A Lagrangian Approach to Turbulent Rayleigh-Bénard Convection

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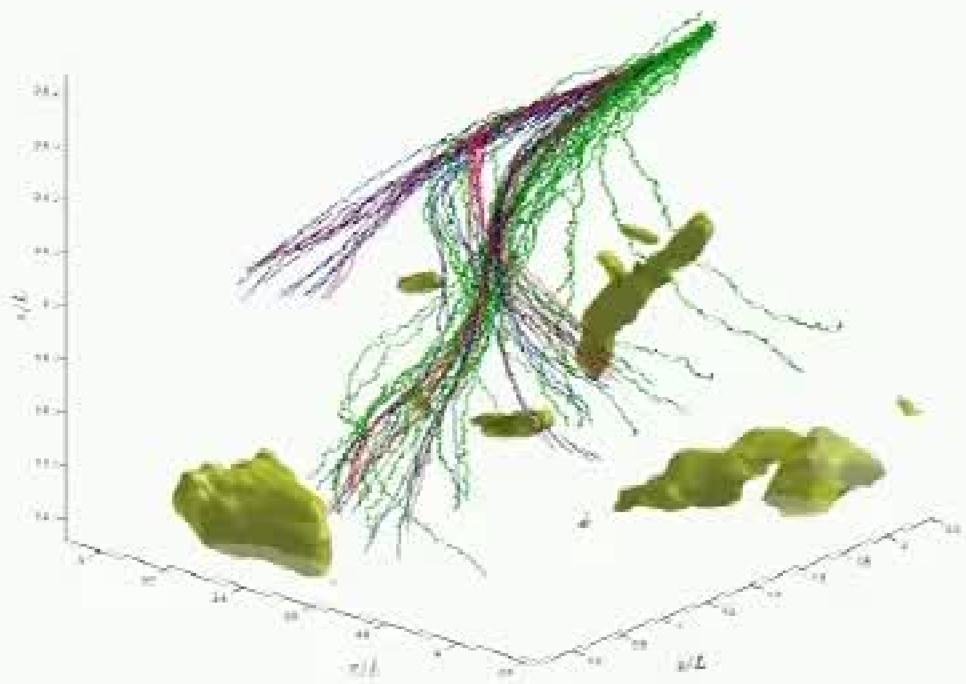
FEYNMAN-KAC REPRESENTATION

$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta + S, \qquad \theta\big|_{t=0} = \theta_0.$$

Representation using stochastic Lagrangian trajectories:

$$\hat{\mathrm{d}}_s\widetilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}) = \mathbf{u}(\widetilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\mathrm{d}s + \sqrt{2\kappa}\,\hat{\mathrm{d}}\mathbf{W}_s, \quad \widetilde{\boldsymbol{\xi}}_{t,t}(\mathbf{x}) = \mathbf{x}$$

$$\theta(\mathbf{x},t) = \mathbb{E}\left[\theta_0(\bar{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t ds\, S(\bar{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\right]$$



Example trajectories calculated in an isotropic Navier-Stokes flow from JHU turbulence database (`http://turbulence.pha.jhu.edu`) for with $\kappa = \nu/Pr$ for $Pr = 10^{-1}$, $Pr = 1$ and $Pr = 10$.

FLUCTUATION-DISSIPATION RELATION

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta + S \qquad \theta\big|_{t=0} = \theta_0$$

By the backward Itô lemma:

$$\begin{aligned} \hat{\mathrm{d}}_s \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s) &= [(\partial_s + \mathbf{u} \cdot \nabla - \kappa \Delta)\theta](\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\mathrm{d}s + \sqrt{2\kappa}\, \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s) \\ &= S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\mathrm{d}s + \sqrt{2\kappa}\, \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s), \end{aligned}$$

Or, introducing a stochastic scalar field:

$$\tilde{\theta}(\mathbf{x}, t) = \theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t ds\, S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)$$

Using the Feynman-Kac representation: $\theta(\mathbf{x}, t) = \mathbb{E}\left[\tilde{\theta}(\mathbf{x}, t)\right]$, we have:

$$\tilde{\theta}(\mathbf{x}, t) - \mathbb{E}\left[\tilde{\theta}(\mathbf{x}, t)\right] = \sqrt{2\kappa} \int_0^t \hat{\mathrm{d}}\mathbf{W}_s \cdot \nabla \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)$$

An application of Itô isometry yields our *local fluctuation-dissipation relation*:

$$\mathrm{Var}\left[\tilde{\theta}(\mathbf{x}, t)\right] = 2\kappa \int_0^t ds\, \mathbb{E}\left[|\nabla \theta(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)|^2\right]$$

LOCAL FLUCTUATION-DISSIPATION RELATION

$$\frac{1}{2}\mathrm{Var}\left[\bar{\theta}(\mathbf{x},t)\right] = \kappa\int_0^t ds\,\mathbb{E}\left[|\nabla\theta(\bar{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)|^2\right]$$

At **short times**, we recover the local scalar dissipation:

$$\lim_{t\to 0}\frac{1}{2t}\mathrm{Var}\left[\tilde{\theta}(\mathbf{x},t)\right] = \kappa|\nabla\theta(\mathbf{x},0)|^2.$$

At **long times**, we prove the local variance becomes independent of space:

$$\lim_{t\to\infty}\frac{1}{2t}\mathrm{Var}\left[\tilde{\theta}(\mathbf{x},t)\right] = \left\langle\kappa|\nabla\theta(\mathbf{x},s)|^2\right\rangle_{\Omega,\infty},$$

where the latter quantity is infinite-time and space average of the dissipation:

$$\left\langle\kappa|\nabla\theta|^2\right\rangle_{\Omega,\infty} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}s\,\frac{1}{V}\int_\Omega \mathrm{d}\mathbf{x}\,\kappa|\nabla\theta(\mathbf{x},s)|^2$$

which exists, along subsequences, provided $\left\langle\kappa|\nabla\theta|^2\right\rangle_\Omega$ remains bounded. This follows for $\kappa>0$, the be ergodicity (and incompressibility) of the processes $\bar{\boldsymbol{\xi}}_{t,s}(\mathbf{x})\in\Omega$ in time s for each fixed $\mathbf{x}\in\Omega$.

GLOBAL FLUCTUATION-DISSIPATION RELATION

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,0}(\mathbf{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,s}(\mathbf{x}), s)\, \mathrm{d}s\right]\right\rangle_\Omega = \kappa \int_0^t ds \left\langle |\nabla \theta^{\nu,\kappa}(s)|^2 \right\rangle_\Omega.$$

Balance between scalar dissipation and the input of scalar fluctuations from the initial scalar field and the scalar sources, as sampled by backwards trajectories.

Recovers Furutsu-Novikov relation when scalar source is a random zero-mean field, delta-correlated in time:

$$\langle \kappa |\nabla \theta(\mathbf{x}, s)|^2 \rangle_{\Omega,\infty,S} = \frac{1}{V}\int_\Omega \mathrm{d}\mathbf{x}\; C_S(\mathbf{x}, \mathbf{x}).$$

Also, for random isotropic uniform gradient initial scalar $\tilde{\theta}_0(\mathbf{x}) = \tilde{\mathbf{G}} \cdot \mathbf{x}$,

$$\kappa \int_0^t ds \left\langle |\nabla \tilde{\theta}(\mathbf{x}, s)|^2 \right\rangle_\Omega = \frac{1}{4} G^2 \mathbb{E}^{1,2} \left\langle \left| \tilde{\boldsymbol{\xi}}^1_{t,0}(\mathbf{x}) - \tilde{\boldsymbol{\xi}}^2_{t,0}(\mathbf{x}) \right|^2 \right\rangle_\Omega$$

where the $1, 2$ averages are taken over two independent ensembles of Brownian motion [Sawford et al. 05].

FLUCTUATION-DISSIPATION RELATION WITH NO WALLS

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,0}(\mathbf{x})) + \int_0^t S(\tilde{\boldsymbol{\xi}}^{\kappa,\nu}_{t,s}(\mathbf{x}), s)\,\mathrm{d}s\right]\right\rangle_\Omega = \kappa\int_0^t ds\left\langle|\nabla\theta^{\nu,\kappa}(s)|^2\right\rangle_\Omega.$$

THEOREM: Spontaneous stochasticity is necessary and sufficient for anomalous dissipation of passive scalars. Necessary for active scalars!

Idea of Proof: A subsequence $\nu_k = \nu_{j_k}$ can be selected together with a corresponding subsequence $\kappa_k \to 0$, so that, e.g.

$$\lim_{k\to\infty}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}^{\nu_k,\kappa_k}_{t,0}(\mathbf{x}))\right]\right\rangle_\Omega = \int d^dx\int d^dx_0\int d^dx_0'\;\theta_0(\mathbf{x}_0)\theta_0(\mathbf{x}_0')$$
$$\times\left[p_2^*(\mathbf{x}_0,0;\mathbf{x}_0',0|\mathbf{x},t) - p^*(\mathbf{x}_0,0|\mathbf{x},t)p^*(\mathbf{x}_0',0|\mathbf{x},t)\right],$$

for all $\theta_0 \in C(\Omega)$, where $p_2^*(\mathbf{x}_0,0,\mathbf{x}_0',0|\mathbf{x},t) \equiv \delta^d(\mathbf{x}_0-\mathbf{x}_0')p^*(\mathbf{x}_0,0|\mathbf{x},t)$.

Note p^* and p_2^* are Young measures on Ω and $\Omega\times\Omega$ respectively, measurably indexed by elements $\mathbf{x}$ of Ω, since they are narrow limits of the Young measures p^{ν_k,κ_k} and $p_2^{\nu_k,\kappa_k}$.
Non-factorization $\Longrightarrow$ spontaneous stochasticity.

Other direction: use Stone-Weierstrass to find $\theta_0 \in C^\infty(\Omega)$ giving variance > 0.

WALL BOUNDED FLOWS WITH IMPOSED SCALAR FLUX

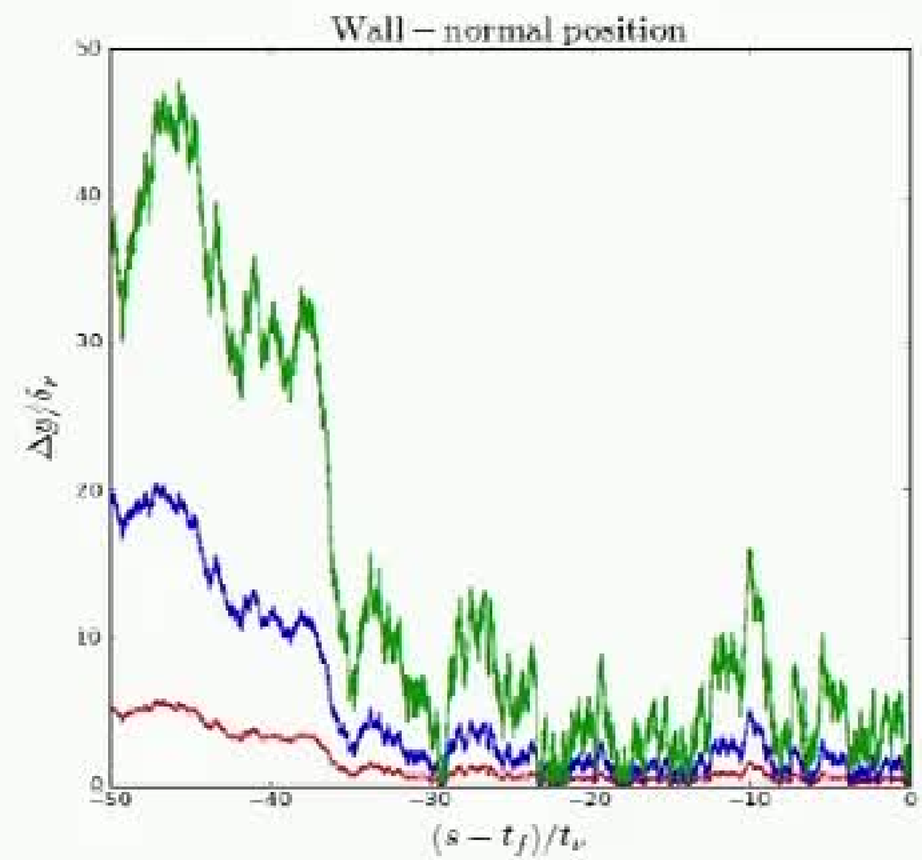
$$\partial_t \theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta + S \qquad \text{for} \quad \mathbf{x}\in\Omega,$$
$$-\kappa\frac{\partial\theta}{\partial n} = g \qquad \text{for} \quad \mathbf{x}\in\partial\Omega.$$

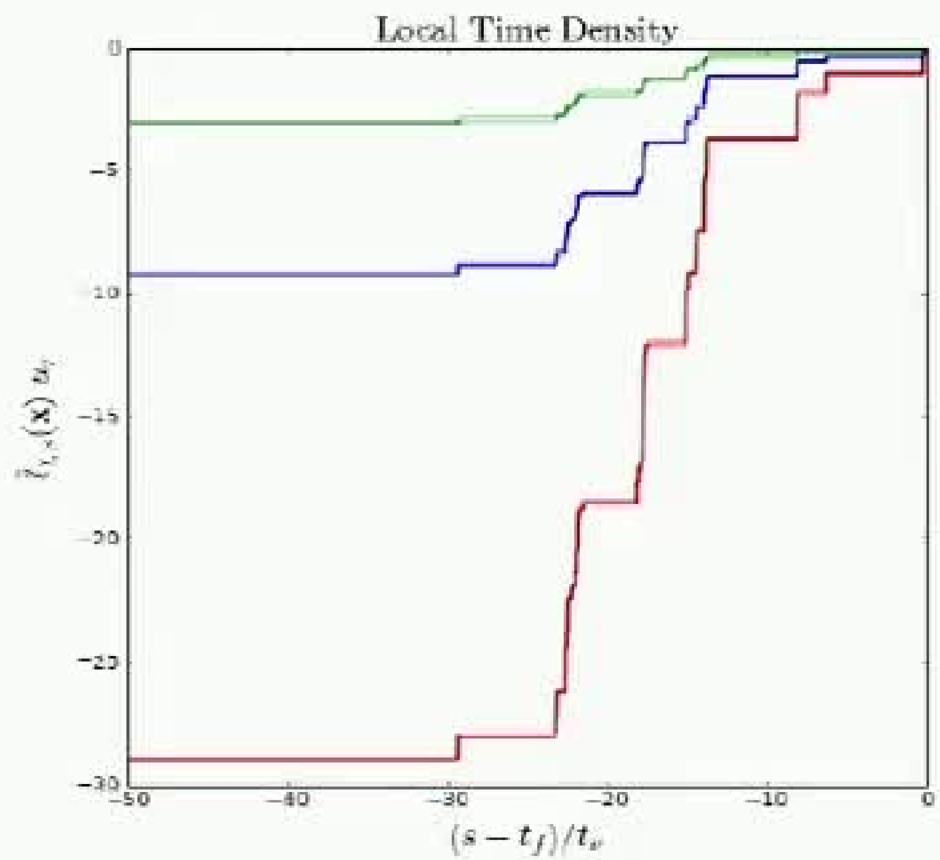
Define stochastic trajectories which reflect off the boundary of the domain:

$$\hat{\mathrm{d}}\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}) = \mathbf{u}^{\nu}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\,\mathrm{d}s + \sqrt{2\kappa}\,\hat{\mathrm{d}}\mathbf{W}_s - \kappa\boldsymbol{n}(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\,\hat{\mathrm{d}}\tilde{\ell}_{t,s}(\mathbf{x})$$

where the boundary local time (a time per unit length) is then defined by

$$\tilde{\ell}_{t,s}(\mathbf{x}) = \int_t^s dr\,\delta(\mathrm{dist}(\tilde{\boldsymbol{\xi}}_{t,r}(\mathbf{x}), \partial\Omega)) \equiv \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^s dr\,\chi_{\partial\Omega_\varepsilon}(\tilde{\boldsymbol{\xi}}_{t,r}(\mathbf{x})), \qquad s<t,$$





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Feynman-Kac formula:

$$\theta(\mathbf{x},t) = \mathbb{E}\left[\theta_0(\bar{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\bar{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s) + \int_0^t g(\bar{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\,\hat{\mathrm{d}}\bar{\ell}_{t,s}\right].$$

Fluctuation dissipation relation:

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\bar{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\bar{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s) + \int_0^t g(\bar{\boldsymbol{\xi}}_{t,s}(\mathbf{x}),s)\hat{\mathrm{d}}\bar{\ell}_{t,s}\right]\right\rangle_\Omega = \kappa\int_0^t \mathrm{d}s\,\langle|\nabla\theta(s)|^2\rangle_\Omega$$

REMARK: FDR also holds for Dirichlet conditions with $g = -\kappa\frac{\partial\theta}{\partial n}$.

WALL BOUNDED FLOWS WITH ZERO FLUX

For zero flux conditions (stirring milk into coffee) our fluctuation-dissipation relation reads

$$\frac{1}{2}\left\langle \mathrm{Var}\left[\theta_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + \int_0^t \mathrm{d}s\, S(\tilde{\boldsymbol{\xi}}_{t,s}(\mathbf{x}), s)\right]\right\rangle_\Omega = \kappa \int_0^t \mathrm{d}s\, \langle |\nabla\theta(s)|^2 \rangle_\Omega$$

and we have equivalence of anomalous dissipation & spontaneous stochasticity.

WALL BOUNDED FLOWS WITH IMPOSED SCALAR FLUX

For general flux conditions the situation is more complicated. For example, consider the heat equation on $\mathbb{R}^+$ with constant flux J at $x = 0$ and $\theta_0, S = 0$. Local time densities may be explicitly calculated and:

$$\theta(x,t) = -J\,\mathbb{E}[\tilde{\ell}^{x=0}_{t,0}(x)] \sim J\sqrt{\frac{t}{\kappa}}\varphi\left(\frac{x}{\sqrt{\kappa t}}\right)$$

for a suitable scaling function φ. Scalar boundary layer of thickness $\sim \sqrt{\kappa t}$ near $x = 0$ where the field diverges as $\sim J\sqrt{t/\kappa}$. Dissipation is non-vanishing (and divergent!) though there is clearly no spontaneous stochasticity:

$$\langle \kappa |\nabla\theta(x,t)|^2\rangle_\Omega \sim J^2\sqrt{\frac{t}{\kappa}} \stackrel{\kappa\to 0}{\longrightarrow} \infty!$$

Thin boundary layers near walls provide another mechanism for non-vanishing dissipation!

There is no longer an equivalence between SS and AD, nevertheless our FDR is still valid and can give important information.

Now to an important application...

RAYLEIGH-BÉNARD CONVECTION:

Eulerian equations of motion:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \beta g\, T \hat{\mathbf{z}}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \kappa \Delta T, \qquad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}\big|_{z=\pm H/2} = 0$$

with fixed temperature boundary conditions for the temperature:

$$T\big|_{z=\pm H/2} = T_{top}/T_{bot}$$ models highly conductive plates

or, fixed flux boundary conditions

$$-\kappa \frac{\partial T}{\partial z}\Big|_{z=\pm H/2} = J$$ imposed flux models poorly conducting plates

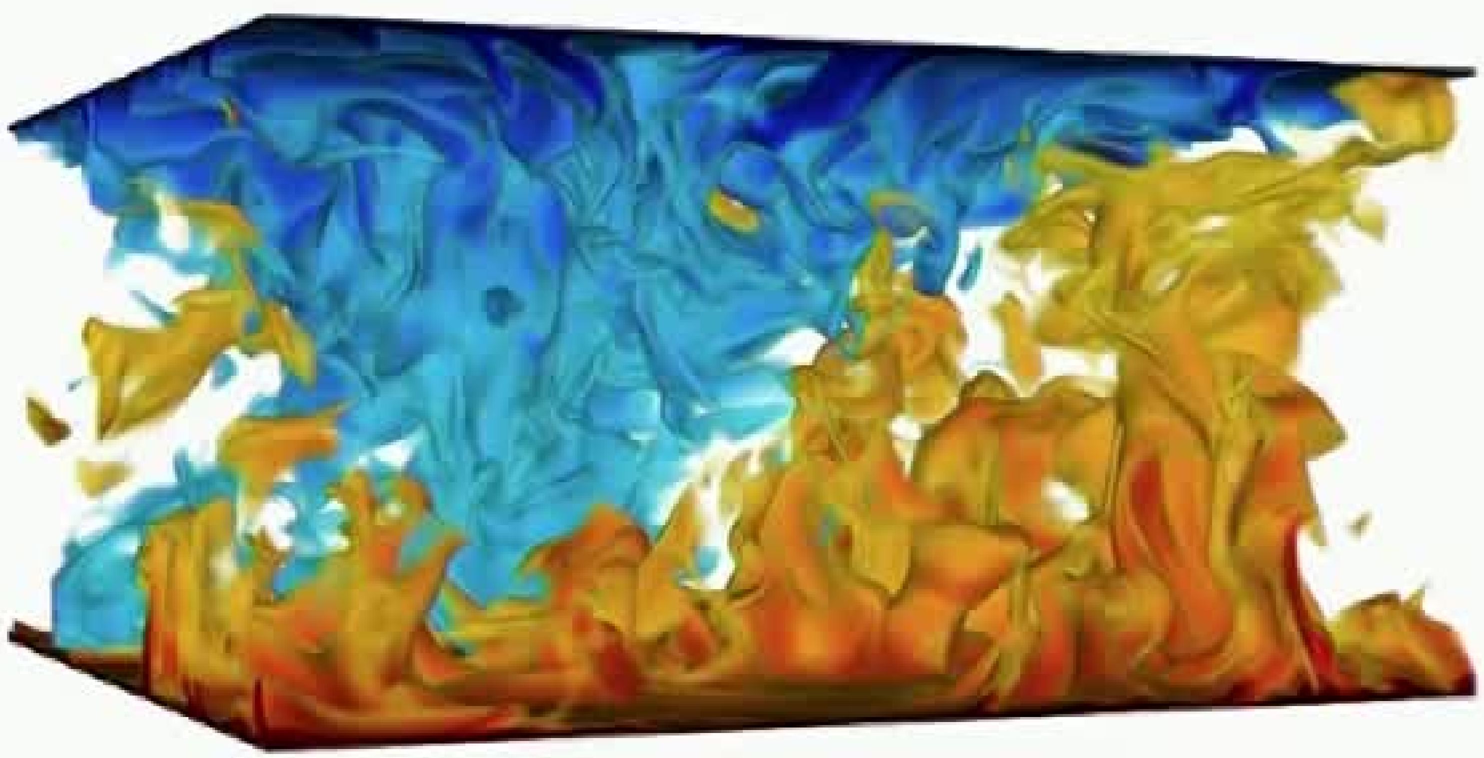


Figure: *fixed temperature RB convection.* Erwin P. van der Poel & Rodolfo Ostilla Manico, Livermore National Laboratories.

ANOMALOUS DISSIPATION AND DIMENSIONAL SCALING THEORIES

Eulerian global balances (identical for temperature-b.c. and flux-b.c.) are:

$$\alpha g\left(J - \frac{\kappa}{H}\Delta T\right) = \nu\langle|\nabla \mathbf{u}|^2\rangle_{V,\infty} = \varepsilon_u,$$

$$\frac{J\Delta T}{H} = \kappa\langle|\nabla T|^2\rangle_{V,\infty} = \varepsilon_T,$$

with the role of J and ΔT (control/response) reversed for the two cases.

Neglecting the vanishing $\frac{1}{Nu}$ correction term in the energy balance,

$$\frac{\varepsilon_u}{U^3/H} = \frac{\varepsilon_T}{(\Delta T)^2 U/H} = \frac{J}{U\Delta T} = \frac{Nu}{\sqrt{Ra\, Pr}}$$

where $U = (\alpha g \Delta T H)^{1/2}$ is the free-fall velocity. Thus, Kraichnan-Spiegel scaling $Nu \sim C \cdot Ra^{1/2} Pr^{1/2}$ holds if and only if at fixed Pr

$$\lim_{\nu,\kappa\to 0} \frac{\varepsilon_u}{U^3/H} = \lim_{\nu,\kappa\to 0} \frac{\varepsilon_T}{(\Delta T)^2 U/H} = C > 0.$$

Lagrangian picture of temperature transport in RB convection?

$$\partial_t T + \mathbf{u}\cdot\nabla T = \kappa \Delta T, \qquad -\kappa \frac{\partial T}{\partial z}\Big|_{z=\pm H/2} = J$$

STOCHASTIC REPRESENTATION: RAYLEIGH-BÉNARD CONVECTION

Local times densities at the top and bottom wall given by

$$\tilde{\ell}^{top}_{t,s}(\mathbf{x}) = \int_t^s dr\, \delta\left(\tilde{\zeta}_{t,r}(\mathbf{x}) - \frac{H}{2}\right), \quad \tilde{\ell}^{bot}_{t,s}(\mathbf{x}) = \int_t^s dr\, \delta\left(\tilde{\zeta}_{t,r}(\mathbf{x}) + \frac{H}{2}\right).$$

Where $\tilde{\boldsymbol{\xi}}_{t,s} = (\tilde{\xi}_{t,s}, \tilde{\eta}_{t,s}, \tilde{\zeta}_{t,s})$. The stochastic representation of the temperature field is:

$$\begin{aligned} T(\mathbf{x}, t) &= \mathbb{E}\left[T_0(\tilde{\boldsymbol{\xi}}_{t,0}(\mathbf{x})) + J\left(\tilde{\ell}^{top}_{t,0}(\mathbf{x}) - \tilde{\ell}^{bot}_{t,0}(\mathbf{x})\right)\right] \\ &= \int d^3x_0\; T_0(\mathbf{x}_0)\; p(\mathbf{x}_0, 0|\mathbf{x}, t) \\ &\quad - J\int_0^t ds\, p_z(H/2, s|\mathbf{x}, t) + J\int_0^t ds\, p_z(-H/2, s|\mathbf{x}, t), \end{aligned}$$

with the conditional probability density for the z-component of the position:

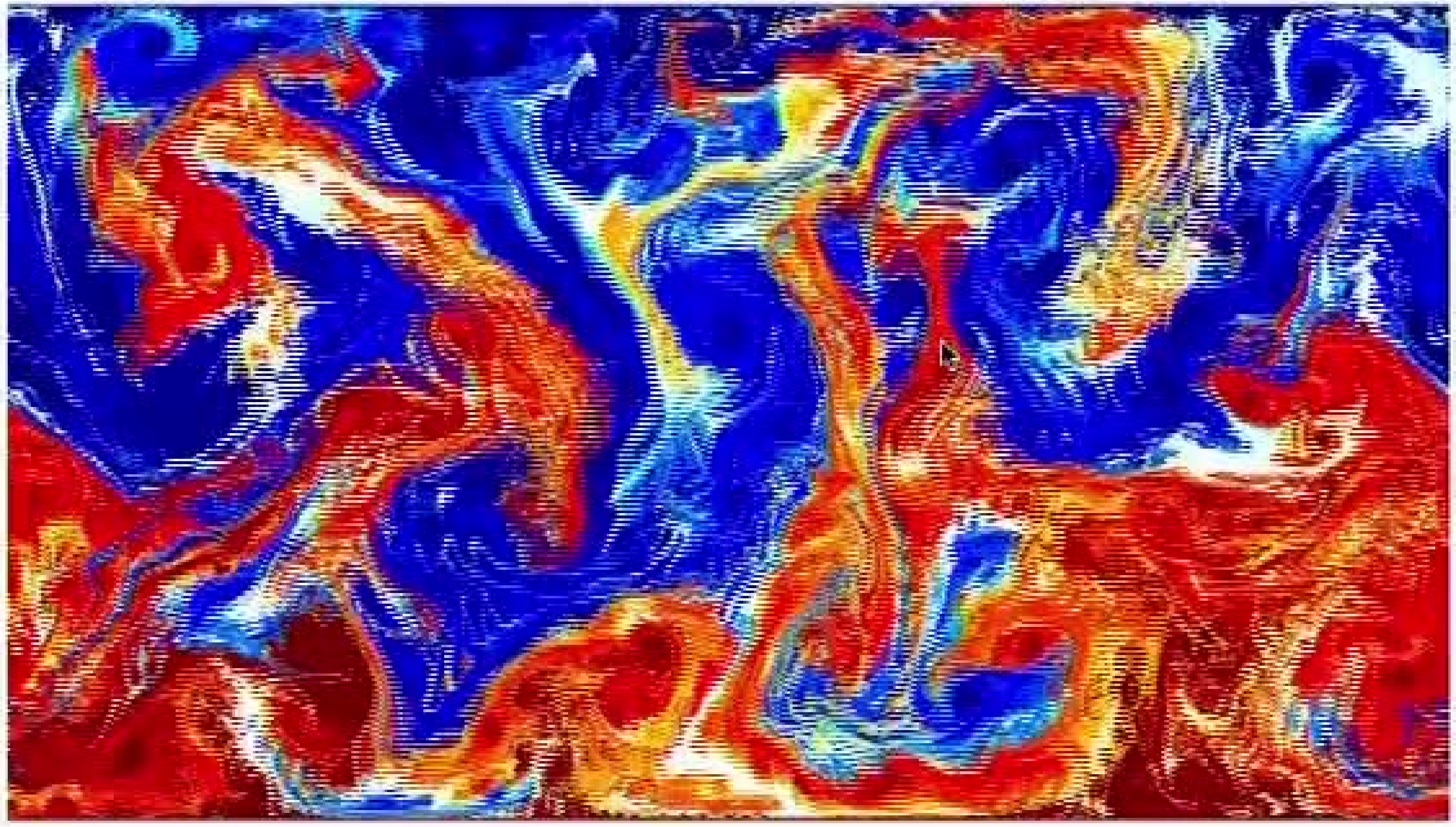
$$p_z(z', s|\mathbf{x}, t) = \iint_S dx'\, dy'\; p(x', y', z', s|\mathbf{x}, t) = \mathbb{E}\left[\delta\left(z' - \tilde{\zeta}_{t,r}(\mathbf{x})\right)\right].$$

Only get a heat input when the particle hits the wall, so we only need the probability density of particle being at the top or bottom wall.

HOW TO MEASURE MIXING TIME

$$\tau_{mix} = \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{bot} - 1) + \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{top} - 1)$$
$$- \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{bot} - 1) - \int_{-\infty}^{0} dt\,(H\langle c(t)\rangle_{top} - 1)$$

Where concentration $\boldsymbol{c}$ solves $\partial_t c + \mathbf{u} \cdot \nabla c = -\kappa \Delta c$ and $\lim_{t \to -\infty} \langle c(t) \rangle_{top/bot} = 1/H$.



MODIFIED FROM:
Two-dimensional convection simulation with $\mathtt{res} = 7680 \times 4320$, $Ra = 10^{13}$, $Pr = 1$, $\Gamma = 16 : 9$.
J. Lülff, M. Wilczek, A. Daitche, 'Turbulence Team Münster' YouTube channel, `http://www.youtube.com/user/turbulenceteamms`, 2012.

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LAGRANGIAN MIXING LENGTH

Let ℓ_T be the distance that a particle can diffuse in the mixing time τ_{mix}:

$$\ell_T = \sqrt{\kappa \tau_{mix}} = H/\sqrt{Nu}.$$

For $Nu \gg 1$ this is much larger than the (outer) thermal boundary layer thickness, or $\ell_T = H/\sqrt{Nu} \gg \delta_T = H/2Nu$. Diffusion of temperature through the thermal boundary layer is not the limiting factor here.

How does ℓ_T compare with the outer kinetic boundary layer thickness $\delta_v = aH/\sqrt{Re}$? *If KS dimensonal scaling holds*, so that $Nu \sim (Ra\, Pr)^{1/2}$, $Re \sim Ra^{1/2}\, Pr^{-1/2}$, then

$$\delta_v/\ell_T \sim (\text{const.}) Pr^{1/2}, \quad \text{independent of } Ra \gg 1$$

However, in all present experiments and simulations $Re \sim Ra^{1/2}$ approximately, but $Nu \sim Ra^x$ with $x < 1/2$. *If the currently observed scaling persists to $Ra \gg 1$ then*

$$\delta_v/\ell_T \sim C(Pr) Ra^{(2x-1)/4} \ll 1, \quad \text{for fixed } Pr \text{ with } Ra \gg 1$$

Under currently observed scaling, τ_{mix} will be much longer than the time for the tracer to mix by pure diffusion across the kinetic & thermal boundary layers!

POSSIBLE TRANSITION TO NEW REGIME (WITH OR WITHOUT KS SCALING)

Define the "Lagrangian mixing zone" as the flow region within distance ℓ_T of the top/bottom walls. If currently observed scaling persists asymptotically for $Ra \gg 1$, then most of the time τ_{mix} will be spent mixing the tracer across the "central region" at distances greater than ℓ_T from these walls.

Why? Presumably the turbulence in the "central region" does not reach to within distance ℓ_T of the wall, so that mixing is slow between the "turbulent core" inside the central region and the Lagrangian mixing zone near the wall.

One expects that turbulence reaches to within distance ℓ_T of the wall when

$$Re_\ell := u(\ell_T)\ell_T/\nu = Re_{crit},$$

where $u(\ell_T)$ is a characteristic velocity at distance ℓ_T from the wall.

[Roche et al. (2010), He et al. (2012)] have claimed to observe a transition to an "ultimate regime" with KS scaling. Based on their data, one can estimate Re_ℓ is of the order of several hundreds at the onset = the order of Re_{crit} where transition is expected.

PUZZLE OF LONG MIXING TIMES

In terms of the free fall time $\tau_{ff} = H/U \approx \tau_{lsc}$:

$$\frac{\tau_{mix}}{\tau_{ff}} = \frac{\sqrt{RaPr}}{Nu}.$$

If Kraichnan-Spiegel scaling is not valid then $\tau_{mix} \gg \tau_{ff}$!

X. He et al. PRL 108, 024502 (2012):

$$Ra = 1.075 \times 10^{15},\ Pr = 0.859,\ Nu = 5631 \Longrightarrow \tau_{mix}/\tau_{free} = 5397.$$

J. J. Niemela & K. R. Sreenivasan, Physica A 315, 203–214 (2002):

$$Ra \sim 10^{15}\ ,\ Pr \sim 2,\quad Nu \sim (0.05)\ Ra^{1/3} \Longrightarrow \tau_{mix}/\tau_{free} \sim 8944.$$

If KS scaling doesn't hold, this ration increases as Ra increases! Why should scalar mixing times be so incredibly large relative to the free-fall time?

Possibilities: Slowing of large-scale circulation so that $\tau_{lsc} \gg \tau_{ff}$, reduced volume and intensity of thermal plumes, etc. [Niemela & Sreenivasan (2002), Emran & Schumacher (2012)], turbulence not reaching sufficiently close to the walls

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