

# Bifurcations and multi-frequency tipping in a periodically forced delay differential equation

**Andrew Keane** and Bernd Krauskopf

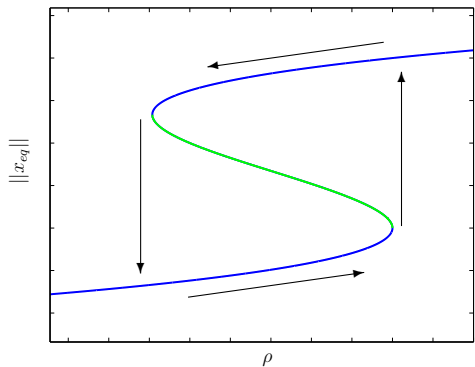
Department of Mathematics, University of Auckland

22nd May 2019



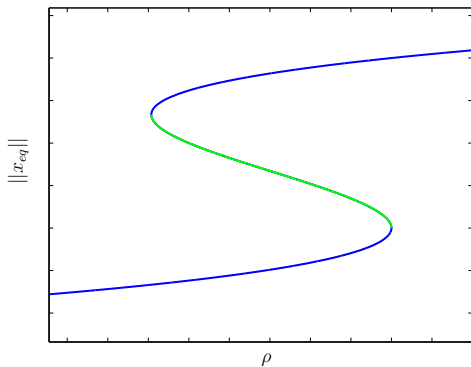
THE UNIVERSITY OF AUCKLAND  
NEW ZEALAND

# Motivation: Hysteresis loops and tipping



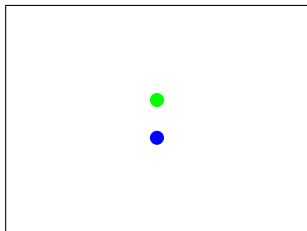
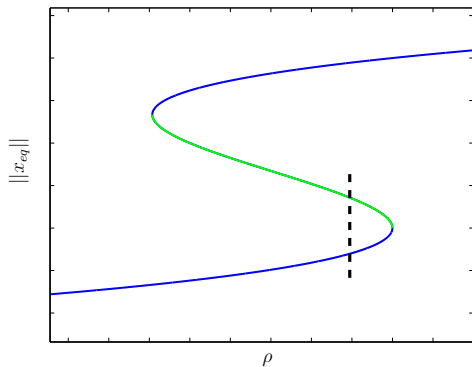
# Motivation: Hysteresis loops and tipping

► Equilibria



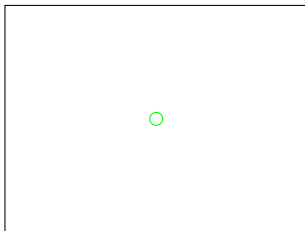
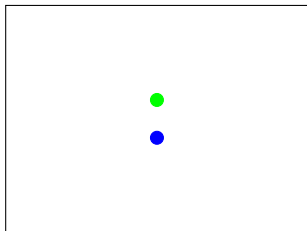
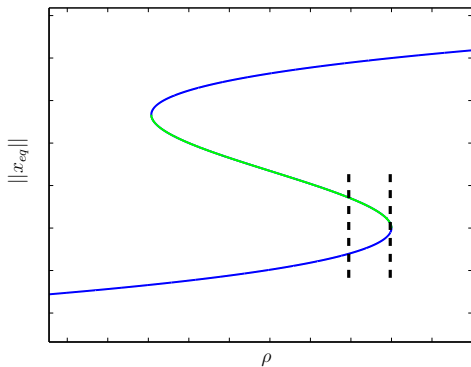
# Motivation: Hysteresis loops and tipping

► Equilibria



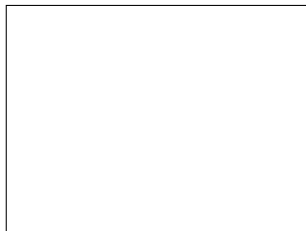
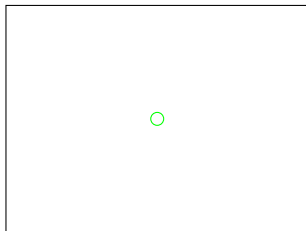
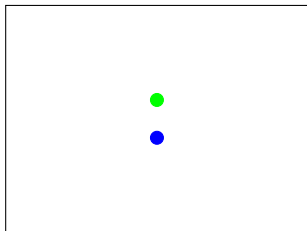
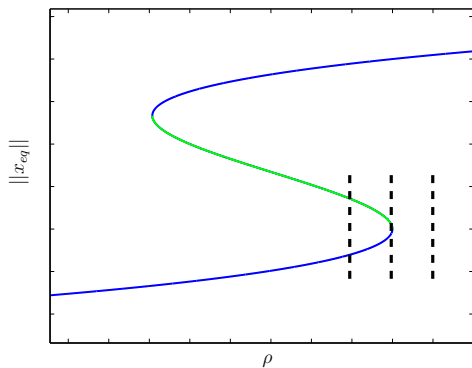
# Motivation: Hysteresis loops and tipping

► Equilibria



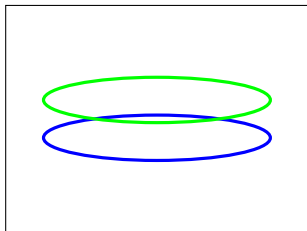
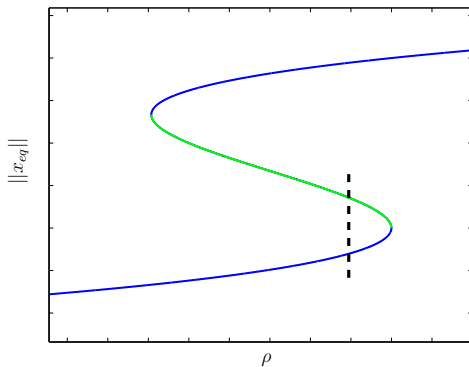
# Motivation: Hysteresis loops and tipping

► Equilibria



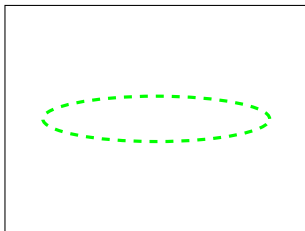
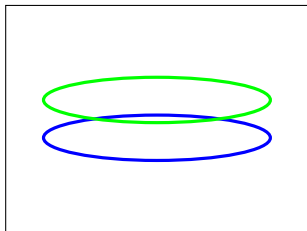
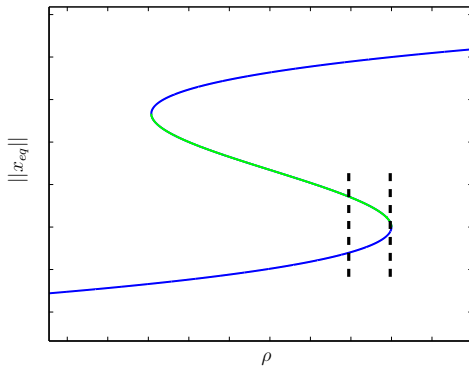
# Motivation: Hysteresis loops and tipping

- ▶ Equilibria
- ▶ Periodic orbits



# Motivation: Hysteresis loops and tipping

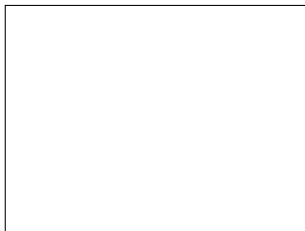
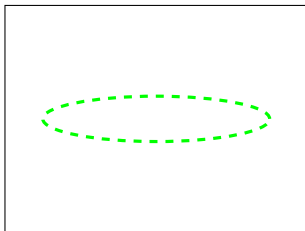
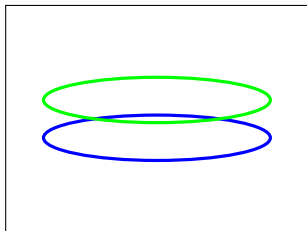
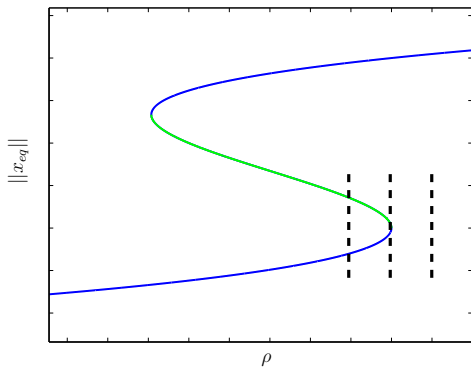
- ▶ Equilibria
- ▶ Periodic orbits





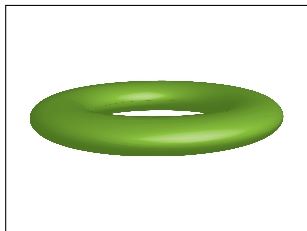
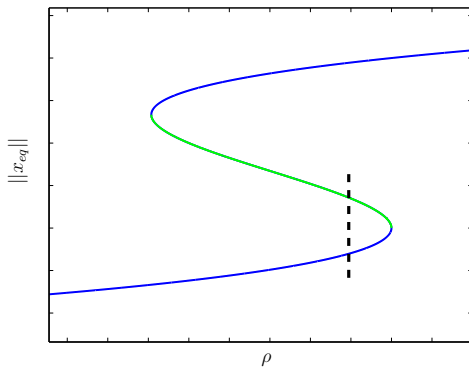
# Motivation: Hysteresis loops and tipping

- ▶ Equilibria
- ▶ Periodic orbits



# Motivation: Hysteresis loops and tipping

- ▶ Equilibria
- ▶ Periodic orbits
- ▶ Tori?



## DDE model for feedback + forcing

$$\dot{h}(t) = -b \tanh(\kappa h(t - \tau_n)) + c \cos(2\pi t)$$

[Ghil *et. al*, Nonlinear Processes in Geophysics (2008)]

# DDE model for feedback + forcing

$$\dot{h}(t) = -b \tanh(\kappa h(t - \tau_n)) + c \cos(2\pi t)$$

[Ghil *et. al*, Nonlinear Processes in Geophysics (2008)]

- ▶ Negative delayed feedback

# DDE model for feedback + forcing

$$\dot{h}(t) = -b \tanh(\kappa h(t - \tau_n)) + c \cos(2\pi t)$$

[Ghil *et. al*, Nonlinear Processes in Geophysics (2008)]

- ▶ Negative delayed feedback
- ▶ Periodic forcing

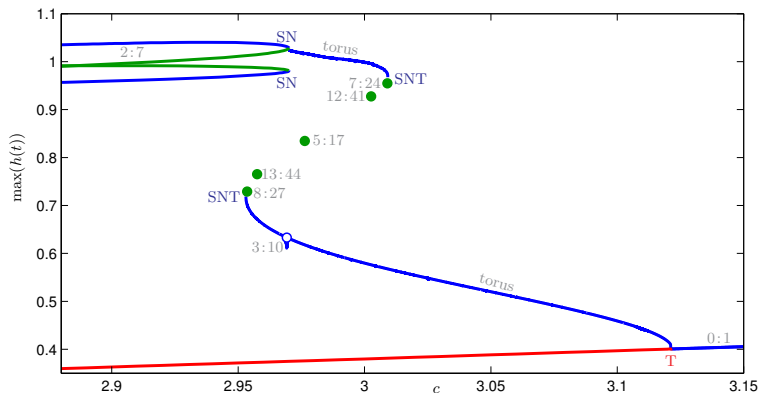
# DDE model for feedback + forcing

$$\dot{h}(t) = -b \tanh(\kappa h(t - \tau_n)) + c \cos(2\pi t)$$

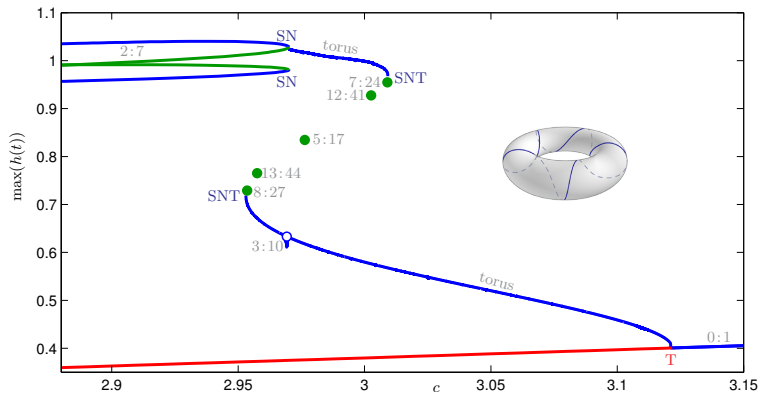
[Ghil *et. al*, Nonlinear Processes in Geophysics (2008)]

- ▶ Negative delayed feedback
- ▶ Periodic forcing
- ▶ Fix  $b = 1$  and  $\kappa = 11$ , vary  $\tau_n$  and  $c$ ,
- ▶ Observe folding tori.

# Hysteresis loop involving torus

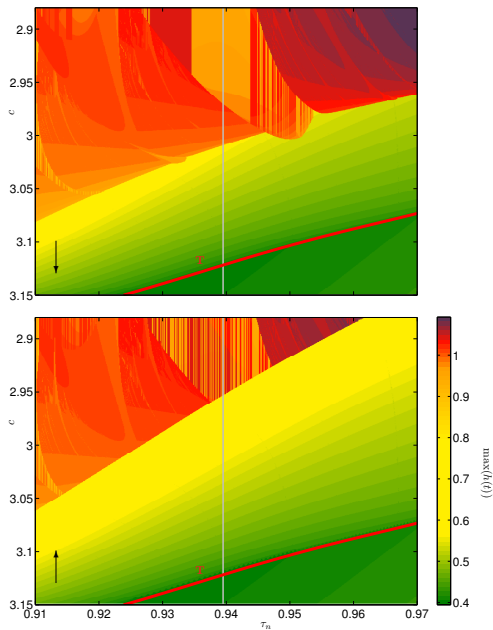


# Hysteresis loop involving torus

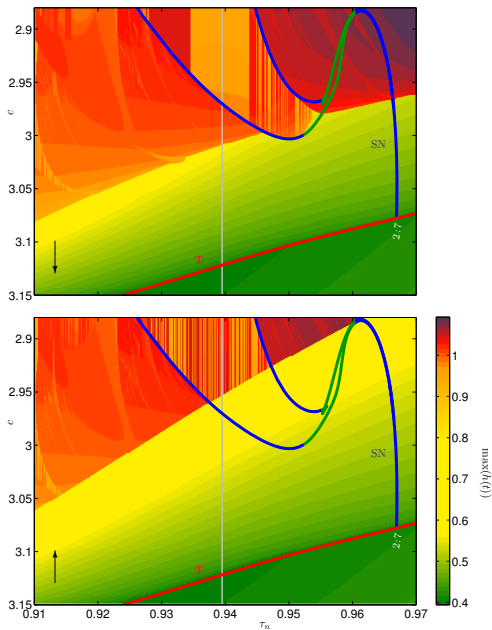




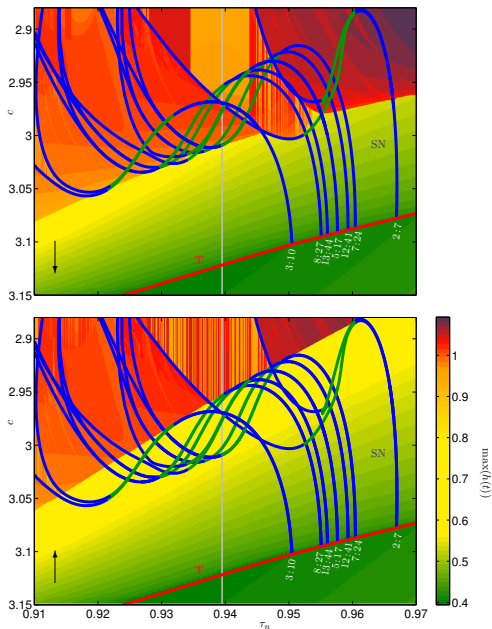
# Folding tori



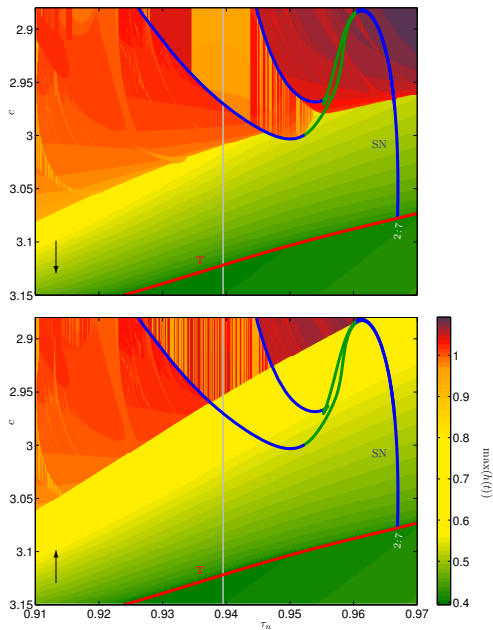
# Folding tori



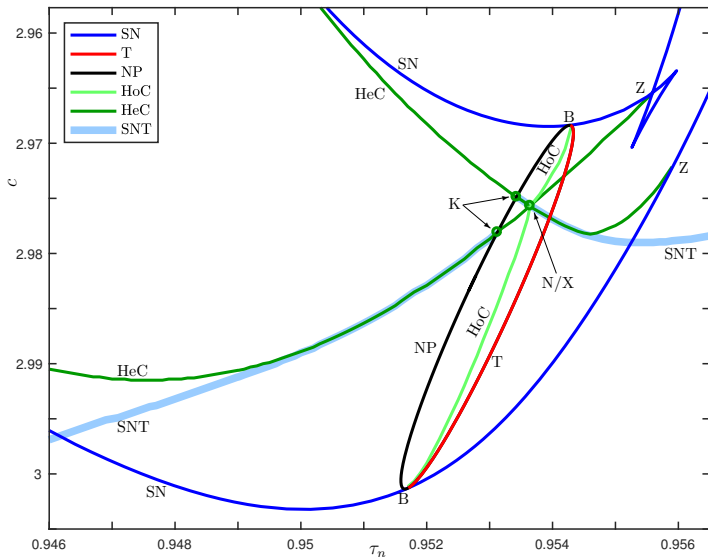
# Folding tori



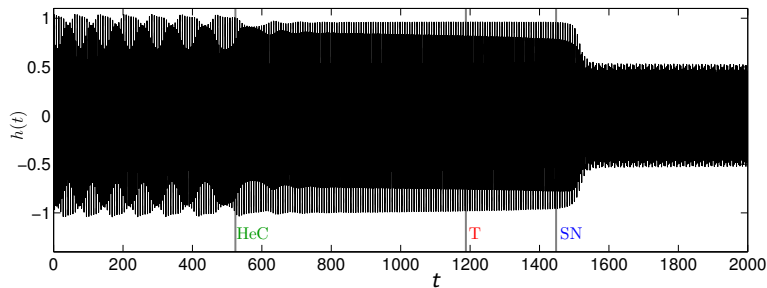
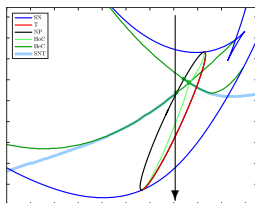
# Folding tori



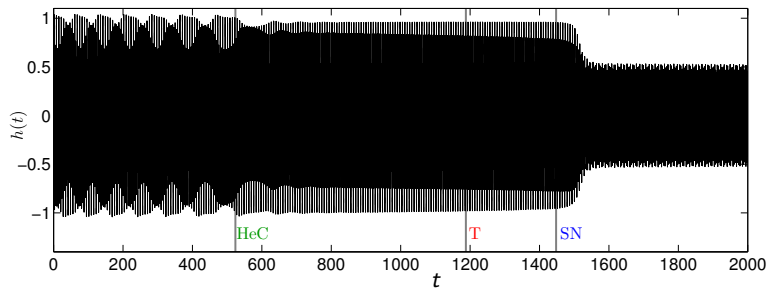
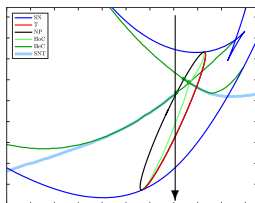
# Folding tori: Chenciner bubble



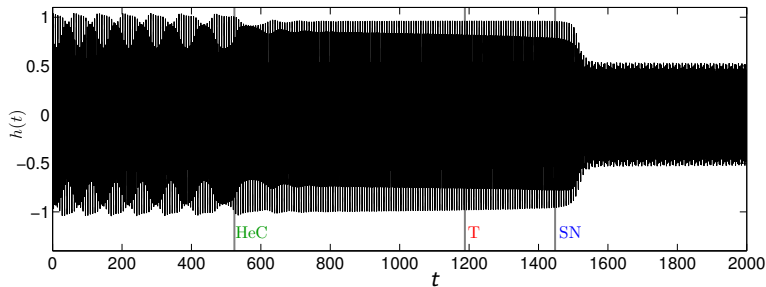
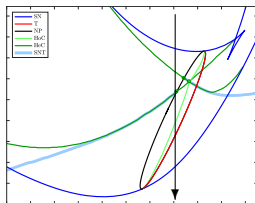
# Folding tori: Transition through the bubble



# Multi-frequency Tipping



# Multi-frequency Tipping

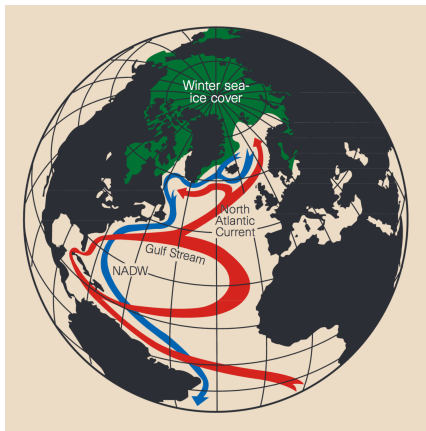


Relevance for (climate) systems?

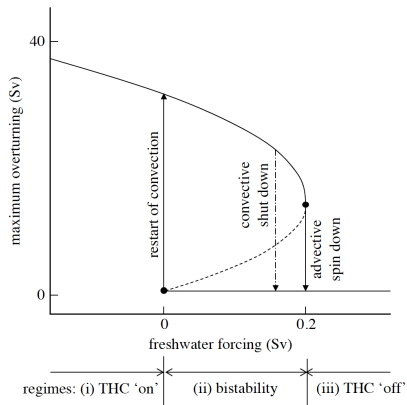


# AMOC

## Atlantic Meridional Overturning Circulation - part of the global thermohaline circulation

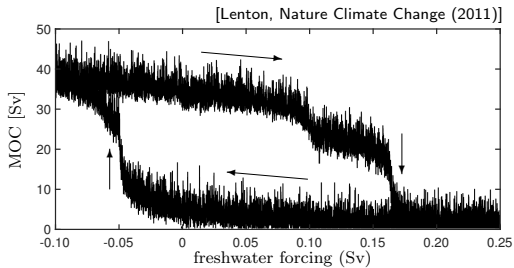


[Rahmstorf, Nature (1997)]



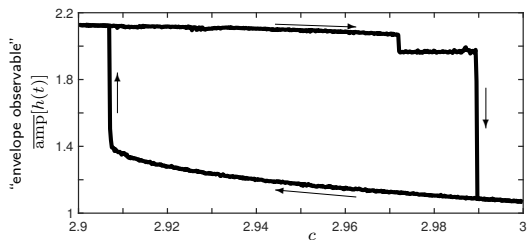
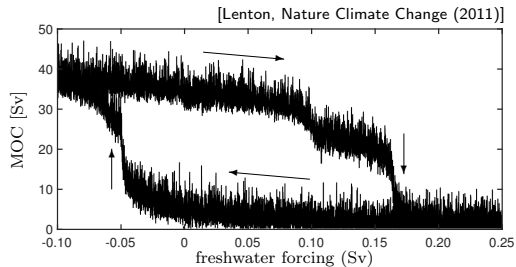
[Lenton *et al.*, Phil. Trans. R. Soc. A (2009)]

# AMOC under freshwater forcing

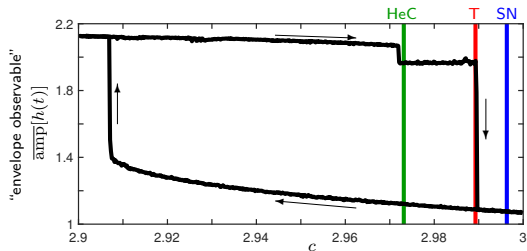
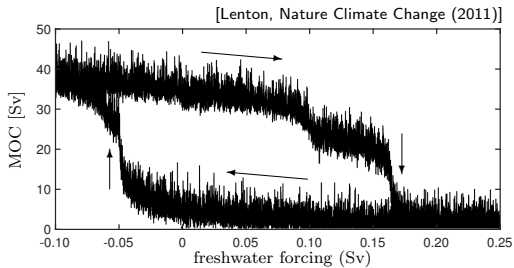


GENIE

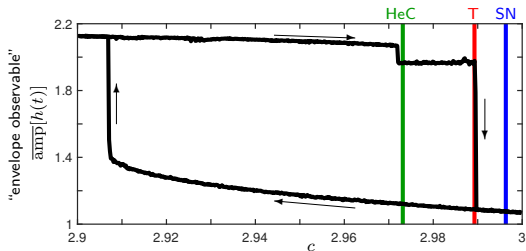
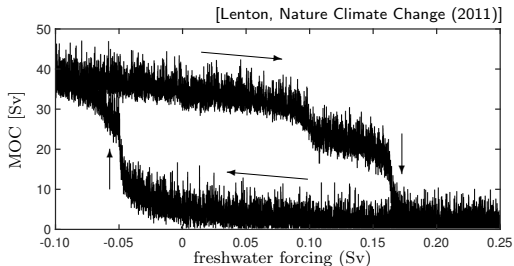
# AMOC under freshwater forcing



# AMOC under freshwater forcing



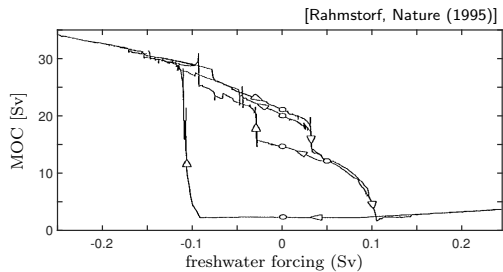
# AMOC under freshwater forcing



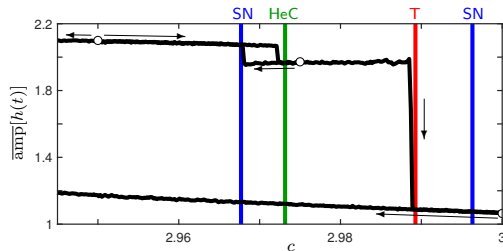
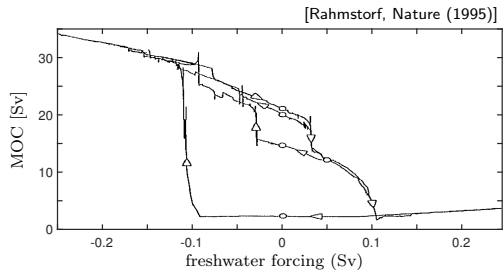
Other examples:

- Rahmstorf, Nature (1995)
- Ganopolski & Rahmstorf, Nature (2001)
- Rahmstorf *et al.*, Geophys Res Lett (2005)

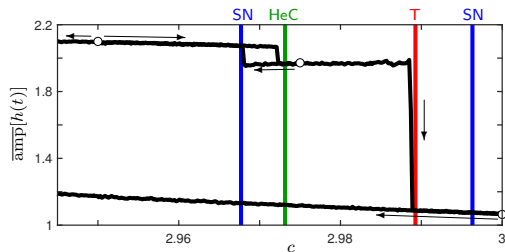
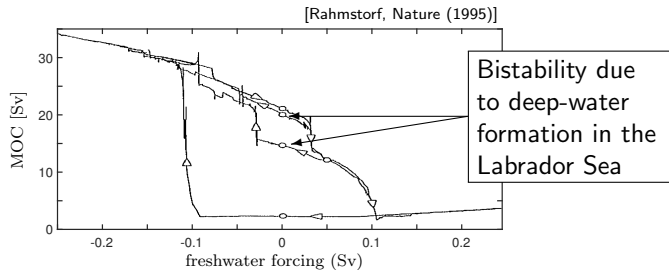
# AMOC under freshwater forcing



# AMOC under freshwater forcing

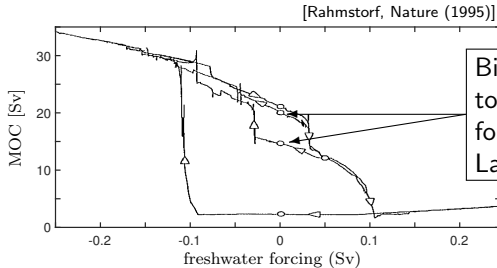


# AMOC under freshwater forcing

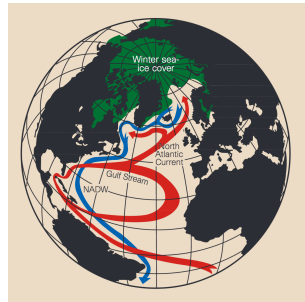
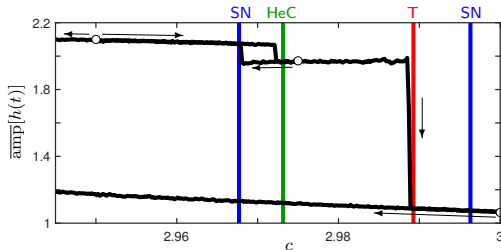




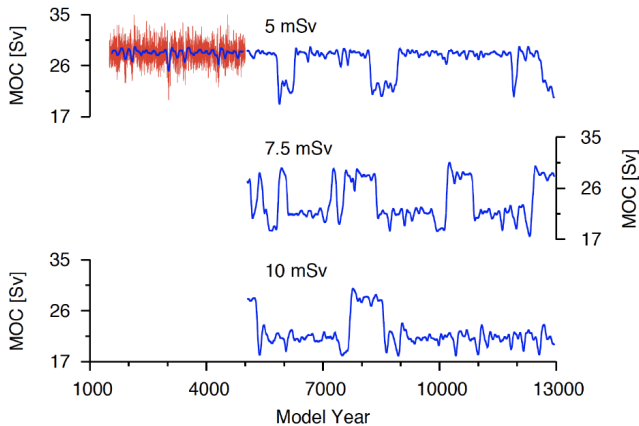
# AMOC under freshwater forcing



Bistability due to deep-water formation in the Labrador Sea

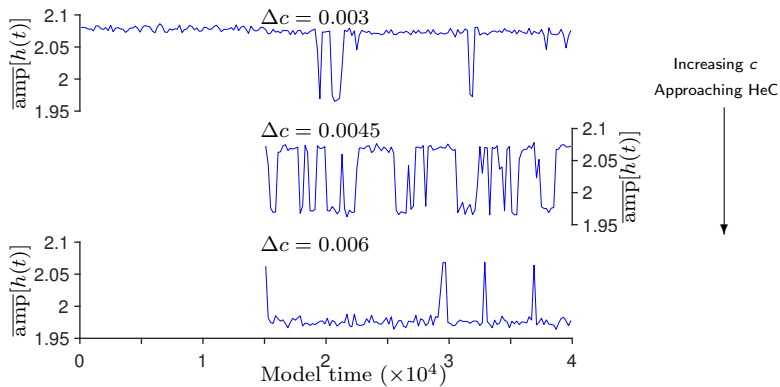


# Transition near Labrador Sea shutdown

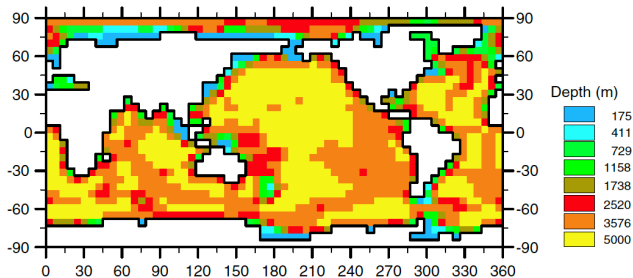


[Schulz *et al.*, *Clim. Past* (2007)]

# Transition near heteroclinic transition

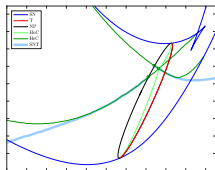
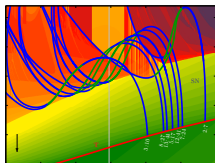


# Ongoing work



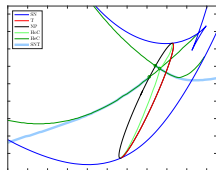
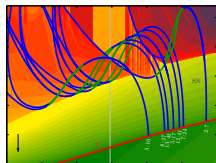
- ▶ GENIE: 3D ocean model + 2D atmosphere + more...  
with help from Andy Ridgwell (Uni. of Bristol),
- ▶ Run freshwater forcing experiments,
- ▶ Detect multistabilities and evidence for different bifurcations.

# Summary



- ▶ Effect of feedback + forcing:  
Multi-frequency Tipping (MFT),
- ▶ Bifurcation structure of folding tori via  
Chenciner bubbles describes dynamics of torus  
break up.
- ▶ Ongoing/future work:
  - ▶ Identifying MFT as a form of tipping,
  - ▶ Analysis of data from complex models,
  - ▶ Interpretation of “observables” from  
dynamical systems perspective.

# Summary



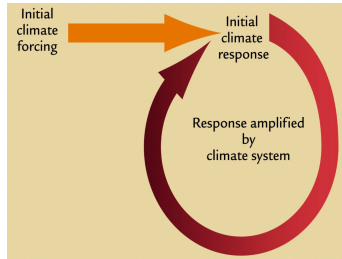
- ▶ Effect of feedback + forcing:  
Multi-frequency Tipping (MFT),
- ▶ Bifurcation structure of folding tori via  
Chenciner bubbles describes dynamics of torus  
break up.
- ▶ Ongoing/future work:
  - ▶ Identifying MFT as a form of tipping,
  - ▶ Analysis of data from complex models,
  - ▶ Interpretation of “observables” from  
dynamical systems perspective.

Thank you for your attention!

# Extra: Feedback loops + forcing

Applications:

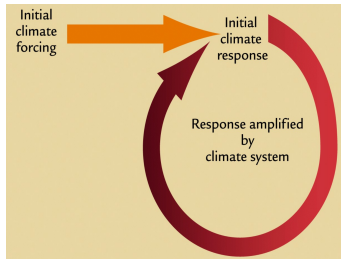
- ▶ Climate
- ▶ Ecology
- ▶ Human motion control
- ▶ Network dynamics
- ▶ Laser systems



# Extra: Feedback loops + forcing

Applications:

- ▶ Climate
- ▶ Ecology
- ▶ Human motion control
- ▶ Network dynamics
- ▶ Laser systems



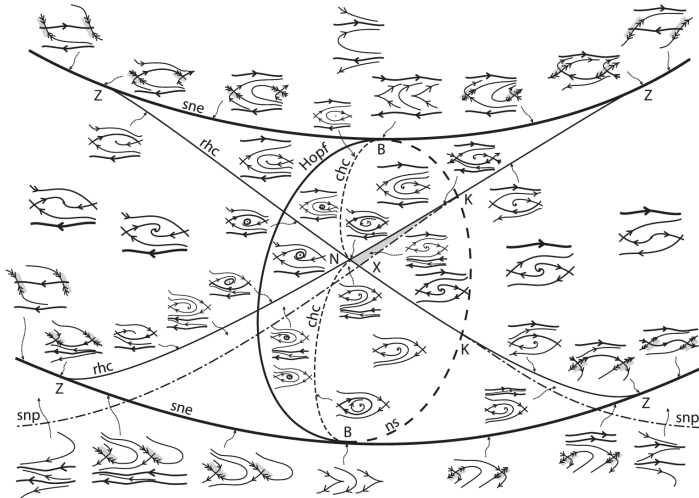
Delay differential equations (DDEs) → convenient representation

- ▶ Describe effects of complex processes
- ▶ Few variables/parameters
- ▶ Infinite-dimensional dynamical system
- ▶ Well-developed theory (for constant delays)



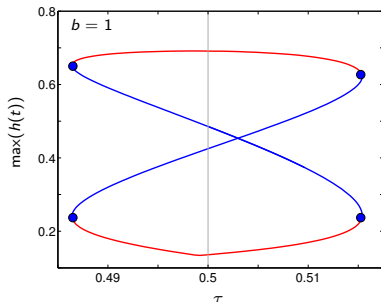
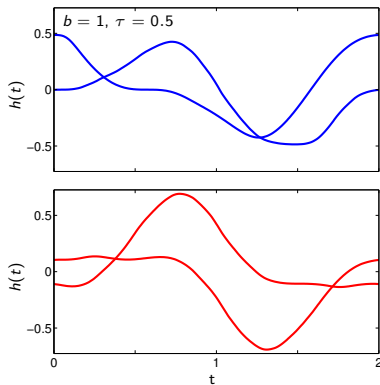
# Extra: Folding tori: Theoretical

Theoretical bifurcation structure (Chenciner bubble)

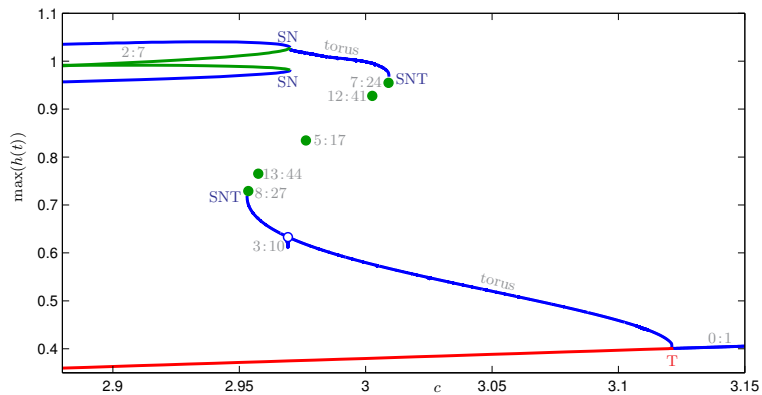


## Extras: Bistability *within* tongues

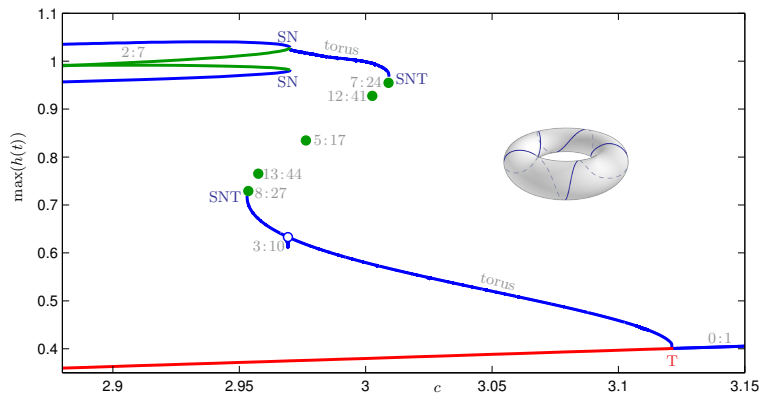
Symmetry of  $p:q$  locked solutions of even  $p$  or  $q$ :  $h_2(t) = -h_1(t + \frac{1}{2})$   
→ 2 stable, 2 unstable solutions



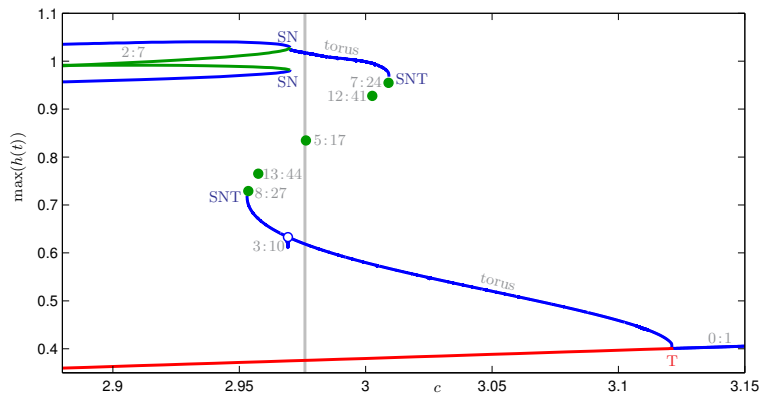
# Extra: Folding tori



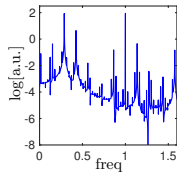
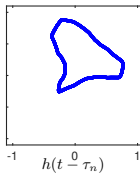
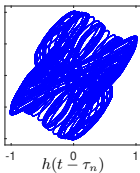
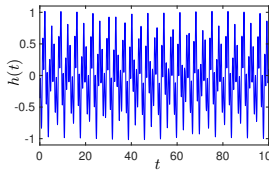
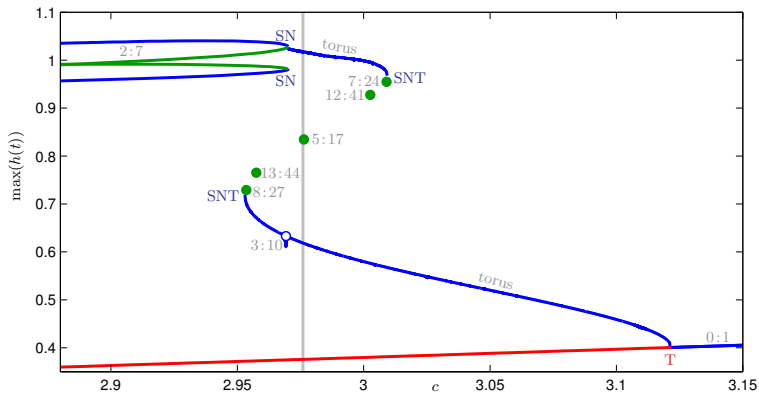
# Extra: Folding tori



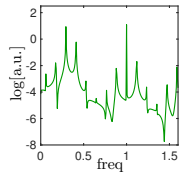
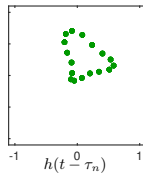
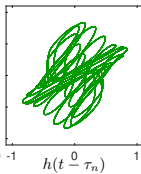
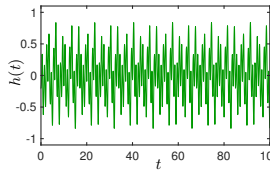
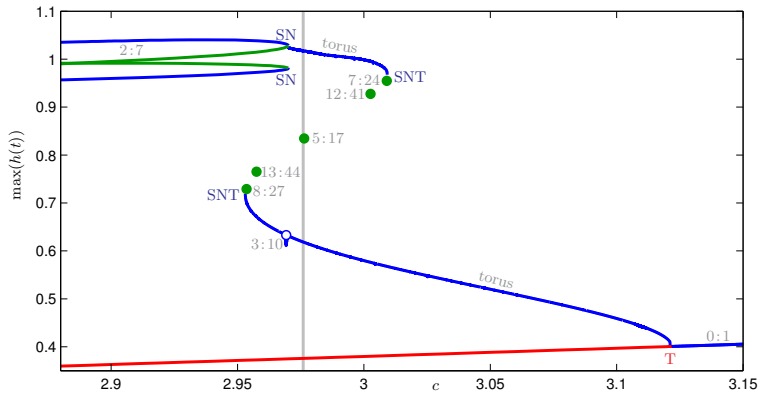
# Extra: Folding tori



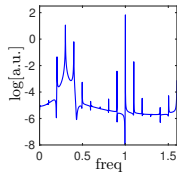
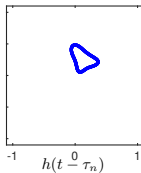
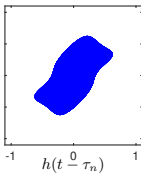
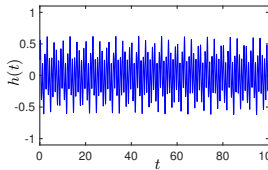
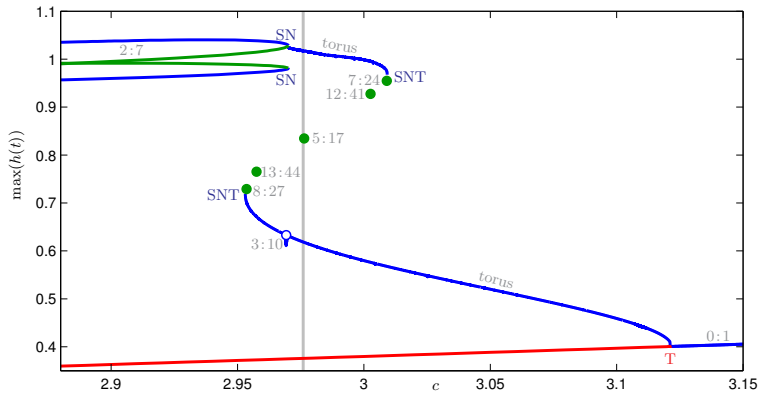
# Extra: Folding tori



# Extra: Folding tori

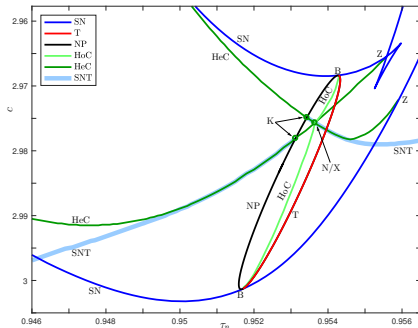
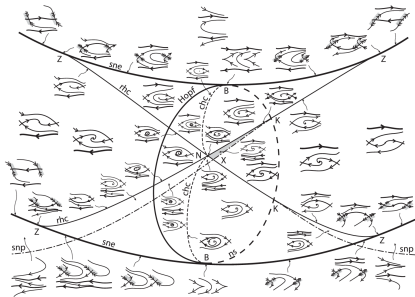


# Extra: Folding tori





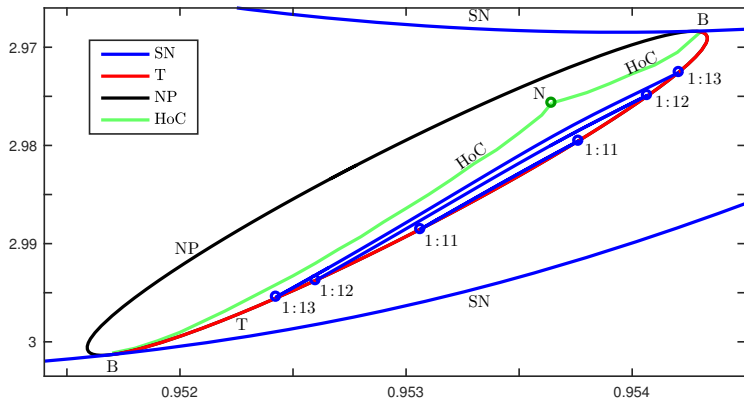
# Extra: Folding tori: DDE model



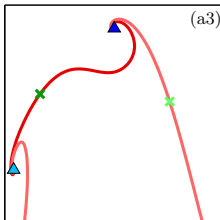
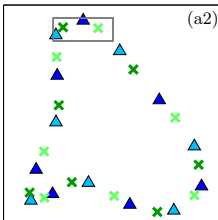
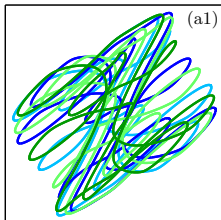
Chenciner bubble structure identified only very recently:

- ▶ 3D map [Neiryck *et al.*, ACM TOMS (2018)]
- ▶ Continuous time case [Keane & Krauskopf, Nonlinearity (2018)]

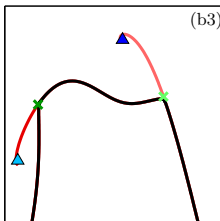
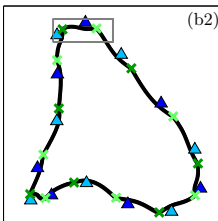
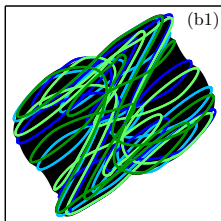
# Extra: Subcritical torus bifurcation



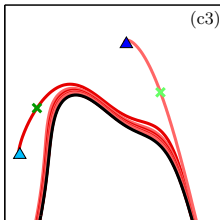
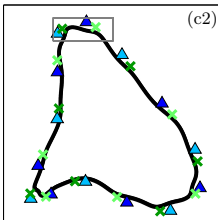
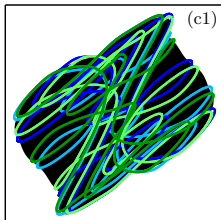
before HeC



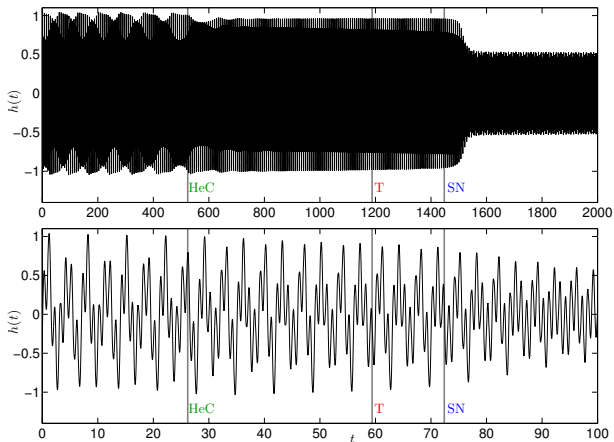
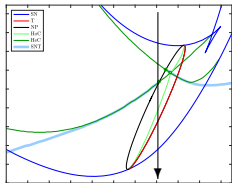
$\approx$  HeC



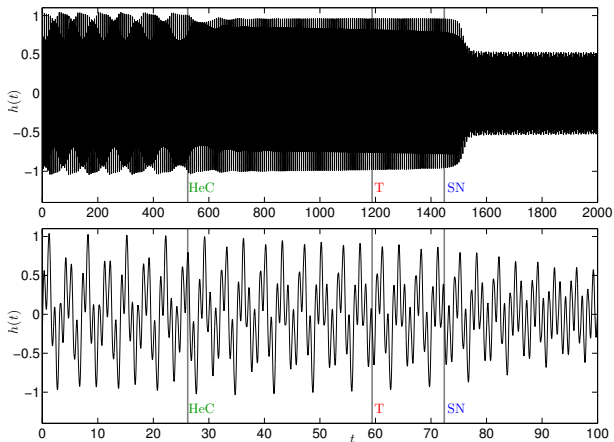
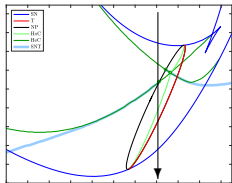
after HeC



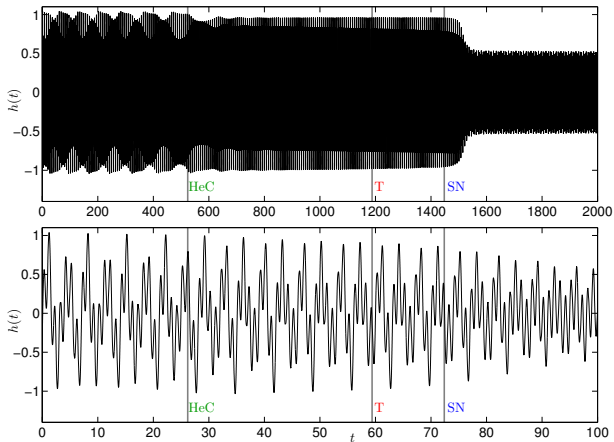
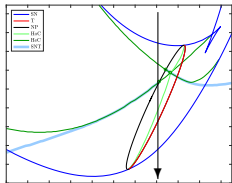
# Extra: Folding tori: Transition through the bubble



# Extra: Multi-frequency Tipping

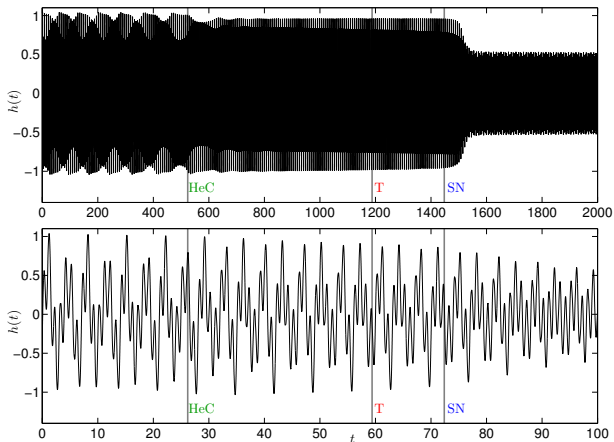
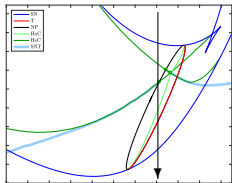


# Extra: Multi-frequency Tipping



Relevance for (climate) systems?

# Extra: Multi-frequency Tipping

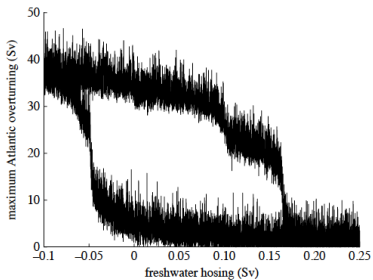


Relevance for (climate) systems?

Major challenges:

- ▶ Relating conceptual model variables to “observables” in complex systems
- ▶ The role of different time scales

# Extra: AMOC under freshwater forcing



Observed hysteresis

[Lenton *et al.*, *Phil. Trans. R. Soc. A* (2009)]

