Dynamics of Clustering in Networks with Repulsive Interaction

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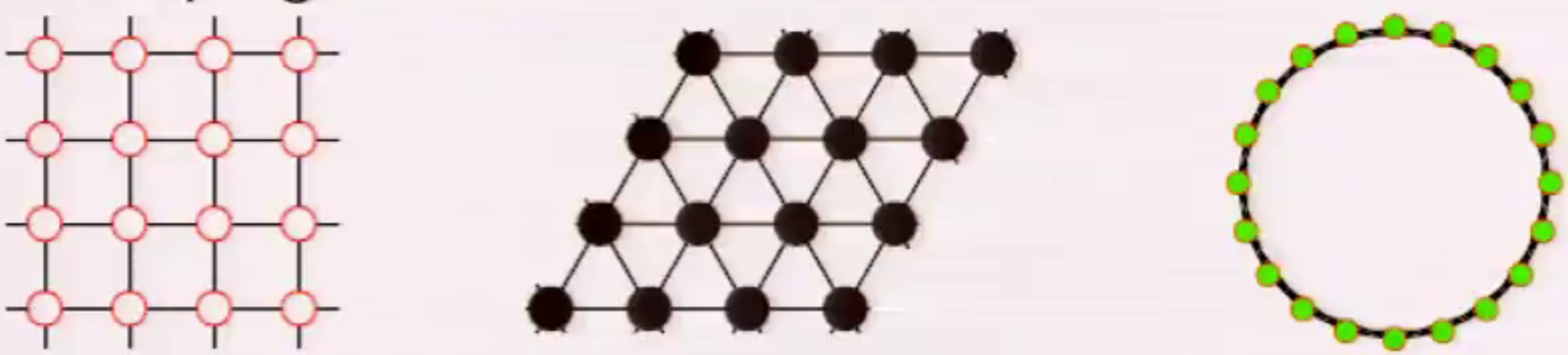
Network of N *identical* one-dimensional elements.

$$\frac{d\varphi_i}{dt} = f(\varphi_i) + \kappa \sum_j A_{ij}\, g(\varphi_i, \varphi_j)$$

with adjacency matrix A_{ij}.

- Global coupling: synchronization,
  in case of coupling through the first Fourier harmonics:
  Strogatz-Watanabe phenomenon
  (existence of N-3 conserved quantities).
- Local coupling: lattices



- Clustering, splay states ...

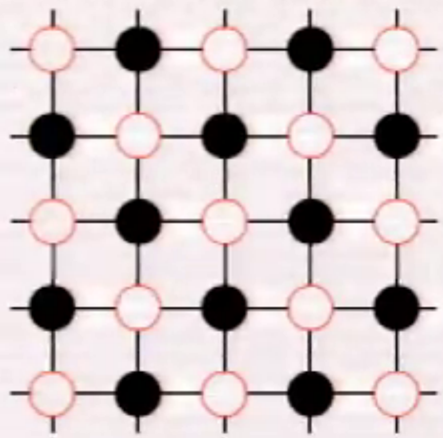
- Working elements: coupled "active rotators" (Kuramoto & Shinomoto, 1979):

$$\frac{d\varphi_i}{dt} = \omega - \sin\varphi_i + \kappa \sum_{j}^{N} A_{ij} \sin(\varphi_j - \varphi_i)$$

- Every unit is connected to M other elements. *(network is a regular graph with degree M).*
- Gradient dynamics: no small-scale oscillations (full-scale rotations along at least one coordinate)
  $\Rightarrow$ no Hopf bifurcations.
- Consider "excitable case": $\omega < 1$.
  For all positive $\kappa > 0$, the synchronous steady state S_0: $\phi_i = \arcsin(\omega)$ $\forall i$, is stable.

- Negative coupling: $\kappa < 0$: neighbors in counterphase are favored
- e.g. checkerboard pattern on a square lattice
  with periodic boundary conditions and even sizes

- But not if at least one of the sizes is odd!



- For a triangular motif (e.g. of a hexagonal lattice)
  at least one link is always frustrated!

- Negative coupling: $\kappa < 0$.
- Steady state S_0 is stable for:

$$\kappa > \kappa_c = -\frac{\sqrt{1-\omega^2}}{M - \lambda_{\min}},$$

where $\lambda_{\min}$ is the minimal eigenvalue of the adjacency matrix A_{ij}.

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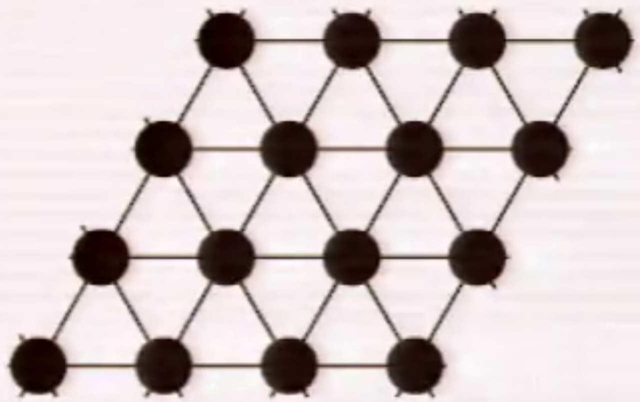
- Empirically: $\lambda_{\min}$ is degenerate (sometimes with rather high multiplicity) for all checked regular graphs.
- Example: hexagonal lattice of L_1 rows and L_2 columns with periodic boundary conditions:

  For $L_1 \neq L_2$: multiplicity equals 2.

  For $L_1 = L_2 = 3k$: multiplicity equals 2.

  For $L_1 = L_2 \neq 3k$: multiplicity equals 6, except

  for $L_1 = N_2 = 4$: multiplicity equals 9.



Shrikhande graph (1959).

Discrete symmetries: translations, rotations, reflections.

$\lambda_{\min} = -2.$

9-dimensional central manifold at κ_c.

What happens at κ_c?

(saddle-node, pitchfork, transcritical, . . .)

- Working elements: N globally coupled active rotators

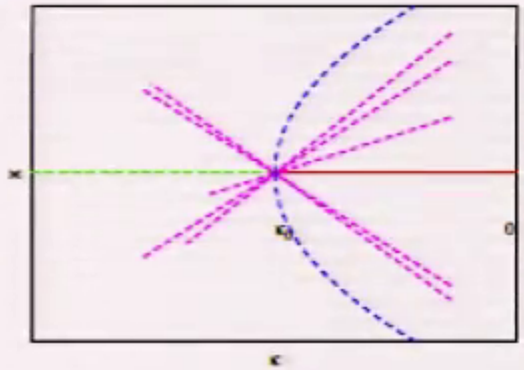
$$\frac{d\varphi_i}{dt} = \omega - \sin\varphi_i + \kappa \sum_{j}^{N} \sin(\varphi_j - \varphi_i)$$

- Synchronous equilibrium $\varphi_i = \arcsin\omega$, $i = 1, \ldots, N$ is stable for

$$\kappa > \kappa_0 = -\frac{\sqrt{1-\omega^2}}{N}$$

- the critical eigenvalue has multiplicity $N-1$.

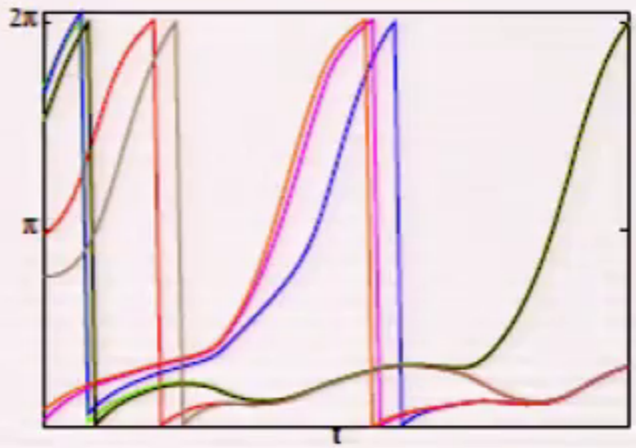
N globally coupled active rotators



- For odd N: $2^{N-1} - 1$ transcritical branches;
- For even N: $2^{N-1} + N!/\left(2(\frac{N}{2}!)^2\right) - 1$ branches,

  whereof $N!/\left((\frac{N}{2}!)^2\right)$ form subcritical pitchforks.
- All saddles with $\dim(W_s) > \dim(W_u)$ lie on transcritical branches;
  All saddles with $\dim(W_s) = \dim(W_u)$ lie on pitchforks.
- For $\kappa < k_0$: no stable equilibria.

For sufficiently negative values of κ, numerical integration shows, that initially distinct and broadly scattered phase values tend to formation of *several oscillating groups*, inside which the values (nearly) coincide.



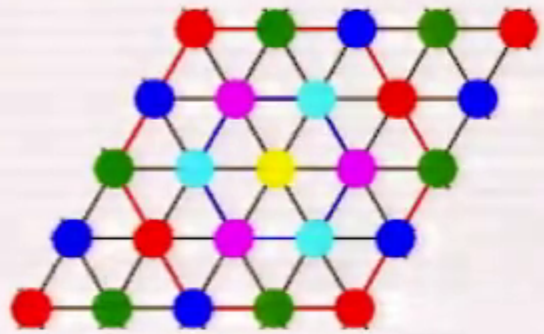
henceforth I refer to such groups as clusters
(dynamical clusters, state clusters...)

Members of the same cluster are not necessarily neighbors on the lattice.

Example: fix $\omega = 0.7 \quad \rightarrow \quad \kappa_c = -0.089267\ldots$.

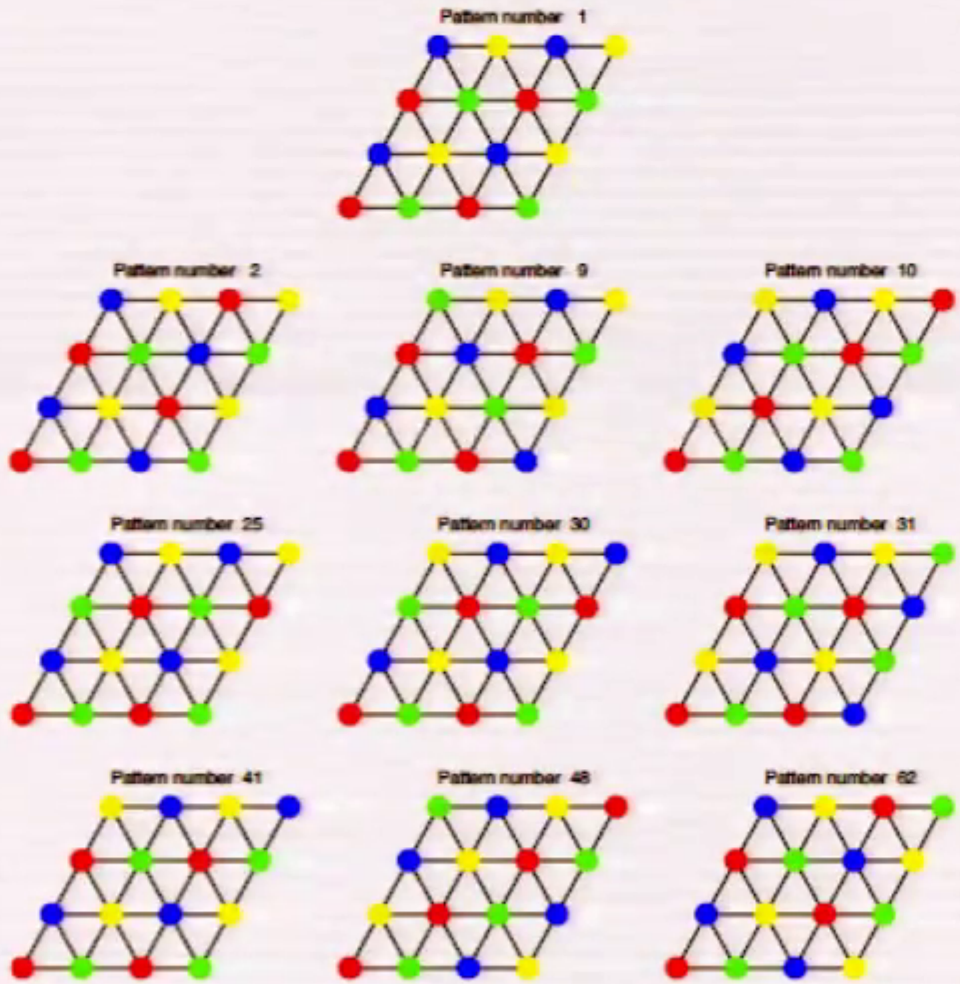
Take $\kappa = -1/3$.

- From 10^5 initial conditions in the hypercube $[0 : 2\pi]^{16}$, 34.87% converge to the limit cycle: the pattern with 6 clusters

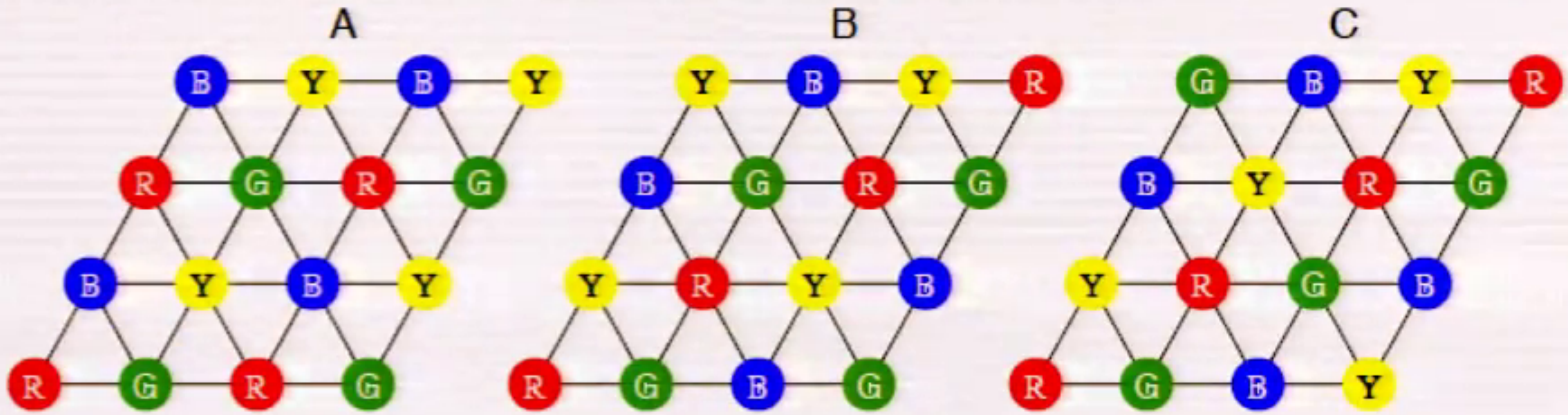


  and its symmetric images (altogether 192).
- 2.17% converge to the limit cycle without clustering: pattern of 16 non-coinciding rotators and its symmetric images.
- 0.3% converge to quasiperiodic oscillations without clustering (smooth curves on the Poincaré hyperplane).
- The rest (63%) converges to periodic oscillations in one of the spatial patterns with 4 clusters.

- ...The rest (63%) converges to periodic oscillations in one of the spatial patterns with 4 clusters.
- There seems to be no distinct value of period: within precision 10^{-4}, over 4×10^{4} different values are resolved.
- Spatial arrangement in these clustering patterns:


Pattern number 1
Pattern number 2
Pattern number 9
Pattern number 10
Pattern number 25
Pattern number 30
Pattern number 31
Pattern number 41
Pattern number 48
Pattern number 62

- Spatial arrangement in these clustering patterns:



- In each configuration, every element is connected to two rotators from every other cluster.
- ⇒ Every cluster is connected to every other cluster with the same coupling strength.
- This is spontaneous onset of global coupling: units are coupled locally, but clusters are coupled globally.

- Introducing collective variables, we arrive at a set of 4 globally coupled identical oscillators.
- $\Rightarrow$ Strogatz-Watanabe phenomenon, 4-3=1 conserved quantity

- Introducing collective variables, we arrive at a set of 4 globally coupled identical oscillators.
- $\Rightarrow$ Strogatz-Watanabe phenomenon, 4-3=1 conserved quantity
- $\Rightarrow$ 1-parameter continuous family of periodic solutions
- Integral:

$$I = \frac{\sin\frac{\phi_1-\phi_2}{2}\sin\frac{\phi_3-\phi_4}{2}}{\sin\frac{\phi_1-\phi_3}{2}\sin\frac{\phi_2-\phi_4}{2}}$$

- The same reduced system of the 4th order (with rescaled coupling) stands behind different configurations of clusters.
- The very same periodic solution of reduced system can be arranged on the lattice in different spatial ways, and have different stability with respect to the cluster-splitting perturbations. (Lou Pecora, yesterday: *"desynchronizing peturbations"*)

- The family of periodic solutions is born at κ_c in a highly degenerate global bifurcation.
- Recall case of global coupling:

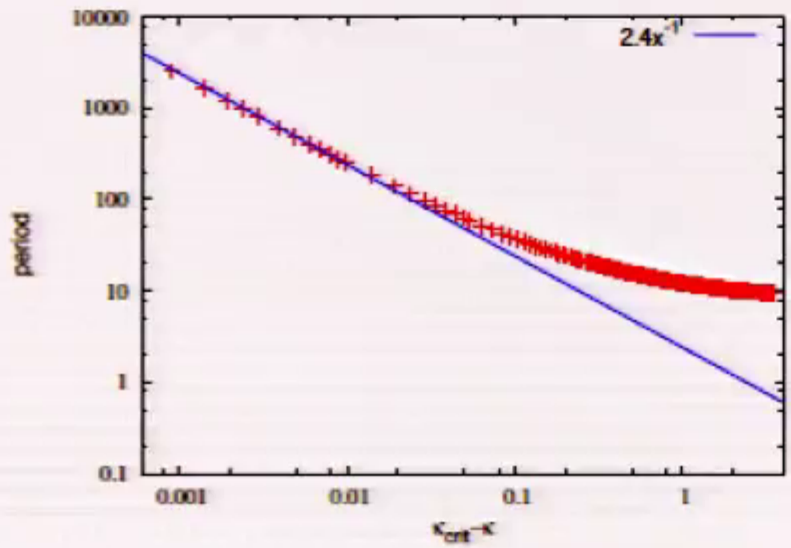
at κ_c the symmetric equilibrium collides with 10 saddles

in the invariant subspace of . . . for every 4-cluster pattern

with all its translations, rotations and reflections.

- The family of periodic solutions is born at κ_c in a highly degenerate global (transcritical heteroclinic) bifurcation.
- Near κ_c the whole newborn continuum of periodic solutions is unstable (with respect to splitting of clusters), but further decrease of κ leads to stabilization of its segments.
- Asymptotics of *(range of)* periods near κ_c:

$$T \sim \frac{1}{\kappa_c - \kappa}$$



Definitions of Peter Ashwin:

The oscillators are *indistinguishable* if they are identical and interchangeable in the sense that they have the same number and strength of inputs.

Oscillators i and j on a trajectory are *frequency synchronized* if

$$\Omega_{ij} := \lim_{T\to\infty} \frac{1}{T}[\theta_i(T) - \theta_j(T)] = 0$$

and the trajectory is frequency synchronized if $\Omega_{ij} = 0$ for all $i \neq j$.

> **A is a *chimera state* if it is a compact recurrent invariant set such that trajectories within A are not frequency synchronized.**

- Steady checkerboard pattern is not the only attractor of the square lattice at negative values of κ.
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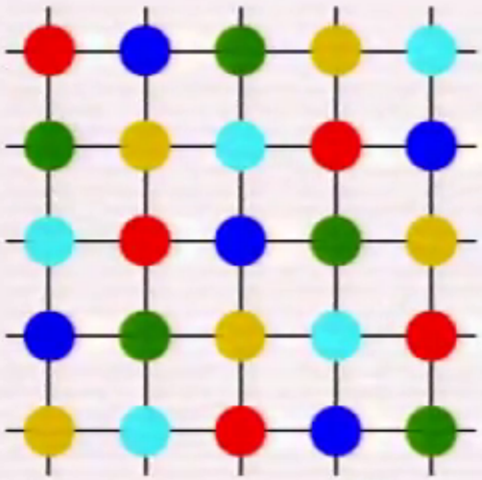
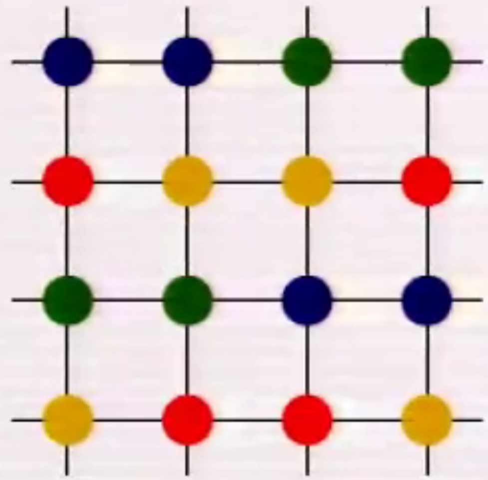


5 clusters

2 integrals



4 clusters

1 integral

note a neighbor from the same cluster!

- Similar effect takes place on a ring with N rotators, where each unit repulsively interacts with M neighbors.

5 clusters

2-parameter continuous family of periodic solutions

- For the effect to take place:
- Number of clusters K should be a divisor of N
- Every element interacts with equal number of elements from every other cluster (except the cluster to which it belongs itself).

  The number n of connections between the element and other elements of the *same* cluster can be anything from 0 to $M+1-K$.
- Hence, K=1+div(M-n) (1+ any of the integer divisors of M-n).

  Integrals, continuous families of solutions etc. arise at K>3.
- For a general hexagonal coupling (M=6) this refers to:

  K=7 at n=0, K=6 at n=1,

  K=5 at n=2, K=4 at n=0 and n=3.
- For the square coupling (M=4) there are just two possibilities:

  K=5 at n=0, and K=4 at n=1.

Summarizing:

- In a network of repulsively coupled identical units on a regular graph with properly related degree and the number of nodes,
- the elements can spontaneously group into clusters of equal size,
- so that every cluster interacts with equal intensity with every other cluster.
- If, additionally, the coupling is restricted to the 1st Fourier harmonics, conserved quantities, and, as a consequence, continuous families of solutions can be encountered.