Stochastic Collocation Methods for Stability Analysis of Dynamical Systems

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Pseudospectra

Eigenvalue problem $J\mathbf{v} = \lambda M\mathbf{v}$

For $M = \boldsymbol{I}$, perturbed problem

$$(J + E)\mathbf{v} = \lambda\mathbf{v}$$

E a perturbation, $\|E\| \leq \epsilon$

Explore pseudospectra $\cup_{\|E\|\leq\epsilon}\sigma(E)$

More in sync with transient growth

Expensive

Eigenvalue problem $J\mathbf{v} = \lambda M\mathbf{v}$, recalling $J = J(\mathbf{u}_\nu)$

Let $\mathbf{u}_\nu + d\mathbf{u}$ be a perturbation of the steady solution $\mathbf{u}_\nu$

Idea: consider perturbed eigenvalue problem $J(\mathbf{u}_\nu + d\mathbf{u})\mathbf{v} = \lambda\, M\mathbf{v}$
Generate perturbation $d\mathbf{u} \equiv d\mathbf{u}(\xi)$ in a systematic way,
depending on some (other) parameters $\xi \equiv (\xi_1, \ldots, \xi_m)^T$

$$J(\mathbf{u}_\nu + d\mathbf{u}(\xi))\,\mathbf{v} = \lambda\, M\mathbf{v} \tag{1}$$

Let $g(\xi) \equiv$ rightmost eigenvalue of (1)

$g^s(\xi) \equiv$ surrogate approximation of $g(\xi)$, cheaper to compute

Pseudo-spectral experiment: study rightmost values of $g^s(\xi)$

Done by *sampling* ξ

Perturbed Eigenvalue Problem

- Steady flow $\vec{u}$ gives

$$\begin{pmatrix} F(\mathbf{u}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} -Q & \alpha B^T \\ \alpha B & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \eta \end{pmatrix}$$

Specify discrete perturbed flow $\mathbf{u} + d\mathbf{u} \longrightarrow$ perturbed eigenvalue problem

$$\begin{pmatrix} F(\mathbf{u}+d\mathbf{u}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} -Q & \alpha B^T \\ \alpha B & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \eta \end{pmatrix}$$

For the perturbation: $d\vec{u} \equiv curl\, \phi(\cdot, \xi)$,

$$\phi(x, \xi) \equiv \sigma \sum_{k=1}^{m} \sqrt{\mu_k}\, \phi_k(x)\, \xi_k$$

Here, $\{\xi_k\}$ are i.i.d. bounded random variables, $\{\mu_k, \phi_k\}$ are dominant eigenpairs of the integral operator associated with the correlation function

$$C[\phi](x, y) = exp\left(-\tfrac{1}{4}\|x-y\|_{\ell_2}^2\right), \qquad x, y \in \mathcal{D}$$

Perturbed eigenvalue problem

$$\begin{pmatrix} F(\mathbf{u}+d\mathbf{u}(\xi)) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} -Q & \alpha B^T \\ \alpha B & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \eta \end{pmatrix}$$

$$F(\mathbf{u}+d\mathbf{u}(\xi)) = \nu A + N + S(\xi) + W$$

Key points:

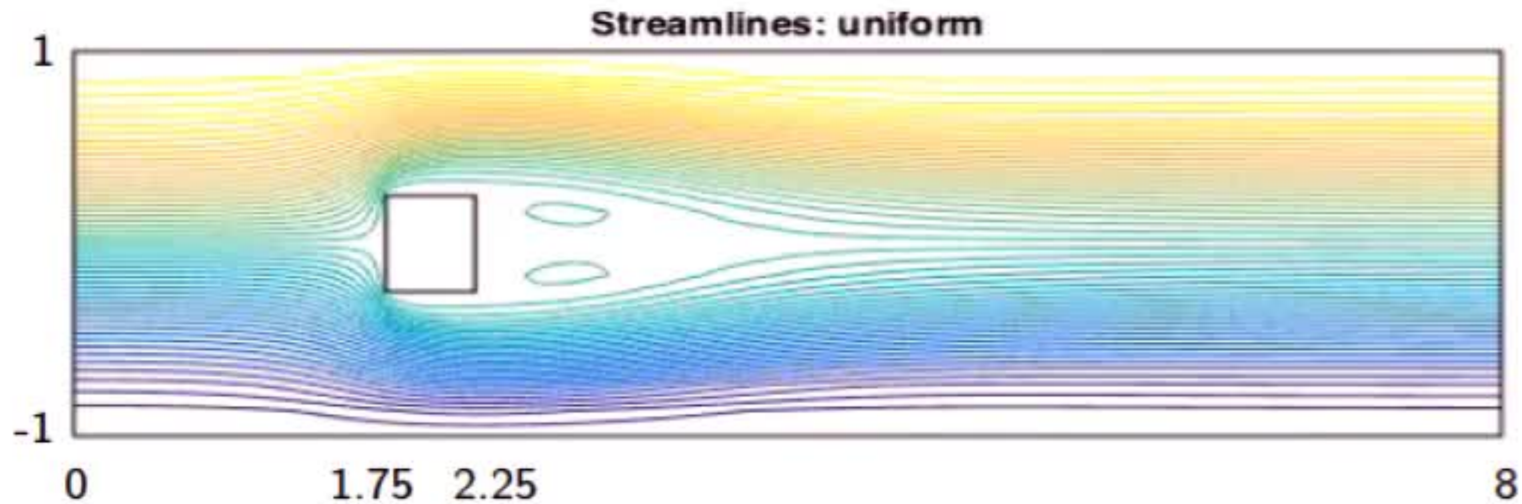
- The perturbation is divergence-free:
  $div\, d\vec{u}(\xi) = div\ curl\, \phi(\xi) = 0$ for any ξ
- The parameter set $\{\xi\}$ can be efficiently sampled using sparse-grid collocation. In experiments, using package `spinterp` (Klimke & Wohlmuth)

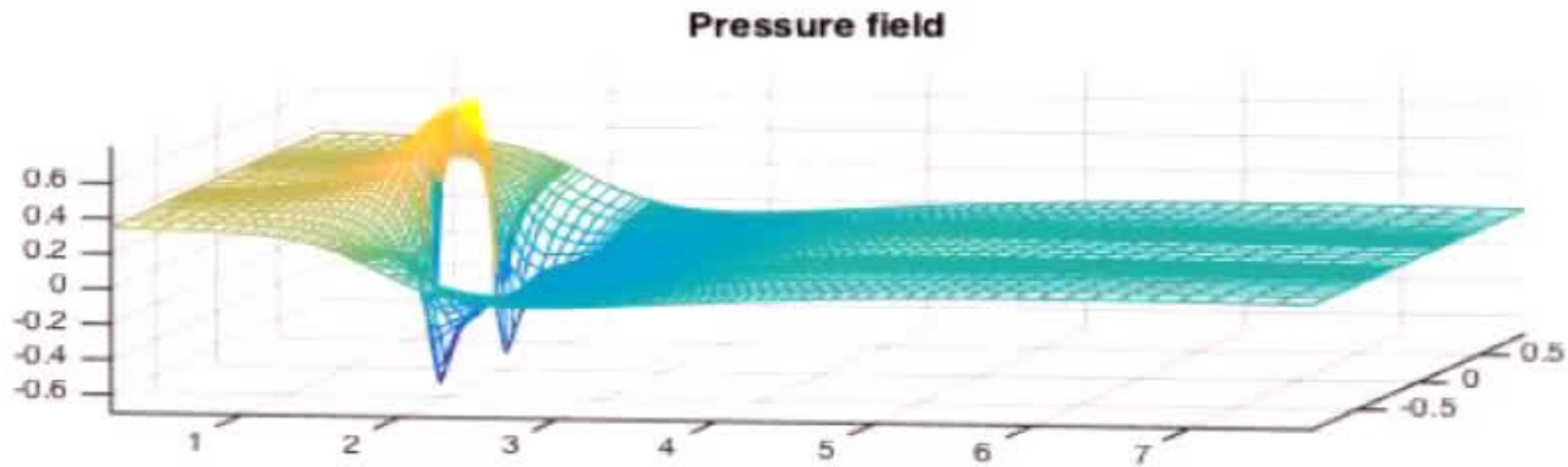
Computational Tests

- **Test problem:** Obstacle channel flow

$Re = \frac{2}{\nu} = 384$
Near critical

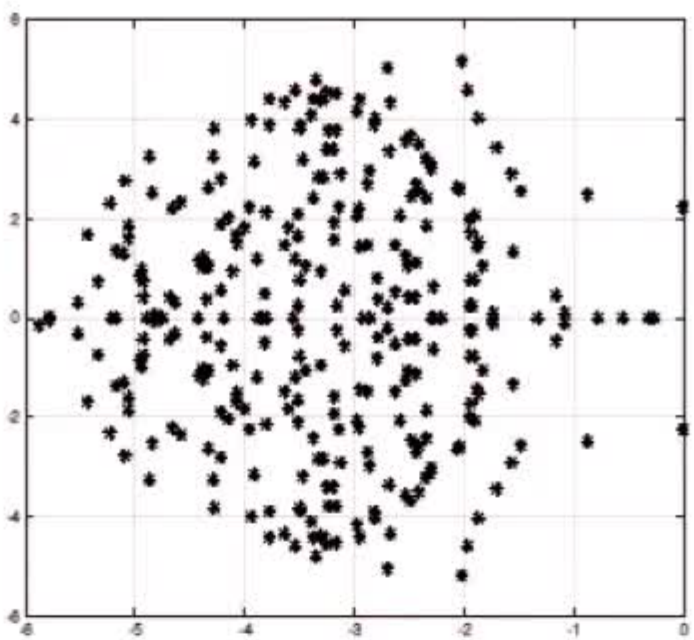
Level 5 non-uniform grid



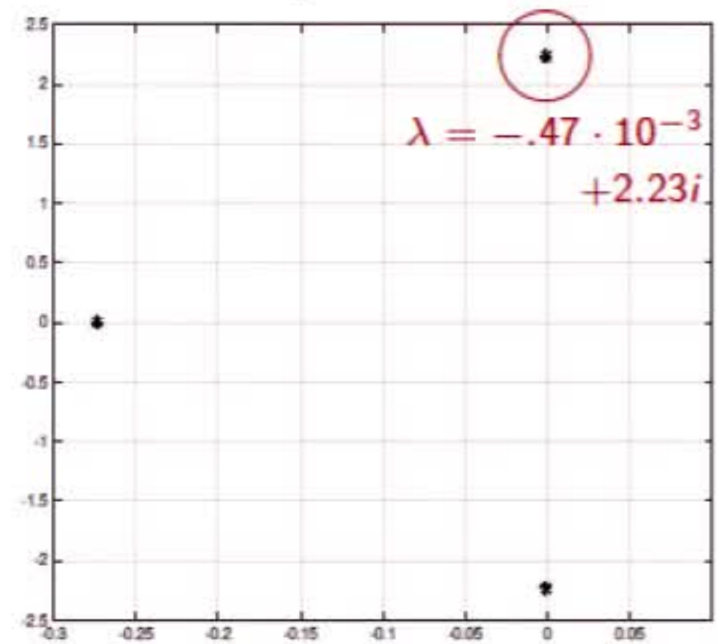


Eigenvalues for steady solution:

300 smallest eigenvalues



Closeup of three rightmost eigenvalues

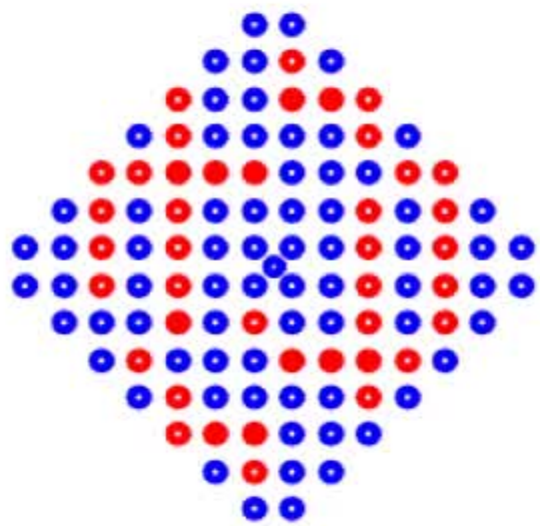


Surrogate Computations

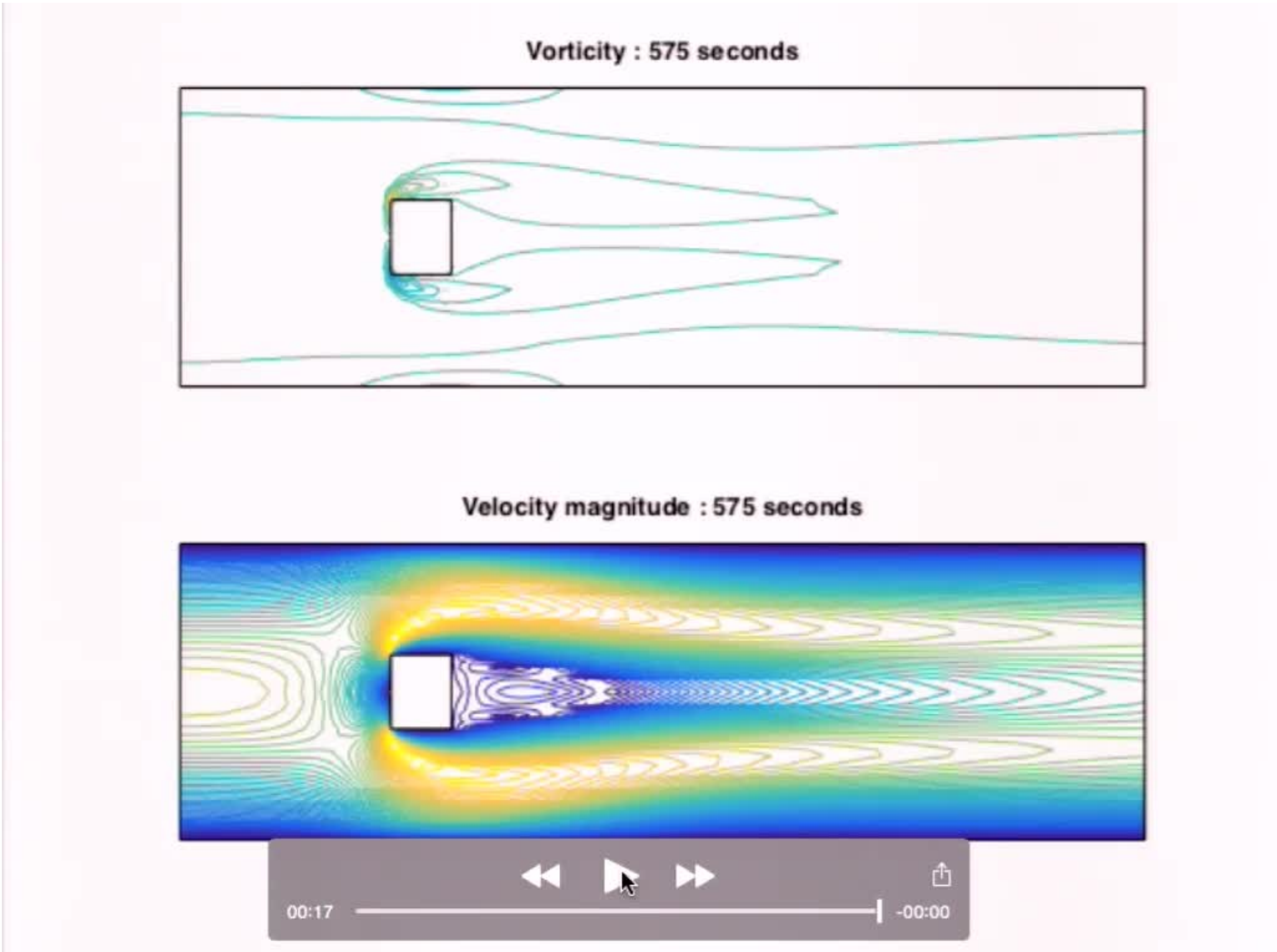
What is needed for surrogate computations?

- Recall $d\vec{u} = curl\ \left(\sigma \sum_{k=1}^{m} \sqrt{\mu_k}\, \phi_k(x)\, \xi_k\right)$; will use $\sigma = .2$
- Dimension of parameter space is $m = 7$
- Use sparse grid of level 2 $\longrightarrow$ $n_\xi = 113$ collocation points
- Main cost: computation of rightmost eigenvalues of 113 perturbed eigenvalue problems
- Simulation: evaluation of surrogate function $g^s(\xi)$ at n_s sample points. Will use $n_s = 10,000$.
  Dirt cheap

Compare with results of transient iteration



- Each dot represents a run of an adaptive time stepping code (stabtrNS) with a collocation point perturbation added at T=500 for 1e-4 time units (12 time steps, starting from an initial time step of 1e-9)
- Open blue circle : stable perturbation (71 cases)
- Open red circle : possibly unstable (30 cases)
- Filled red circle : unstable perturbation (11 cases)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