Sharp Korn inequalities in thin domains: The *first and a half* Korn inequality

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Korn's Inequalities

Assume $\Omega \subset \mathbb{R}^n$ is open, bounded, connected and Lipschitz and $u \in H^1(\Omega, \mathbb{R}^n)$, where $u = (u_1, u_2, ..., u_n)$ and $\nabla u = \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j=1}^n$.
Set

$$e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \qquad e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}.$$

Denote

$$\mathrm{skew}(\mathbb{R}^n) = \{L = Ax + b \;:\; A \in M^{n\times n}, A^T = -A, b \in \mathbb{R}^n\}.$$

Assume V is a closed subspace of $H^1(\Omega, \mathbb{R}^n)$ such that

$$V \cap \mathrm{skew}(\mathbb{R}^n) = \{0\}.$$

Korn's Inequalities

Korn's First and Second Inequalities

1. There exists a constant K_1 depending only on Ω such that

$$K_1(\Omega)\int_\Omega |\nabla u|^2 \leq \left(\int_\Omega |u|^2 + \int_\Omega |e(u)|^2\right), \quad \text{for any} \quad u \in H^1(\Omega, \mathbb{R}^n)$$

2. There exists a constant K_2 depending only on Ω and V such that

$$K_2(\Omega, V)\int_\Omega |\nabla u|^2 \leq \int_\Omega |e(u)|^2, \quad \text{for any} \quad u \in V$$

.

3. There exists a constant $K > 0$ depending only on Ω such that for any $u \in H^1(\Omega, \mathbb{R}^n)$, there exists a skew-symmetric matrix A_u such that

$$K(\Omega)\int_\Omega |\nabla u - A_u|^2 \leq \int_\Omega |e(u)|^2$$

Inequalities of Our Interest

We are interested in sharp Korn inequalities.

Question: How do $K_1(\Omega)$ and $K_2(V,\Omega)$ depend on Ω and V, when Ω is thin?

- Ω is a thin domain (strips, rods, shells,...) with thickness h, then $K_1(\Omega) \sim h^\alpha$ and $K_2(V,\Omega) \sim h^\beta$ as $h \to 0$. Find α and β.
- If for instance β is known and $K_2(V,\Omega) \approx C(V,\Omega)h^\beta$, when h is sufficiently small, then what is $C(V,\Omega)$?

Goal: Find the optimal constants in Korn's inequalities.

Examples

Example 1 (Zero boundary conditions). If $V = \{u \in H^1(\Omega, \mathbb{R}^n) \ : \ u(x) = 0 \text{ on } \partial\Omega\}$, then

$$K_2(V, \Omega) = \frac{1}{2}.$$

Example 2 (Thin rectangle). If $\Omega = [0, h] \times [0, l]$, $V = \{u \in H^1(\Omega, \mathbb{R}^2) \ : \ u(x, 0) = u(x, l) = 0\}$, then

$$K_2(V, \Omega) \approx Ch^2.$$

Motivation

Why optimal constants?

The problem we were interested in: *Buckling of cylindrical shells under axial compression, (2011).*

- Critical buckling load, deformation modes?
- **Koiter's formula** (1945). $\lambda(h) = Ch$, where h is the thickness of the shell, and C depends on the material. The buckling modes are given by **"Koiter's circle".**
- It was known, that the buckling load is highly sensitive to imperfections (shape, load).
- We aim to derive Koiter's formula and understand the *sensitivity to imperfections* applying the theory of buckling of slender structures, **Grabovsky, Truskinovsky (2007).**

Motivation

If

$$C_h = \{(r, \theta, z) \; : \; r \in [R, R+h], \quad \theta \in [0, 2\pi], \quad z \in [0, L]\},$$

and

$$u = u_r \bar{e}_r + u_\theta \bar{e}_\theta + u_z \bar{e}_z,$$

in cylindrical coordinates, the we impose the B.C.:

- Fixed bottom boundary conditions:

  $$u_r(r, \theta, 0) = u_\theta(r, \theta, 0) = u_z(r, \theta, 0) = u_r(r, \theta, L) = u_\theta(r, \theta, L) = 0,$$

  V_1,
- Breathing cylinder

  $$u_\theta(r, \theta, 0) = u_z(r, \theta, 0) = u_\theta(r, \theta, L) = 0, u_z(r, \theta, L) = c,$$

  V_2.

Motivation, Problem

The theory of Grabovsky and Truskinovsky implies

$$\lambda(h) \geq cK(C_h),$$

where $K(C_h)$ is the optimal Korn's constant in the second Korn inequality for V_1 or V_2.

Whether one has $K(V_i, C_h) \sim h$?

Answer: NO! $K(V_i, C_h) \sim h\sqrt{h}$.

Theorem (Grabovsky, H., 2012)

If

$$\lim_{h\to 0} \frac{K(C_h)}{\lambda_{cl}(h)} = 0,$$

then the constitutively linearized quotient captures both, the critical load and the buckling modes.

Korn's inequalities for perfect cylindrical shells

Theorem (Grabovsky, H., 2012)

There exist absolute constants $C_i > 0, i = 1, 2$ such that for any $u \in V_i$, there holds

$$\int_{C_h} |\nabla u|^2 \leq \frac{C_i}{h\sqrt{h}} \int_{C_h} |e(u)|^2.$$

These estimates are sharp, in the sense that the power of h is optimal.

If $u = u_r \bar{e}_r + u_\theta \bar{e}_\theta + u_z \bar{e}_z$, then

$$\nabla u = \begin{bmatrix} u_{r,r} & \frac{u_{r,\theta} - u_\theta}{r} & u_{r,z} \\ u_{\theta,r} & \frac{u_{\theta,\theta} + u_r}{r} & u_{\theta,z} \\ u_{z,r} & \frac{u_{z,\theta}}{r} & u_{z,z} \end{bmatrix}.$$

Korn's inequalities for perfect cylindrical shells

Ansatz. We assume $R = 1$, then

$$\begin{cases} \phi_r^h(r,\theta,z) = & -W_{,\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) \\ \phi_\theta^h(r,\theta,z) = & r\sqrt[4]{h}W_{,\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) + \frac{r-1}{\sqrt[4]{h}}W_{,\eta\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \\ \phi_z^h(r,\theta,z) = & (r-1)W_{,\eta\eta z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) - \sqrt{h}W_{,z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \end{cases}$$

where the function $W(\eta, z)$ is a smooth compactly supported function on $(-1,1)\times(0,L)$, while the function $\phi^h(\theta, z)$ is extended as a 2π-periodic function in $\theta \in \mathbb{R}$.

Remarks on the Korn inequality, strategy

If $u = u_r \bar{e}_r + u_\theta \bar{e}_\theta + u_z \bar{e}_z$, then

$$\nabla u = \begin{bmatrix} u_{r,r} & \frac{u_{r,\theta} - u_\theta}{r} & u_{r,z} \\ u_{\theta,r} & \frac{u_{\theta,\theta} + u_r}{r} & u_{\theta,z} \\ u_{z,r} & \frac{u_{z,\theta}}{r} & u_{z,z} \end{bmatrix}.$$

Prove the inequality block by block, which means fixing r, θ and z and proving $2D$ inequalities. For $r, \theta, z = \mathrm{const}$, we have the blocks

$$\begin{bmatrix} - & - & - \\ - & \frac{u_{\theta,\theta} + u_r}{r} & u_{\theta,z} \\ - & \frac{u_{z,\theta}}{r} & u_{z,z} \end{bmatrix}, \quad \begin{bmatrix} u_{r,r} & - & u_{r,z} \\ - & - & - \\ u_{z,r} & - & u_{z,z} \end{bmatrix}, \quad \begin{bmatrix} u_{r,r} & \frac{u_{r,\theta} - u_\theta}{r} & - \\ u_{\theta,r} & \frac{u_{\theta,\theta} + u_r}{r} & - \\ - & - & - \end{bmatrix},$$

respectively.

Available Tools

We needed Korn's inequalities with constants decaying like $h\sqrt{h}$ or slower!
For instance the cross section $\theta = \text{const}$ gives a Korn's second inequality on a thin rectangle:

$$\begin{bmatrix} u_{r,r} & - & u_{r,z} \\ - & - & - \\ u_{z,r} & - & u_{z,z} \end{bmatrix}.$$

What is available?

- $\theta = \text{const}$ gives a thin rectangle, Korn's second inequality on rectangles, $K \sim h^2$ **Ryzhak 2001?**: not applicable.
- $z = \text{const}$ gives a thin annulus, again optimal constant scales like h^2, **Dafermos 1968, for normalization conditions**: not applicable.
- Uniform Korn-Poincaré inequality in thin domains, **Lewicka, Müller 2011, tangential boundary conditions**: not applicable.

New inequalities are needed.

A Korn type inequality

Standard approach: It is sufficient to prove a second Korn inequality subject to Dirichlet type boundary conditions for harmonic displacements.

Theorem (Grabovsky, H., 2012)

Suppose $w(x,y)$ is harmonic in $[0,h]\times[0,L]$, and satisfies $w(x,0)=w(x,L)$. Then

$$\|w_y\|^2 \leq \frac{2\sqrt{3}}{h}\|w\|\|w_x\| + \|w_x\|^2.$$

The equality is attained at

$$w(x) = \cosh\left(\frac{\pi}{L}\left(x-\frac{h}{2}\right)\right)\sin\left(\frac{\pi y}{L}\right),$$

The first and a half Korn inequality for rectangles

Theorem (Grabobsky, H., 2012)

Suppose that the vector field $U = (u, v) \in H^1(\Omega, \mathbb{R}^2)$, *where* $\Omega = [0, h] \times [0, L]$, *satisfies* $u(x, 0) = u(x, L)$. *Then for any* $h \in (0, 1)$ *and any* $L > 0$ *there holds:*

$$\|\nabla U\|^2 \leq 100 \left(\frac{\|u\| \cdot \|e(U)\|}{h} + \|e(U)\|^2 \right).$$

There are no boundary conditions imposed on $v(x, y)$.

- This implies both the first (via Schwartz inequality) and the second (via Friedrichs inequality) Korn inequalities.
- The scaling of the constant is as needed.

The first and a half Korn inequality for cylindrical shells

Theorem (Grabobsky, H., 2012)

Suppose $U \in V_1$ or $U \in V_2$. Then there exists a universal constant $C > 0$ such, that for any $h \in (0,1)$ and any $L > 0$ there holds:

$$\|\nabla U\|^2 \leq C \left(\frac{\|u_r\| \cdot \|e(U)\|}{h} + \|e(U)\|^2 \right).$$

- This implies the second Korn inequality, but with h^2.
- Combine with $\|u_r\|^3 \leq C\|\nabla U\|^2 \cdot \|e(U)\|$.

Extensions

An extension to $\mathbb{R}^n$ for thin domains with nonconstant thickness.

Assume the operator

$$L(u) = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

with constant coefficients satisfies

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j \geq \lambda |x|^2 \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{1}$$

where $\lambda > 0$, and,

$$\sum_{i=1}^{n} |a_{ij}| \leq \Lambda \quad \text{for all} \quad 1 \leq j \leq n. \tag{2}$$

Extensions

For $x = (x_1, x_2, \dots, x_n)$, let $x' = (x_2, \dots, x_n)$.

Theorem (H., 2014)

Let $\omega \subset \mathbb{R}^{n-1}$ *be a bounded and simply-connected Lipschitz domain, let* $x_1 = \varphi(x')\colon \omega \to \mathbb{R}$ *be a positive Lipschitz function with* $H = \sup_{x' \in \omega} \varphi(x')$ *and* $h = \inf_{x' \in \omega} \varphi(x') > 0$*. Denote* $\Omega = \{x \in \mathbb{R}^n \ : \ x' \in \omega, \ 0 < x_1 < \varphi(x')\}$ *and assume that the operator* $L(u) = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$ *with constant coefficients satisfies conditions (1) and (2). Then there exists a constant* C *depending on* n, Λ, λ, $L = \mathrm{Lip}(\varphi)$ *and the ratio* $m = H/h$ *such that any* $u \in C^3(\bar{\Omega})$ *solution of* $L(u) = 0$ *satisfying the boundary conditions* $u(x) = 0$ *on the portion* $\Gamma = \{x \in \partial\Omega \ : \ x' \in \partial\omega\}$ *of the boundary of* Ω *fulfills the inequality*

$$\|\nabla u\|^2 \leq C \left(\frac{\|u\| \cdot \|u_{x_1}\|}{h} + \|u_{x_1}\|^2 \right).$$

$$C = C(n, \lambda, \Lambda, L, m).$$

Extensions

Theorem (H., 2014)

Let $l > 0$, let $\varphi_1 \in C^1[0,l]$ and let φ_2 and φ_1' be Lipschitz functions defined on $[0,l]$. Assume furthermore that $0 < h = \min_{y\in[0,l]}(\varphi_2(y) - \varphi_1(y))$ and $H = \min_{y\in[0,l]}(\varphi_2(y) - \varphi_1(y))$. Denote $\Omega = \{(x,y) \in \mathbb{R}^2 \ : \ y \in (0,l),\ \varphi_1(y) < x < \varphi_2(y)\}$. Then there exists a constant C depending on $m = H/h$, $\rho_1 = \|\varphi_1'\|_{L^\infty(\Omega)}$, $\rho_2 = \|\varphi_2'\|_{L^\infty(\Omega)}$ and $\rho_1' = \|\varphi_1''\|_{L^\infty(\Omega)}$ such that if the first component of the displacement $U = (u,v) \in W^{1,2}(\Omega)$ satisfies the boundary conditions $u(x) = 0$ on the boundary portion $\Gamma = \{(x,y) \in \partial\Omega \ : \ y = 0 \text{ or } y = l\}$ in the trace sense, then the strong second Korn inequality holds:

$$\|\nabla U\|^2 \le \left(\frac{\|u\|\cdot\|e(U)\|}{h} + \|e(U)\|^2\right).$$

$$C = C(m, \rho_1, \rho_2, \rho_1').$$

The estimate is sharp.

Extensions

Theorem (H., 2014)

Let $L > 0$, φ_1, φ_2, Ω, h, H, m, ρ_1, ρ_2 and ρ_1' be as in the previous theorem. Then there exists a constant C depending on m, ρ_1, ρ_2 and ρ_1' such that if the first component u of the displacement $U=(u,v)\in W^{1,2}(\Omega)$ is L-periodic, then the second Korn inequality holds:

$$\|\nabla U\|^2 \leq \left(\frac{\|u\|\cdot\|e(U)\|}{h} + \|e(U)\|^2\right).$$

$$C = C(m,\rho_1,\rho_2,\rho_1').$$

$L-$periodicity is the periodicity of both the function and the gradient.

Recent progress

Consider a shell in the (r, θ, z) variables (θ and z are the principal directions):

$$C_h = \left[-\frac{h}{2}, \frac{h}{2}\right] \times [0, s] \times [0, L],$$

with $\kappa_z = 0$. (this yields a zero Gaussian curvature). If $U = (u_r, u_\theta, u_z)$, then

$$\nabla U = \begin{bmatrix} u_{r,r} & \frac{1}{A_\theta} u_{r,\theta} - \kappa_\theta u_\theta & \frac{1}{A_z} u_{r,z} \\ u_{\theta,r} & \frac{1}{A_\theta} u_{\theta,\theta} + \frac{A_{\theta,z}}{A_\theta A_z} u_z + \kappa_\theta u_r & \frac{1}{A_z} u_{\theta,z} \\ u_{z,z} & \frac{1}{A_\theta} u_{z,\theta} - \frac{A_{\theta,z}}{A_\theta A_z} u_\theta & \frac{1}{A_z} u_{z,z} \end{bmatrix}.$$

Assume

$$K = \sup |\kappa_\theta| < \infty, \quad K_1 = \sup |\kappa_{\theta,\theta}| < \infty,$$

$$0 < a_\theta \le A_\theta \le b_\theta, \quad 0 < a_z \le A_z \le b_z, \quad |\nabla A_z|, |\nabla A_\theta| \le A.$$

where $a_\theta, a_z, b_\theta, b_z, A$ are constants.

This includes cut cones and straight cylinders with arbitrary cross sections.

Recent progress

The spaces V_1 and V_2 are the same as before. C will be a constant depending only on the constants $K, k, K_1, a_z, a_\theta, b_z, b_\theta$ and A.

Theorem (Grabovsky, H., 2015)

For any $h \in (0,1)$ and any $U \in V_i$, there holds:

$$\|\nabla U\|^2 \leq C \left(\frac{\|u_r\| \cdot \|e(U)\|}{h} + \|e(U)\|^2 + \|u_r\|^2 \right).$$

Recent progress

Theorem (Grabovsky, H., 2015)

If $\kappa_\theta > 0$*, then for any* $h \in (0,1)$ *and any* $U \in V_i$*, there holds:*

$$\|\nabla U\|^2 \leq \frac{C}{h\sqrt{h}} \|e(U)\|^2.$$

If $\kappa_\theta = 0$ *in a box* $[\theta_1, \theta_2] \times [z_1, z_2]$*, then*

$$\|\nabla U\|^2 \leq \frac{C}{h^2} \|e(U)\|^2.$$

Work in progress

Assume Ω_h is a shell of revolution given by

$$C_h = \left[r(z) - \frac{h}{2}, r(z) + \frac{h}{2}\right] \times [0, 2\pi] \times [0, L].$$

Then (conjecture)

- If $\kappa_z < 0$, then for any $h \in (0,1)$ and any $U \in V_i$, there holds:

$$\|\nabla U\|^2 \leq \frac{C}{h^{4/3}} \|e(U)\|^2.$$

- If $\kappa_z > 0$, then

$$\|\nabla U\|^2 \leq \frac{C}{h} \|e(U)\|^2.$$

Theorem (Grabovsky, H., 2015)

For any $h \in (0,1)$ and any $U \in V_i$, there holds:

$$\|\nabla U\|^2 \leq C \left(\frac{\|U\| \cdot \|e(U)\|}{h} + \|e(U)\|^2 + \|U\|^2 \right).$$