Swarming by Nature and by Design

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Sonia Kovalevsky Lecture

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Thanks to support from ARO, NSF, and ONR and to contributions from numerous collaborators.

Students, Postdocs, and Collaborators

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- Yanghong Huang
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- Dejan Slepcev
- Chad Topaz
- Thomas Laurent

- Carolyn Boettcher, Raytheon
- Lincoln Chayes, UCLA Math
- Emilio Frazzoli, MIT Aeronautics
- Mark Lewis, Alberta Math Biology
- Richard Murray, Cal Tech Control Dyn. Sys.
- Ira Schwartz, Naval Research Lab
- Jose Carrillo, Barcelona
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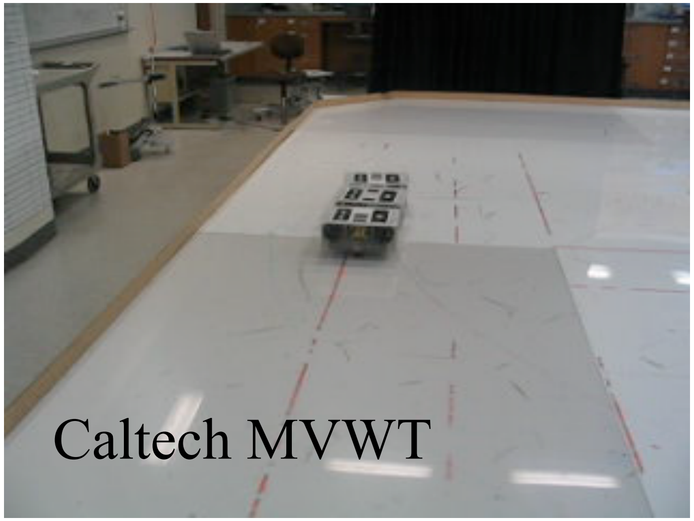
- 8 Journal Publications
- 13 Refereed Conference Proceedings

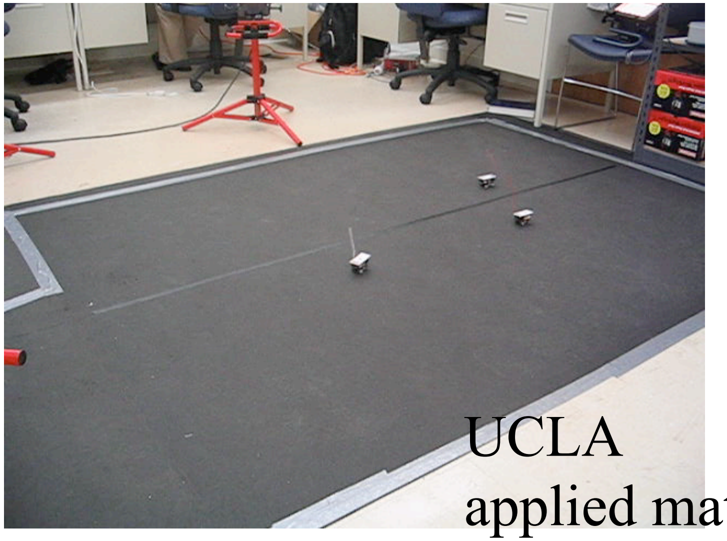
- Physics, Mathematics, Control Theory, Robotics literature.

Properties of biological aggregations

- Large-scale coordinated movement
- No centralized control
- Interaction length scale (sight, smell, etc.) << group size
- Sharp boundaries and "constant" population density
- Observed in insects, fish, birds, mammals…
- Also important for cooperative control robotics.



Caltech MVWT



UCLA applied math lab

Propagation of constant density groups in 2D

Assumptions:

Topaz and Bertozzi (SIAM J. Appl. Math., 2004)

- Conserved population
- Velocity depends nonlocally, linearly on density

$$\rho_t + \nabla \cdot (\vec{v}\rho) = 0 \qquad \vec{v}(\vec{x},t) = \vec{K} * \rho = \int_{\mathbb{R}^2} \vec{K}(|\vec{x}-\vec{y}|)\rho(\vec{y},t)d\vec{y}$$

$$K = \nabla^{\perp}\Psi + \nabla\Phi$$

incompressibility

Potential flow

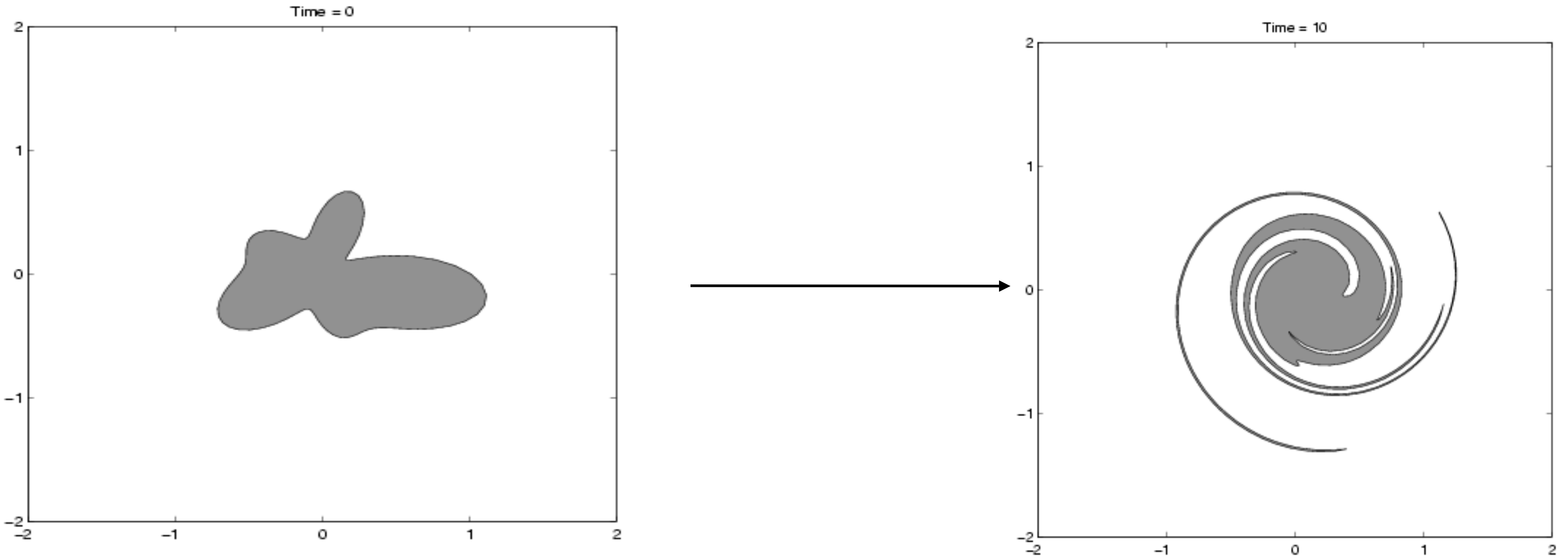
Incompressible flow dynamics

Topaz and Bertozzi (SIAM J. Appl. Math., 2004)

Assumptions:

- Conserved population
- Velocity depends nonlocally, linearly on density

$$\rho_t + \nabla \cdot (\vec{v}\rho) = 0 \qquad \vec{v}(\vec{x},t) = \vec{K} * \rho = \int_{\mathbb{R}^2} \vec{K}(|\vec{x}-\vec{y}|)\rho(\vec{y},t)d\vec{y}$$



Incompressibility leads to rotation in 2D

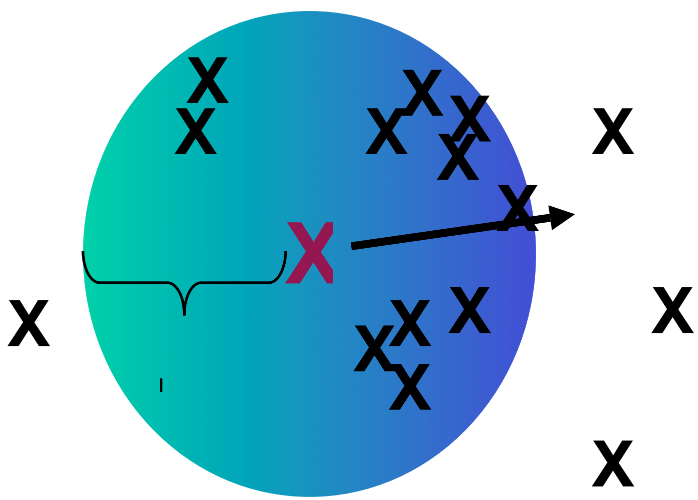
Mathematical model

Topaz, Bertozzi, and Lewis
Bull. Math. Bio. 2006

$$\rho_t + \nabla \cdot \{\rho\,[\nabla(K * \rho) - r\rho\nabla\rho]\} = 0$$

Social attraction:

- Sense averaged nearby pop.
- Climb gradients
- K spatially decaying, isotropic
- Weight 1, length scale 1



Mathematical model

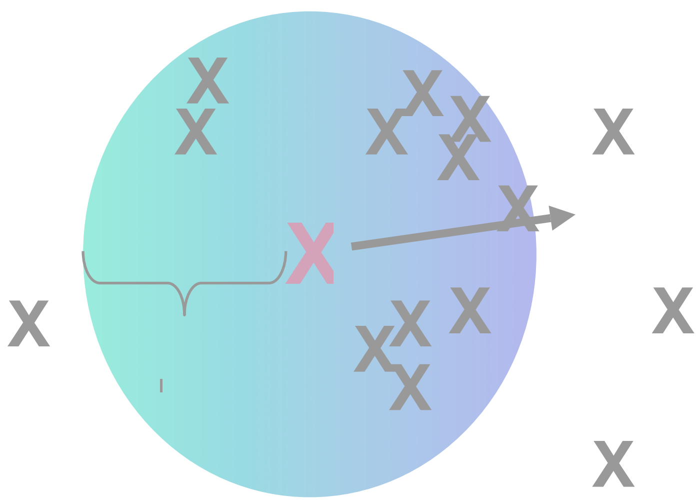
$$\rho_t + \nabla \cdot \{\rho[\nabla(K * \rho) - \boxed{r\rho\nabla\rho}]\} = 0$$

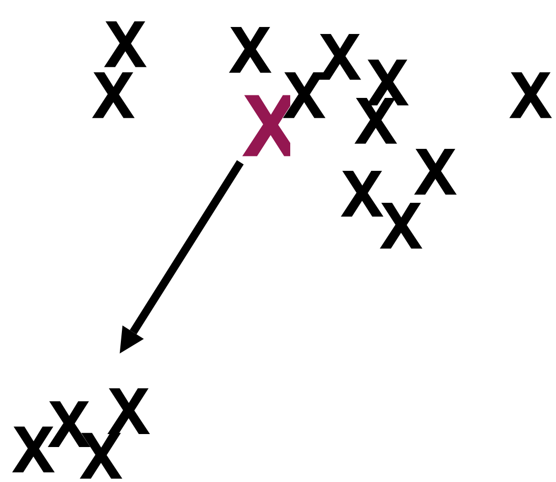
Social attraction:

- Sense averaged nearby pop.
- Climb gradients
- K spatially decaying, isotropic
- Weight 1, length scale 1

Dispersal (overcrowding):

- Descend pop. gradients
- Short length scale (local)
- Strength ~ density
- Characteristic speed r

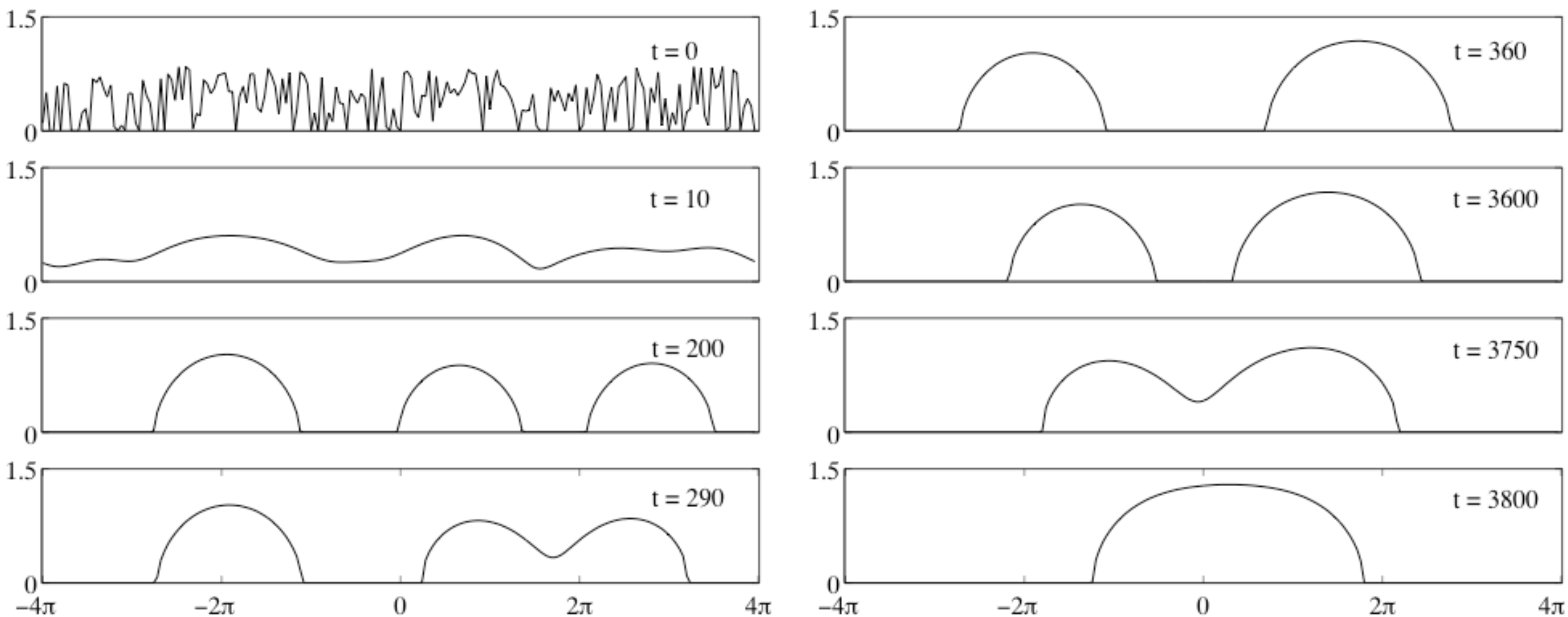




Coarsening dynamics

<u>Example</u>

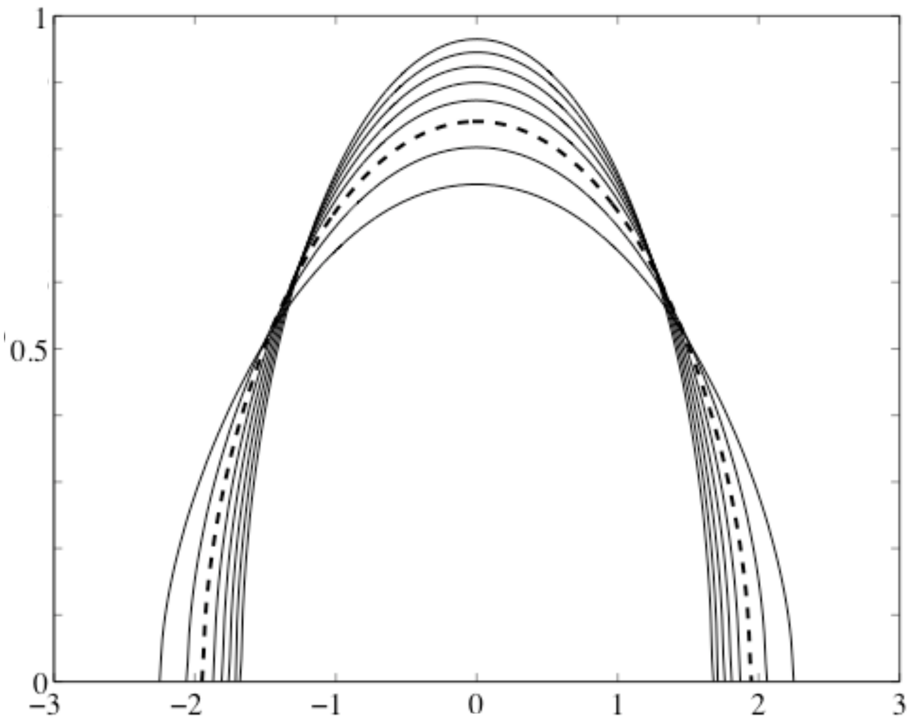
box length L = 8π • velocity ratio r = 1 • mass M = 10



__Example__

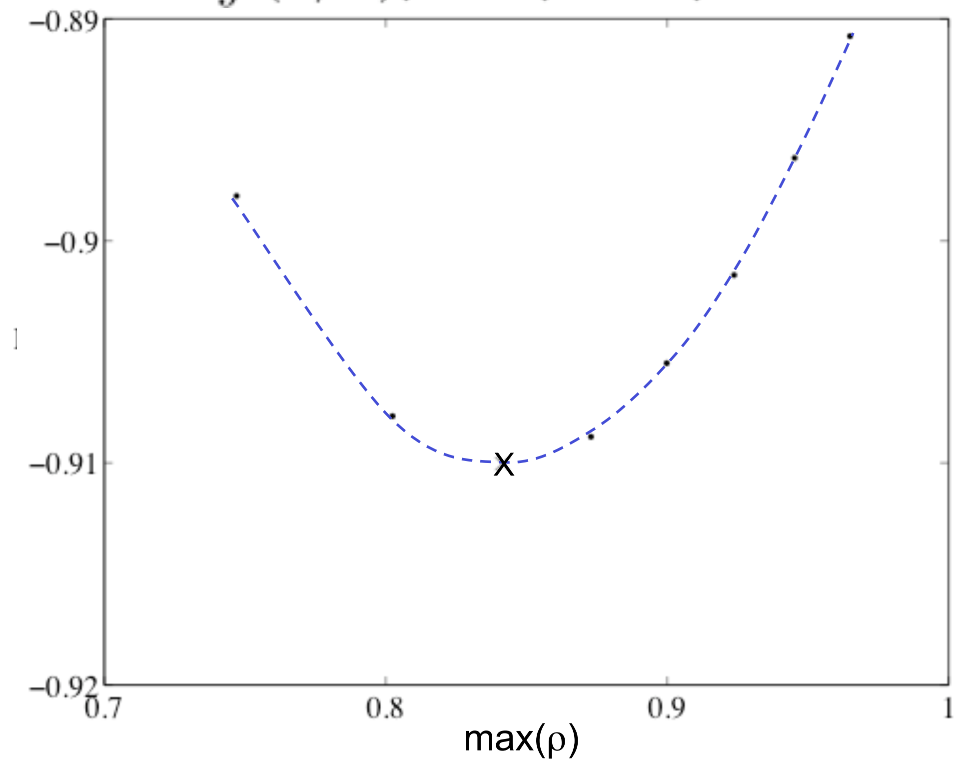
box length L = 2π • velocity ratio r = 1 • mass M = 2.51

Steady-state density profiles



Energy

$$\int (r/3)\rho^3 - \rho K * \rho \, d\vec{x}$$



Large aggregation limit

How to understand? Minimize energy

$$E(\rho) = \int_D \frac{r}{3}\rho^3 - \rho K * \rho \; d\vec{x}$$

over all possible rectangular density profiles.

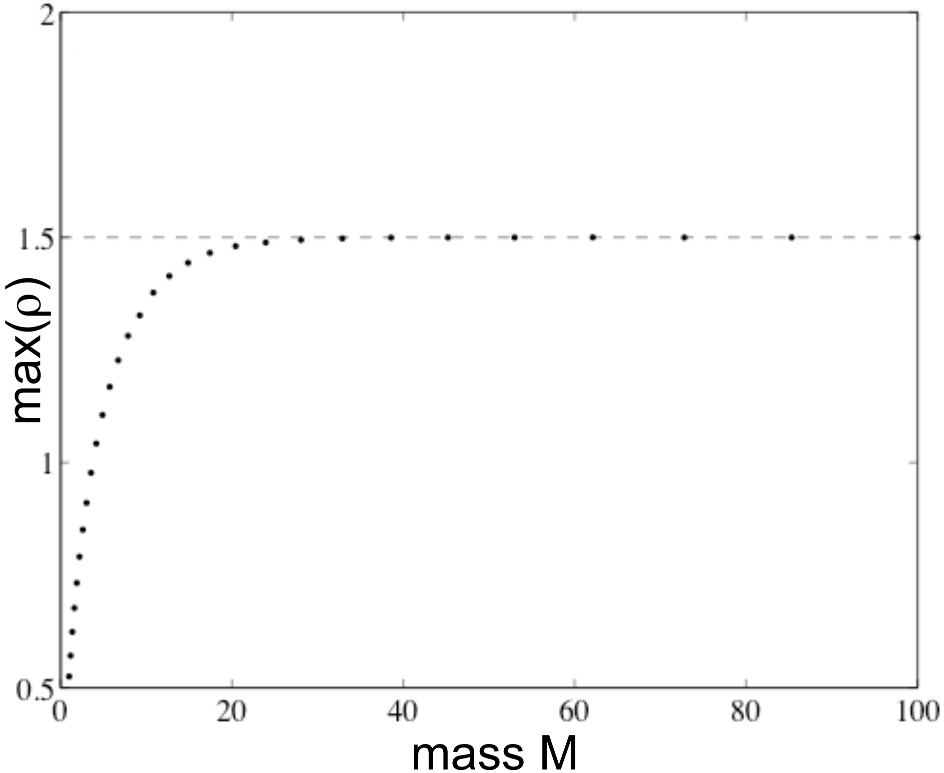
Results:

- Energetically preferred swarm has density 1.5r
- Preferred size is L/(1.5r)
- Independent of particular choice of K
- Generalizes to 2d – currently working on coarsening and boundary motion

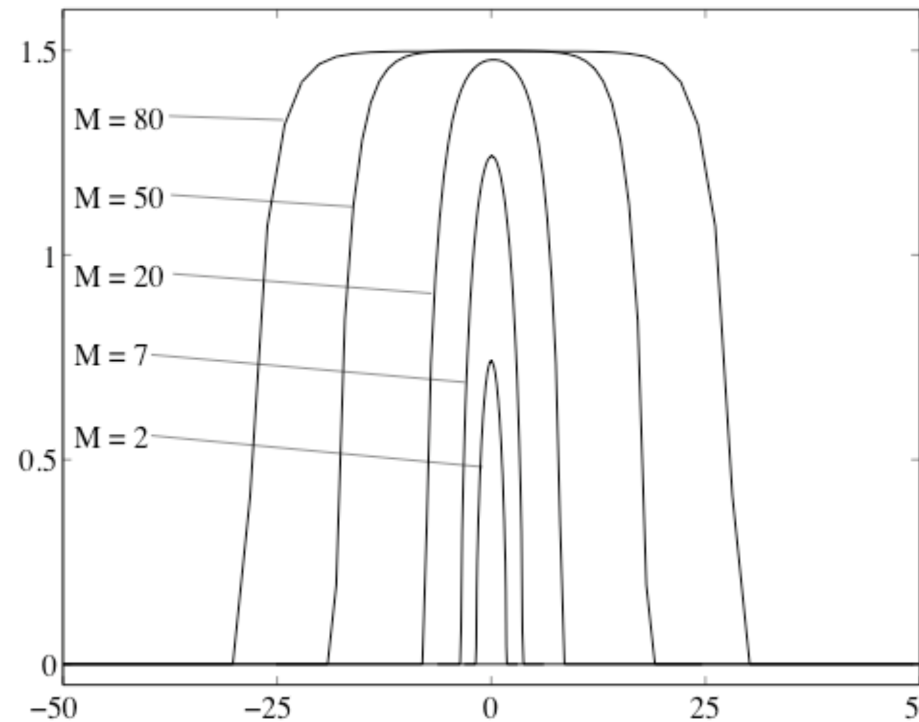
<u>Example</u>

velocity ratio r = 1

Peak density



Density profiles



Bertozzi and Laurent
Comm. Math. Phys. 2007

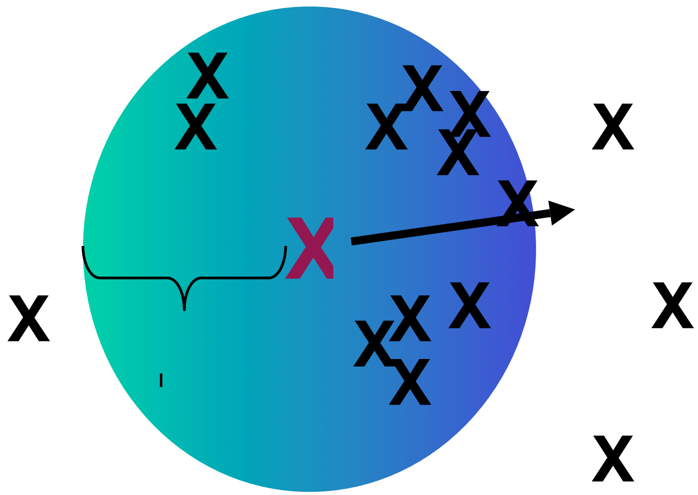
Finite time singularities- pointy potentials

$$\rho_t + \nabla \cdot (\rho \nabla K * \rho) = 0$$

- Previous Results
- For smooth *K* the solution blows up in infinite time
- For n=1, and `pointy' K (biological kernel: $K=e^{-|x|}$) blows up in finite time due to delta in K_{xx}

2007 result:
For `pointy' kernel one can have smooth initial data that blows up in finite time *in any space dimension*.

Proof: uses Lyapunov function and some potential theory estimates.



Bertozzi and Brandman extension to L-infty initial data to appear in *Comm. Math. Sci. 2009*

Bertozzi Carrillo, Laurent
Nonlinearity `featured article' 2009

Finite time singularities- general potentials

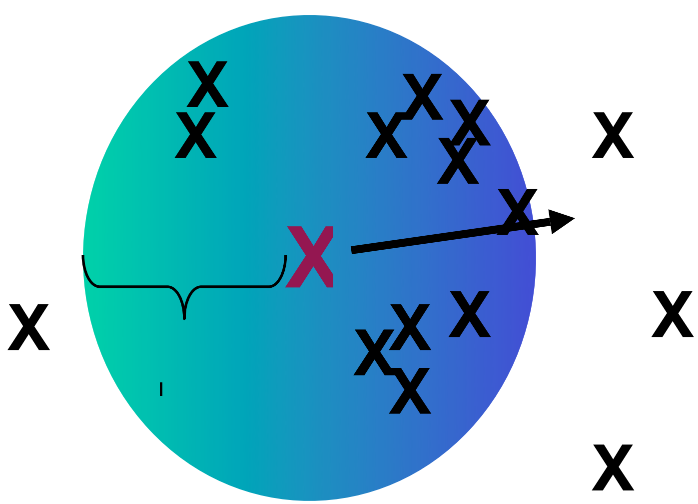
$$\rho_t + \nabla \cdot (\rho \nabla K * \rho) = 0$$

- Previous Results
- For smooth *K* the solution blows up in infinite time
- For `pointy' K (biological kernel such as K=e$^{-|x|}$) blows up in finite time for special radial data in any space dimension.

New result:
Osgood condition

$$\int_0^L \frac{1}{K'(r)} dr < \infty$$

is a necessary and sufficient condition for finite time blowup in any space dimension
(under mild monotonicity conditions).



Moreover-finite time blowup for pointy potential can not be described by `first kind' similarity solution in dimensions N=3,5,7,...

Shape of singularity- pointy potential

``Finite time blowup for `pointy' potential, K=|x|, can not be described by `first kind' similarity solution in dimensions N=3,5,7,...'' - Bertozzi, Carrillo, Laurent

- Similarity solution of form

$$\rho(x,t) = \frac{1}{(T-t)^\alpha} w\left(\frac{x}{(T-t)^\beta}\right)$$

- The equation implies $\alpha = (n-1)\beta + 1$
- Conservation of mass would imply $\alpha = n\beta$ - NO
- Second kind similarity solution - no mass conservation
- Experimentally, the exponents vary smoothly with dimension of space, and there is no mass concentration in the blowup....

Simulations by Y. Huang

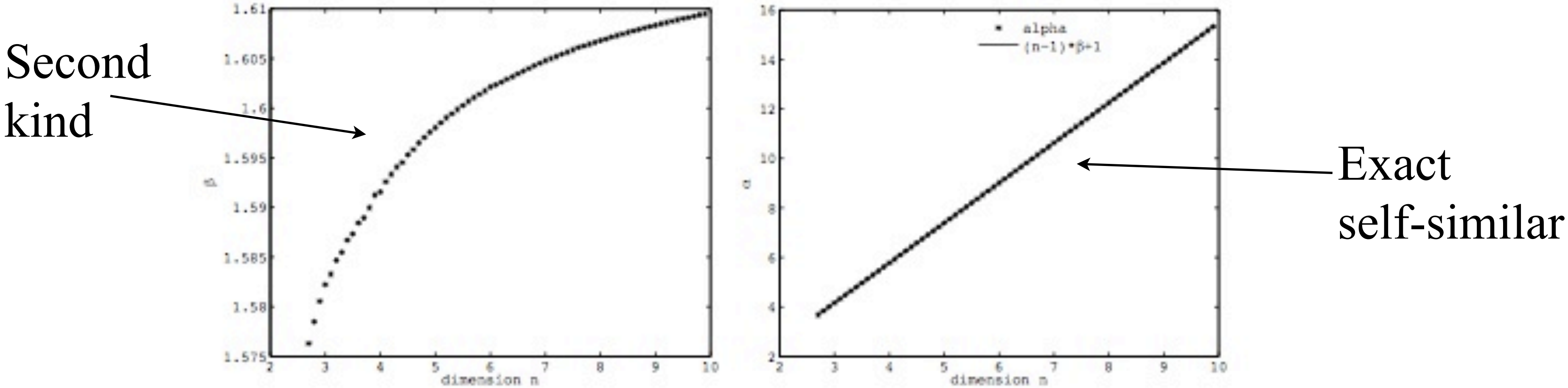


Figure 3: The exponents characterizing the blowup in different spatial dimensions: β(Left) and α(right). The comparison of the estimated α is in perfect agreement with the relation (11).

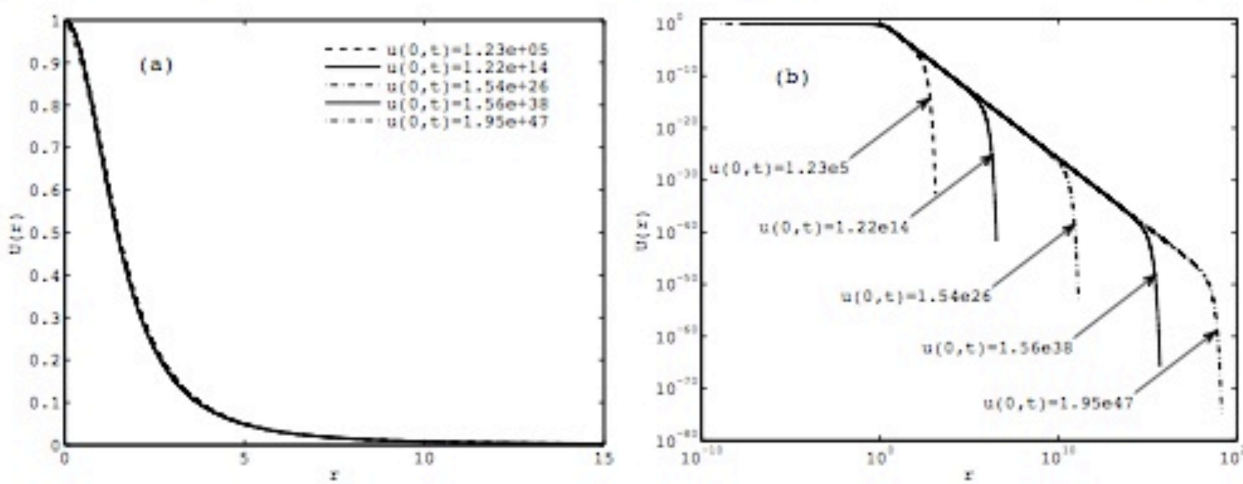
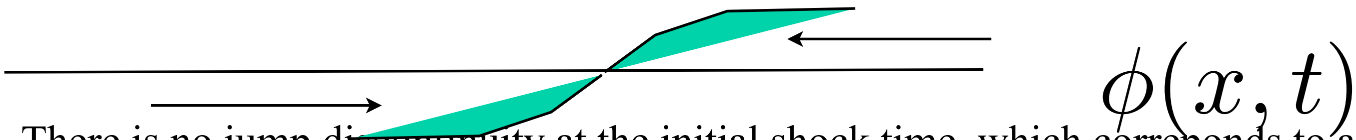


Figure 4: The convergence of the normalized profiles in dimension three. (a) Near the origin, all the profiles are indistinguishable. (b) Far away from the origin, the blowup dynamics adjusts the algebraic decay of the tail.

Shape of singularity- pointy potential

- CONNECTION TO BURGERS SHOCKS
- In one dimension, K(x) = |x|, even initial data, the problem can be transformed exactly to *Burgers equation* for the integral of u.

$$\psi = \int_0^x u(x')dx', \quad \phi = C - 2\psi, \quad \phi_t + \phi\phi_x = 0.$$

- *Burgers equation* for odd initial data has an exact similarity solution for the blowup - it is an initial shock formation, with a 1/3 power singularity at x=0.
- 

- There is no jump discontinuity at the initial shock time, which corresponds to a zero-mass blowup for the aggregation problem. However immediately after the initial shock formation a jump discontinuity opens up - corresponds to mass concentration in the aggregation problem instantaneously after the initial blowup.
- This Burgers solution is (a) self-similar, (b) of `second kind', and (c) generic for odd initial data. There is a one parameter family of such solutions (also true in higher D).
- For the original u equation, this corresponds to beta = 3/2.

L^p well-posedness for general potential

- Local existence of solutions in L^p provided that

$$\nabla K \in W^{1,q}(R^N)$$

- where q is the Holder conjugate of p (characteristics).
  - Existence proof constructs solutions using characteristics, in a similar fashion to weak L^∞ solutions (B. and Brandman *Comm. Math. Sci.* - special issue).
- Global existence of the same solutions in L^p provided that K satisfies the *Osgood condition* (derivation of a priori bound for L^p norm - similar to refined potential theory estimates in BCL 2009).
- When *Osgood condition* is violated, solutions blow up in finite time - implies blowup in L^p for all $p>p_c$.

Bertozzi, Laurent, and Rosado
manuscript in preparation 2009

L^p well-posedness for general potential

- *Ill-posedness* of the problem in L^p for p less than the Holder-critical p_c associated with the potential K.
- *Ill-posedness* results because one can construct examples in which mass concentrates instantaneously (for all $t>0$).
- For $p> p_c$, uniqueness in L^p can be proved for initial data also having bounded second moment, the proof uses ideas from *optimal transport*.
- The problem is *globally well-posed* with measure-valued data (preprint of Carrillo, DiFrancesco, Figalli, Laurent, and Slepcev - using optimal transport ideas).
- Even so, for *non-Osgood* potentials K, there is loss of information as time increases.
- Analogous to information loss in the case of compressive shocks for scalar conservation laws.

Discrete Swarms: A simple model for the mill vortex

M. D'Orsogna, Y.-L. Chuang, A. L. Bertozzi, and L. Chayes,
Physical Review Letters 2006

Discrete:

$$m_i \frac{\partial v_i}{\partial t} = (\alpha - \beta |v_i|^2) v_i - \nabla_i \sum_j V(|x_i - x_j|)$$ Rayleigh friction $\beta |v|^2 = \alpha$

$$V(|x_i - x_j|) = -C_a e^{|x_i - x_j|/l_a} + C_r e^{|x_i - x_j|/l_r}$$ Morse potential

Adapted from Levine, Van Rappel Phys. Rev. E 2000

•without self-propulsion and drag this is Hamiltonian
•Many-body dynamics given by statistical mechanics
•Proper thermodynamics in the H-stable range
•Additional Brownian motion plays the role of a temperature

•What about the non-conservative case?
•Self-propulsion and drag can play the role of a temperature
•non-H-stable case leads to interesting swarming dynamics

H-Stability for thermodynamic systems

Thermodynamic Stability for N particle system

Mathematically treat the limit $N \to \infty$

So that free energy per particle:

$$\lim_{N \to \infty} \frac{E(N)}{N}$$ exists

NO PARTICLE COLLAPSE IN ONE POINT

(H-stability)

NO INTERACTIONS AT TOO LARGE DISTANCES

(Tempered potential : decay faster than $r^{-3+\varepsilon}$)

H-instability 'catastrophic' collapse regime

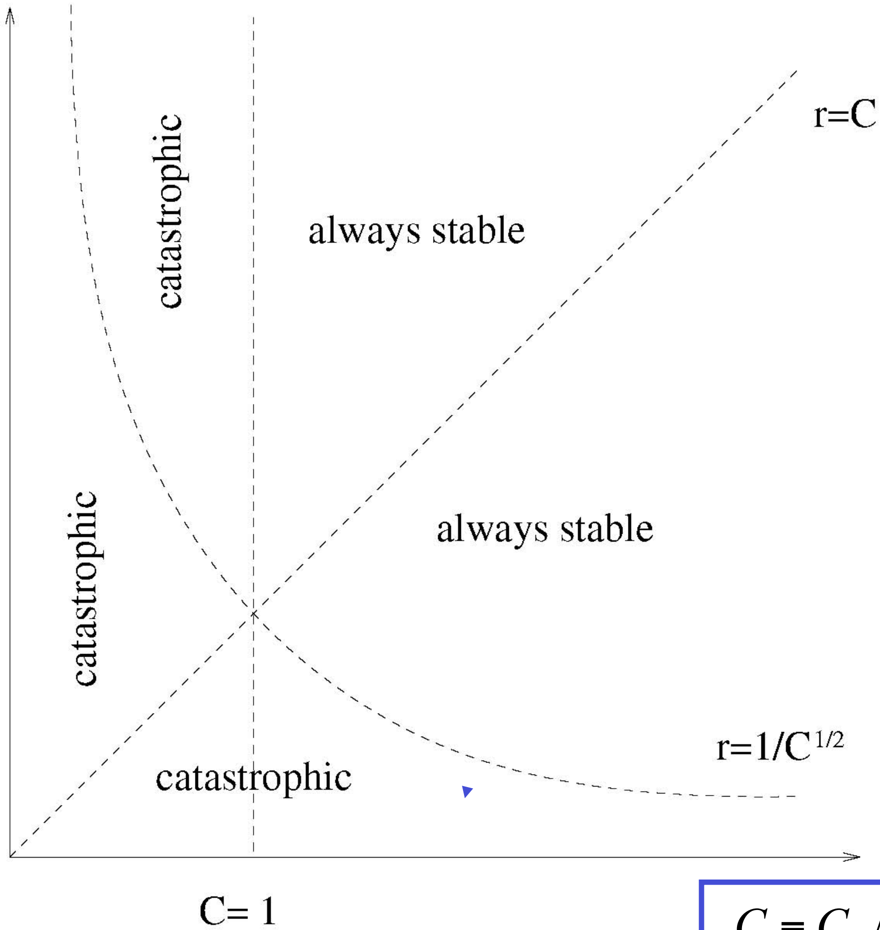
D.Ruelle, Statistical Mechanics, Rigorous results
A. Procacci, Cluster expansion methods in rigorous S.M.

Morse Potential 2D

$\beta |\vec{v}_i|^2 = \alpha$

$\vec{v}_i$ rotation!

$r \equiv l_r / l_a$



$C \equiv C_r / C_a$

Catastrophic:
particle collapse as

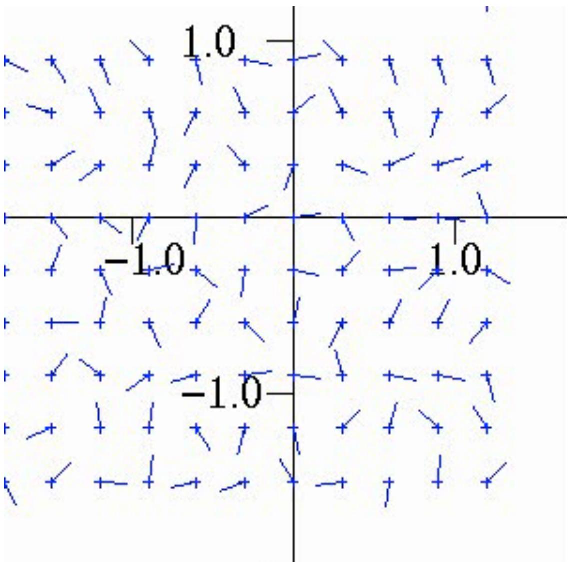
$N \to \infty$

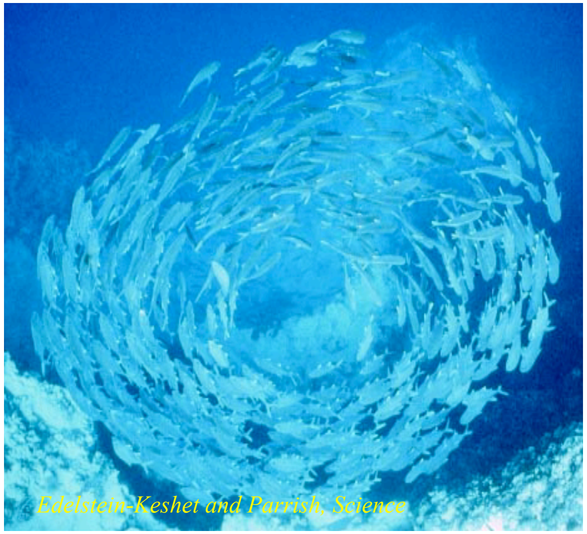
Stable:
particles occupy
macroscopic volume as

$N \to \infty$

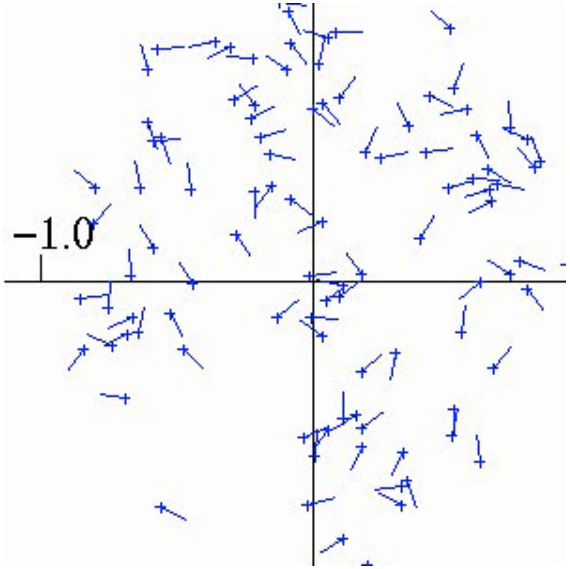
Interacting particle models for swarm dynamics

H-unstable potential





H-stable potential



D'Orsogna et al PRL 2006, Chuang et al Physica D 2007

- FEATURES OF SWARMING IN NATURE: Large-scale coordinated movement,
- No centralized control
- Interaction length scale (sight, smell, etc.) << group size
- Sharp boundaries and "constant" population density, Observed in insects, fish, birds, mammals…

Catastrophic vs H Stable

Discrete:

H Stable

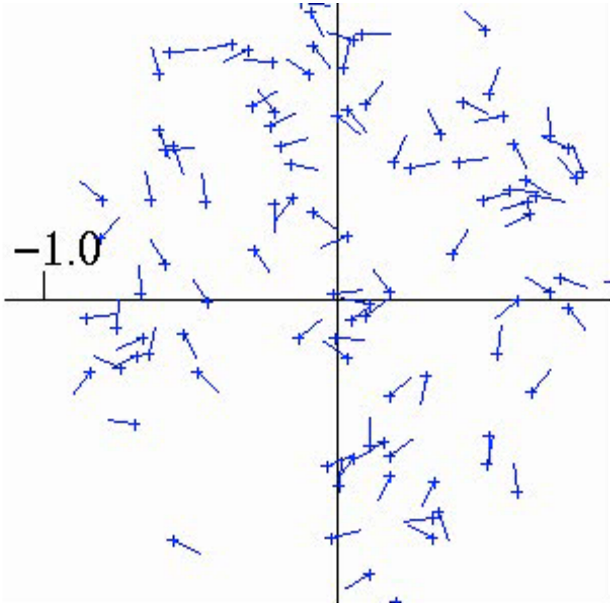
$$\alpha = \beta = 0.5$$
$$C_a = 1.0,\ C_r = 40.0$$
$$l_a = 0.6,\quad l_r = 0.1$$

Catastrophic

$$\alpha = 0.8, \beta = 0.5$$
$$C_a = 0.5, C_r = 1.0$$
$$l_a = 2.0, l_r = 0.5$$

Catastrophic vs H Stable

Discrete:



H Stable

$\alpha = \beta = 0.5$

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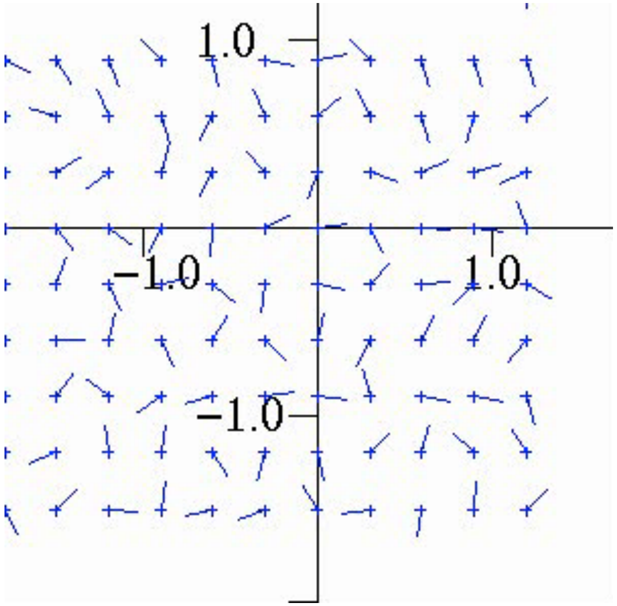

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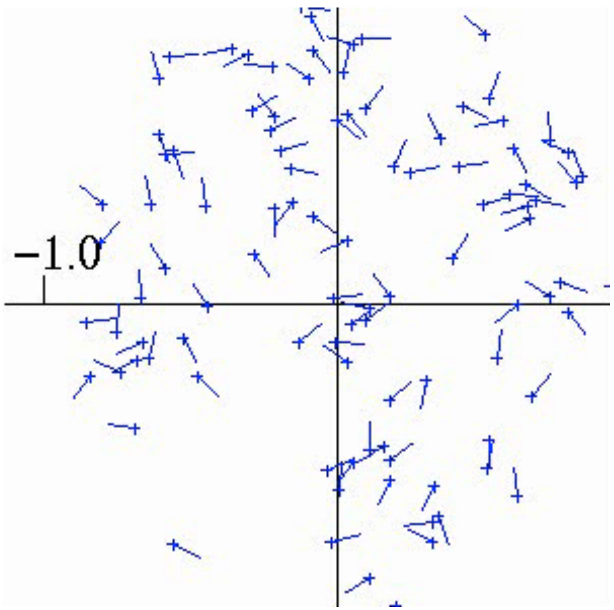
Catastrophic vs H Stable

Discrete:



H Stable

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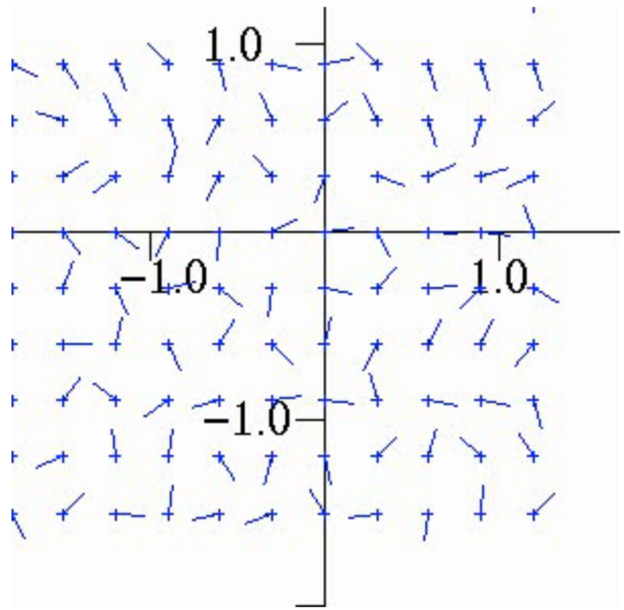


Catastrophic

$\alpha = 0.8, \beta = 0.5$

$C_a = 0.5, C_r = 1.0$

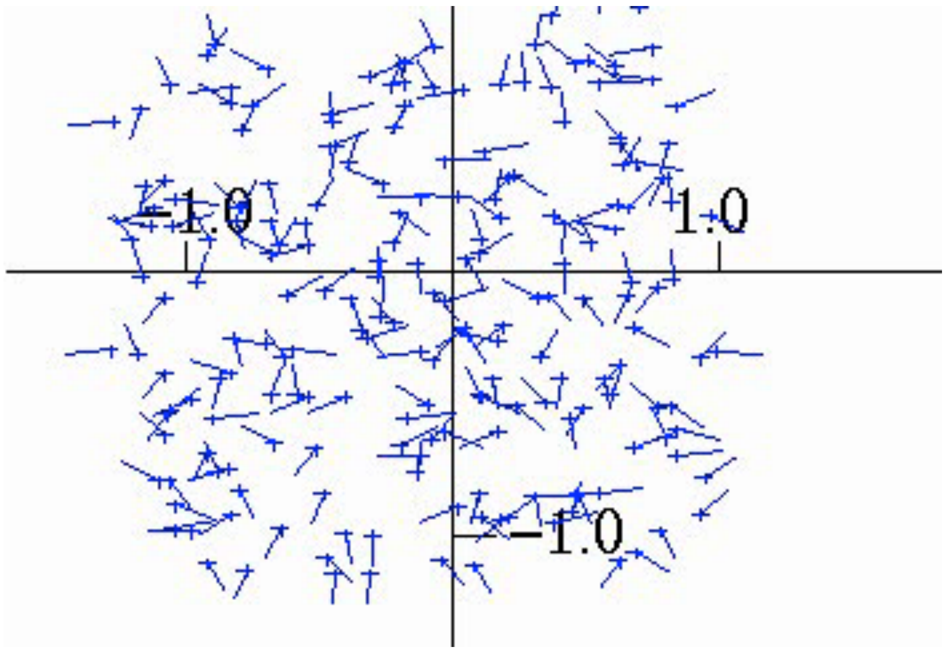
$l_a = 2.0, l_r = 0.5$

Double spiral

$\beta\,|\vec{v}_i|^2 = \alpha$

$\vec{v}_i$ rotation!

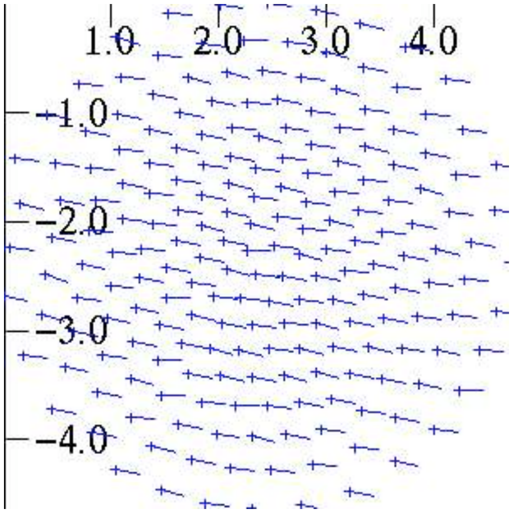
Discrete:



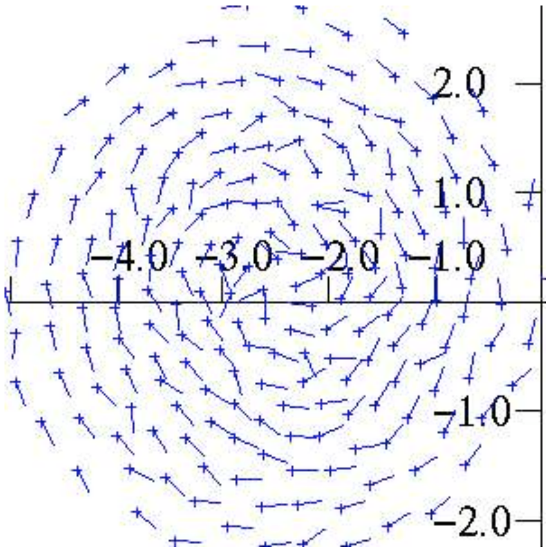
$$\alpha = 3, \beta = 0.5$$
$$C_a = 0.5, C_r = 1.0$$
$$l_a = 2.0, l_r = 0.5$$

Run3
H-unstable

H-stable dynamics



$\alpha = 0.7$



$\alpha = 0.7$

Well defined spacings

$N = 200,\ \beta = 0.5$

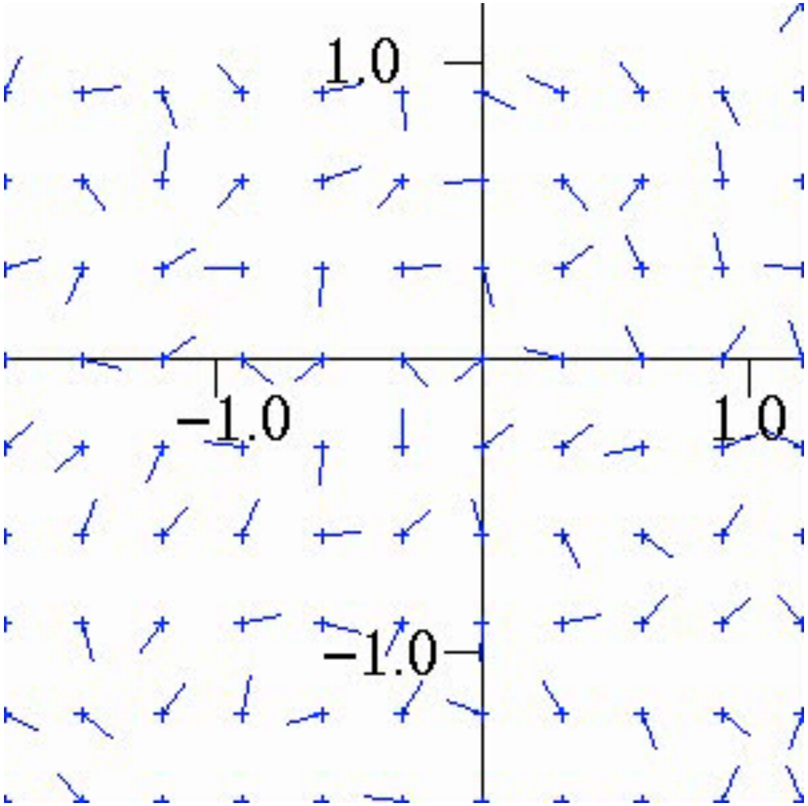
$C_a = 1.0,\ C_r = 37.0$

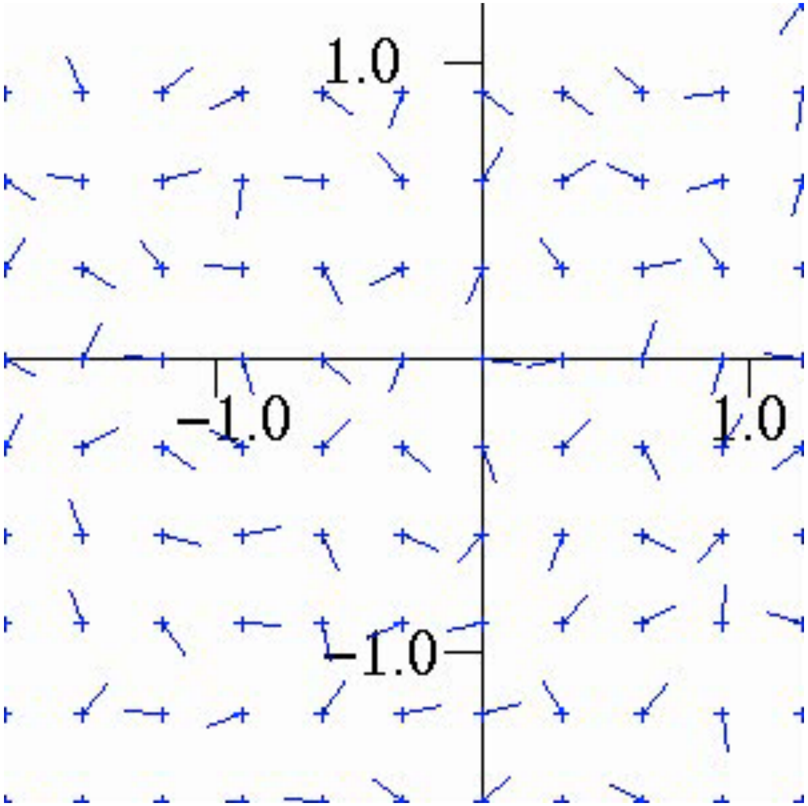
$l_a = 0.7,\quad l_r = 0.1$

Large alpha fly apart (infinite b.c)

Constant angular velocity?

Ring and Clump formation

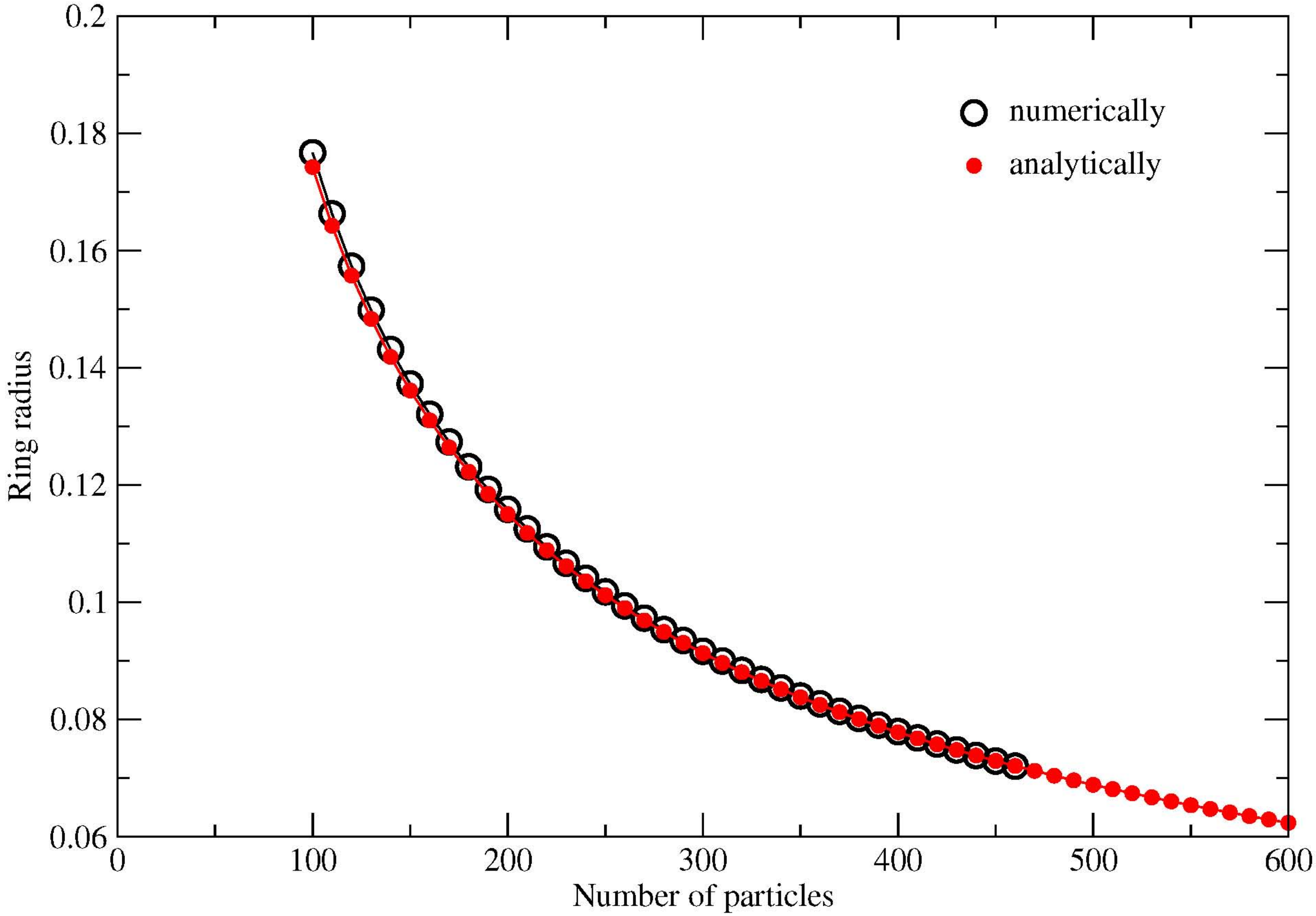




$\alpha = 0.8, \beta = 0.5$
$C_a = 2.0, C_r = 0.5$
$l_a = 2.0, l_r = 0.5$

$\alpha = 0.8, \beta = 0.5$
$C_a = 2.0, C_r = 0.6$
$l_a = 2.0, l_r = 0.5$

Ring formation



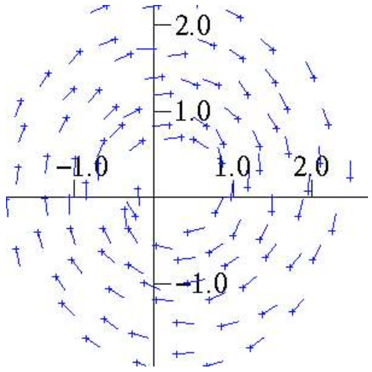
Continuum limit of particle swarms

YL Chuang, M. R. D'Orsogna, A. L. Bertozzi, and L. Chayes, Physica D 2007

set rotational velocities

$$\vec{v} = \sqrt{\frac{\alpha}{\beta}}(-\sin\theta, \cos\theta)$$

$$\int_0^\infty \rho(R)\, U(r-R)\, dR = D - \frac{\alpha}{\beta}\ln r$$



Steady state: Density implicitly defined

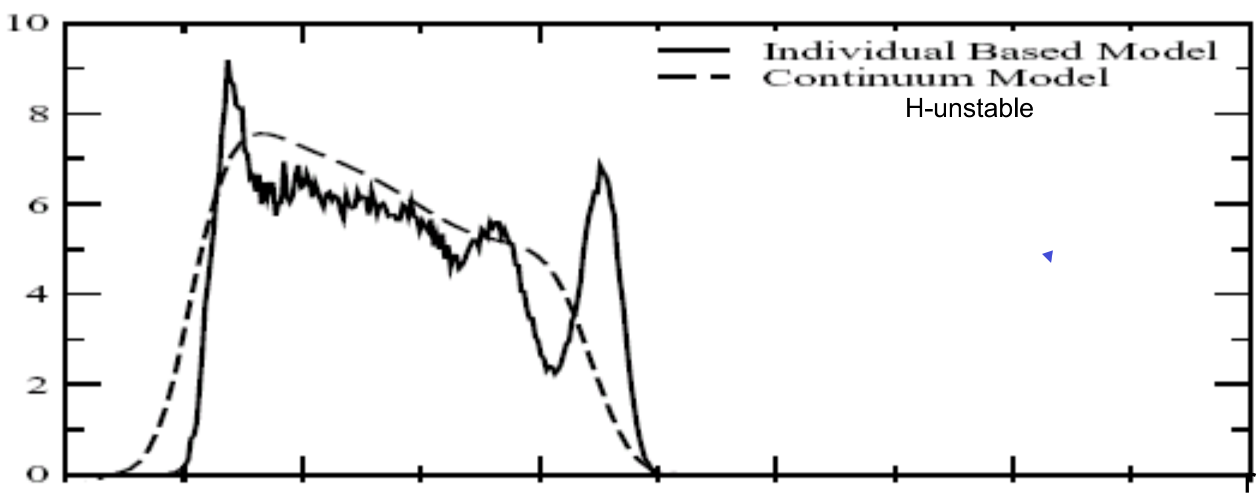
Constant speed

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

$$\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = \alpha\vec{u} - \beta|\vec{u}|^2\vec{u} - \frac{1}{m^2}\nabla\int V(\vec{x}-\vec{y})\rho(\vec{y},t)d\vec{y}$$

ρ(r)

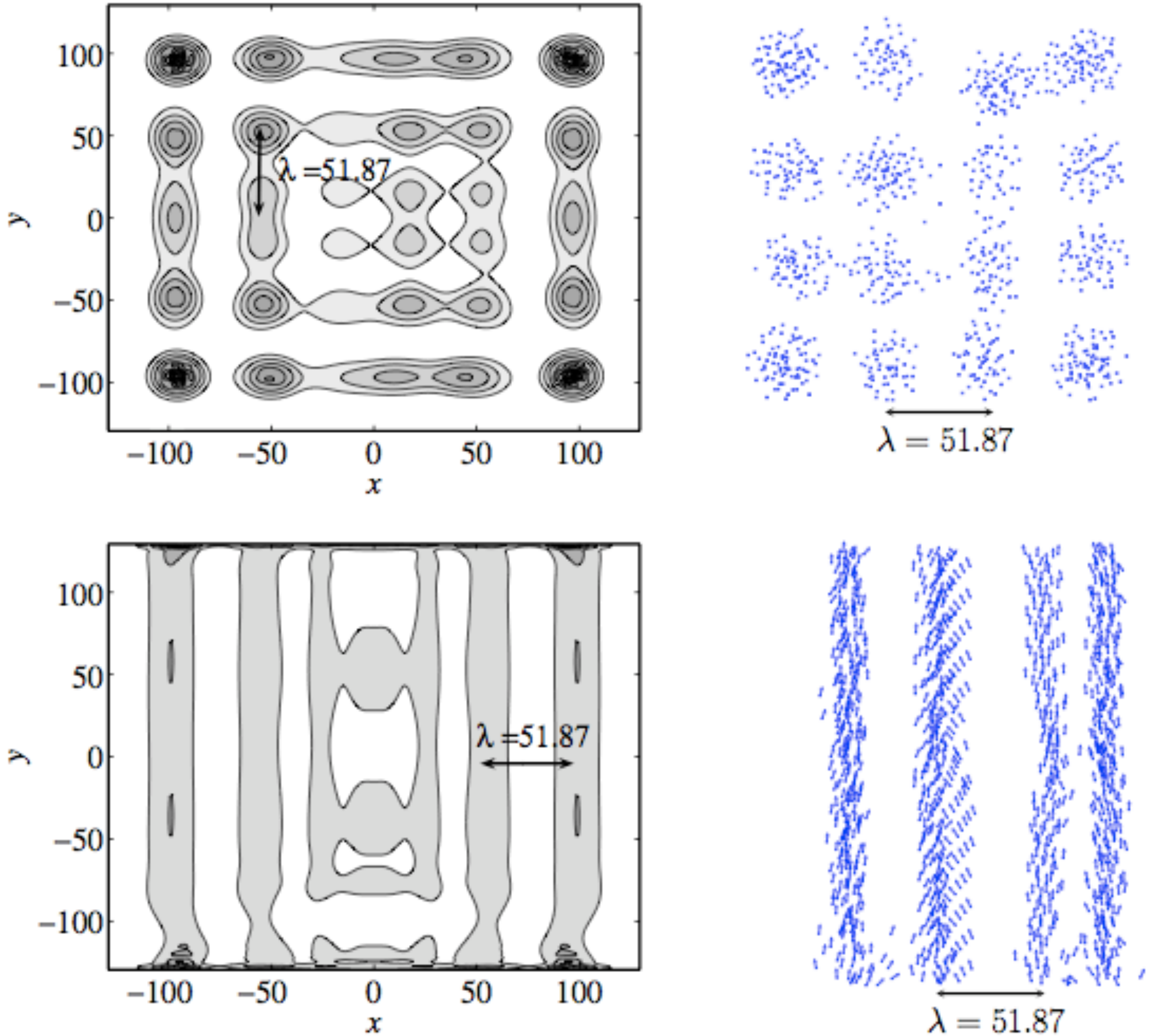
Dynamic continuum model may be valid for catastrophic case but not H-stable.



Constant speed, not a constant angular velocity

Comparison continuum vs discrete for catastrophic potentials

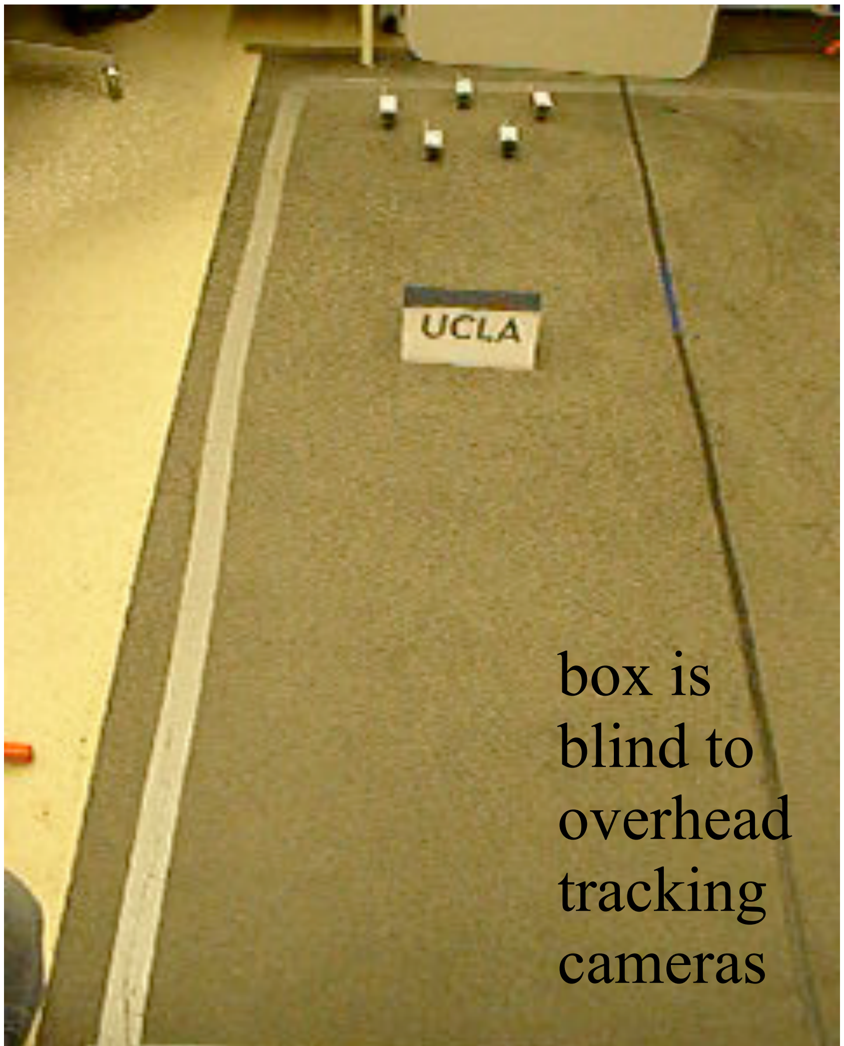
-



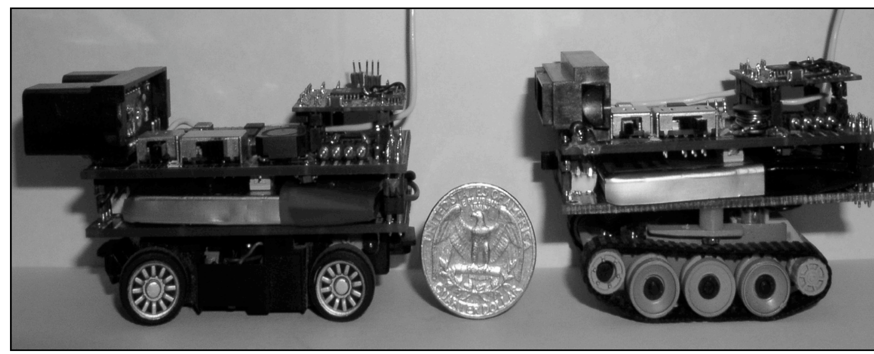
Why H-unstable for continuum limit?

- H-unstable - for large swarms, the characteristic distance between neighboring particles is much smaller than the interaction length of the potential so that
- $$\nabla \int V(\vec{x}-\vec{y})\rho(\vec{y},t)d\vec{y} \sim \nabla \sum_i V(x-y_i) \ .$$
- In H-stable regime the two lengthscales are comparable (by definition).

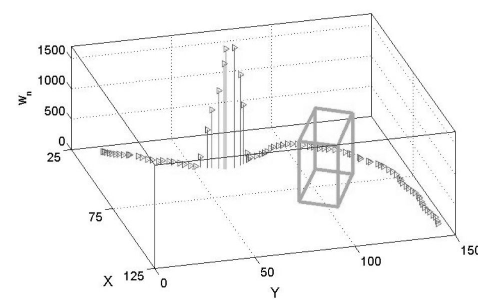
Demonstration of cooperative steering - *Leung et al ACC 2007.*



Obstacle is camouflaged from overhead vision tracking system – cars use crude onboard sensors to detect location and avoid obstacle while maintaining some group cohesion. Their goal to to end up at target location on far side of obstacle.



Algorithm by D. Morgan and I. Schwartz, NRL, related to discrete swarming model.

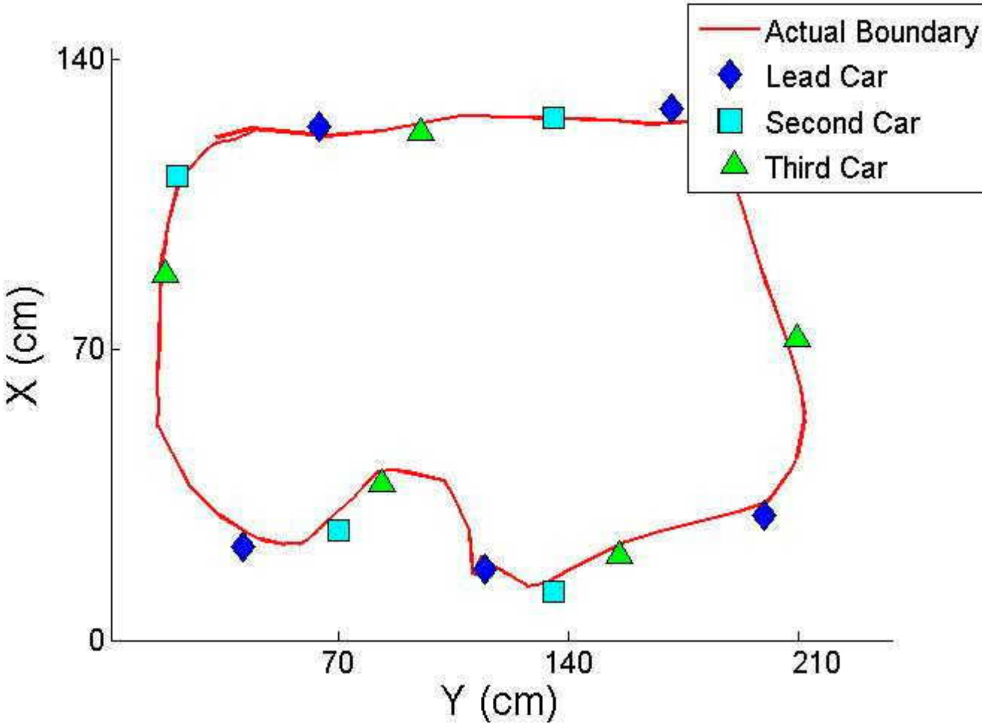


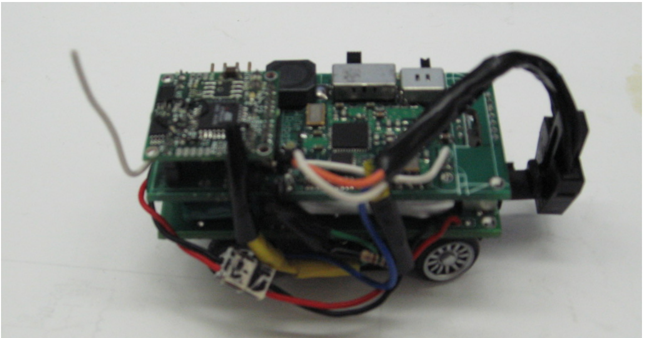
Processed IR sensor output for obstacle avoidance shown along path.

Biomimetic Boundary Tracking

Joshi et al ACC 2009.

- boundary tracking like ants following pheremone trails.
- uses sensor data.
- geometric motion returns vehicle to the path.
- cooperative steering (convoy).
- statistical signal filtering is important.
- idea also applied to edge detection in hyperspectral imagery and AFM.





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