Locally conservative parameter-robust finite element methods for poroelasticity

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Poroelasticity theory

- In poroelasticity theory, poroelastic medium saturated by a viscous fluid is modeled as a mixture of solid and fluid phases
- For small deformations and slow dynamics, the problem is a linear PDE coupling linear elasticity and Darcy flow equations

$\boldsymbol{u} : \Omega \to \mathbb{R}^n$: solid displacement, $p : \Omega \to \mathbb{R}$: pore pressure

$2\mu\epsilon(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u}\underline{\boldsymbol{I}} - \alpha p \underline{\boldsymbol{I}}$: stress tensor

μ, λ : Lamé parameters , $\alpha > 0$: Biot–Willis coefficient

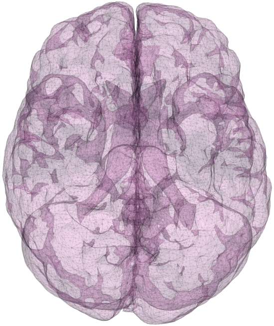
Governing equations

$$
\begin{aligned}
-\operatorname{div}(2\mu\epsilon(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u}\underline{\boldsymbol{I}} - \alpha p \underline{\boldsymbol{I}}) &= \boldsymbol{f}, \\
s_0 \dot{p} + \alpha \operatorname{div} \dot{\boldsymbol{u}} - \operatorname{div}(\kappa \nabla p) &= g,
\end{aligned}
$$

with $s_0 \geq 0$: storage coefficient κ : permeability

Applications

- Geomechanics
- Reservoir modeling in petroleum engineering
- Biological tissue modeling (bone, articular cartilage)
- Recently, multiple-network poroelasticity models are used for modeling of human brain



Finite element methods for poroelasticity (review)

Formulations

- displacement – pressure (Murad et al. (1996), Riviere et al. (2017), Chen et al. (2013))
- displacement – flux – pressure (Phillips, Wheeler (2008), Yi (2013), Lee (2018), Zikatanov et al. (2016-))
- stress – displacement – flux – pressure (Starke et al.(2005), Yi (2014), Lee (2016), Fu (2018))
- displacement – fluid content – pressure (Feng et al. (2016))
- displacement – total pressure – fluid pressure (Lee et al. (2017), Ruiz-Baier et al. (2016))

Solver strategy

- monolithic
- iterative coupling
- operator splitting (or partitioned scheme)

Finite element methods for poroelasticity (review)

Iterative coupling algorithms

- Coupling elasticity equation and Poisson equation solves iteratively
- Intrinsically block preconditioned iterative methods of monolithic methods
- The number of sufficient iteration is unknown and is sensitive to parameters
- Convergence rate is difficult to derive

Operator splitting algorithms

- Elaborate combination of time schemes (backward Euler, Crank-Nicolson) or additional stabilization
- Only one solve of each subproblem at each time step
- Only first order convergence in time is known
- Time step size is limited by parameter values

Finite element methods for poroelasticity (review)

Previous works

- iterative coupling
  Kim (2010), Wheeler et al., Kumar et al.
- operator splitting
  Bukac et al. (2015) (conditionally stable)
  Riviere et al. (2017) (discrete acceleration term for stability)

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Question

Can we develop unconditionally stable operator splitting methods for poroelasticity models?

Finite element methods for poroelasticity (review)

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Question

Can we develop unconditionally stable operator splitting methods for poroelasticity models?

Wait, how about MPET?

Multiple-network (quasi-static) poroelasticity model (MPET)

$\boldsymbol{u}$: solid displacement $\qquad p_i$: pore pressure of i-th pore network

$$-\operatorname{div}(2\mu\epsilon(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u}\underline{\boldsymbol{I}} + \sum_{i=1}^{N} \alpha_i \nabla p_i \underline{\boldsymbol{I}}) = f,$$

$$s_i \dot{p}_i + \alpha_i \operatorname{div} \dot{\boldsymbol{u}} - \operatorname{div} K_i \nabla p_i + \xi_i(\boldsymbol{p}) = g_i, \qquad 1 \leq i \leq N,$$

where $\boldsymbol{p} = (p_1, \ldots, p_N)$ and fluid exchanges are given by

$$\xi_i(\boldsymbol{p}) = \sum_{j=1}^{N} \xi_{j \leftarrow i}(p_j - p_i), \qquad \xi_{j \leftarrow i} \geq 0, \quad \xi_{j \leftarrow i} = \xi_{i \leftarrow j}$$

These MPET models are used to model multiple pore-network of human brain (Tully, Vardakis, Ventikos, etc.)

Multiple-network (quasi-static) poroelasticity model (MPET)

Challenges

- In the model, λ can be large because soft biological tissues are almost incompressible
- The system is a saddle point problem but the Babuska-Brezzi condition is difficult to check due to many pressure variables

To circumvent these difficulties, we introduce total pressure p_t

$$p_t := \lambda \operatorname{div} \boldsymbol{u} - \sum_{i=1}^{N} \alpha_i p_i.$$

Multiple-network (quasi-static) poroelasticity (MPET) model

The system

$$
\begin{aligned}
\operatorname{div} \boldsymbol{u} - \lambda^{-1} p_t - \lambda^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{p} &= 0, \\
-\operatorname{div} (2\mu\varepsilon(\boldsymbol{u}) + p_t \underline{\boldsymbol{I}}) &= \boldsymbol{f}, \\
s_i \dot{p}_i + \alpha_i \lambda^{-1} (\dot{p}_t + \boldsymbol{\alpha} \cdot \dot{\boldsymbol{p}}) - \nabla \cdot (K_i \nabla p_i) + \xi_i(\boldsymbol{p}) &= g_i \quad i = 1, \ldots, N
\end{aligned}
$$

has an energy-type estimates and its stability can be proved.

Multiple-network (quasi-static) poroelasticity (MPET) model

The system

$$\begin{aligned}\operatorname{div}\boldsymbol{u} - \lambda^{-1}p_t - \lambda^{-1}\boldsymbol{\alpha}\cdot\boldsymbol{p} &= 0,\\ -\operatorname{div}(2\mu\varepsilon(\boldsymbol{u}) + p_t\underline{\boldsymbol{I}}) &= \boldsymbol{f},\\ s_i\dot{p}_i + \alpha_i\lambda^{-1}(\dot{p}_t + \boldsymbol{\alpha}\cdot\dot{\boldsymbol{p}}) - \nabla\cdot(K_i\nabla p_i) + \xi_i(\boldsymbol{p}) &= g_i \quad i = 1,\ldots,N\end{aligned}$$

has an energy-type estimates and its stability can be proved.

- If stable mixed finite elements for the Stokes equation for $(\boldsymbol{u}, p_t)$ and Lagrange finite elements are used for p_i, $i = 1,\ldots,N$, then the discretization gives an optimal approximate solution
- The solution is robust for large λ and small s_i's. [L.-Mardal-Rognes-Piersanti]

Motivation for operator splitting algorithms

In human brain model, $N = 4$ and the system

$$\begin{aligned} \operatorname{div} \boldsymbol{u} - \lambda^{-1} p_t - \lambda^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{p} &= 0, \\ -\operatorname{div}(2\mu\varepsilon(\boldsymbol{u}) + p_t \underline{\boldsymbol{I}}) &= \boldsymbol{f}, \\ s_i \dot{p}_i + \alpha_i \lambda^{-1} (\dot{p}_t + \boldsymbol{\alpha} \cdot \dot{\boldsymbol{p}}) - \nabla \cdot (K_i \nabla p_i) + \xi_i(\boldsymbol{p}) &= g_i \quad i = 1, \ldots, N. \end{aligned}$$

is computationally expensive.

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In large scale simulations, memory limit can be more severe than computation time, so reducing sizes of the linear algebraic system is advantageous for large scale computations

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is computationally expensive.

In large scale simulations, memory limit can be more severe than computation time, so reducing sizes of the linear algebraic system is advantageous for large scale computations

Question
Can we develop unconditionally stable operator splitting methods for multiple-network poroelasticity models?

Operator splitting algorithm 1 (elasticity then diffusion)

Step 1: initial Data

Prepare initial data $(\boldsymbol{u}_h^0, p_{t,h}^0, \boldsymbol{p}_h^0)$ and the first time step solution $(\boldsymbol{u}_h^1, p_{t,h}^1, \boldsymbol{p}_h^1)$ (e.g., by monolithic approach)

Step 2: solve elasticity (Lame) equation (find $(\boldsymbol{u}_h^{n+1}, p_{t,h}^{n+1})$)

$$
\begin{aligned}
-\operatorname{div}(2\mu\epsilon(\boldsymbol{u}_h^{n+1})) - \nabla p_{t,h}^{n+1}\boldsymbol{I} &= \boldsymbol{f}^{n+1} \\
\operatorname{div}\boldsymbol{u}_h^{n+1} - \lambda^{-1}p_{t,h}^{n+1} &= \lambda^{-1}\boldsymbol{\alpha}\cdot\boldsymbol{p}_h^n
\end{aligned}
$$

Step 3: solve heat equation system (find $\boldsymbol{p}_h^{n+1}$)

$$
s_i\frac{p_{i,h}^{n+1} - p_{i,h}^n}{\Delta t} + \alpha_i\lambda^{-1}\boldsymbol{\alpha}\cdot\frac{\boldsymbol{p}_h^{n+1} - \boldsymbol{p}_h^n}{\Delta t} - \operatorname{div}\left(K_i\nabla\boldsymbol{p}_h^{n+1}\right) + \xi_i(\boldsymbol{p}_h^{n+1}) = g^{n+1} - \frac{\alpha_i}{\lambda}\frac{p_{t,h}^{n+1} - p_{t,h}^n}{\Delta t}
$$

Operator splitting algorithm 2 (diffusion then elasticity)

Step 1: initial Data

Prepare initial data $(\boldsymbol{u}_h^0, p_{t,h}^0, \boldsymbol{p}_h^0)$ and the first time step solution $(\boldsymbol{u}_h^1, p_{t,h}^1, \boldsymbol{p}_h^1)$ (e.g., by monolithic approach)

Step 2: solve heat equation system (find $\boldsymbol{p}_h^{n+1}$)

$$s_i \frac{p_{i,h}^{n+1} - p_{i,h}^n}{\Delta t} + \alpha_i \lambda^{-1} \boldsymbol{\alpha} \cdot \frac{\boldsymbol{p}_h^{n+1} - \boldsymbol{p}_h^n}{\Delta t} - \operatorname{div}\left(K_i \nabla \frac{\boldsymbol{p}_h^n + \boldsymbol{p}_h^{n+1}}{2} \right) + \xi_i \left(\frac{\boldsymbol{p}_h^{n+1} + \boldsymbol{p}_h^n}{2} \right) = \frac{g^n + g^{n+1}}{2} - \frac{\alpha_i}{\lambda} \frac{p_{t,h}^n - p_{t,h}^{n-1}}{\Delta t}$$

Step 3: solve elasticity (Lame) equation (find $(\boldsymbol{u}_h^{n+1}, p_{t,h}^{n+1})$)

$$-\operatorname{div}(2\mu\epsilon(\boldsymbol{u}_h^{n+1})) - \nabla p_{t,h}^{n+1} \boldsymbol{I} = \boldsymbol{f}^{n+1}$$

$$\operatorname{div} \boldsymbol{u}_h^{n+1} - \lambda^{-1} p_{t,h}^{n+1} = \lambda^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{p}_h^{n+1}$$

Comparison of two operator splitting algorithms 2

elasticity then diffusion

- First order convergence in time
- Local mass conservation holds
  (with discontinuous Galerkin or enriched Galerkin methods)

diffusion then elasticity

- Second order convergence in time
- Local mass conservation does NOT hold due to time step discrepancy

Variational form (exact solution)

$$\left\langle 2\mu\epsilon(\boldsymbol{u}^{n+1}), \epsilon(\boldsymbol{v})\right\rangle + \left\langle p_t^{n+1}, \operatorname{div} \boldsymbol{v}\right\rangle = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h$$

$$\left\langle \operatorname{div} \boldsymbol{u}^{n+1}, q_t\right\rangle - \left\langle \lambda^{-1} p_t^{n+1}, q_t\right\rangle = \left\langle \lambda^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{p}^{n+1}, q_t\right\rangle, \quad \forall q_t \in Q_{t,h}$$

$$\left\langle s_i \dot{p}_i^{n+1}, q_i\right\rangle + \left\langle \alpha_i \lambda^{-1} \boldsymbol{\alpha} \cdot \dot{\boldsymbol{p}}^{n+1} + \xi_i\left(\boldsymbol{p}^{n+1}\right), q_i\right\rangle + a_{h,i}\left(p_i^{n+1}, q_i\right)$$
$$= -\left\langle \alpha_i \lambda^{-1} \dot{p}_t^{n+1}, q_i\right\rangle \qquad \forall\, q_i \in Q_{i,h}, 1 \le i \le N.$$

$a_{h,i}(v,w)$: discrete bilinear form for $\langle K_i \nabla v, \nabla w\rangle$

$$a_{h,i}(v,w)$$
$$= (K_i \nabla v, \nabla w) - \left(\langle \{\!\{K_i \nabla v\}\!\}, [\![w]\!]\rangle_{\mathcal{E}_h^i \cup \mathcal{E}_h^D} + \langle [\![v]\!], \{\!\{K_i \nabla w\}\!\}\rangle_{\mathcal{E}_h^i \cup \mathcal{E}_h^D}\right)$$
$$+ \left\langle \gamma h_e^{-1} [\![v]\!], [\![w]\!]\right\rangle_{\mathcal{E}_h^i \cup \mathcal{E}_h^D} \quad \text{(symmetric interior penalty DG)}$$

Error analysis of algorithm 1 (elasticity then diffusion)

Variational form (discrete solution)

$$\left\langle 2\mu\epsilon(\boldsymbol{u}_h^{n+1}), \epsilon(\boldsymbol{v}) \right\rangle + \left\langle p_{t,h}^{n+1}, \operatorname{div} \boldsymbol{v} \right\rangle = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h$$

$$\left\langle \operatorname{div} \boldsymbol{u}_h^{n+1}, q_t \right\rangle - \left\langle \lambda^{-1} p_{t,h}^{n+1}, q_t \right\rangle = \left\langle \lambda^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{p}_h^n, q_t \right\rangle, \quad \forall q_t \in Q_{t,h}$$

$$\left\langle s_i \left(\frac{p_{i,h}^{n+1} - p_{i,h}^n}{\Delta t} \right), q_i \right\rangle$$

$$+ \left\langle \alpha_i \lambda^{-1} \boldsymbol{\alpha} \cdot \left(\frac{\boldsymbol{p}_h^{n+1} - \boldsymbol{p}_h^n}{\Delta t} \right) + \xi_i \left(\boldsymbol{p}_h^{n+1} \right), q_i \right\rangle$$

$$+ a_{h,i} \left(p_{i,h}^{n+1}, q_i \right)$$

$$= - \left\langle \alpha_i \lambda^{-1} \left(\frac{p_{t,h}^{n+1} - p_{t,h}^n}{\Delta t} \right), q_i \right\rangle \qquad \forall q_i \in Q_{i,h}, 1 \leq i \leq N.$$

Variational form (error)

$$e_\sigma^n := \sigma^n - \sigma_h^n$$

$$\left\langle 2\mu\epsilon(e_{\boldsymbol{u}}^{n+1}), \epsilon(\boldsymbol{v}) \right\rangle + \left\langle e_{p_t}^{n+1}, \operatorname{div}\boldsymbol{v} \right\rangle = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h$$

$$\left\langle \operatorname{div} e_{\boldsymbol{u}}^{n+1}, q_t \right\rangle - \left\langle \lambda^{-1} e_{p_t}^{n+1}, q_t \right\rangle = \left\langle \lambda^{-1}\boldsymbol{\alpha} \cdot \left(\boldsymbol{p}^{n+1} - \boldsymbol{p}_h^n \right), q_t \right\rangle, \quad \forall q_t \in Q_{t,h}$$

$$\left\langle s_i \left(\dot{p}_i^{n+1} - \frac{p_{i,h}^{n+1} - p_{i,h}^n}{\Delta t} \right), q_i \right\rangle$$

$$+ \left\langle \alpha_i \lambda^{-1} \boldsymbol{\alpha} \cdot \left(\dot{\boldsymbol{p}}^{n+1} - \frac{\boldsymbol{p}_h^{n+1} - \boldsymbol{p}_h^n}{\Delta t} \right) + \xi_i \left(e_{\boldsymbol{p}}^{n+1} \right), q_i \right\rangle + a_{h,i}\left(e_{p_i}^{n+1}, q_i \right)$$

$$= - \left\langle \alpha_i \lambda^{-1} \left(\dot{p}_t^{n+1} - \frac{p_{t,h}^{n+1} - p_{t,h}^n}{\Delta t} \right), q_i \right\rangle \qquad \forall q_i \in Q_{i,h}, 1 \le i \le N.$$

Error analysis of algorithm 1 (elasticity then diffusion)

Elliptic projection as interpolation

$(\Pi_h \boldsymbol{u}^{n+1}, \Pi_h p_t^{n+1})$ is the solution of

$$
\begin{aligned}
&\left\langle 2\mu\epsilon(\Pi_h \boldsymbol{u}^{n+1}), \epsilon(\boldsymbol{v}) \right\rangle + \left\langle \Pi_h p_t^{n+1}, \operatorname{div} \boldsymbol{v} \right\rangle \\
&\quad = \left\langle 2\mu\epsilon(\boldsymbol{u}^{n+1}), \epsilon(\boldsymbol{v}) \right\rangle + \left\langle p_t^{n+1}, \operatorname{div} \boldsymbol{v} \right\rangle, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h \\
&\left\langle \operatorname{div} \Pi_h \boldsymbol{u}^{n+1}, q_t \right\rangle - \left\langle \lambda^{-1} \Pi_h p_t^{n+1}, q_t \right\rangle \\
&\quad = \left\langle \operatorname{div} \boldsymbol{u}^{n+1}, q_t \right\rangle - \left\langle \lambda^{-1} p_t^{n+1}, q_t \right\rangle, \quad \forall q_t \in Q_{t,h}
\end{aligned}
$$

Error analysis of algorithm 1 (elasticity then diffusion)

Elliptic projection as interpolation

$\Pi_h \boldsymbol{p}^{n+1}$ is the solution of the system

$$\begin{aligned} &\left\langle \xi_i \left(\Pi_h \boldsymbol{p}^{n+1} \right), q_i \right\rangle + a_{h,i} \left(\Pi_h p_i^{n+1}, q_i \right) \\ &\quad = \left\langle \xi_i \left(\boldsymbol{p}^{n+1} \right), q_i \right\rangle + a_{h,i} \left(p_i^{n+1}, q_i \right) \qquad \forall\, q_i \in Q_{i,h}, 1 \le i \le N. \end{aligned}$$

Splitting of errors:

$$e_\sigma^n = \sigma^n - \sigma_h^n = (\sigma^n - \Pi_h \sigma^n) + (\Pi_h \sigma^n - \sigma_h^n) =: e_\sigma^{I,n} + e_\sigma^{h,n}$$

Variational form (error)

$$\left\langle 2\mu\epsilon(e_{\boldsymbol{u}}^{h,n+1}), \epsilon(\boldsymbol{v})\right\rangle + \left\langle e_{p_t}^{h,n+1}, \operatorname{div}\boldsymbol{v}\right\rangle = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h$$

$$\left\langle \operatorname{div} e_{\boldsymbol{u}}^{h,n+1}, q_t\right\rangle - \left\langle \lambda^{-1} e_{p_t}^{h,n+1}, q_t\right\rangle = \left\langle \lambda^{-1}\boldsymbol{\alpha}\cdot\left(\boldsymbol{p}^{n+1} - \boldsymbol{p}_h^n\right), q_t\right\rangle, \quad \forall q_t \in Q$$

$$\left\langle s_i\left(\dot{p}_i^{n+1} - \frac{p_{i,h}^{n+1} - p_{i,h}^n}{\Delta t}\right), q_i\right\rangle$$

$$+ \left\langle \alpha_i\lambda^{-1}\boldsymbol{\alpha}\cdot\left(\dot{\boldsymbol{p}}^{n+1} - \frac{\boldsymbol{p}_h^{n+1} - \boldsymbol{p}_h^n}{\Delta t}\right) + \xi_i\left(e_{\boldsymbol{p}}^{h,n+1}\right), q_i\right\rangle$$

$$+ a_{h,i}\left(e_{p_i}^{h,n+1}, q_i\right)$$

$$= -\left\langle \alpha_i\lambda^{-1}\left(\dot{p}_t^{n+1} - \frac{p_{t,h}^{n+1} - p_{t,h}^n}{\Delta t}\right), q_i\right\rangle \qquad \forall q_i \in Q_{i,h}, 1 \le i \le N.$$

Variational form (error)

$$\left\langle 2\mu\epsilon(e_{\boldsymbol{u}}^{h,n+1}), \epsilon(\boldsymbol{v})\right\rangle + \left\langle e_{p_t}^{h,n+1}, \operatorname{div}\boldsymbol{v}\right\rangle = 0,$$

$$\left\langle \operatorname{div} e_{\boldsymbol{u}}^{h,n+1}, q_t\right\rangle - \left\langle \lambda^{-1} e_{p_t}^{h,n+1}, q_t\right\rangle = \left\langle \lambda^{-1}\boldsymbol{\alpha}\cdot\left(\boldsymbol{p}^{n+1} - \boldsymbol{p}_h^n\right), q_t\right\rangle,$$

$$\left\langle s_i\left(\dot{p}_i^{n+1} - \frac{\Pi_h p_i^{n+1} - \Pi_h p_i^n}{\Delta t} + \frac{e_{p_i}^{h,n+1} - e_{p_i}^{h,n}}{\Delta t}\right), q_i\right\rangle$$

$$+\left\langle \alpha_i\lambda^{-1}\boldsymbol{\alpha}\cdot\left(\dot{\boldsymbol{p}}^{n+1} - \frac{\Pi_h\boldsymbol{p}^{n+1} - \Pi_h\boldsymbol{p}^n}{\Delta t} + \frac{e_{\boldsymbol{p}}^{h,n+1} - e_{\boldsymbol{p}}^{h,n}}{\Delta t}\right) + \xi_i\left(e_{\boldsymbol{p}}^{h,n+1}\right)\right.$$

$$+\, a_{h,i}\left(e_{p_i}^{h,n+1}, q_i\right)$$

$$= -\left\langle \alpha_i\lambda^{-1}\left(\dot{p}_t^{n+1} - \frac{\Pi_h p_t^{n+1} - \Pi_h p_t^n}{\Delta t} + \frac{e_{p_t}^{h,n+1} - e_{p_t}^{h,n}}{\Delta t}\right), q_i\right\rangle$$

Variational form (error)

$$
\begin{aligned}
&\left\langle 2\mu\epsilon(e_{\boldsymbol{u}}^{h,n+1}), \epsilon(\boldsymbol{v})\right\rangle + \left\langle e_{p_t}^{h,n+1}, \operatorname{div}\boldsymbol{v}\right\rangle = 0,\\
&\left\langle \operatorname{div} e_{\boldsymbol{u}}^{h,n+1}, q_t\right\rangle - \left\langle \lambda^{-1} e_{p_t}^{h,n+1}, q_t\right\rangle = \left\langle \lambda^{-1}\boldsymbol{\alpha}\cdot\left(\boldsymbol{p}^{n+1} - \Pi_h\boldsymbol{p}^n + e_{\boldsymbol{p}}^{h,n}\right), q_t\right\rangle,\\
&\left\langle s_i\left(\frac{e_{p_i}^{h,n+1} - e_{p_i}^{h,n}}{\Delta t}\right), q_i\right\rangle\\
&\quad + \left\langle \alpha_i\lambda^{-1}\boldsymbol{\alpha}\cdot\left(\frac{e_{\boldsymbol{p}}^{h,n+1} - e_{\boldsymbol{p}}^{h,n}}{\Delta t}\right) + \xi_i\left(e_{\boldsymbol{p}}^{h,n+1}\right), q_i\right\rangle\\
&\quad + a_{h,i}\left(e_{p_i}^{h,n+1}, q_i\right)\\
&= -\left\langle \alpha_i\lambda^{-1}\left(\frac{e_{p_t}^{h,n+1} - e_{p_t}^{h,n}}{\Delta t}\right), q_i\right\rangle - \left\langle s_i I_{1,i}^n + \alpha_i\lambda^{-1}\boldsymbol{\alpha}\cdot I_2^n + \alpha_i\lambda^{-1} I_3^n, q_i\right\rangle
\end{aligned}
$$

Error analysis of algorithm 1 (elasticity then diffusion)

Interpolation errors

$$I_{1,i}^n = \dot{p}_i^{n+1} - \frac{\Pi_h p_i^{n+1} - \Pi_h p_i^n}{\Delta t}$$
$$I_2^n = \dot{\boldsymbol{p}}^{n+1} - \frac{\Pi_h \boldsymbol{p}^{n+1} - \Pi_h \boldsymbol{p}^n}{\Delta t}$$
$$I_3^n = \dot{p}_t^{n+1} - \frac{\Pi_h p_t^{n+1} - \Pi_h p_t^n}{\Delta t}$$

Analysis challenges
The index (time step) discrepancy is an obstacle to obtain an energy-type estimate in a standard way.
To overcome this difficulty, we consider estimates of difference terms

$$D_\sigma^n := e_\sigma^{h,n+1} - e_\sigma^{h,n}$$

Variational form (error)

$$\left\langle 2\mu\epsilon(D_{\boldsymbol{u}}^{n+1}), \epsilon(\boldsymbol{v})\right\rangle + \left\langle D_{p_t}^{n+1}, \operatorname{div}\boldsymbol{v}\right\rangle = 0,$$

$$\left\langle \operatorname{div} D_{\boldsymbol{u}}^{n+1}, q_t\right\rangle - \left\langle \lambda^{-1} D_{p_t}^{n+1}, q_t\right\rangle$$

$$= \left\langle \lambda^{-1}\boldsymbol{\alpha}\cdot\left(I_0^n + D_{\boldsymbol{p}}^n\right), q_t\right\rangle,$$

$$\left\langle s_i D_{p_i}^n, q_i\right\rangle + \left\langle \alpha_i\lambda^{-1}\boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^n + \Delta t\xi_i\left(e_{\boldsymbol{p}}^{h,n+1}\right), q_i\right\rangle$$

$$+ \Delta t a_{h,i}\left(e_{p_i}^{h,n+1}, q_i\right)$$

$$= -\left\langle \alpha_i\lambda^{-1} D_{p_t}^n, q_i\right\rangle - \Delta t\left\langle s_i I_{1,i}^n + \alpha_i\lambda^{-1}\boldsymbol{\alpha}\cdot I_2^n + \alpha_i\lambda^{-1} I_3^n, q_i\right\rangle$$

where

$$I_0^n = \boldsymbol{p}^{n+1} - \boldsymbol{p}^n - \Pi_h\boldsymbol{p}^n - \Pi_h\boldsymbol{p}^{n-1}$$

Error analysis of algorithm 1 (elasticity then diffusion)

From the inf-sup stability of $(\boldsymbol{V}_h, Q_{t,h})$, there exists $\boldsymbol{w}^n \in \boldsymbol{V}_h$ such that

$$\left\langle \operatorname{div} \boldsymbol{w}^n, D_{p_t}^n \right\rangle = \|D_{p_t}^n\|_{Q_t}^2, \qquad \|\boldsymbol{w}^n\|_{\boldsymbol{V}} \lesssim \|D_{p_t}^n\|_{Q_t}.$$

where

$$\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 = \langle 2\mu\epsilon(\boldsymbol{v}), \epsilon(\boldsymbol{v})\rangle, \quad \|q_t\|_{Q_t}^2 = \left\langle (2\mu)^{-1} q_t, q_t \right\rangle$$

By taking $\boldsymbol{v} = D_{\boldsymbol{u}}^n + \delta \boldsymbol{w}^n$ and $q_t = -D_{p_t}^n$ with $\delta > 0$ independent of h, we can get

$$\|D_{\boldsymbol{u}}^n\|_{\boldsymbol{V}}^2 + C\|D_{p_t}^n\|_{Q_t}^2 + \|D_{p_t}^n\|_{\lambda^{-1}}^2 \le \left\langle \lambda^{-1}\boldsymbol{\alpha} \cdot I_0^n, D_{p_t}^n \right\rangle + \left\langle \lambda^{-1}\boldsymbol{\alpha} \cdot D_{\boldsymbol{p}}^n, D_{p_t}^n \right\rangle$$

with $C > 0$ independent of h

Error analysis of algorithm 1 (elasticity then diffusion)

Taking $q_i = D_{p_i}^n$ for $1 \le i \le N$,

$$\left\langle \operatorname{div} \boldsymbol{w}^n, D_{p_t}^n \right\rangle = \|D_{p_t}^n\|_{Q_t}^2, \qquad \|\boldsymbol{w}^n\|_{\boldsymbol{V}} \lesssim \|D_{p_t}^n\|_{Q_t}.$$

where

$$\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 = \langle 2\mu\epsilon(\boldsymbol{v}), \epsilon(\boldsymbol{v})\rangle, \quad \|q_t\|_{Q_t}^2 = \left\langle (2\mu)^{-1} q_t, q_t \right\rangle$$

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$$\|D_{\boldsymbol{u}}^n\|_{\boldsymbol{V}}^2 + C\|D_{p_t}^n\|_{Q_t}^2 + \|D_{p_t}^n\|_{\lambda^{-1}}^2 \le \left\langle \lambda^{-1}\boldsymbol{\alpha} \cdot I_0^n, D_{p_t}^n \right\rangle + \left\langle \lambda^{-1}\boldsymbol{\alpha} \cdot D_{\boldsymbol{p}}^n, D_{p_t}^n \right\rangle$$

with $C > 0$ independent of h

Error analysis of algorithm 1 (elasticity then diffusion)

Defining

$$\mathcal{A}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{N} \left(\langle \xi_i(\boldsymbol{p}), q_i \rangle + a_{h,i}(p_i, q_i) \right)$$

the sum of the previous inequalities yield

$$\begin{aligned}
&\|D_{\boldsymbol{u}}^n\|_{\boldsymbol{V}}^2 + C\|D_{p_t}^n\|_{Q_t}^2 + \|D_{p_t}^n\|_{\lambda^{-1}}^2 + \|\boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^n\|_{\lambda^{-1}}^2 \\
&\quad + \sum_{i=1}^{N}\left[\|D_{p_i}^n\|_{s_i}^2\right] + \Delta t\mathcal{A}(e_{\boldsymbol{p}}^{h,n+1}, e_{\boldsymbol{p}}^{h,n+1}) \\
&\quad \le \left\langle \lambda^{-1}\boldsymbol{\alpha}\cdot I_0^n, D_{p_t}^n \right\rangle + \left\langle \lambda^{-1}\boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^n, D_{p_t}^n \right\rangle + \Delta t\mathcal{A}(e_{\boldsymbol{p}}^{h,n+1}, e_{\boldsymbol{p}}^{h,n}) \\
&\qquad + \Delta t\left(\sum_i \left\langle s_i I_{1,i}^n, D_{p_i}^n \right\rangle + \left\langle \lambda^{-1}(I_2^n + I_3^n), \boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^n \right\rangle \right) \\
&\qquad - \left\langle \lambda^{-1} D_{p_t}^{n-1}, \boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^n \right\rangle
\end{aligned}$$

After Young's inequality,

$$\begin{aligned}
&\|D_{\boldsymbol{u}}^n\|_{\boldsymbol{V}}^2 + \frac{C}{2}\|D_{p_t}^n\|_{\lambda^{-1}}^2 + C_1(\|\boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^n\|_{\lambda^{-1}}^2) \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\left[\|D_{p_i}^n\|_{s_i}^2\right] + \frac{1}{2}\Delta t\mathcal{A}(e_{\boldsymbol{p}}^{h,n+1}, e_{\boldsymbol{p}}^{h,n+1}) \\
&\quad \leq \frac{1}{4\epsilon}\left(\|I_0^n\|_{\lambda^{-1}}^2 + \sum_{j=1}^{3}(\Delta t)^2\|I_j^n\|^2\right) + \frac{C}{4}\|D_{p_t}^{n-1}\|_{\lambda^{-1}}^2 \\
&\quad + \frac{1}{2}\Delta t\mathcal{A}(e_{\boldsymbol{p}}^{h,n}, e_{\boldsymbol{p}}^{h,n})
\end{aligned}$$

Error analysis of algorithm 1 (elasticity then diffusion)

The summation over n gives (with C_0 depending only on C)

$$\sum_{l=1}^{n}\left[\|D_{\boldsymbol{u}}^l\|_{\boldsymbol{V}}^2 + C_0(\|D_{p_t}^l\|_{\lambda^{-1}}^2 + \|\boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^l\|_{\lambda^{-1}}^2) + \frac{1}{2}\sum_{i=1}^{N}\left[\|D_{p_i}^l\|_{s_i}^2\right]\right]$$
$$+ \frac{1}{2}\Delta t\mathcal{A}(e_{\boldsymbol{p}}^{h,n+1}, e_{\boldsymbol{p}}^{h,n+1})$$
$$\leq \frac{1}{4\epsilon}\sum_{l=1}^{n}\left(\|I_0^l\|_{\lambda^{-1}}^2 + \sum_{j=1}^{3}(\Delta t)^2\|I_j^l\|^2\right)$$
$$+ \frac{1}{2}\Delta t\mathcal{A}(e_{\boldsymbol{p}}^{h,0}, e_{\boldsymbol{p}}^{h,0})$$

As a consequence, we have $O(\Delta t(\Delta t + h^k))$ estimate of

$$\sum_{l=1}^{n}\left[\|D_{\boldsymbol{u}}^l\|_{\boldsymbol{V}}^2 + \|D_{p_t}^l\|_{\lambda^{-1}}^2 + \|\boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^l\|_{\lambda^{-1}}^2 + \frac{1}{2}\sum_{i=1}^{N}\|D_{p_i}^l\|_{s_i}^2\right]$$

Error analysis of algorithm 1 (elasticity then diffusion)

To estimate $e_{\boldsymbol{p}}^{h,n}$, take $q_i = e_{p_i}^{h,n+1}$ in

$$\left\langle s_i D_{p_i}^n, q_i \right\rangle + \left\langle \alpha_i \lambda^{-1} \boldsymbol{\alpha} \cdot D_{\boldsymbol{p}}^{h,n} + \Delta t \xi_i \left(e_{\boldsymbol{p}}^{h,n+1} \right), q_i \right\rangle + \Delta t a_{h,i} \left(e_{p_i}^{h,n+1}, q_i \right) = - \left\langle \alpha_i \lambda^{-1} D_{p_t}^n, q_i \right\rangle - \Delta t \left\langle s_i I_{1,i}^n + \alpha_i \lambda^{-1} \boldsymbol{\alpha} \cdot I_2^n + \alpha_i \lambda^{-1} I_3^n, q_i \right\rangle$$

which gives

$$\sum_{i=1}^N \| e_{p_i}^{h,n+1} \|_{s_i}^2 + \| \boldsymbol{\alpha} \cdot e_{\boldsymbol{p}}^{h,n+1} \|_{\lambda^{-1}}^2 + \Delta t \mathcal{A}(e_{\boldsymbol{p}}^{h,n+1}, e_{\boldsymbol{p}}^{h,n+1}) = - \sum_{i=1}^N \left\langle s_i e_{p_i}^{h,n}, e_{p_i}^{h,n+1} \right\rangle + \left\langle \lambda^{-1} e_{\boldsymbol{p}}^{h,n}, \boldsymbol{\alpha} \cdot e_{\boldsymbol{p}}^{h,n+1} \right\rangle - \left\langle \lambda^{-1} D_{p_t}^n, \boldsymbol{\alpha} \cdot e_{\boldsymbol{p}}^{h,n+1} \right\rangle - \Delta t \sum_{i=1}^N \left\langle s_i I_{1,i}^n + \alpha_i \lambda^{-1} \boldsymbol{\alpha} \cdot I_2^n + \alpha_i \lambda^{-1} I_3^n, e_{p_i}^{h,n+1} \right\rangle$$

Error analysis of algorithm 1 (elasticity then diffusion)

By (discrete) Poincare inequality,

$$\|\boldsymbol{\alpha} \cdot e_{\boldsymbol{p}}^{h,n+1}\|_{\lambda^{-1}}^2 \le C_P^2 \mathcal{A}(e_{\boldsymbol{p}}^{h,n+1}, e_{\boldsymbol{p}}^{h,n+1})$$

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N} \|e_{p_i}^{h,n+1}\|_{s_i}^2 + \frac{1}{2}\|\boldsymbol{\alpha} \cdot e_{\boldsymbol{p}}^{h,n+1}\|_{\lambda^{-1}}^2 \\
&= \frac{1}{2}\sum_{i=1}^{N} \|e_{p_i}^{h,n}\|_{s_i}^2 + \frac{1}{2}\|\boldsymbol{\alpha} \cdot e_{\boldsymbol{p}}^{h,n}\|_{\lambda^{-1}}^2 + \frac{1}{\epsilon \Delta t}\|D_{p_t}^n\|_{\lambda^{-1}}^2 \\
&\quad + \sum_{i=1}^{N} \|I_1, I_2, I_3\|^2
\end{aligned}$$

Since we estimated $\sum_{l=1}^{n} \|D_{p_t}^l\|_{\lambda^{-1}}^2$ by $O(\Delta t(\Delta t + h^k))$, we can estimate $\sum_{i=1}^{N} \|e_{p_i}^{h,n+1}\|_{s_i}^2 + \|\boldsymbol{\alpha} \cdot e_{\boldsymbol{p}}^{h,n+1}\|_{\lambda^{-1}}^2$ by $O(\Delta t + h^k)$

Summary of error estimates

Streamline of error estimate

- Estimate $\sum_{l=1}^{n}\left[\|D_{p_t}^l, \boldsymbol{\alpha}\cdot D_{\boldsymbol{p}}^l\|_{\lambda^{-1}}^2\right]$
- Estimate $\|\boldsymbol{\alpha}\cdot e_{\boldsymbol{p}}^{h,n}\|_{\lambda^{-1}}$
- Estimate $\|e_{\boldsymbol{u}}^{h,n}\|_{\boldsymbol{V}}$ and $\|e_{p_t}^{h,n}\|_{Q_t}$ from the Lamé problem
- Estimate $a_{h,i}(e_{p_i}^{h,n}, e_{p_i}^{h,n})$ from the coupled heat equations

Error analysis of the second operator splitting method

Similar idea works and $O((\Delta t)^2 + h^k)$ estimate can be obtained

Numerical results (only the second algorithm, Biot)

Discretization with Taylor-Hood element ($P_2 - P_1$) for the elasticity problem and P_1 element for the Poisson equation
$\Delta t = h (= 1/N)$

N	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1}$	Rate	$\|p_t - p_{t,h}\|_{L^2}$	Rate	$\|p - p_h\|_{H^1}$	Rate	$\|p - p_h\|_{L^2}$	Rate
4	$4.95e-02$	–	$7.44e-01$	–	$1.38e-01$	–	$8.62e-03$	–
8	$1.18e-02$	2.06	$1.77e-01$	2.07	$6.95e-02$	1.00	$2.20e-03$	1.97
16	$2.92e-03$	2.02	$4.37e-02$	2.02	$3.48e-02$	1.00	$5.55e-04$	1.99
32	$7.27e-04$	2.00	$1.09e-02$	2.00	$1.74e-02$	1.00	$1.39e-04$	2.00
64	$1.82e-04$	2.00	$2.72e-03$	2.00	$8.69e-03$	1.00	$3.48e-05$	2.00

Table: $N \times N$ mesh of unit square dividing each square by a diagonal.

When $N = 64$, $\dim V_h = 33282$, $\dim Q_{t,h} = 4225$, $\dim Q_h = 4225$

Numerical results (only the second algorithm, Biot)

Discretization with stabilized P_1–P_0 element for the elasticity problem and P_1 element for the Poisson equation
$\Delta t = h(= 1/N)$

N	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1}$	Rate	$\|p_t - p_{t,h}\|_{L^2}$	Rate	$\|p - p_h\|_{H^1}$	Rate	$\|p - p_h\|_{L^2}$	Rate
4	$4.56e-01$	–	$6.77e+00$	–	$1.39e-01$	–	$8.81e-03$	–
8	$2.30e-01$	0.99	$3.42e+00$	0.98	$6.95e-02$	1.00	$2.29e-03$	1.95
16	$1.15e-01$	1.00	$1.72e+00$	0.99	$3.48e-02$	1.00	$5.79e-04$	1.98
32	$5.74e-02$	1.00	$8.62e-01$	1.00	$1.74e-02$	1.00	$1.45e-04$	1.99
64	$2.87e-02$	1.00	$4.31e-01$	1.00	$8.69e-03$	1.00	$3.63e-05$	2.00

Table: $N \times N$ mesh of unit square dividing each square by a diagonal.

When $N = 64$, $\dim V_h = 8450$, $\dim Q_{t,h} = 8192$, $\dim Q_h = 4225$

Numerical results (only the second algorithm, Biot)

Discretization with stabilized P_1–P_1 element for the elasticity problem and P_1 element for the Poisson equation
$\Delta t = h (= 1/N)$

N	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1}$	Rate	$\|p_t - p_{t,h}\|_{L^2}$	Rate	$\|p - p_h\|_{H^1}$	Rate	$\|p - p_h\|_{L^2}$	Rate
4	$4.56e-01$	–	$1.22e+00$	–	$1.39e-01$	–	$8.74e-03$	–
8	$2.29e-01$	0.99	$2.96e-01$	2.04	$6.95e-02$	1.00	$2.25e-03$	1.96
16	$1.15e-01$	1.00	$7.42e-02$	2.00	$3.48e-02$	1.00	$5.69e-04$	1.99
32	$5.74e-02$	1.00	$1.91e-02$	1.96	$1.74e-02$	1.00	$1.43e-04$	2.00
64	$2.87e-02$	1.00	$5.06e-03$	1.91	$8.69e-03$	1.00	$3.57e-05$	2.00

Table: $N \times N$ mesh of unit square dividing each square by a diagonal.

When $N = 64$, $\dim V_h = 8450$, $\dim Q_{t,h} = 4225$, $\dim Q_h = 4225$

Conclusion

1. We proposed two unconditionally stable operator splitting algorithms for MPET (also Biot) model
2. The first method is locally conservative with first order in time convergence
3. The second method is NOT locally conservative with second order in time convergence
4. The methods can be optimized with well-known solvers for the heat equation and the Lame equation

To do

1. Locally conservative operator splitting method with higher order convergence in time
2. Preconditioning

Thank you!