambda'] \leq 2\mu$.

Under these conditions, the slowest decaying mode (as the density approaches its steady state P_0) will oscillate with period $2\pi/\omega$, and decay with time constant $1/|\mu|$.

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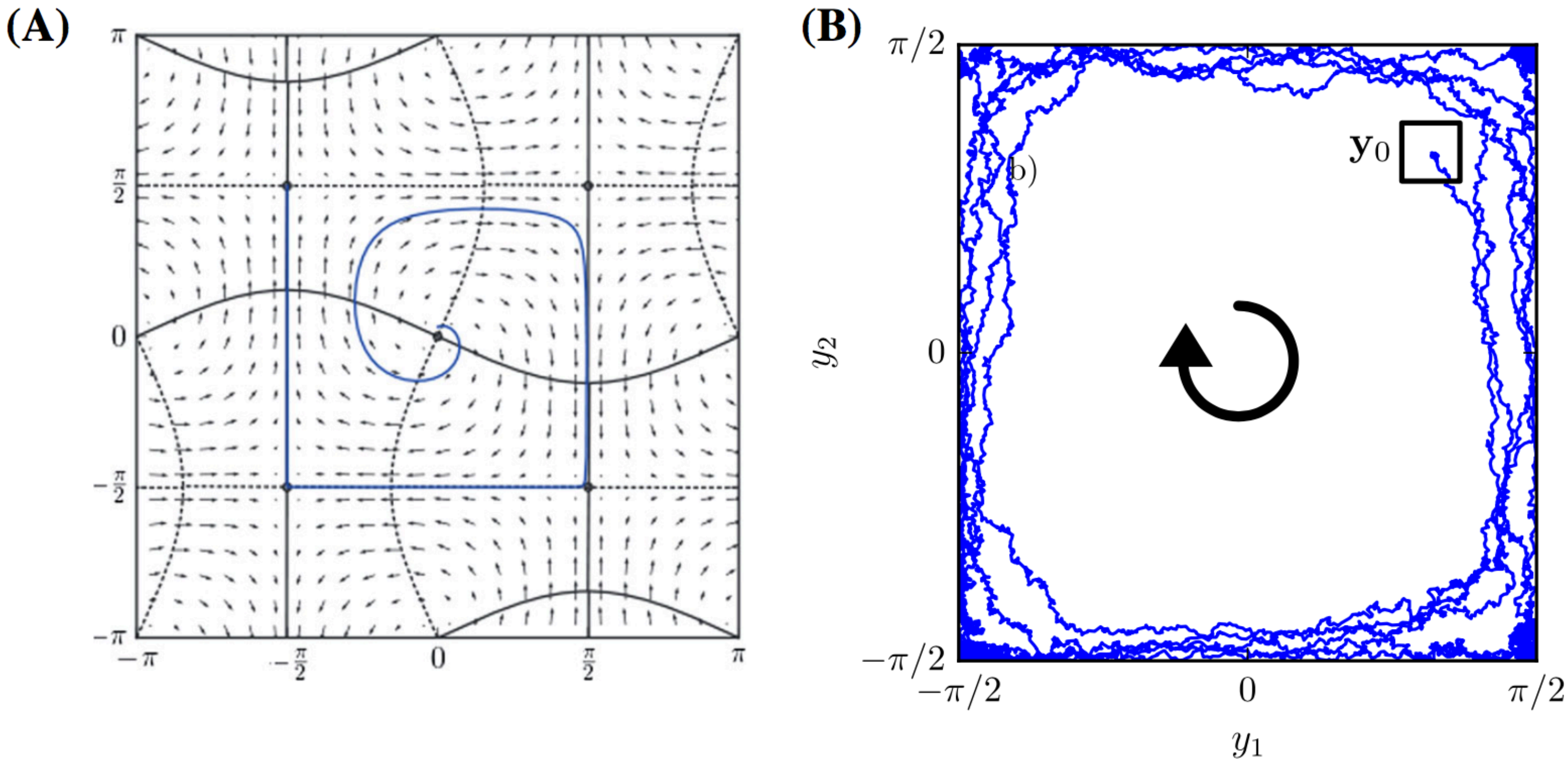
Example: Heteroclinic "Oscillator"



$$\dot{y}_1 = \cos(y_1)\sin(y_2) + \alpha\sin(2y_1) + \sqrt{2D}\xi_1(t)$$
$$\dot{y}_2 = -\sin(y_1)\cos(y_2) + \alpha\sin(2y_2) + \sqrt{2D}\xi_2(t)$$

Phase Resetting in an Asymptotically Phaseless System:
On the Phase Response of Limit Cycles Verging on a Heteroclinic Orbit.
Shaw, Park, Chiel, Thomas. SIADS. 2012

Power Spectrum of a Noisy System Close to a Heteroclinic Orbit
Giner-Baldo, Thomas, Lindner. J. Stat. Phys. in press 2017

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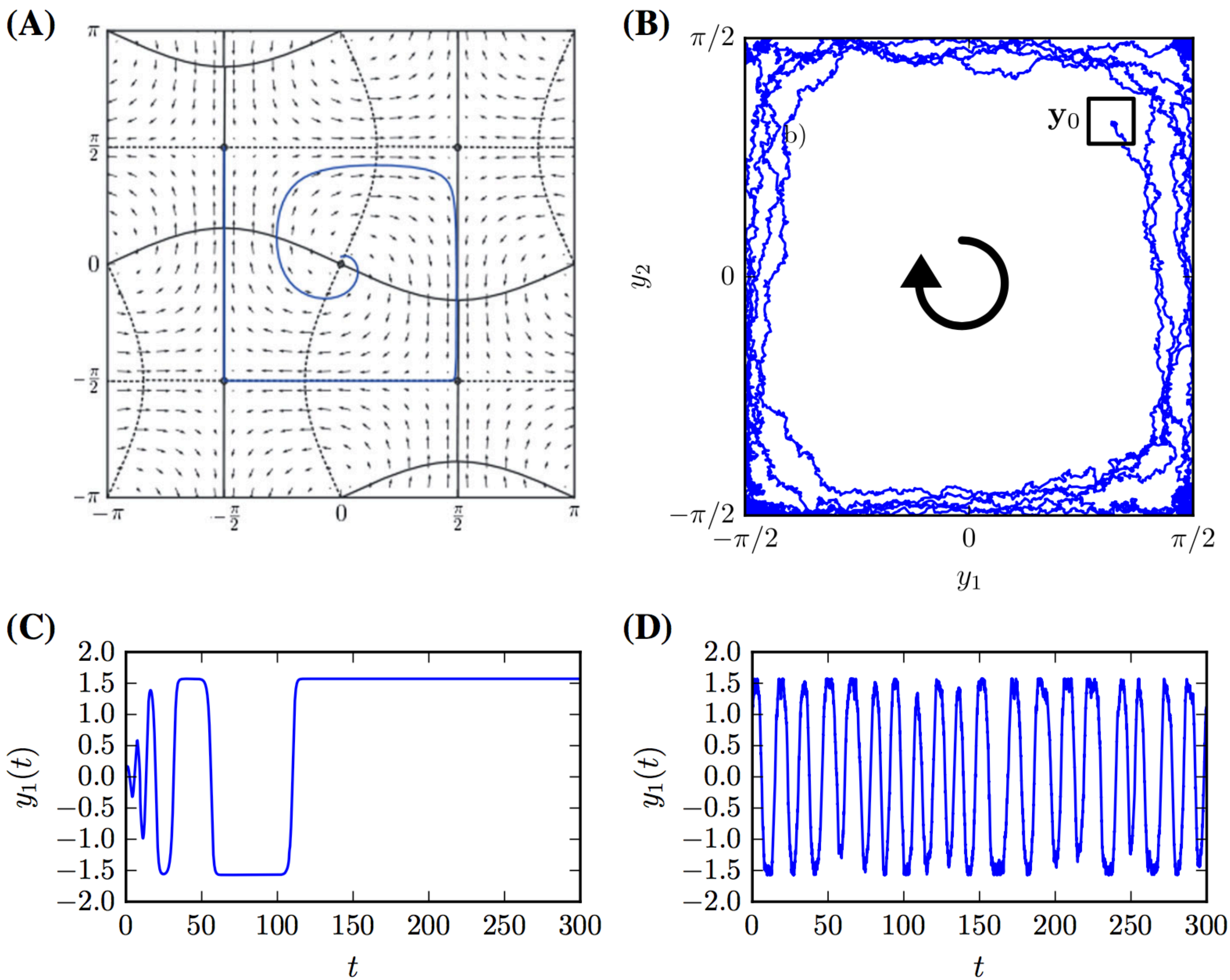

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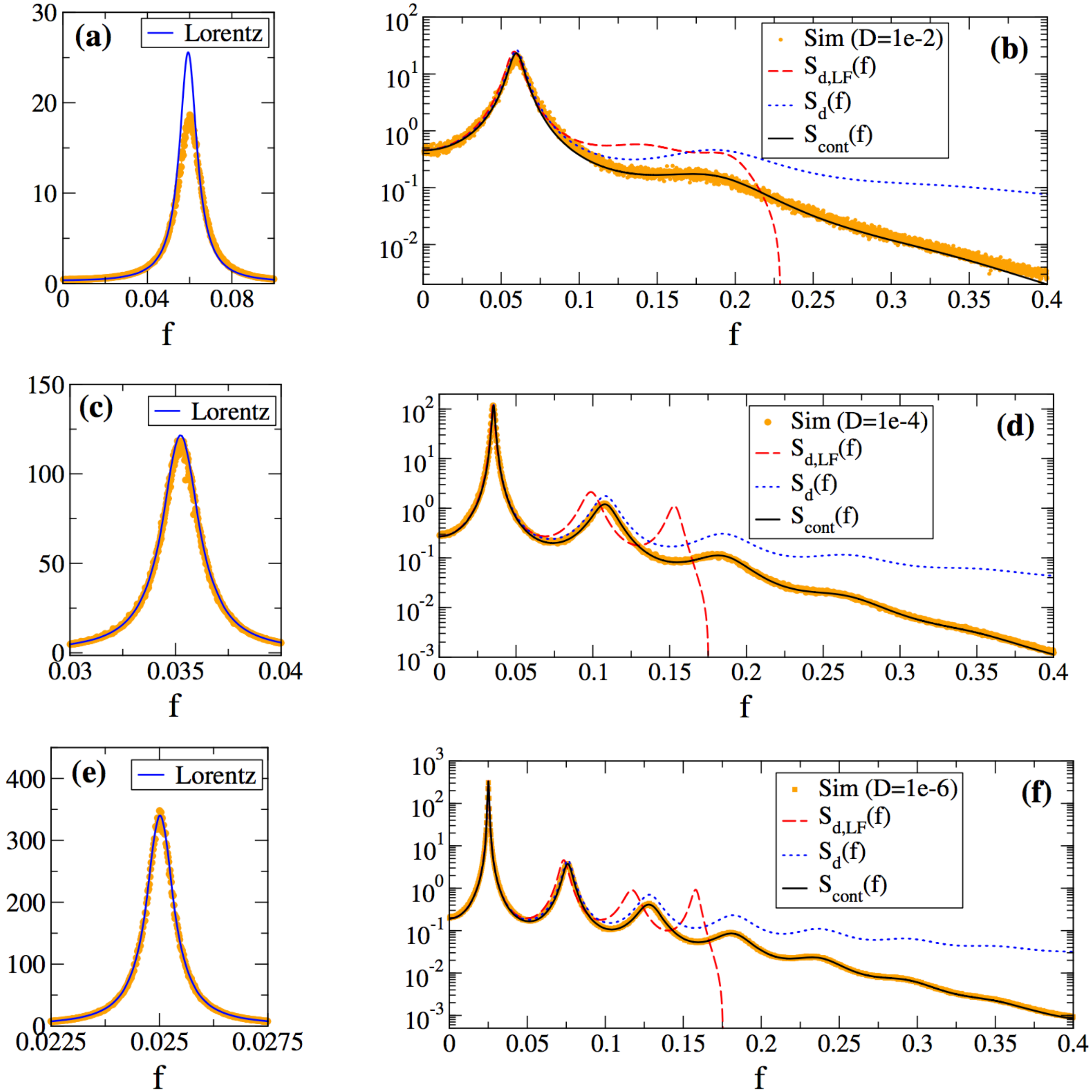


Power Spectrum of a Noisy System Close to a Heteroclinic Orbit

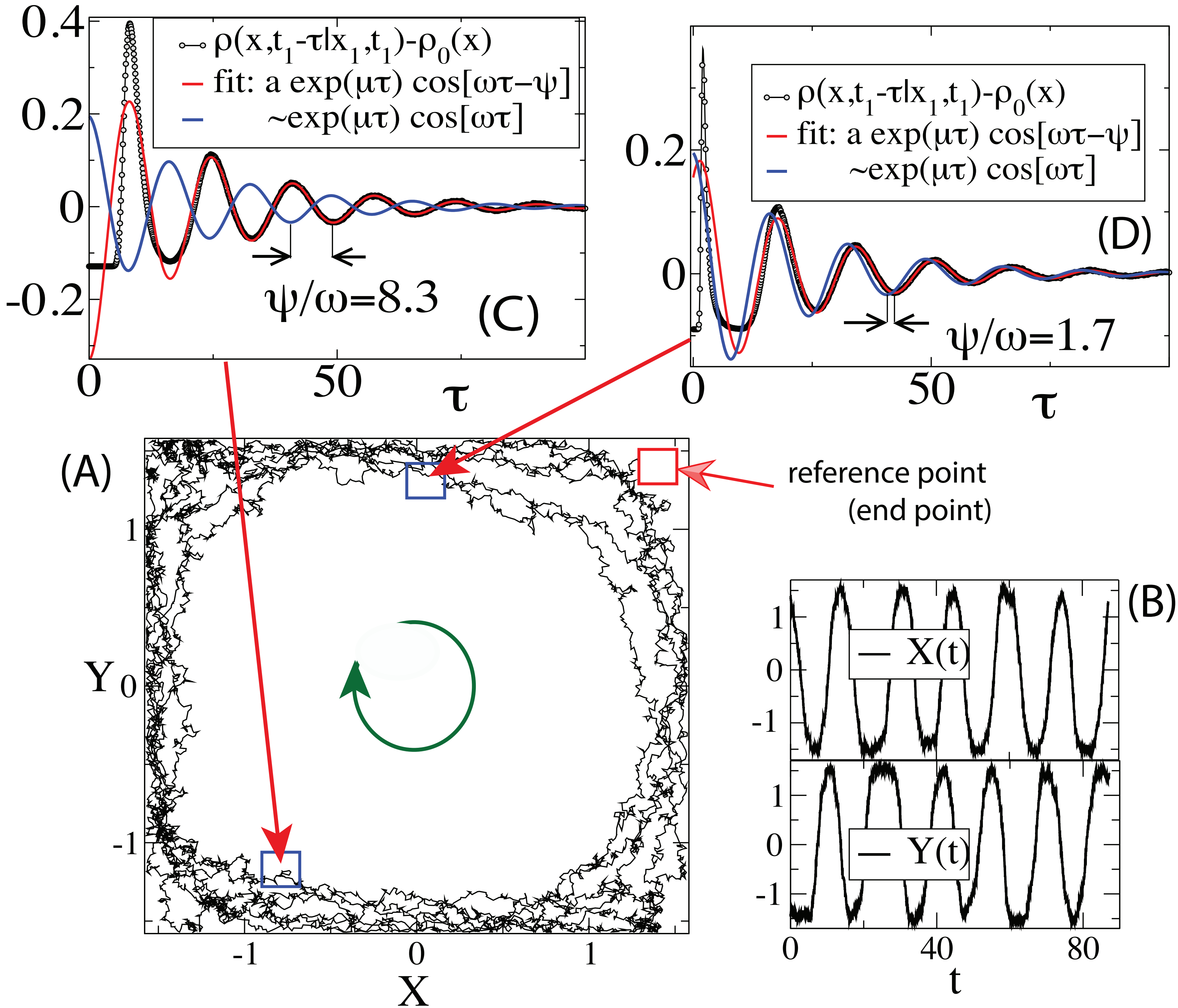
Giner-Baldo, Thomas, Lindner. J. Stat. Phys. in press 2017

Gedankenexperiment

Fix a final condition $\mathbf{y}$ and measure the time varying histogram at earlier times $s < t$ at another point $\mathbf{x}$, i.e. $\rho(\mathbf{x}, s|\mathbf{y}, t)$.
For a stationary ensemble, $\rho(\mathbf{x}, s|\mathbf{y}, t) \to P_0(x)$ as $s \to -\infty$. The convergence follows a decaying oscillation:

$$\frac{\rho(\mathbf{x}, s|\mathbf{y}, t) - P_0(\mathbf{x})}{2u(\mathbf{x})v(\mathbf{y})P_0(\mathbf{x})} \simeq \frac{e^{\mu(t-s)}}{P_0(\mathbf{y})} \cos\left(\omega(t-s) + \psi(\mathbf{x}) - \phi(\mathbf{y})\right). \quad (5)$$

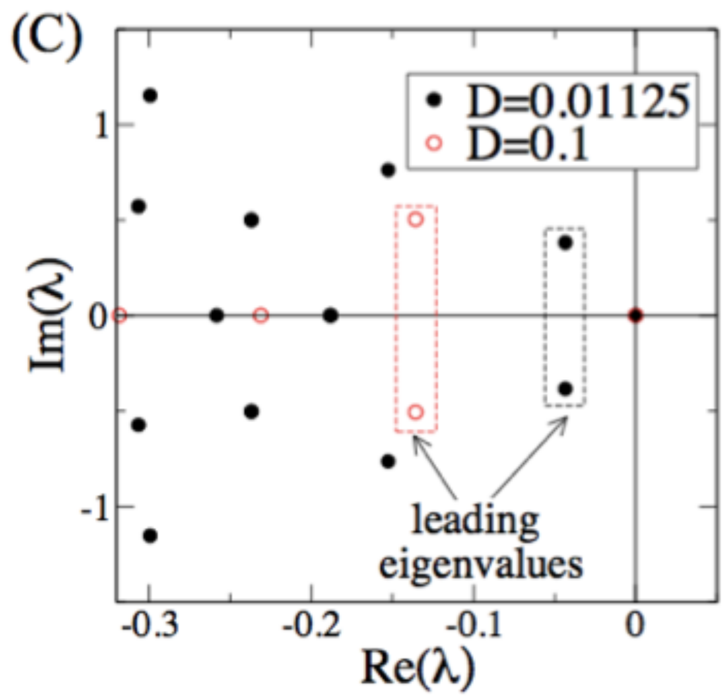
The phase offset upon shifting the "initial point" $\mathbf{x}$ defines the asymptotic phase $\psi(\mathbf{x})$ of the oscillator.
This phase comes from the slowest decaying eigenfunction of the *backward* operator.



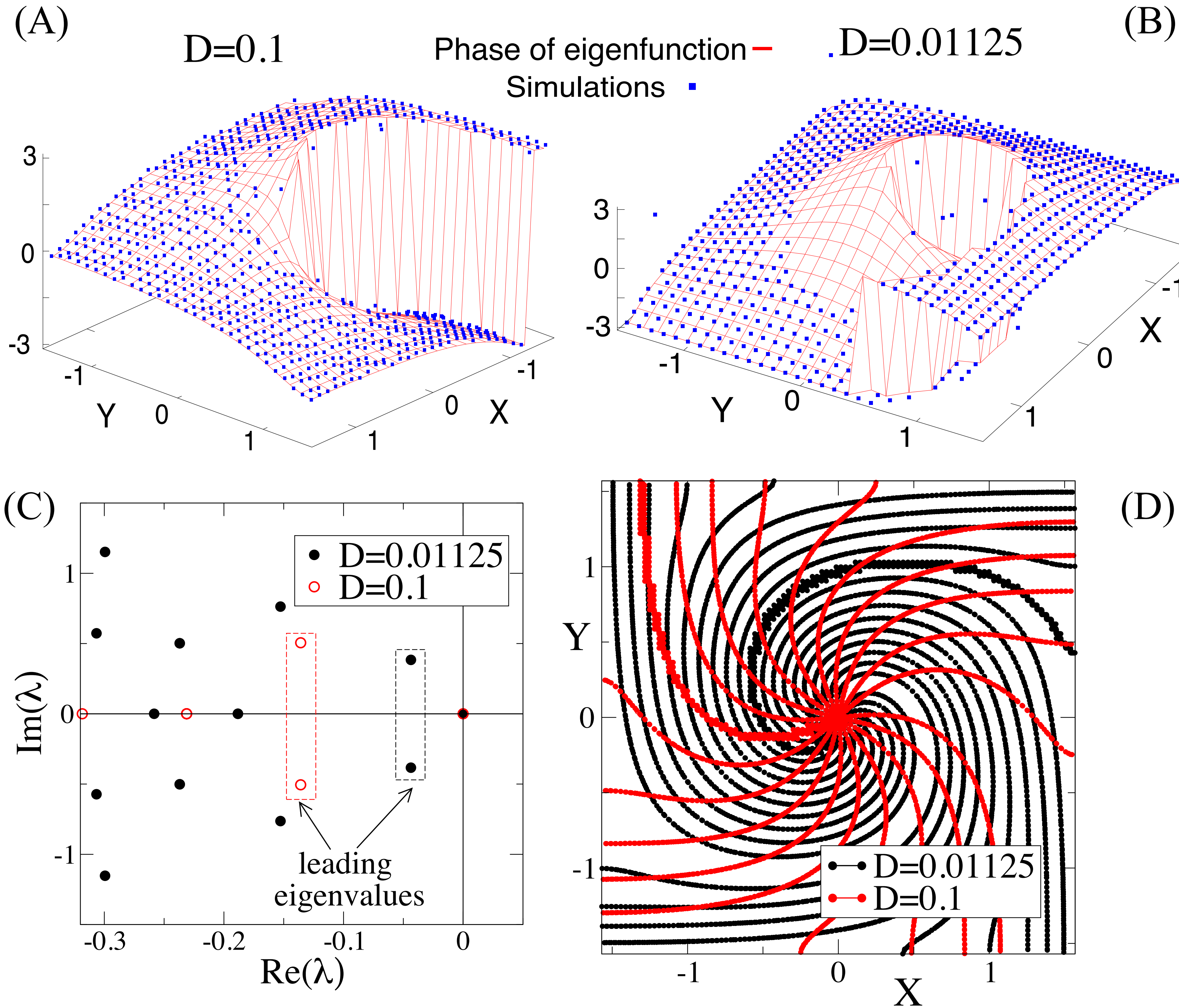
ρ(x,t1-τ|x1,t1)-ρ0(x)
fit: a exp(μτ) cos[ωτ−ψ]
~exp(μτ) cos[ωτ]
ψ/ω=8.3
(C)
τ
(D)
ψ/ω=1.7
(A)
reference point
(end point)
(B)
X(t)
Y(t)
t
X
Y

Eigenvalues of the backward (adjoint) Fokker-Planck operator

$$\begin{aligned}\mathcal{L}^\dagger &= [\cos(x)\sin(y) + \alpha\sin(2x)]\partial_x + D\partial_x^2 \\ &+ [-\sin(x)\cos(y) + \alpha\sin(2y)]\partial_y + D\partial_y^2. \qquad (7)\end{aligned}$$



Asymptotic Phase for Stochastic Oscillators. Thomas and Lindner. Phys. Rev. Lett. 2014


(A)
D=0.1
Phase of eigenfunction
Simulations
D=0.01125
(B)
3
0
-3
-1
Y
0
1
X
(C)
D=0.01125
D=0.1
leading
eigenvalues
Im(λ)
Re(λ)
-0.3
-0.2
-0.1
(D)
D=0.01125
D=0.1

Conclusions I

- Asymptotic phase cannot be defined in the usual way for stochastic systems. Individual trajectories do not converge asymptotically to one another, and all densities

$$\rho(\mathbf{y}, t) \to P_0(\mathbf{y}), \text{ as } t \to \infty$$

regardless of initial density.

- Convergence to the unique invariant distribution P_0 is asymptotic to a decaying oscillation

$$\rho(\mathbf{x}, s|\mathbf{y}, t) - P_0(\mathbf{y}) \sim e^{\mu(t-s)} \cos(\omega(t-s) + \psi(\mathbf{x}) - \phi(\mathbf{y})), \text{ as } (t-s) \to \infty$$

with $\lambda = \mu + i\omega$ the slowest decaying eigenvalue of $\mathcal{L}$.

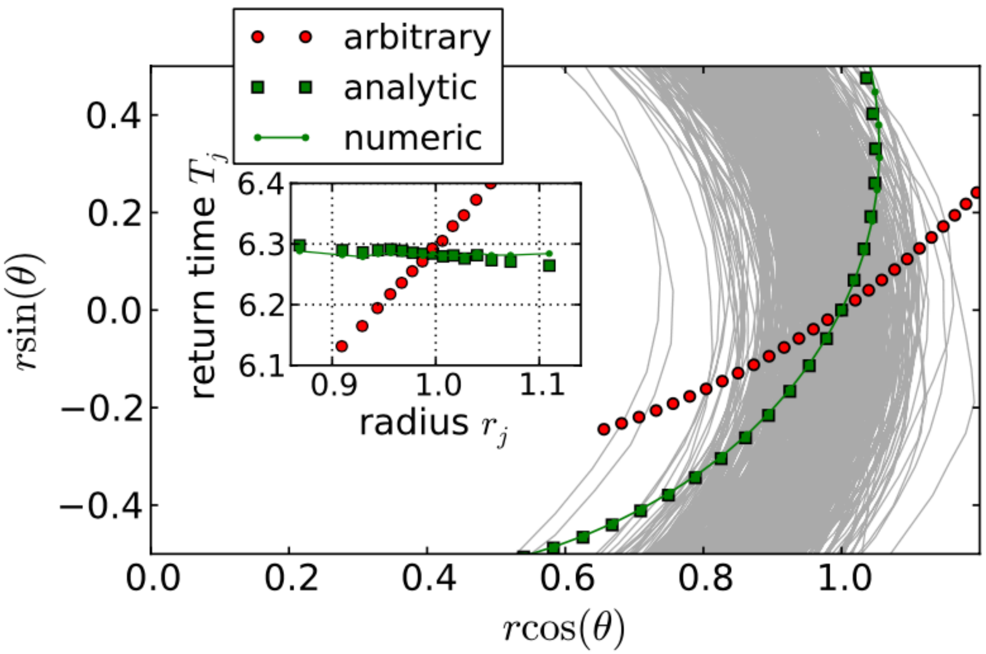
- The argument $\psi(\mathbf{x})$ of the corresponding eigenfunction, $Q^*(\mathbf{x}) = u(\mathbf{x})e^{i\psi(\mathbf{x})}$, of the adjoint operator $\mathcal{L}^\dagger[Q^*] = \lambda Q^*$ generalizes the asymptotic phase of a deterministic oscillator.

PRL 110, 204102 (2013) PHYSICAL REVIEW LETTERS week ending 17 MAY 2013

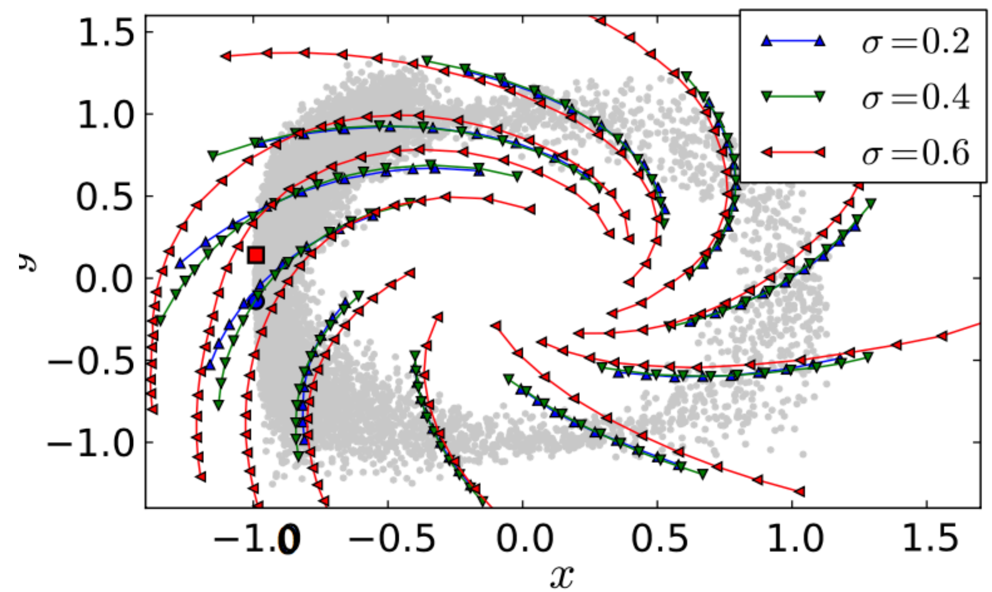
Phase Description of Stochastic Oscillations

Justus T. C. Schwabedal* and Arkady Pikovsky
Department of Physics and Astronomy, Potsdam University, 14476 Potsdam, Germany
(Received 29 January 2013; published 13 May 2013)

We introduce an invariant phase description of stochastic oscillations by generalizing the concept of standard isophases. The average isophases are constructed as sections in the state space, having a constant mean first return time. The approach allows us to obtain a global phase variable of noisy oscillations, even in the cases where the phase is ill defined in the deterministic limit. A simple numerical method for finding the isophases is illustrated for noise-induced switching between two coexisting limit cycles, and for noise-induced oscillation in an excitable system. We also discuss how to determine isophases of observed irregular oscillations, providing a basis for a refined phase description in data analysis.



Stuart-Landau perturbed by Ornstein Uhlenbeck process



Stuart-Landau perturbed by y-polarized noise

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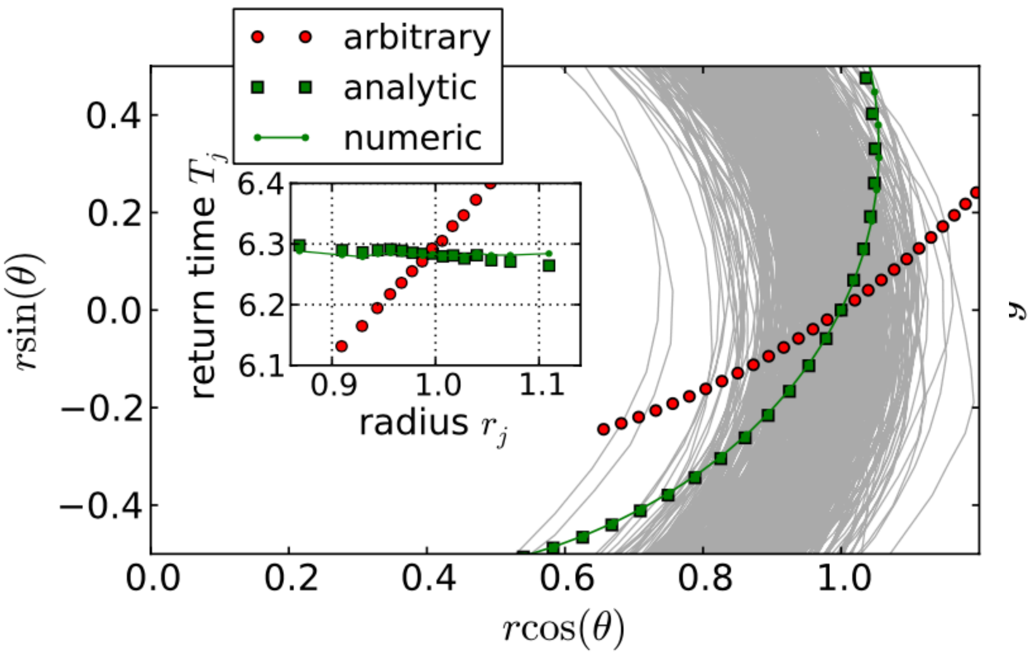

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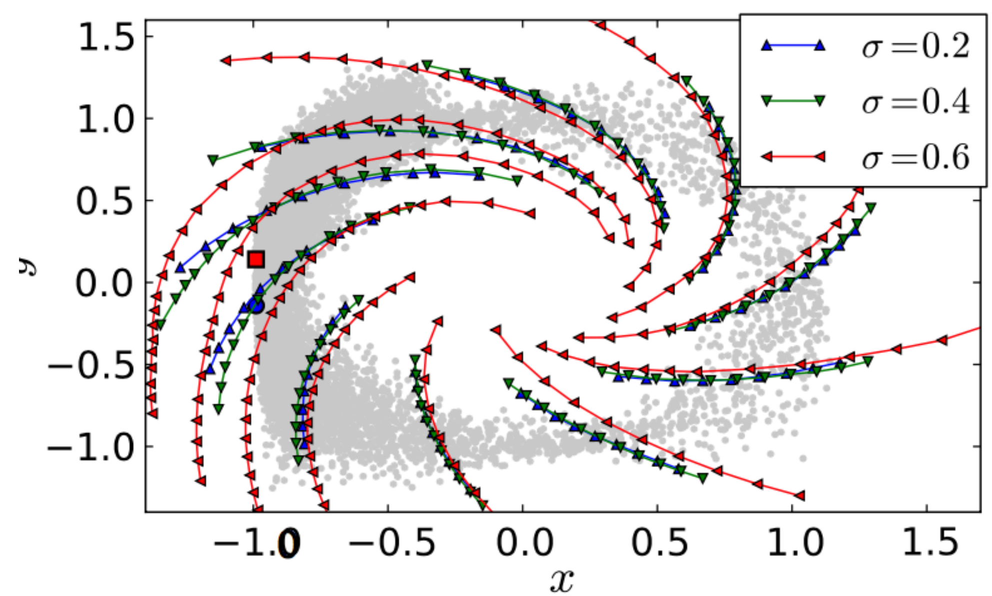

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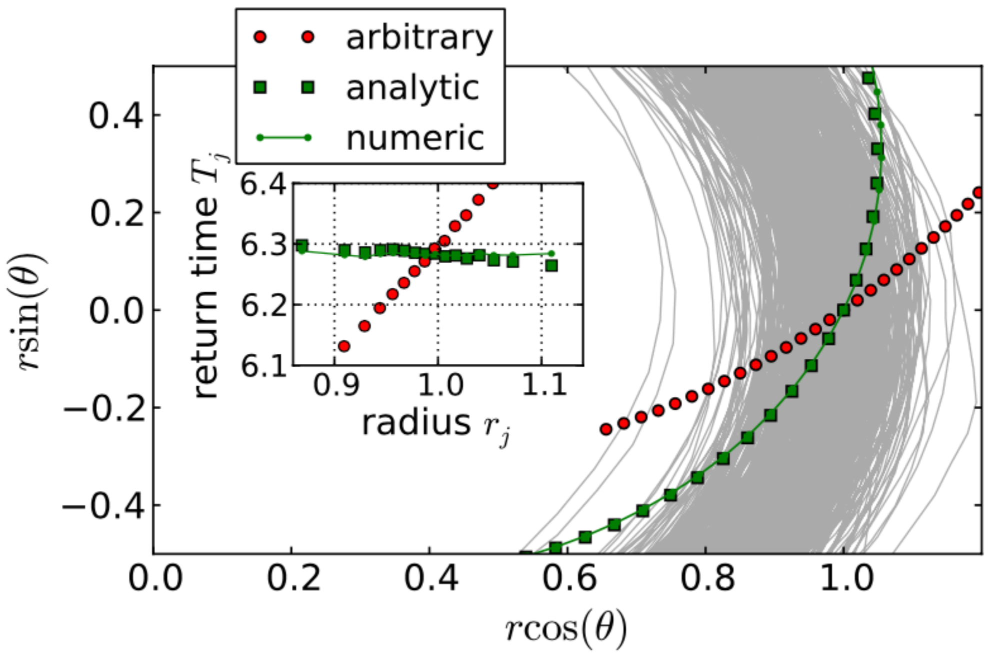

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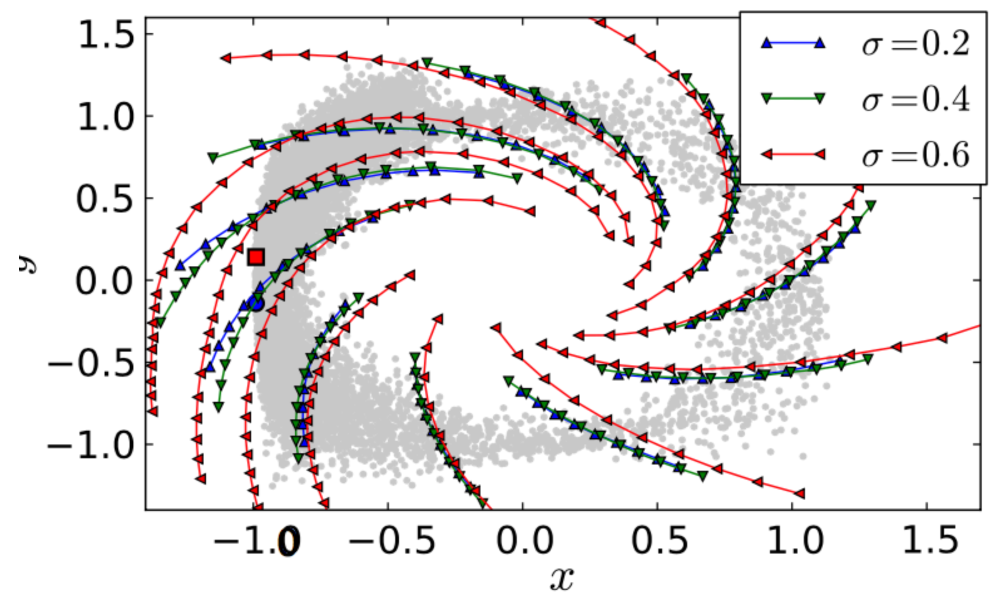

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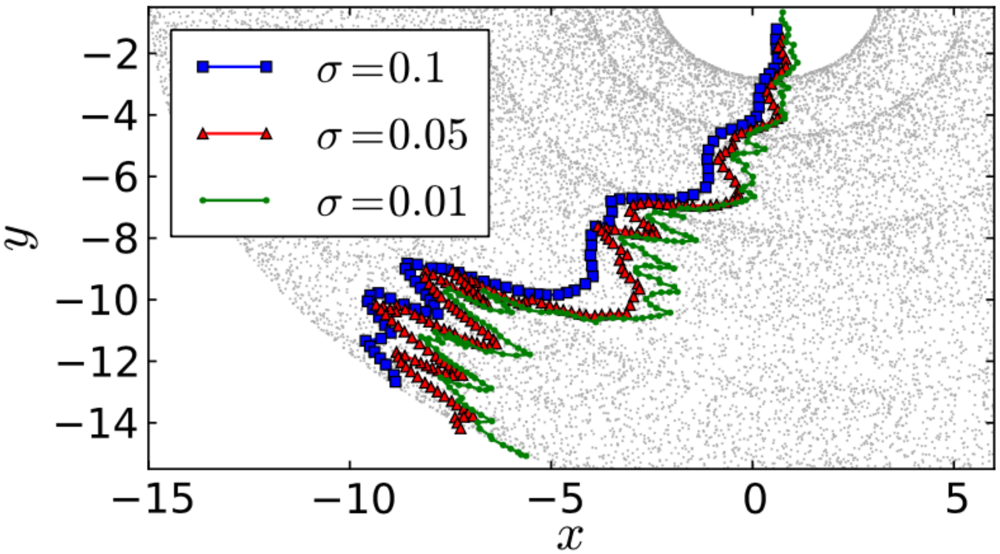
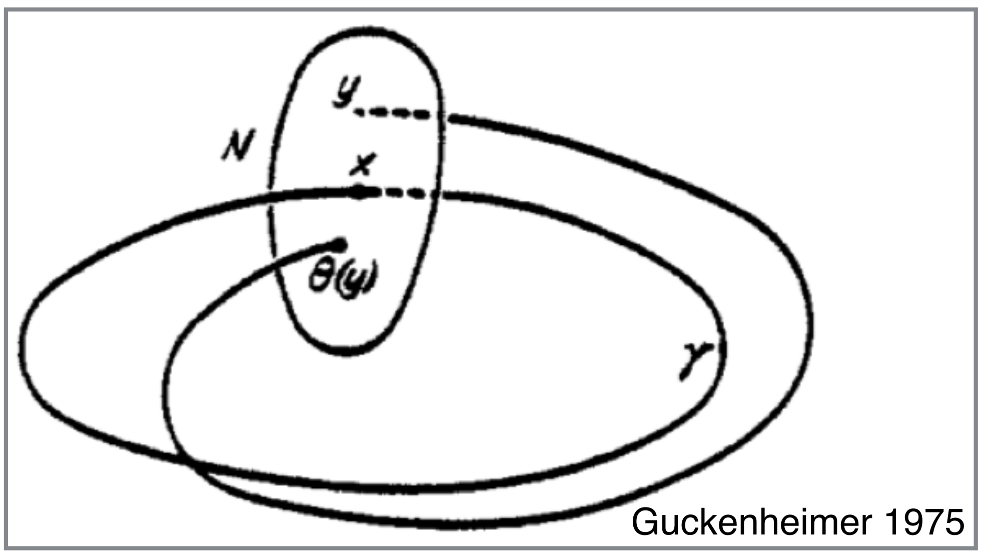




Guckenheimer 1975

For a noisy system we define the isophase surface J as a Poincaré surface of section, for which the mean first return time $J \to J$, after performing one full oscillation, is a constant T, which can be interpreted as the average oscillation period. In order for isophases to be well defined, oscillations have to be well defined as well: for example in polar coordinates, the radius variable must never become zero, so that one can reliably recognize each "oscillation." Random processes for which this is not the case should be treated with care.

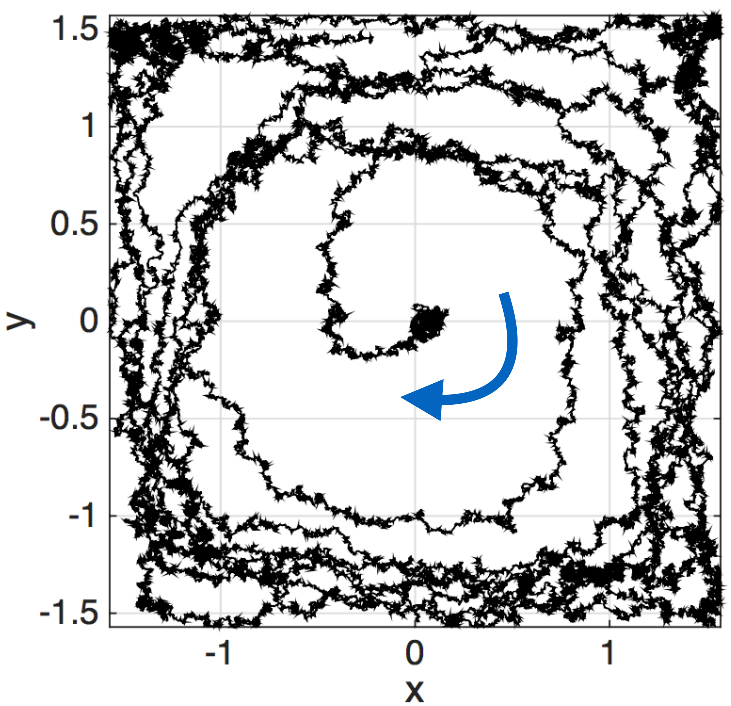
Analytical calculations of the mean first return time (MFRT) are a complex problem in dimensions larger than one; therefore, below we apply a simple numerical algorithm for construction of the isophases: an initial Poincaré section is iteratively altered until all mean return times are approximately equal. In two-dimensional systems for which isophases are lines, we represent Poincaré sections by a linear interpolation in between a set of knots. For each knot x_j, the average return time T_j is computed via the Monte Carlo simulation. According to the mismatch of T_j and the mean period $\langle T \rangle$, the knot x_j is advanced or retarded. The procedure is repeated with all knots, until it converges and all return times T_j are nearly equal to $\langle T \rangle$.

Reformulation of the "Average Isophase" Construction

Mean first passage time $T(x)$ from x to an absorbing boundary $\mathcal{S}$:

$$\mathcal{L}_x^\dagger[T(x)] = -1. \quad T(x) = 0,\, x \in \mathcal{S}. \quad n \cdot \nabla T(x) = 0, x \in \mathcal{S}_{\text{refl}}.$$

To establish the boundary conditions, we unwrap the oscillator.



Heteroclinic oscillator

Alexander Cao, MS Thesis 2017

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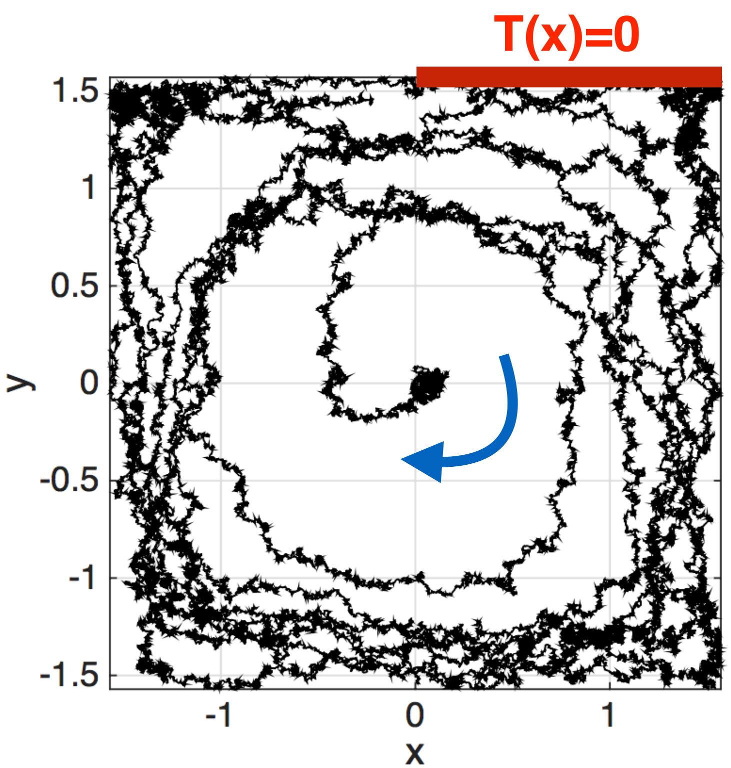

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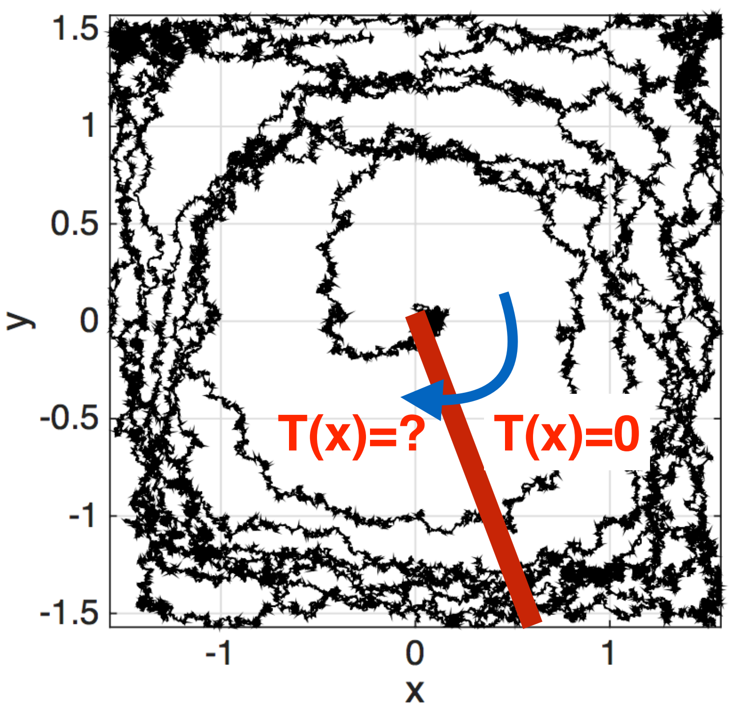

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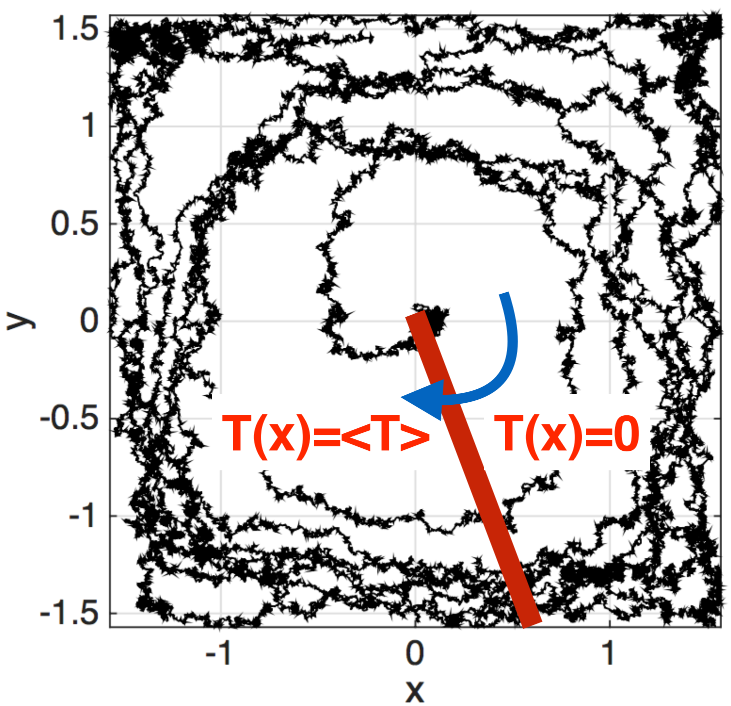

Heteroclinic oscillator

Mean first passage time function T(x) should jump by mean period <T> at section.

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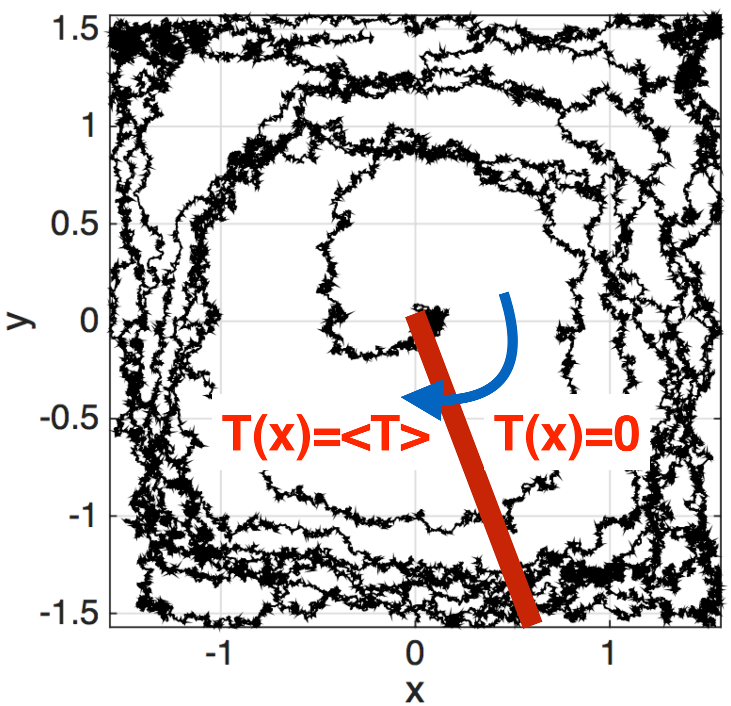
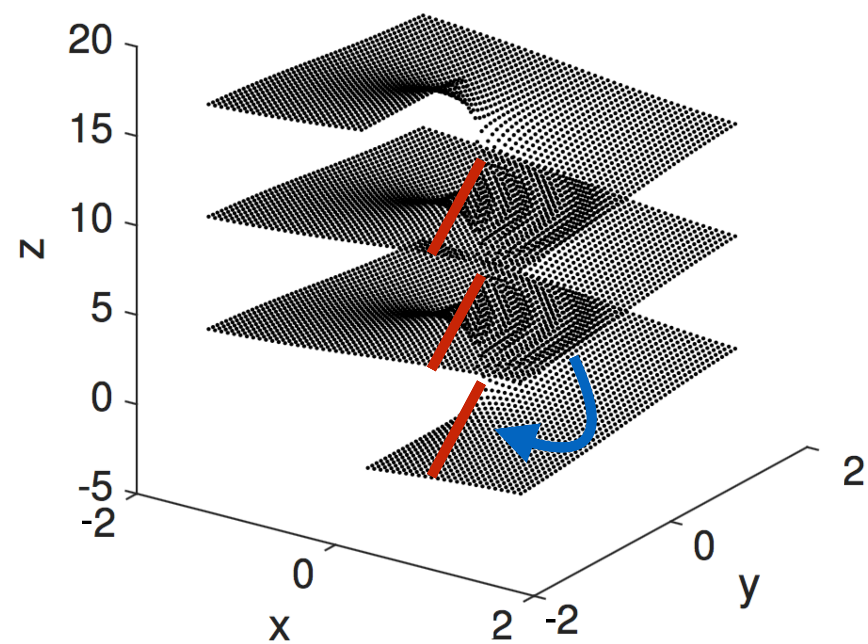


Heteroclinic oscillator



Multileaf construction.

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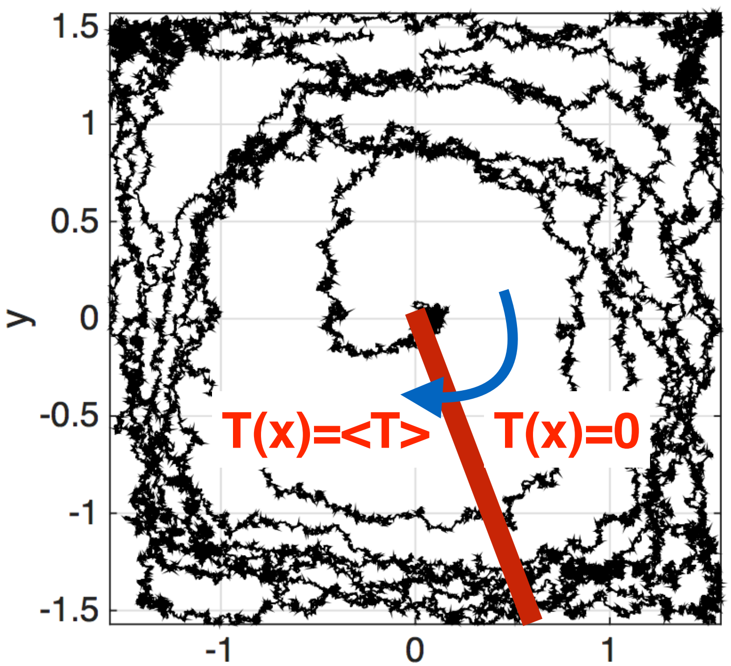

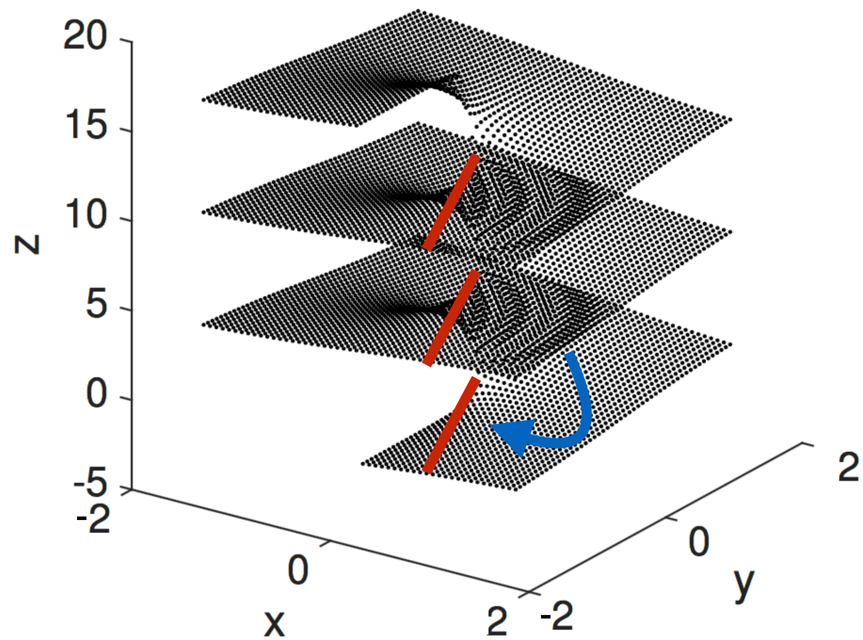

1. Obtain stationary density $\mathcal{L}[P_0] = 0$ and mean period $\bar{T}$.
2. Solve $\mathcal{L}_{\mathbf{x}}^\dagger[T(\mathbf{x})] = -1$, where $T(\mathbf{x})$ jumps by $\bar{T}$ across an (arbitrary) section to the center.

Reformulation of the "Average Isophase" Construction

Noisy heteroclinic oscillator

$$\dot{y}_1 = \cos(y_1)\sin(y_2) + \alpha \sin(2y_1) + \sqrt{2D}\xi_1(t)$$
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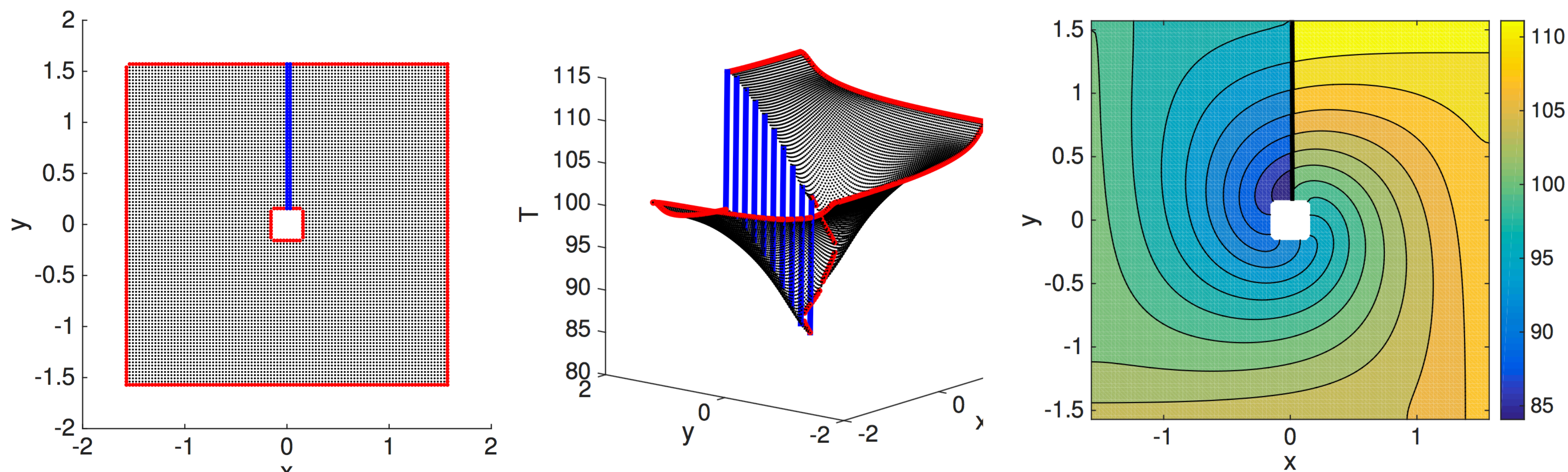

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Reformulation of the "Average Isophase" Construction

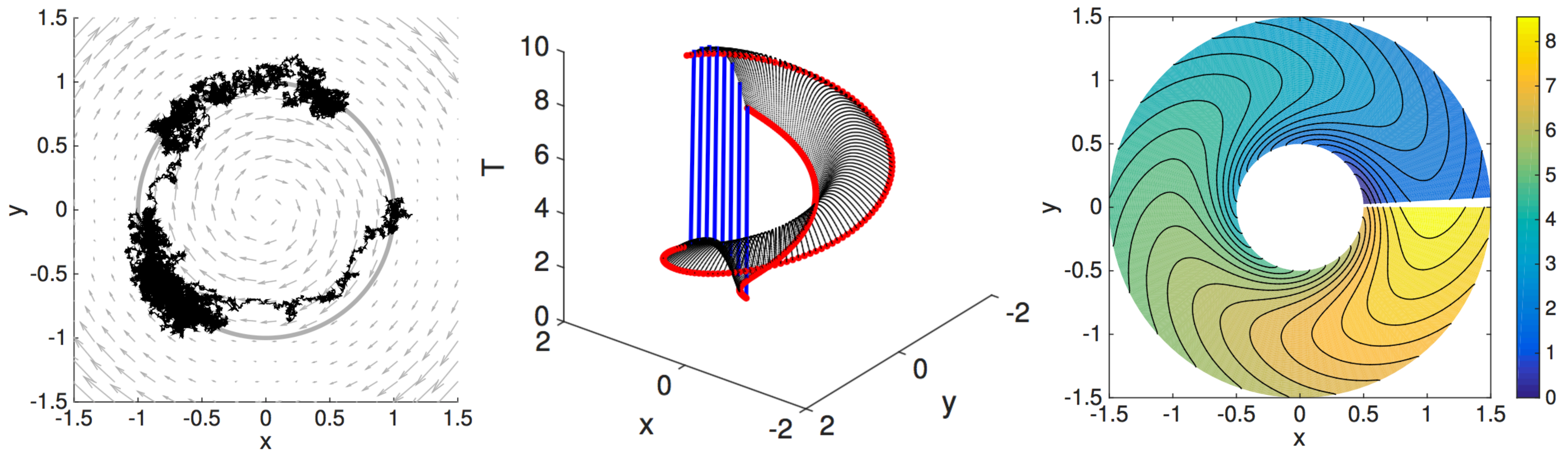
Newby and Schwemmer's "Antirotating" Oscillator

$$\dot{x} = -\omega y + \gamma x(1-\rho^2) + c\gamma y Q(\rho) + \sqrt{2D}\xi_x(t)$$

$$\dot{y} = \omega x + \gamma y(1-\rho^2) - c\gamma x Q(\rho) + \sqrt{2D}\xi_y(t)$$

$$\rho = \sqrt{x^2+y^2}, \quad Q_1(\rho) = \rho^2 - 1$$

Effects of moderate noise on a limit cycle oscillator: counterrotation and bistability.
Newby and Schwemmer. Phys. Rev. Lett., 2014



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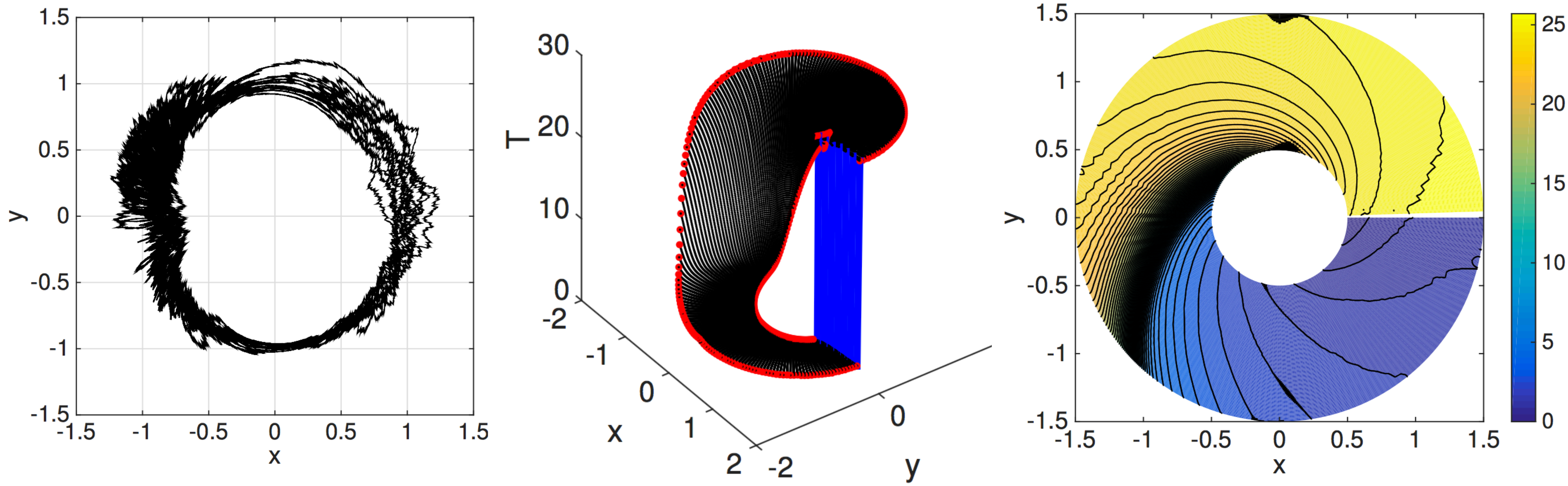
Stuart-Landau perturbed by y-polarized noise



$$\dot{r} = f + \sigma r \cos\theta \xi(t), \quad \dot{\theta} = g + \sigma \sin\theta \xi(t)$$

$$f = r(1 - r^2) + \frac{r\sigma^2}{2}\left(\cos^2\theta - \sin^2\theta\right)$$

$$g = \omega + r\cos\theta - \kappa r^2 + \frac{\sigma^2}{2}\cos\theta\sin\theta.$$

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Conclusions II

- Schwabedal and Pikovsky (2013) suggested defining isochrons as sections with uniform mean first return times.
- We interpret such "average isophase" isochrons as level curves of a MFPT function $T(x)$ satisfying a 2nd order elliptic PDE $\mathcal{L}^\dagger[T] = -1$ with a jump boundary condition. This formulation solves both the geometry *and* timing of the isochrons.
- Rigorous existence and uniqueness theory remains to be established, but numerical implementation (finite difference) appears robust.

Defining the Phase for Stochastic Oscillators

Does $T(\mathbf{x}) = (\psi_0 - \psi(\mathbf{x})) \cdot \bar{T}/2\pi$? That is, are the average isophase function $T(\mathbf{x})$ and the asymptotic phase $\psi(\mathbf{x})$ obtained from the slowest decaying complex eigenfunction $Q(\mathbf{x}) = u(x)e^{i\psi(\mathbf{x})}$ of $\mathcal{L}^\dagger$ equivalent?

- From Ito, $E[dT(\mathbf{x})] = \mathcal{L}^\dagger[T(\mathbf{x})] = -1 = \text{const}$. Similarly

$$E[d\psi(\mathbf{x})] = \mathcal{L}^\dagger[\psi(\mathbf{x})] = \frac{2\pi}{\omega} + \sum_{ij} B_{ij}(\partial_i \ln u)(\partial_j \psi) \neq \text{const.}$$

- Lemma 1 (Cao 2017): For additive isotropic noise $B_{ij} = \delta_{ij}$, if the eigenfunction Q is complex analytic, then average isophase and stochastic asymptotic phase are equivalent.
- Lemma 2 (Cao 2017): For a 1D oscillator,

$$dX = f(X)\,dt + \sqrt{2D(X)}\,dW(t), \quad X \in [0, 2\pi)$$

if ψ and T coincide then D and f are both constant.

Bibliography & Acknowledgments

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