Dynamical Systems Analysis of the Maasch–Saltzman Model for Glacial Cycles

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Snowbird, DS19
May 22, 2019

- Joint work
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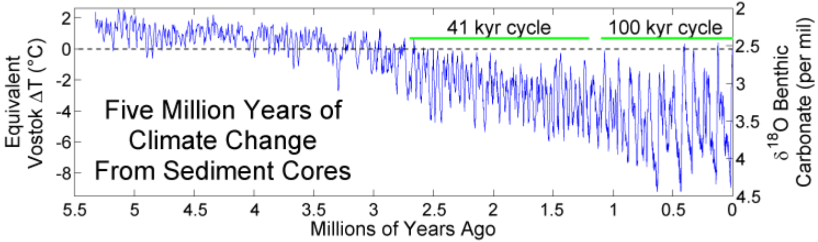
  - Theodore Vo (Florida State University)
- References
  - H. Engler, H.G. Kaper, T.J. Kaper, and Th. Vo, *Modeling the Dynamics of Glacial Cycles*, in: "Mathematics of Planet Earth," H.G. Kaper and F.S. Roberts (eds.), Springer Verlag (to be published, 2019)
  - — , *Dynamical systems analysis of the Maasch–Saltzman model for glacial cycles*, Physica D **359** (2017) 1–20

Outline of the Talk

- Background
  - Glacial-interglacial cycles
- Conceptual climate models
  - Maasch & Saltzman, 1990
- Dimension reduction
  - Time scales, symmetry
- Slow–fast model
  - Symmetric version, slow manifold
  - Breaking the symmetry
- MS-90 model
  - Global bifurcation analysis
  - Center manifold reduction
- Conclusions

Temperature Record, 5.5 Myr BP – Present



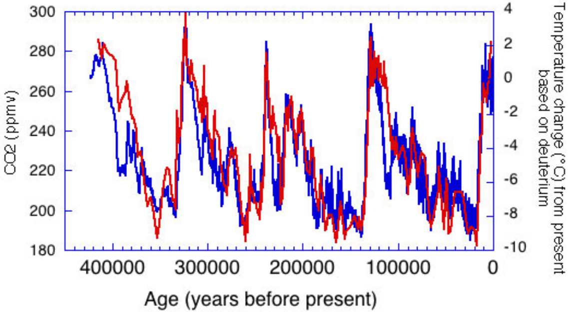
- Reconstructed from proxy data
- Oxygen isotope ratio, $\delta^{18}\mathrm{O} = \mathrm{O}^{18} / \mathrm{O}^{16}$

Pleistocene Epoch: 2.6 Myr–10K yr BP

- Early Pleistocene
  - Oscillatory behavior, period approximately 41 Kyr
  - Correlates with period of the *obliquity* of Earth's orbit
- Mid-Pleistocene Transition
  - Period changes from 41 Kyr to 100 Kyr
  - Amplitude increases
- Late Pleistocene
  - Oscillatory behavior, period approximately 100 Kyr
  - Correlates with period of the *precession* of Earth's orbit
  - But signal is too weak to explain 100 Kyr cycles

Conceptual Models – Saltzman et al., 1980s

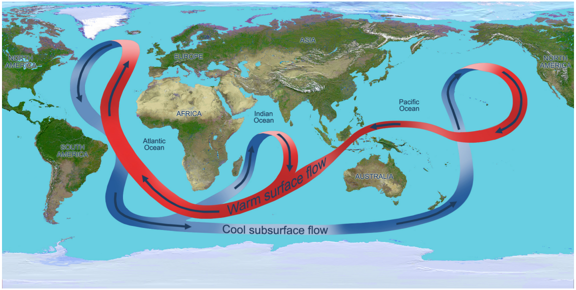
- Key observation #1
  - Strong correlation of global temperature change and change in concentration of atmospheric CO_2
    - Vostok ice core data, 420 Kyr record



Temperature (blue) and atmospheric CO_2 (red)

Conceptual Models – Saltzman et al., 1980s

- Key observation #2
  - Removal of CO_2 from the atmosphere
    - Ocean dynamics
    - Thermohaline circulation
    - North Atlantic Deep Water



The Great Conveyor Belt

MS-90 Model (J. Geophys. Res. 1990)

- State variables (anomalies, dimensionless, rescaled)

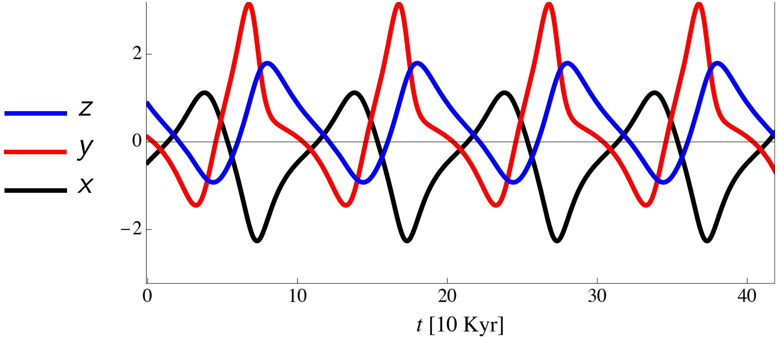
x	Total global ice mass
y	Atmospheric CO_2 concentration
z	North Atlantic Deep Water (NADW)

- Internal dynamics (no external forcing)

$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= (r - z^2)y - (p - sz)z \\ \dot{z} &= -qx - qz \end{aligned}}$$

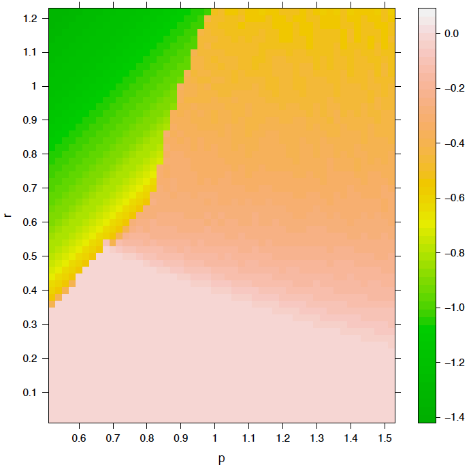
  - Time t, measured in units of 10 Kyr
  - Parameters p, q, r, s, all positive, $q > 1$
  - Bifurcation parameters, (p, r)

Computational Result



- Maasch & Saltzman: $p = 1.0$, $q = 1.2$, $r = 0.8$, $s = 0.8$
- Limit cycle, period 100 Kyr
- Approximately correct shape and order of events
  - Slow glaciation followed by rapid deglaciation
  - Deglaciation happens during temperature spike
  - Build-up of NADW during interglacial stage

Numerical Exploration – Equilibrium or Limit Cycle



- Fix $q = 1.2$, $s = 0.8$
- Vary $(p, r) \in \Omega$
- Integrate system of nonlinear ODEs, random initial data
- Color map $\overline{x}(p, r) = \limsup_t x(t)$
- *Could come from equilibrium point or limit cycle*

Dimension Reduction

Autonomous MS $\xrightarrow{s=0}$ **Symmetric MS**

$$\boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - z^2)y - (p - sz)z\\ \dot{z} &= -qx - qz\end{aligned}} \qquad \boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - z^2)y - pz\\ \dot{z} &= -qx - qz\end{aligned}}$$

$\downarrow\ q \gg 1$ $\qquad$ $\downarrow\ q \gg 1$

Asymmetric 2-D $\xrightarrow{s=0}$ **Symmetric 2-D**

$$\boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - x^2)y + (p + sx)x\end{aligned}} \qquad \boxed{\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - x^2)y + px\end{aligned}}$$

Slow–Fast System

- $q > 1$, ratio of time scales
- Assume $q \gg 1$, define $\varepsilon = 1/q$, so $0 < \varepsilon \ll 1$

$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= ry - pz + (s - y)z^2 \\ \varepsilon\dot{z} &= -x - z \end{aligned}}$$

- Slow–fast system: x and y slow variables, z fast variable
- $\varepsilon = 0$: Invariant, normally attracting manifold

$$\mathcal{M}_0 = \{z = -x\}$$

- $0 < \varepsilon \ll 1$: (*Fenichel Theory*) Family of invariant, normally attracting manifolds

$$\mathcal{M}_\varepsilon = \{z = h_\varepsilon(x, y)\}$$

Slow Manifold and Invariance Equation

- Describe $\mathcal{M}_\varepsilon$ with *invariance equation*

$$\varepsilon \frac{d}{dt} h_\varepsilon(x,y) = -x - h_\varepsilon(x,y)$$

- Expand, $h_\varepsilon(x,y) = h_0(x,y) + \varepsilon h_1(x,y) + \varepsilon^2 h_2(x,y) + \cdots$
- Find h_i by setting coefficients of successive powers of ε equal to 0

$$\begin{aligned}
h_0(x,y) &= -x \\
h_1(x,y) &= -(x+y) \\
h_2(x,y) &= -(x+y) + (ry + px + (s-y)x^2) \\
&\cdots
\end{aligned}$$

Slow–Fast System on $\mathcal{M}_\varepsilon$

- Dynamical system on $\mathcal{M}_\varepsilon$

$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= ry - ph_\varepsilon(x,y) + (s-y)(h_\varepsilon(x,y))^2 \end{aligned}}$$

- Assume symmetry, $s = 0$
- Use zero-order approximation, $h_\varepsilon(x,y) = -x$
- Symmetric 2-D dynamical system

$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= px + (r - x^2)y \end{aligned}}$$

- Equivalent to Duffing–Van der Pol equation

$$\ddot{x} + g(x)\dot{x} + f(x) = 0$$

where $f(x) = x(x^2 - (r-p))$, $g(x) = x^2 - (r-1)$

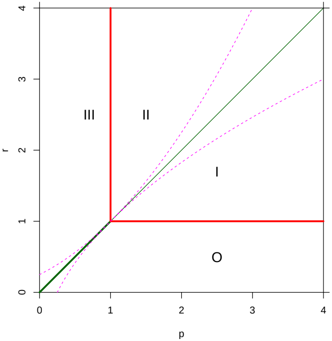
Symmetric 2-D System – Zero Order in ε

- Equilibrium states
  - Trivial state, $P_0 : (x, y) = (0, 0)$ for all (p, r)
  - Two nontrivial states if $r > p$,

$$P_1 : (x, y) = (\sqrt{r-p}, -\sqrt{r-p}), \quad \text{“cold” state}$$
$$P_2 : (x, y) = (-\sqrt{r-p}, \sqrt{r-p}), \quad \text{“warm” state}$$

  - Generated in a pitchfork bifurcation along $r = p$

Linear Stability Analysis



- P_0 stable in O
  - $\{p > 1, r = 1\}$
  - Supercritical Hopf bifurcation
- $P_{1,2}$ stable in III
  - $\{p = 1, r > 1\}$
  - Subcritical Hopf bifurcation

- Organizing Center
  - Bogdanov–Takens singularity at Q: $(p, r) = (1, 1)$
  - All bifurcation curves emanate from Q

Focus on Organizing Center

- Blow up parameters, $0 < \eta \ll 1$

$$\begin{aligned} r - p &= \eta^2 \mu \\ r - 1 &= \eta^2 \lambda \end{aligned} \implies r - 1 = m\,(p-1), \;\; m = \frac{\lambda}{\lambda - \mu}$$

- Rescale variables

$$t = \eta\tau, \quad x = \eta u, \quad -(x+y) = \eta^2 v$$

- Dynamical system near organizing center

$$\boxed{\begin{aligned} \dot{u} &= v \\ \dot{v} &= \mu u - u^3 + \eta(\lambda - u^2) v \end{aligned}}$$

- Perturbed Hamiltonian system $(\eta > 0)$

$$H(u,v) = \tfrac{1}{2} v^2 + \tfrac{1}{4} u^4 - \tfrac{1}{2}\mu\, u^2$$

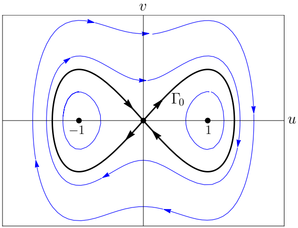
- Interesting case: $\mu > 0$ (wlog $\mu = 1$)

Melnikov Theory

- Unperturbed system
  - Hamiltonian

    $$H(u,v) = \tfrac{1}{2}v^2 + \tfrac{1}{4}u^4 - \tfrac{1}{2}u^2$$

  - Phase portrait $\Longrightarrow$

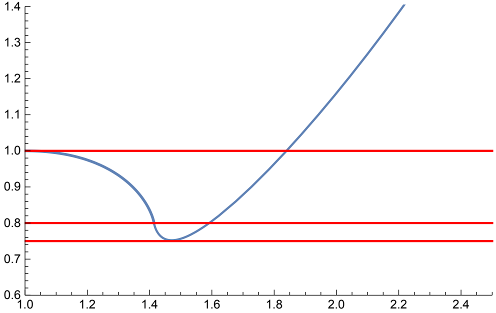


- Homoclinic and periodic orbits Γ through $(u_0, 0)$, $u_0 > 1$
- Melnikov function $\boxed{M(\lambda, u_0) = \oint_\Gamma (\lambda - u^2)\, v(u)\, du}$
  - Zero set of $M(\lambda, u_0)$ defines the locus of all bifurcations near the organizing center

Bifurcation Set

- Zero set of $M(\lambda, u_0)$

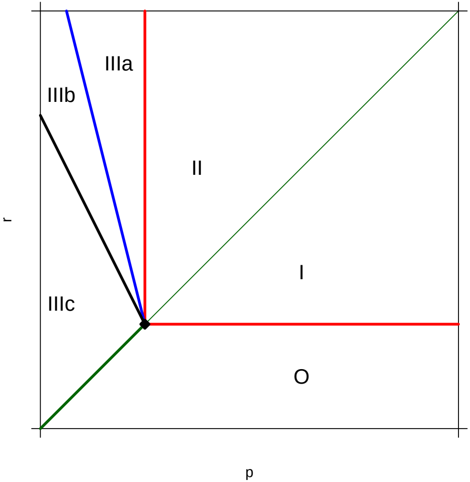
$$M(\lambda, u_0) = 0 \implies \lambda = R(u_0), \quad u_0 > 1$$

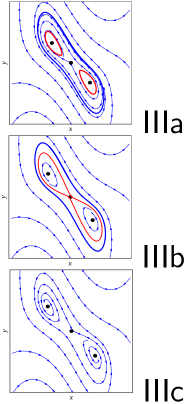


- Bifurcation set

$$\{(p, r) : r - 1 = m\,(p - 1)\}, \quad m = \frac{\lambda}{\lambda - 1}$$

Decomposition of Region III (sketch)

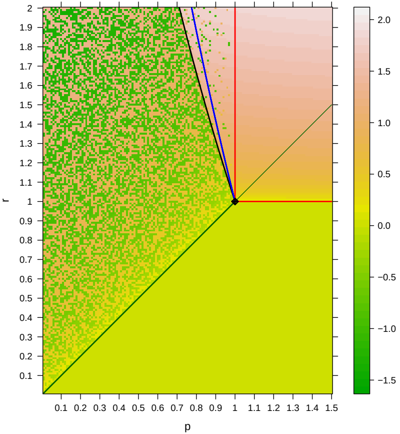




Symmetric 2-D System – Limit Cycles

- Trivial state P_0
  - Stable in O
  - Loses stability at transition O $\rightarrow$ I
  - Supercritical Hopf bifurcation, generates limit cycles
  - Amplitude increases as (p, r) moves through I and II
  - Limit cycles persist in IIIa and IIIb
  - Limit cycles disappear at transition IIIb $\rightarrow$ IIIc
- Nontrivial states P_1, P_2
  - Emerge as (p, r) transits from I $\rightarrow$ II
  - Unstable in II, stable in III
  - Subcritical Hopf bifurcation
  - Generate unstable limit cycles in IIIa and IIIb
  - Affect the basins of attraction of stable limit cycles
- Stable limit cycles throughout O, I, II, IIIa, IIIb

Symmetric 2-D System – Limit Cycles

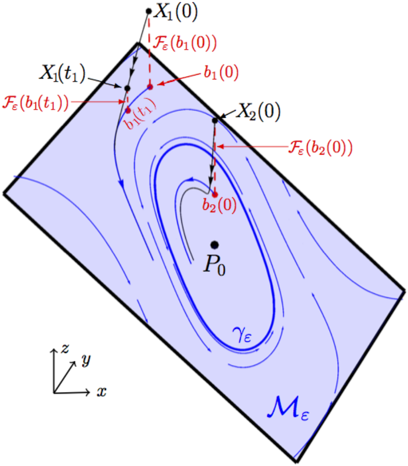


- Integrate ODEs, random initial data
- Use AUTO to find bifurcation curves
  - Hopf
  - Homoclinic
  - Saddle-node of limit cycles
- Color map

$$\overline{x}(p, r) = \limsup_{t} x(t)$$

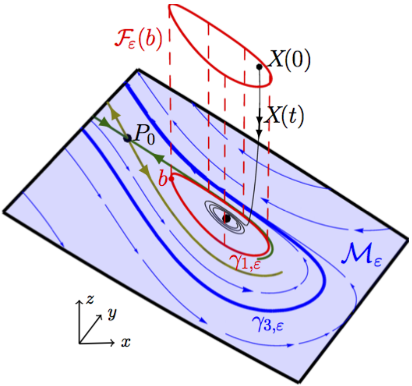
Slow–Fast Decomposition of Typical Solutions

- Solution $X(t)$ starts at $X(0)$ on the stable fast fiber $\mathcal{F}_\varepsilon(b(0))$
- $X(t)$ decomposes into
  - a fast component decaying along $\mathcal{F}_\varepsilon(b(t))$ and
  - a slow component which moves with the base point $b(t)$.
- Thus, $b(t) \in \mathcal{M}_\varepsilon$ represents $X(t)$ faithfully



Basin of Attraction $\mathcal{B}_1$ of P_1

- (p, r) in Region IIIa (between Hopf and homoclinic bifurcation curves)
- P_1 stable, surrounded by unstable limit cycle $\gamma_{1,\varepsilon}$ in $\mathcal{M}_\varepsilon$
- Also shown: large stable limit cycle $\gamma_{3,\varepsilon}$
- The fast stable fibers with base points on $\gamma_{1,\varepsilon}$ form the boundary of $\mathcal{B}_1$.



Symmetric 2-D System – First Order in ε

- Dynamical system on $\mathcal{M}_\varepsilon$

$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= ry - ph_\varepsilon(x,y) + (s-y)(h_\varepsilon(x,y))^2 \end{aligned}}$$

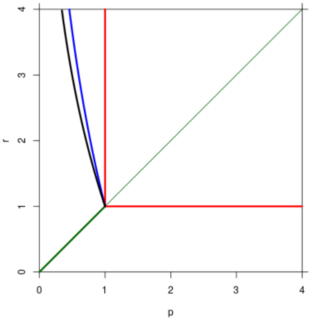
- Assume symmetry, $s = 0$
- Use first-order approximation, $h_\varepsilon(x,y) = -x - \varepsilon(x+y)$
- Symmetric 2-D dynamical system

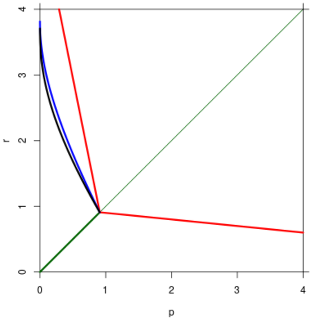
$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= (1+\varepsilon)p\,x + (r + \varepsilon p - (1+2\varepsilon)x^2 - 2\varepsilon xy)y \end{aligned}}$$

- Equilibrium states as before: P_0, and $P_{1,2}$ if $r > p$
- Linear stability analysis, Bogdanov–Takens singularity
- Organizing center at $(p,r) = ((1+\varepsilon)^{-1}, (1+\varepsilon)^{-1})$
- All bifurcation curves emanate from the organizing center

Symmetric Slow–Fast System – Bifurcations

$\varepsilon = 0$ $(q = \infty)$ $\varepsilon = 0.1$ $(q = 10)$





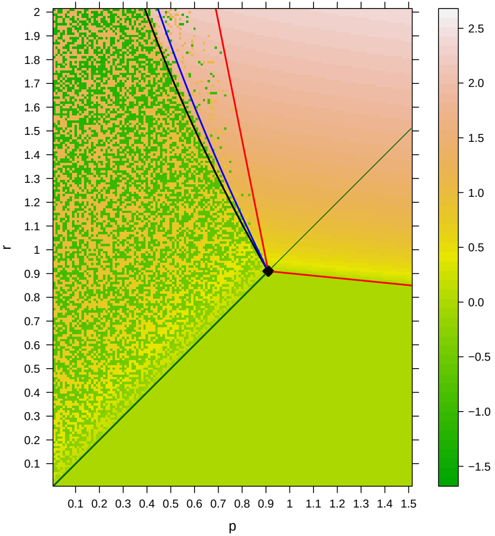
- Hopf bifurcations
- Homoclinic bifurcations
- Saddle-node bifurcations of limit cycles

Slow–Fast System: Limit Cycles

- Integrate symmetric 2-D system, random initial data
- $\varepsilon = 0.1$ ($q = 10$)
- Color map

$$\overline{x}(p, r) = \limsup_t x(t)$$

- Bifurcation curves of the reduced system, using $h_\varepsilon = h_0 + \varepsilon h_1 + \varepsilon^2 h_2$



Breaking the Symmetry

- Two-dimensional model with asymmetry ($s > 0$)
- Zero order in ε

$$\boxed{\begin{aligned}\dot{x} &= -x - y,\\ \dot{y} &= (p + s\,x)x + (r - x^2)y\end{aligned}}$$

- Equilibrium states
  - Trivial state, $P_0 : (x, y) = (0, 0)$ for all (p, r, s)
  - Two nontrivial states if $r > p - \frac{1}{4}s^2$

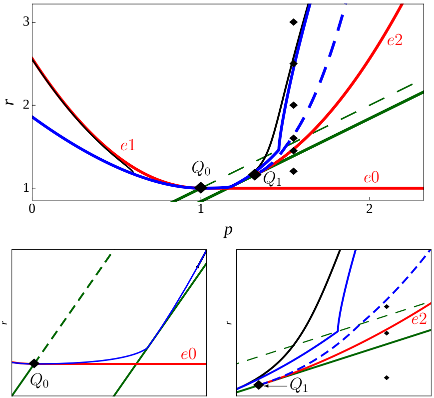
$$P_1 = x_1^*\,(1, -1), \quad x_1^* = -\tfrac{1}{2}s + \tfrac{1}{2}\sqrt{s^2 + 4(r - p)}$$

$$P_2 = x_2^*\,(1, -1). \quad x_2^* = -\tfrac{1}{2}s - \tfrac{1}{2}\sqrt{s^2 + 4(r - p)}$$

- Bogdanov–Takens singularities

$$Q_0 = (1, 1), \quad Q_1 = (1 + \tfrac{1}{2}s^2, 1 + \tfrac{1}{4}s^2)$$

Stability Boundaries and Bifurcation Curves

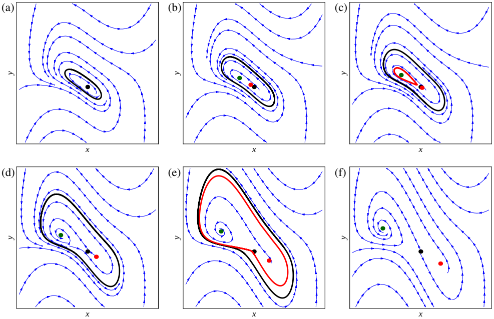


2.pdf

$s = 0.8$

- Use AUTO to find bifurcation curves
  - Hopf
  - Homoclinic
  - Saddle-node of limit cycles

Phase Portraits



$$r = 1.2 \to 1.45 \to 1.6 \to 2.0 \to 2.5 \to 3.0$$

$$s = 0.8, \ p = 1.55$$

Back to the MS-90 System

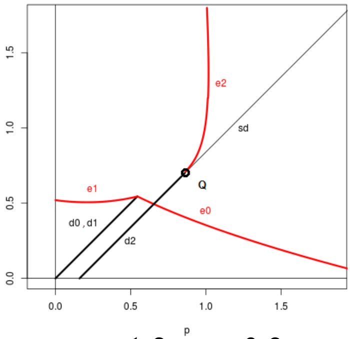
$$\boxed{\begin{aligned} \dot{x} &= -x - y \\ \dot{y} &= (r - z^2)y - (p - sz)z \\ \dot{z} &= -qx - qz \end{aligned}}$$

- Equilibrium states
  - Trivial state, $P_0 : (x, y, z) = (0, 0, 0)$
  - Two nontrivial states if $\rho = s^2 + 4(r - p) > 0$

$$P_1 : (x, y, z) = x_1^* (1, -1, -1), \quad x_1^* = \tfrac{1}{2}(-s + \sqrt{\rho})$$
$$P_2 : (x, y, z) = x_2^* (1, -1, -1), \quad x_2^* = \tfrac{1}{2}(-s - \sqrt{\rho})$$

Linear Stability Analysis

- P_0 stable below **d0/d1** ($r = p$) and **e0**
- P_1, P_2 exist above **d2/sd** ($\rho = 0$)
- Hopf bifurcations off $P_{0,1,2}$ on **e0, e1, e2**
  - Supercritical on **e0**
  - Subcritical on **e1**
  - Sub-/supercritical on **e2**



$q = 1.2$, $s = 0.8$

- Two organizing centers
  - Q_0, where **e0/e1** and **d0/d1** meet
  - Q_1, where **e2** and **d2** meet)

Center Manifold Reduction near Q

- Shift Q to $(0,0)$: $(p,r) \mapsto (\tilde{p},\tilde{r})$
- Dynamical system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ n \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & \frac{q}{1+q} & -\frac{q}{1+q} \\ -q & 0 & -q \end{pmatrix}$$

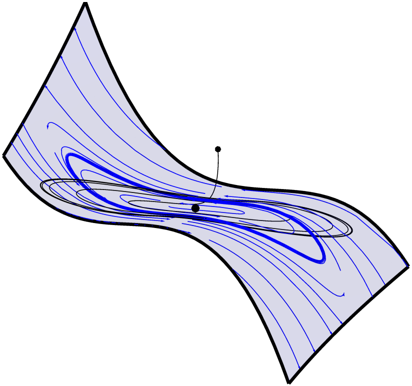
  - n contains all nonlinearities, $n = n(x, y, z, \tilde{p}, q, \tilde{r}, s)$
- Transform to Jordan normal form, $(x, y, z) \mapsto (u, v, w)$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = J \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{n} \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

  - $\tilde{n}$ more complicated, but still of rank 1
- Center manifold, $\mathcal{W}^c = \{w = h(u, v, \ldots)\}$

Dynamics on Center Manifold

- Blue surface is $\mathcal{W}^c$, blue streamlines indicate the flow on $\mathcal{W}^c$, thick blue line is the stable limit cycle there
- The black curve is a solution of the MS-90 system

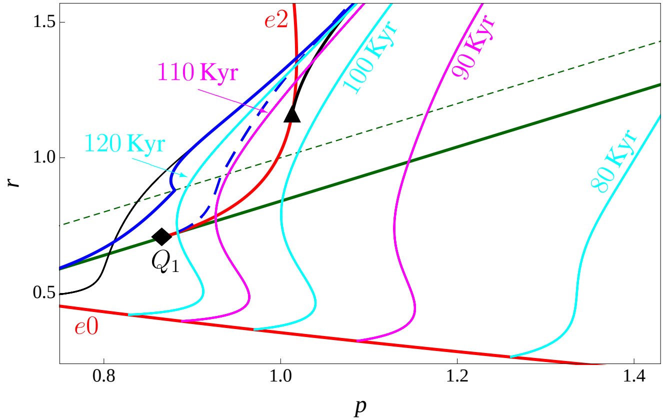


$q = 1.2,\ s = 0,$
$\tilde{p} = 0.15,\ \tilde{r} = 0.1$

Region of Validity

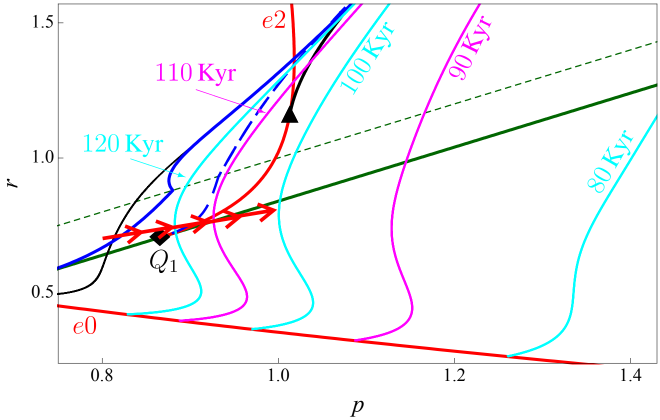
- Expect both Lyapunov type numbers to become less than 1 if $q < q_c$ for some critical value $q_c = q_c(p, r, s)$
- $\mathcal{W}^c$ loses smoothness
- Compute q_c numerically
  - Compare the real parts of the eigenvalues at equilibrium points on $\mathcal{W}^c$ to λ_3
  - $q \approx q_c$ when one of these ratios becomes 1
- $q_c < 1$ for $0 < p < 2, 0 < r < \frac{3}{2}$
- **The reduced systems provide reliable qualitative information about the full dynamics, over the entire parameter range.**

Isoperiod Curves



Mid-Pleistocene Transition

- Slow passage through Hopf bifurcation (red path)
  - Move (p, r) from $(0.8, 0.7)$ to $(1.0, 0.8)$ over 2 Myr
  - Arrive at limit cycles with the right period (100 Kyr)



Conclusions

- Maasch–Saltzman model
  - Rich dynamics: multiple equilibria, limit cycles, bifurcation phenomena, symmetry breaking
  - Long-term dynamics occur on or near two-dimensional invariant manifolds: slow manifolds ($q \gg 1$) or center manifolds ($q > q_c > 1$)
  - Bogdanov–Takens points are organizing centers for the dynamics
- Pleistocene climate
  - Model can be tuned to yield 41 Kyr cycles of early Pleistocene under orbital forcing
  - Slow passage through Hopf bifurcations explains Mid-Pleistocene Transition
  - Internal dynamics generate 100 Kyr cycles of late Pleistocene

THANK YOU!