Model-based learning in imaging

Carola-Bibiane Schönlieb

Department for Applied Mathematics and Theoretical Physics
Cantab Capital Institute for the Mathematics of Information
EPSRC Centre for Mathematical Imaging in Healthcare
Alan Turing Institute
University of Cambridge, UK

SIAM annual meeting, Pittsburgh, 11 July 2017







Topics in model-based learning in imaging

Carola-Bibiane Schönlieb

Department for Applied Mathematics and Theoretical Physics
Cantab Capital Institute for the Mathematics of Information
EPSRC Centre for Mathematical Imaging in Healthcare
Alan Turing Institute
University of Cambridge, UK

SIAM annual meeting, Pittsburgh, 11 July 2017







Outline

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

1 Variational regularisation and PDEs for imaging

Outline

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

1. Variational regularisation and PDEs for imaging
2. Learning of variational models in imaging

Outline

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

1. Variational regularisation and PDEs for imaging
2. Learning of variational models in imaging
3. Conclusions and outlook . . . and maybe a fun example

Outline

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

1. Variational regularisation and PDEs for imaging
2. Learning of variational models in imaging
3. Conclusions and outlook . . . and maybe a fun example

The central inverse problem

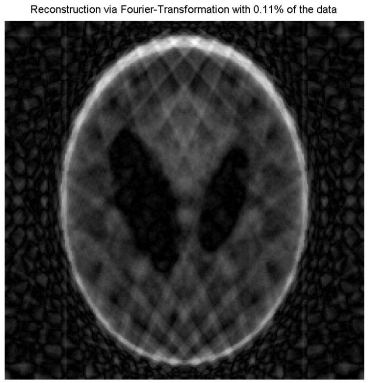
CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

General task: **restore** u from an **observed datum** g where

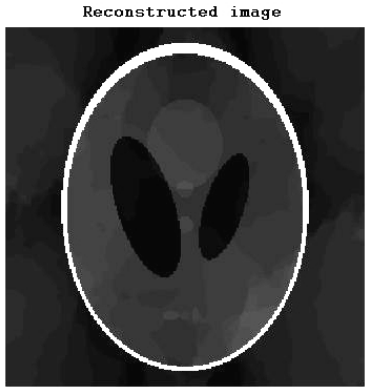
$$g = \underbrace{Tu}_{\text{forward model}} + \underbrace{n}_{\text{noise}}.$$

For most problems this equation is ill-posed!

Variational Magnetic Resonance Imaging



Zero-filling solution



Total variation solution

MRI: measured datum is a sampled Fourier transform; nonadaptive compressed acquisition, i.e., compressed sensing

$$g = (\mathcal{F}u)_{|\Lambda} + n$$

Goal: identify a piecewise constant function u consistent to the datum

Sparsity in $\nabla u \Rightarrow$ NP-hard problem!

Convex relaxation $\Rightarrow \ell_1$- and total variation minimization

$$\alpha\|\nabla u\|_1 + \frac{1}{2}\|(\mathcal{F}u)_{|\Lambda} - g\|^2 \to \min_u,$$

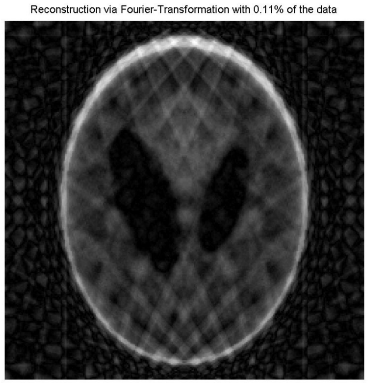
Rudin, Osher, Fatemi, Physica D '92; Candes, Romberg, Tao, IEEE Trans Inf Theory '06

Shepp-Logan Phantom

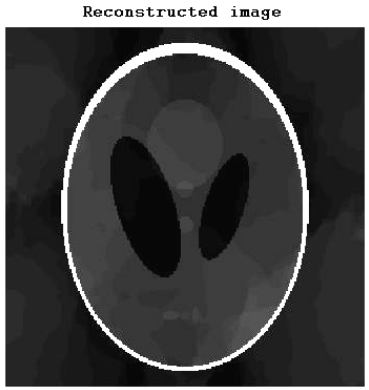
CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



Variational Magnetic Resonance Imaging



Zero-filling solution



Total variation solution

MRI: measured datum is a sampled Fourier transform; nonadaptive compressed acquisition, i.e., compressed sensing

$$g = (\mathcal{F}u)_{|\Lambda} + n$$

Goal: identify a piecewise constant function u consistent to the datum

Sparsity in $\nabla u \Rightarrow$ NP-hard problem!

Convex relaxation $\Rightarrow \ell_1$- and total variation minimization

$$\alpha\|\nabla u\|_1 + \frac{1}{2}\|(\mathcal{F}u)_{|\Lambda} - g\|^2 \to \min_u,$$

Rudin, Osher, Fatemi, Physica D '92; Candes, Romberg, Tao, IEEE Trans Inf Theory '06

The variational approach

General task: **restore** $\mathbf{u}$ from an **observed datum** $\mathbf{g}$ where

$$g = \underbrace{Tu}_{\text{forward model}} + \underbrace{n}_{\text{noise}}.$$

Variational approach: Compute u as a minimizer of

$$\mathcal{J}(u) = \alpha \underbrace{R(u)}_{\text{regularization}} + \underbrace{D(Tu, g)}_{\text{data fidelity}} \to \min_{u \in B},$$

where

- $R(u)$ is a prior/regularizer that models a-priori information on u weighted by positive α, e.g., $R(u) = \|\nabla u\|_{L^1}$
- $D(\cdot, \cdot)$ is a distance function, e.g. $D(Tu, g) = \|Tu - g\|_2^2$ and B suitable Banach space, e.g., $B = BV(\Omega)$.

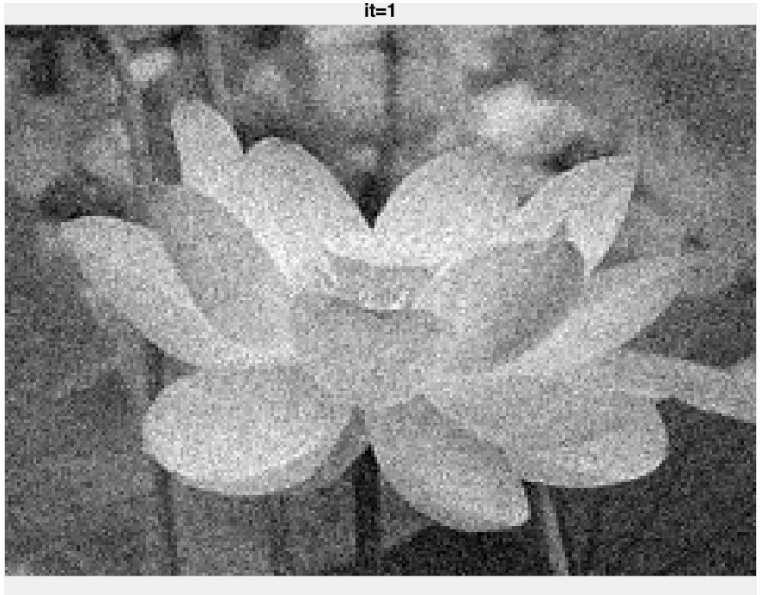
PDE connection

Nonlinear image smoothing with total variation regularisation

$$\alpha\|\nabla u\|_1 + \frac{1}{2}\|u-g\|^2 \to \min_u$$

with steepest descent

$$u_t = \alpha\ \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + (u-g), \quad \text{in } \Omega,$$



Rudin, Osher, Fatemi, Physica D '92; Perona, Malik, IEEE Trans Patt Anal Mach Int '90

Modelling idea . . .

Taking advantage of the property of solutions of such PDEs and variational problems to be characterised by a few relevant features with certain geometric properties, e.g. curvature, which are recovered by adaptive nonlinear iterations.

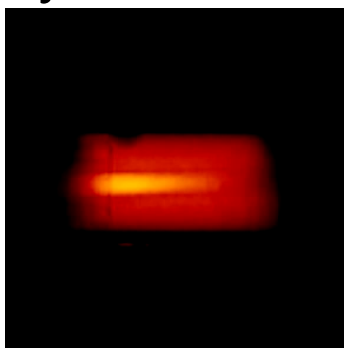
What can we do with this?

Mathematical imaging by examples

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

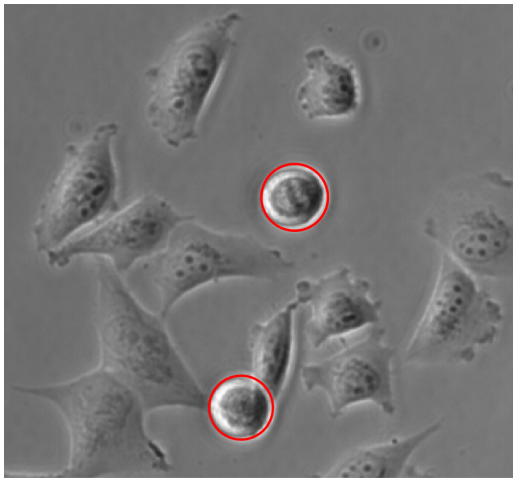
- MR reconstruction for quantifying fluid flow and high-resolution medical imaging click

Dynamic MRI



- Image segmentation for mitosis analysis and forest conservation click

MitosisAnalyser



Mathematical imaging by examples

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- Arts restoration click

Neidthart frescoes



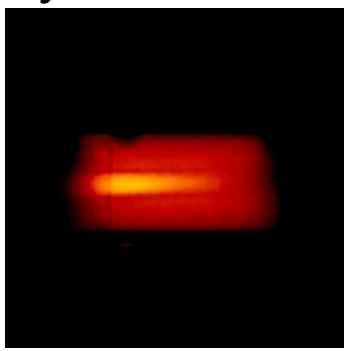
. . . and a fun example if there is time . . .

Mathematical imaging by examples

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

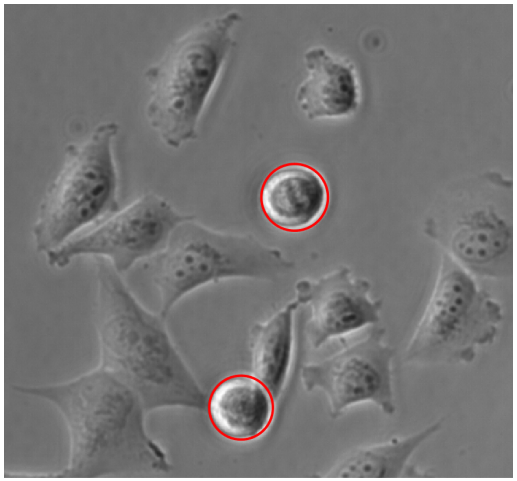
- MR reconstruction for quantifying fluid flow and high-resolution medical imaging

Dynamic MRI

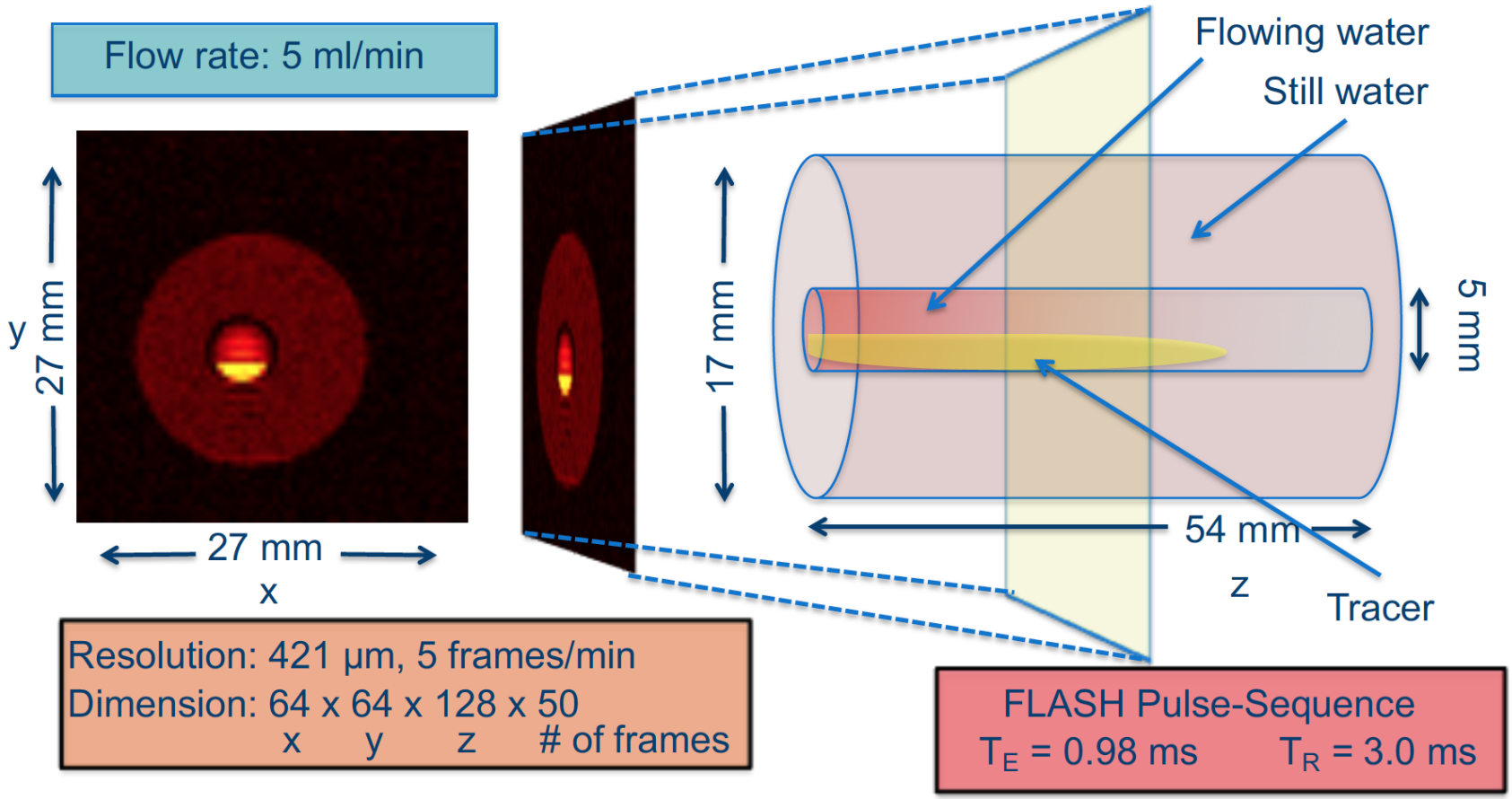


- Image segmentation for mitosis analysis and forest conservation

MitosisAnalyser



Dynamic contrast-enhanced MRI



Data acquisition: D. Allen, J. Beech, V. Kersemans, S. Smart (Uni. Oxford), M. Mantle (Uni. Cambridge)

Dynamic contrast-enhanced MRI

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Joint work with Martin Benning, Andrew Sederman, Lynn Gladden



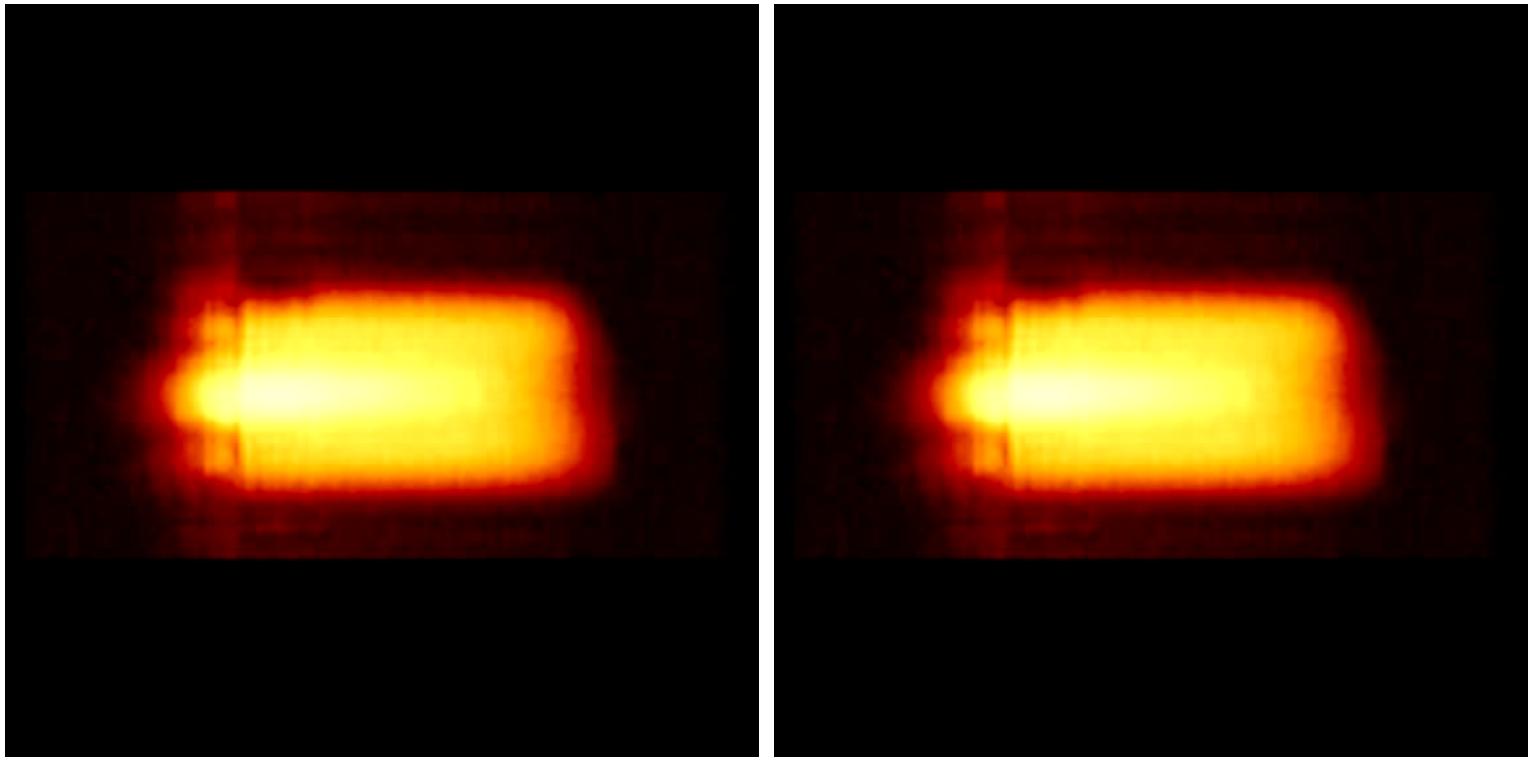


Conundrum: capture dynamics by undersampling in space to increase sampling density in time.

4D reconstruction from sub-sampled data

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Fourier inversion from 20% line-sampling



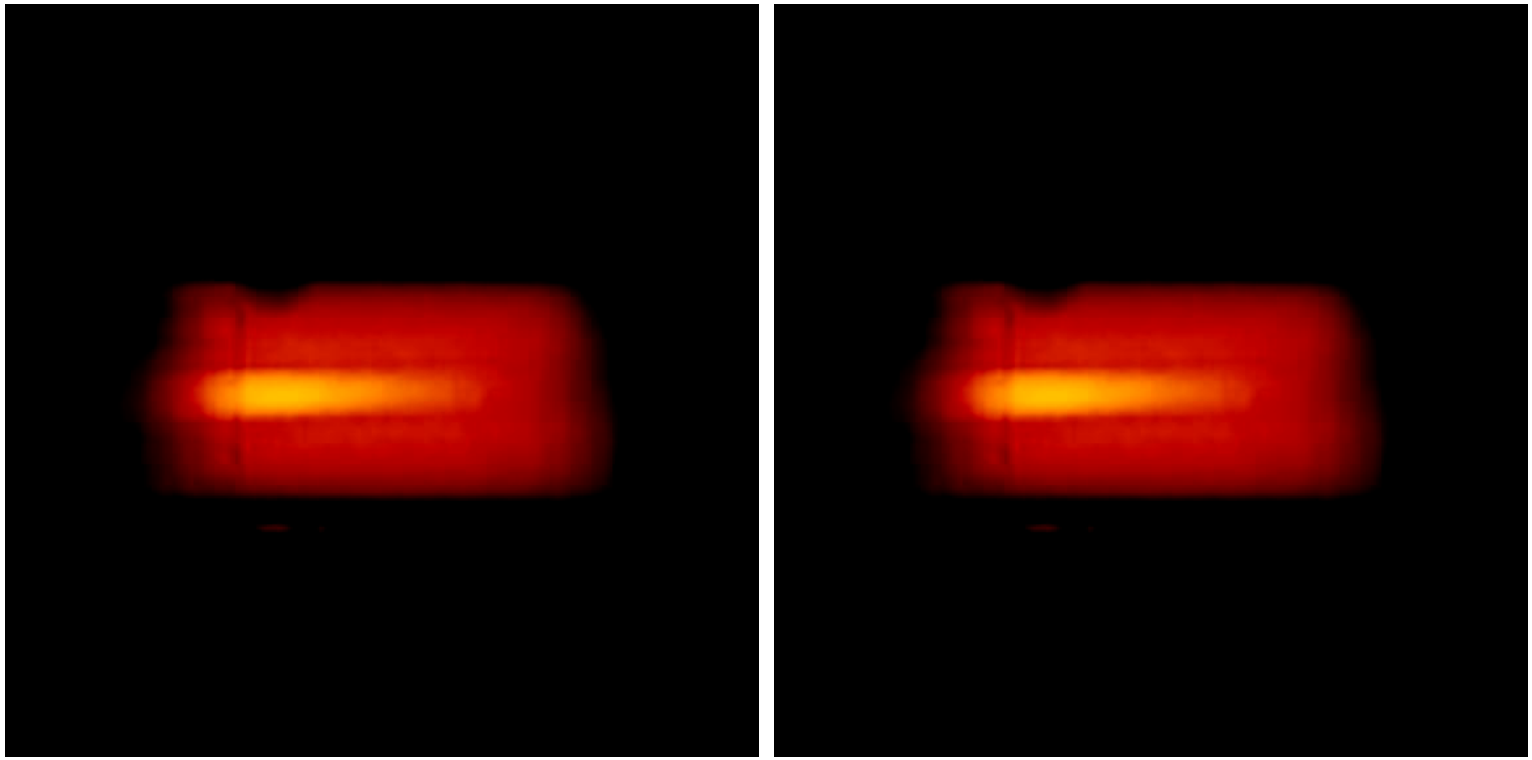
$$\text{Acquire } \mathcal{S}\mathcal{F}(u) \Rightarrow \text{ reconstruct } u(x,t)$$

M. Benning, A. Sederman, CBS, L. Gladden, et al.

4D reconstruction from sub-sampled data

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

4D Bregman-TV Reconstruction from 20% line-sampling



$$\text{Acquire } \mathcal{SF}(u) \Rightarrow \text{ reconstruct } u(x,t) = \text{argmin}_v \alpha \, \|\nabla v\|_1 + \|\mathcal{SF}(v) - g\|_2^2$$

M. Benning, A. Sederman, CBS, L. Gladden, et al.; M. Benning, L. Gladden, D. Holland, CBS, and T. Valkonen, J. Magnetic Resonance 238, pp. 26 - 43, 2014; Osher, Burger et al. '05; M. Lustig et al. '07

Dynamic modelling

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Joint work with Martin Burger, Hendrik Dirks, Lukas Lang, Jocelyn Etienne and Guy Blanchard







Dynamic modelling

Optical Flow

Assumption: **constant brightness**

$$u(x,t) = u(x + \delta_t \mathbf{v}(x,t), t + \delta_t)$$

Leads to:
optical flow equation:

$$\mathbf{v} \cdot \nabla u + u_t = 0.$$

Mass preservation

Assumption: **continuous flow, no teleportation.**

$$\frac{d}{dt} \int_\Omega u(x,t) = 0$$

Leads to:
continuity equation:

$$\nabla \cdot (\mathbf{v} u) + u_t = 0.$$

Challenge: underdetermined problem $\Rightarrow$ regularisation.

Dynamically informed reconstruction

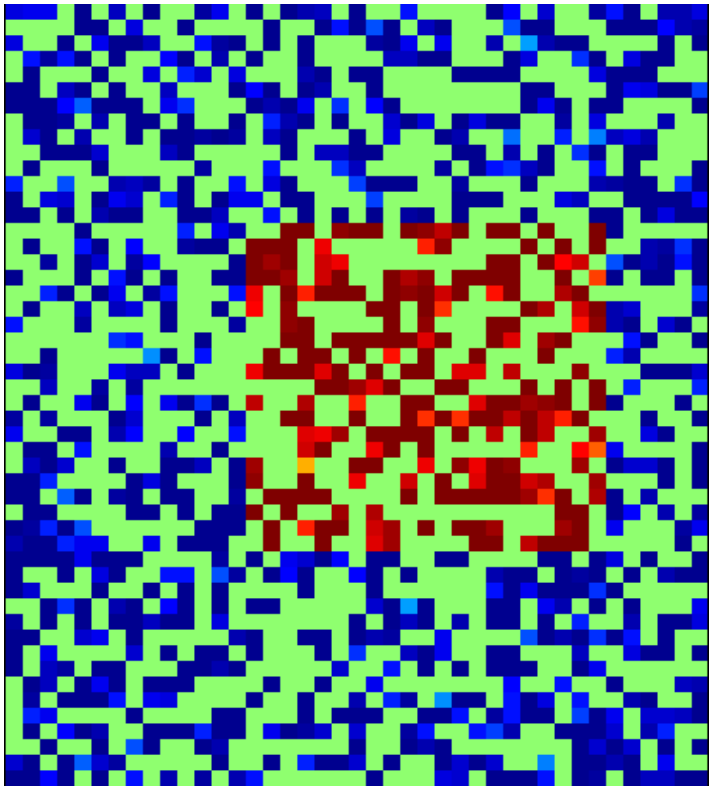


Figure: 50% destroyed

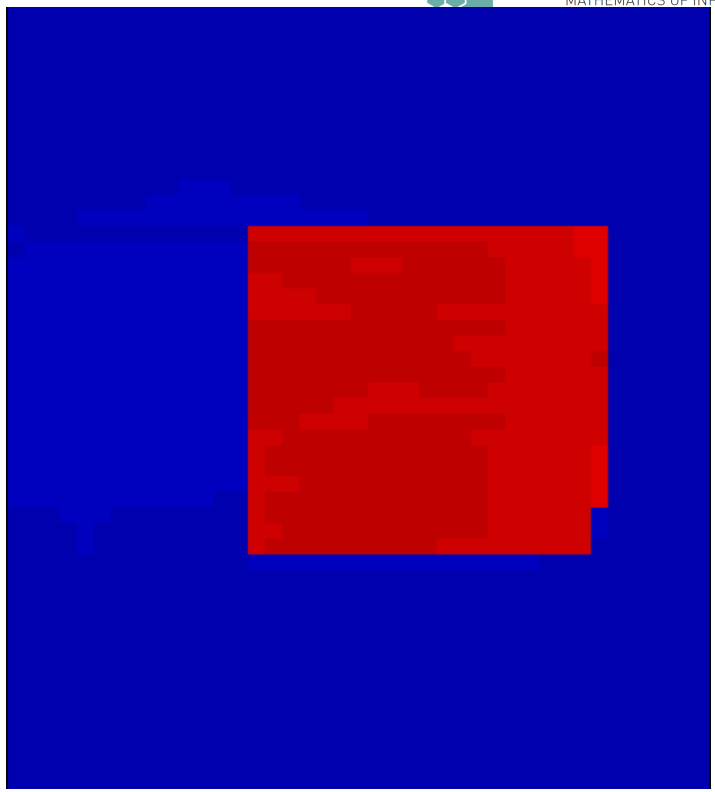


Figure: Restoration

M. Burger, H. Dirks, CBS, arXiv:1607.03255; L. Lang, J. Etienne, G. Blanchard, CBS, in preparation; A. Hauptmann et al. '17; F. Lucka et al. '17

Reconstruction without dynamics

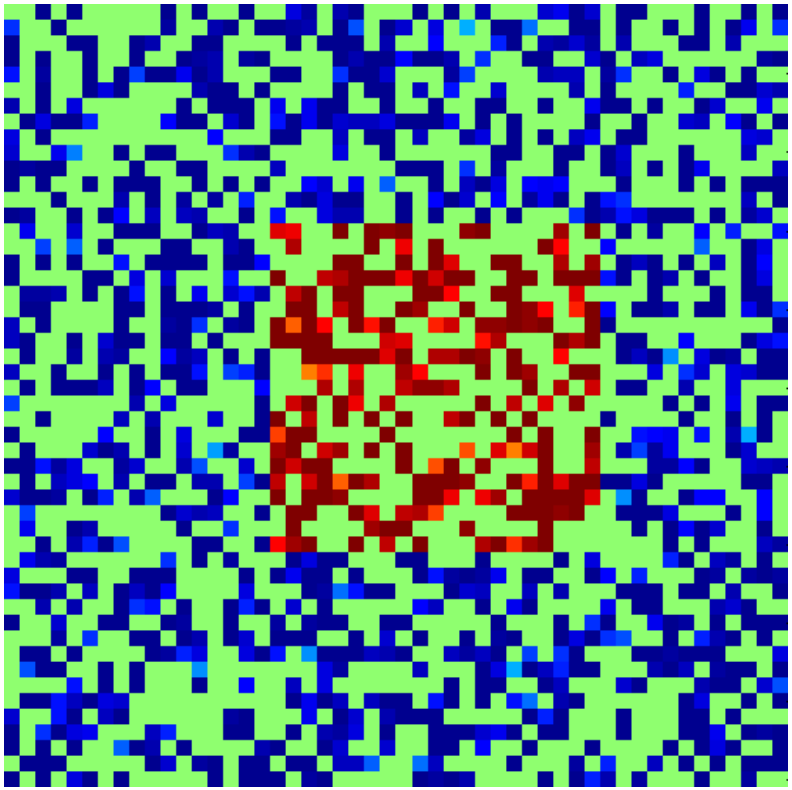


Figure: 50% destroyed

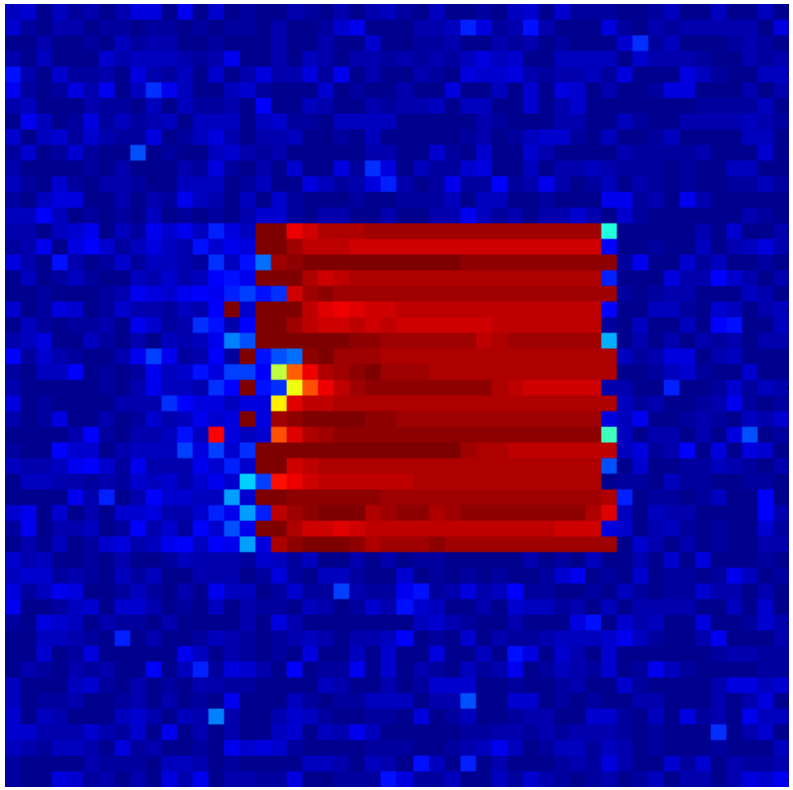


Figure: Restoration

M. Burger, H. Dirks, CBS, arXiv:1607.03255; L. Lang, J. Etienne, G. Blanchard, CBS, in preparation; A. Hauptmann et al. '17; F. Lucka et al. '17

Multi-Contrast MRI Reconstruction

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Work of Simon Arridge, Matthias Ehrhardt and Marta Betcke



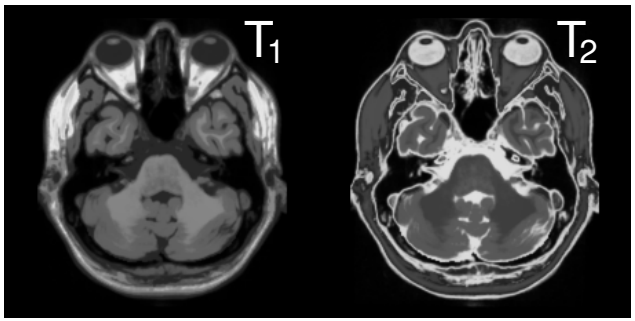


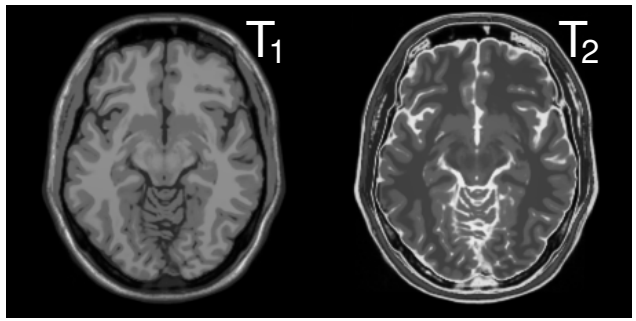
Idea: make use of correlation in MR contrasts and reconstruct jointly.

Multi-Contrast MRI Reconstruction

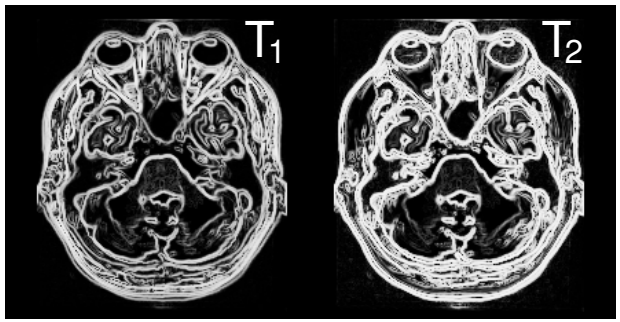
CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

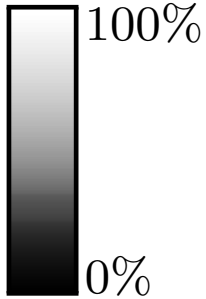
- versatile technology with many **different contrasts**
- MRI contrasts show **similar structures** due to same anatomy
- **exploit redundancy** and reconstruct with less data
- **shorter scan**: patient comfort, save time, dynamic imaging



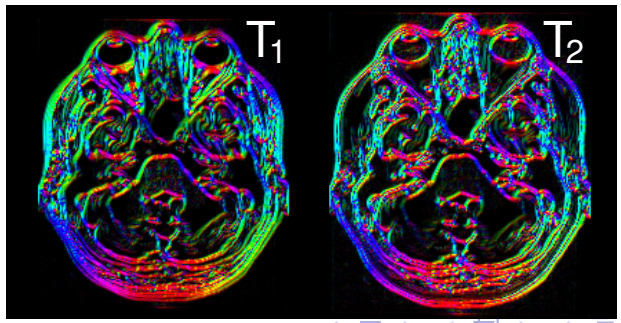


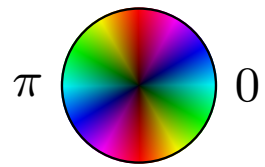
- Difficult to compare images of different contrasts
- Base image structure on location or direction of gradients





direction





Multi-Contrast MRI Reconstruction

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

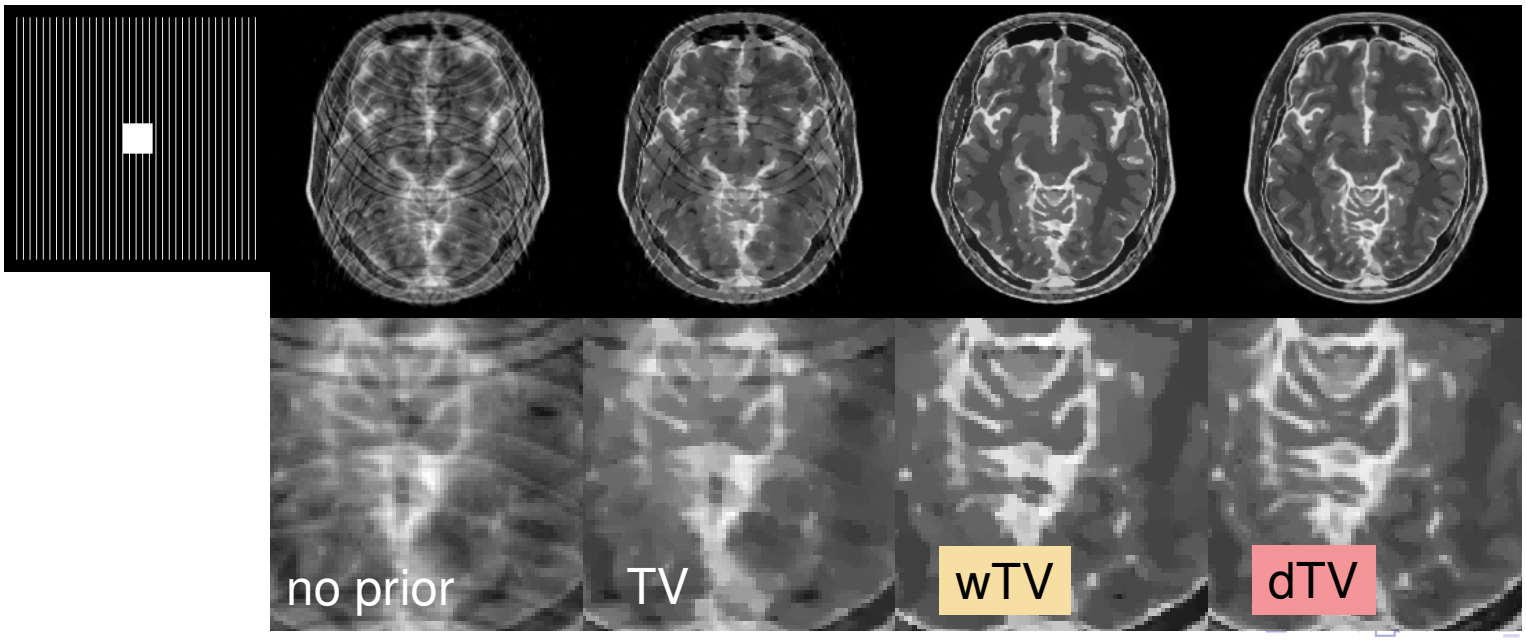
$$\operatorname*{argmin}_u \left\{ \frac{1}{2}|Au - b|^2 + \alpha \int_\Omega |\mathcal{D}(x)\nabla u(x)| dx \right\}$$

Structure either based on location [Arridge et al., 2007]:

$$\mathcal{D}(x) = \eta / |\nabla v(x)|_\eta, \quad |\nabla v(x)|_\eta = \sqrt{|\nabla v(x)|^2 + \eta^2}, \qquad (\text{wTV})$$

or direction [Ehrhardt and Betcke, 2015, arxiv]:

$$\mathcal{D}(x) = I - \xi(x)\xi^T(x), \quad \xi(x) = \nabla v(x) / |\nabla v(x)|_\eta. \qquad (\text{dTV})$$

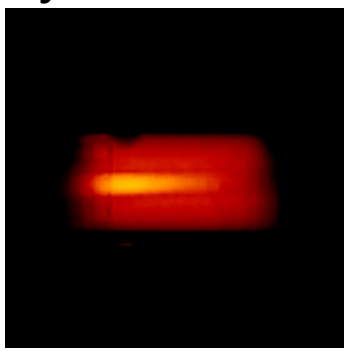


Mathematical imaging by examples

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

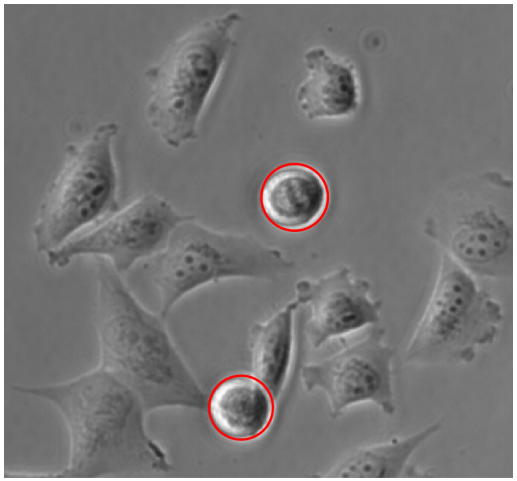
- MR reconstruction for quantifying fluid flow and high-resolution medical imaging

Dynamic MRI

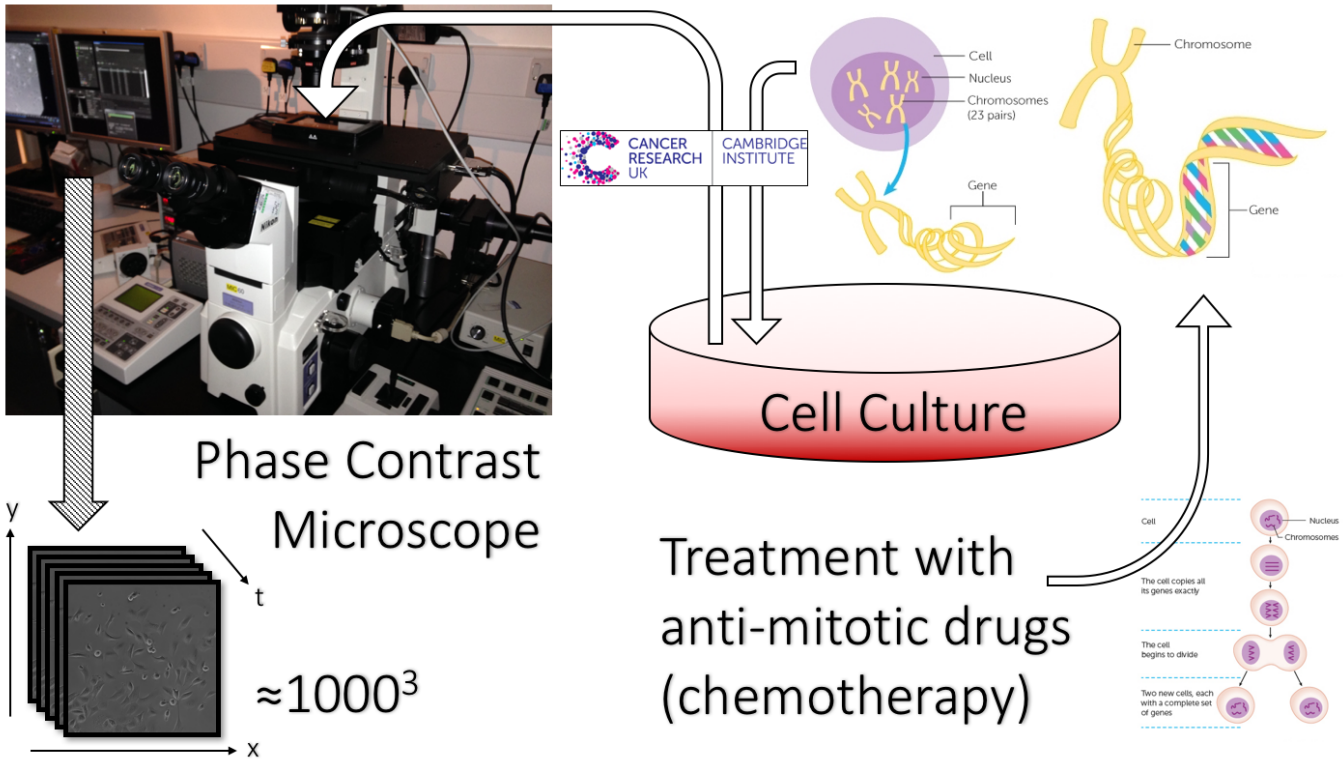


- Image segmentation for mitosis analysis and forest conservation

MitosisAnalyser



Mitosis Analysis for Cancer Research



Aim: Development of mathematical imaging tools for automated analysis of the acquired image sequences

→ **Mitosis duration** and distribution of **cell fates**

[J. Grah, M. Burger, S. Reichelt, C.-B. Schönlieb et al. MitosisAnalyser 2016]

Mitosis Analysis for Cancer Research

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Joint work with Joana Grah, Martin Burger and Stefanie Reichelt

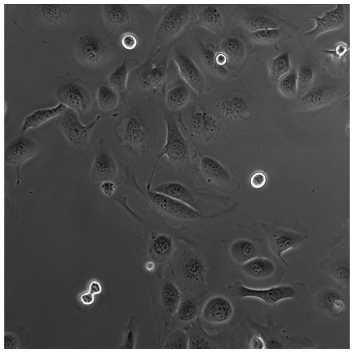




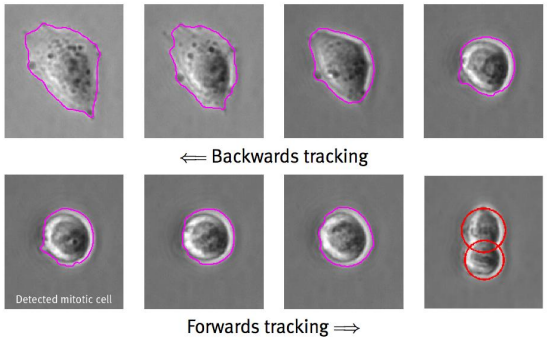
Automated Mathematical Mitosis Analysis

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

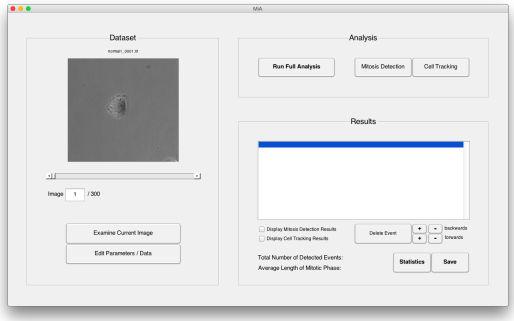
Step 1: Mitosis Detection



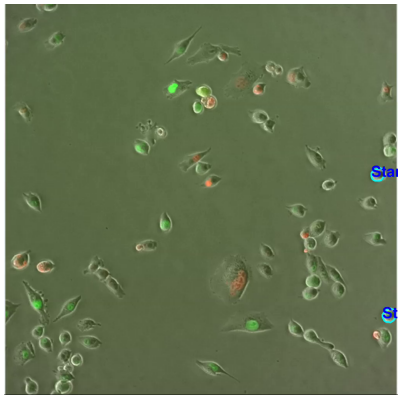
Step 2 & 3: Cell Tracking & Cell Fate



Overall: MitosisAnalyser



Example: Multi-modal experiment (**F**luorescent **U**biquitination-based **C**ell **C**ycle **I**ndicator)



	Control			Treatment 1			Treatment 2		
	1	2	3	4	5	6	7	8	9
Number of events	27	21	43	29	21	32	34	15	35
Avg mitosis duration (min)	101	86	80	77	107	106	110	53	111
Total avg mitosis duration	89			97			91		

[J. Grah, S. B. Koo, M. Burger, S. Reichelt, CBS et al. 2016]

Step 2: cell tracking

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Minimise energy:

$$E(\phi) = \underbrace{\lambda_1 \int_\Omega (1 - H(\phi)) (c_{\text{in}} - |v|)^2 \, dx + \lambda_2 \int_\Omega (H(\phi)) (c_{\text{out}} - |v|)^2 \, dx}_{\text{partition into two regions with different normal velocities}}$$

$$+ \underbrace{\lambda_3 \int_\Omega g_{\text{grad}} (|\nabla f_\sigma(x)|) \; \delta(\phi(x)) \; |\nabla \phi(x)| \, dx}_{\text{stop contour at edges based on the gradient}}$$

$$+ \underbrace{\lambda_4 \int_\Omega g_{\sigma_{\text{loc}}} (f_\sigma(x)) \; \delta(\phi(x)) \; |\nabla \phi(x)| \, dx}_{\text{stop contour at edges based on the local std}}$$

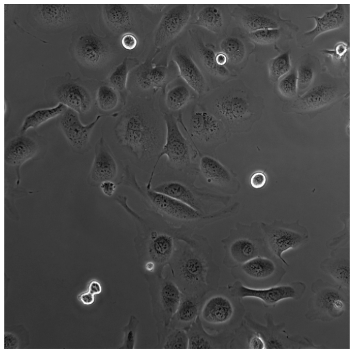
$$+ \underbrace{\mu \int_\Omega \delta(\phi(x)) \, |\nabla \phi(x)| \, dx}_{\text{small contour length}}$$

[Grah 14'; Moeller '13; Chan, Vese 98']

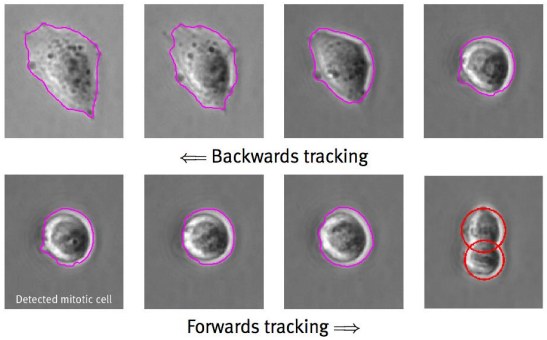
Automated Mathematical Mitosis Analysis

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

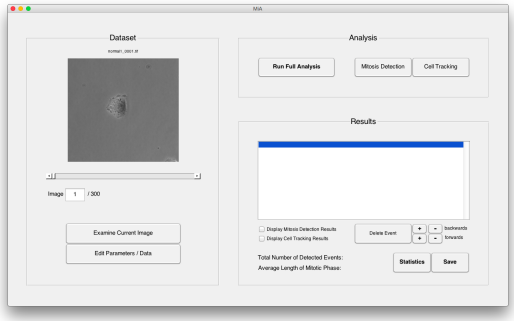
Step 1: Mitosis Detection



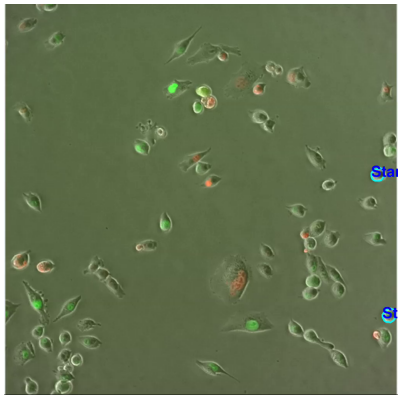
Step 2 & 3: Cell Tracking & Cell Fate



Overall: MitosisAnalyser



Example: Multi-modal experiment (**F**luorescent **U**biquitination-based **C**ell **C**ycle **I**ndicator)



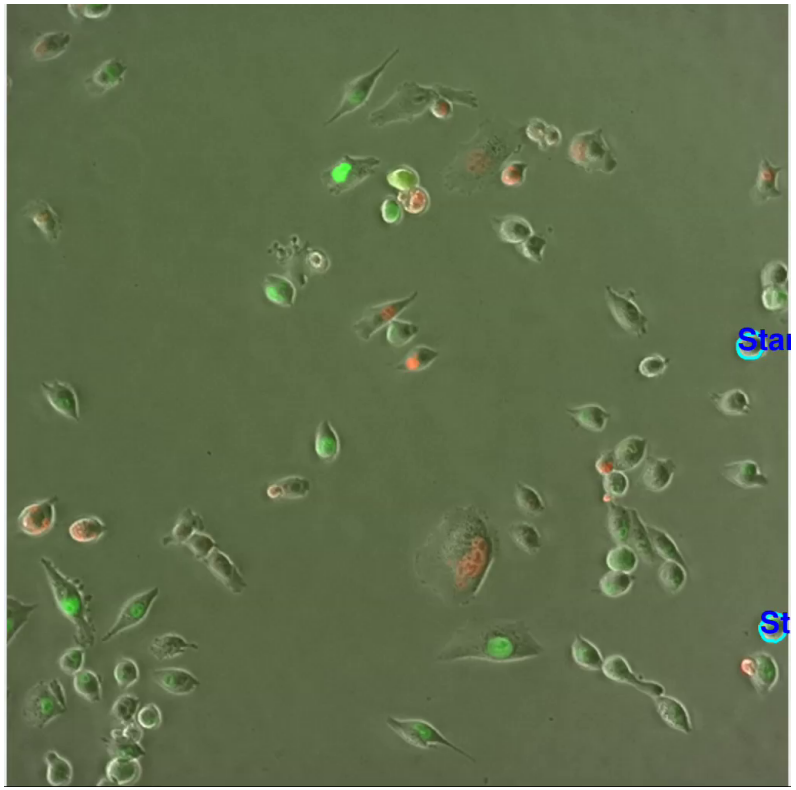
	Control			Treatment 1			Treatment 2		
	1	2	3	4	5	6	7	8	9
Number of events	27	21	43	29	21	32	34	15	35
Avg mitosis duration (min)	101	86	80	77	107	106	110	53	111
Total avg mitosis duration	89			97			91		

[J. Grah, S. B. Koo, M. Burger, S. Reichelt, CBS et al. 2016]

Automated Mathematical Mitosis Analysis

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

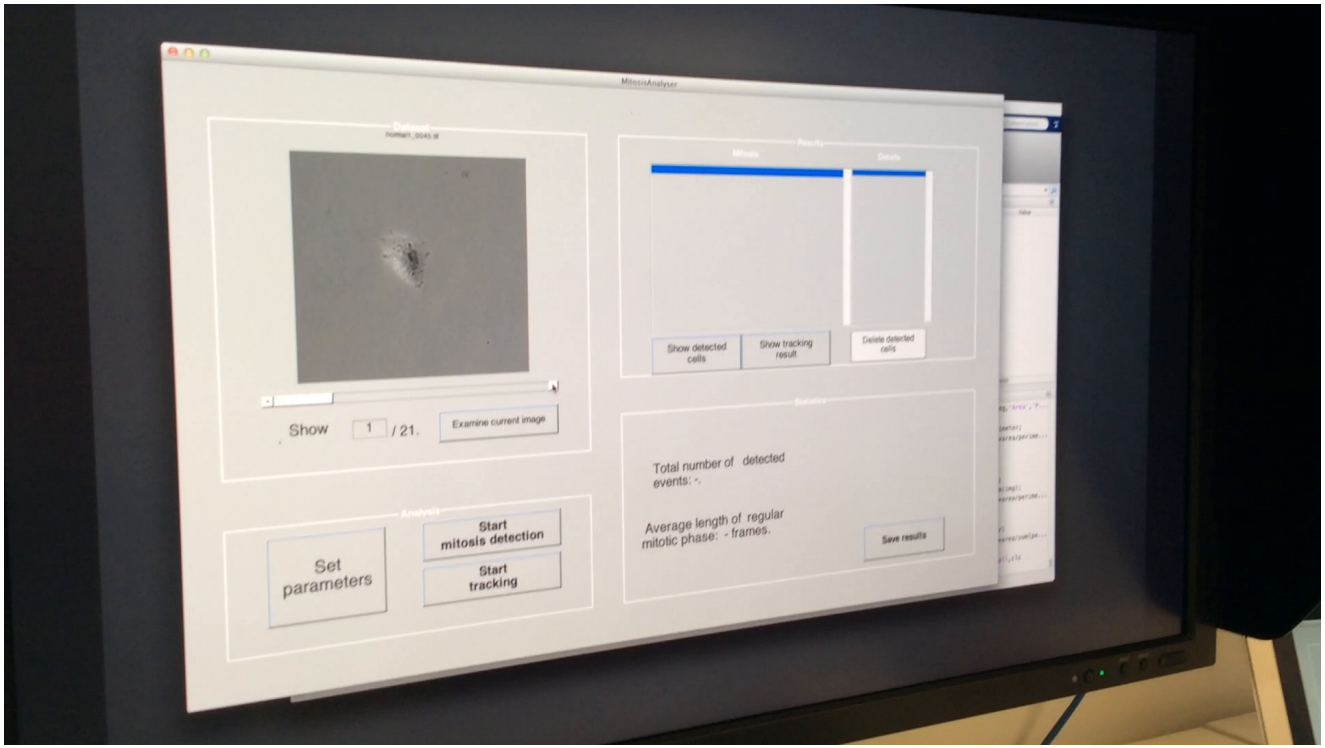
Overall: Multi-modal experiment



[J. Grah, S. B. Koo, M. Burger, S. Reichelt, CBS et al. 2016]

MitosisAnalyser

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



[J. Grah, M. Burger, S. Reichelt, CBS et al., Methods, 115, 2017] **Software:**
`github.com/JoanaGrah/MitosisAnalyser` forward

Tree segmentation for forest conservation

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- Analyse how and why the world's forests are changing
- Use insights gained from such research to inform conservation policy



Tree segmentation for forest conservation

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Joint work with Juheon Lee, Xiaohao Cai and David Coomes



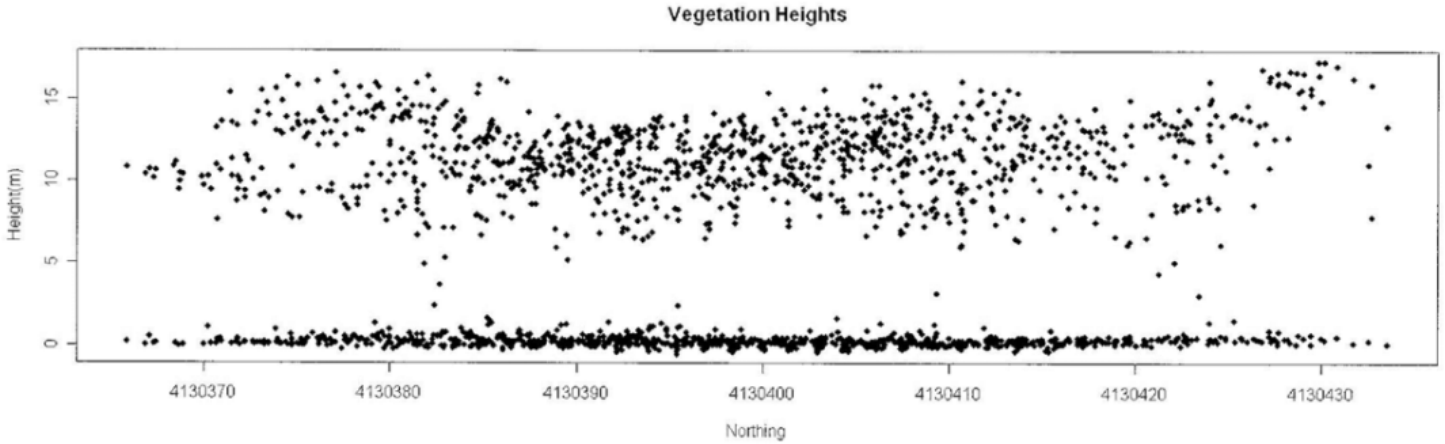


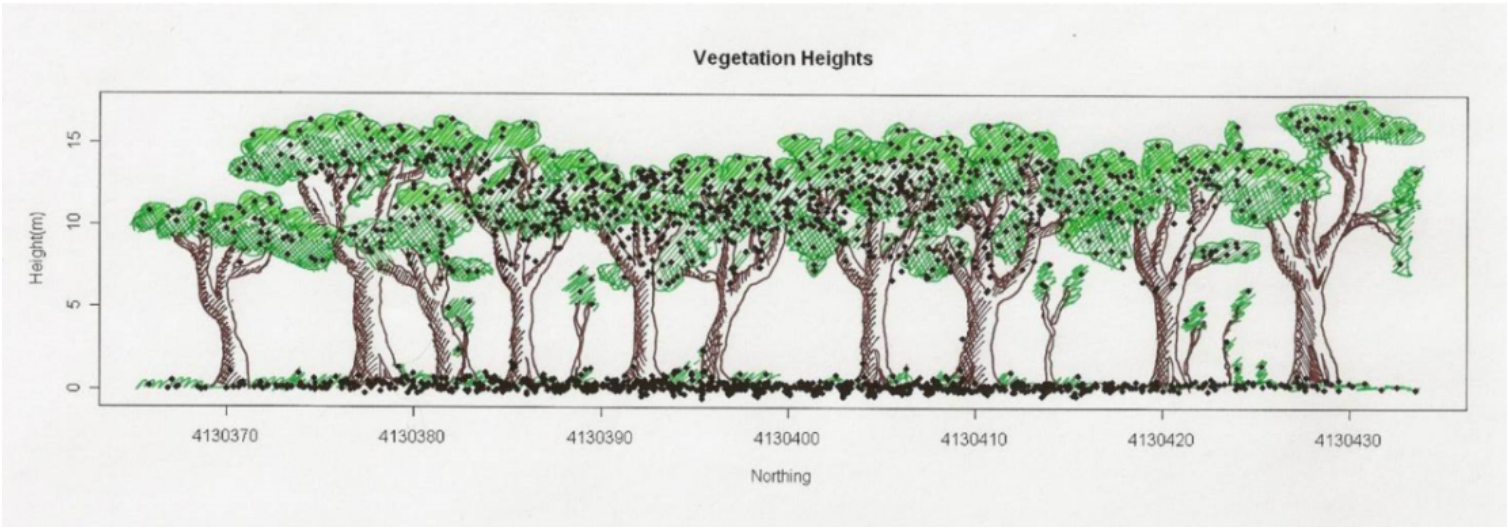


Tree segmentation for forest conservation

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Multi-sensor data: LiDAR



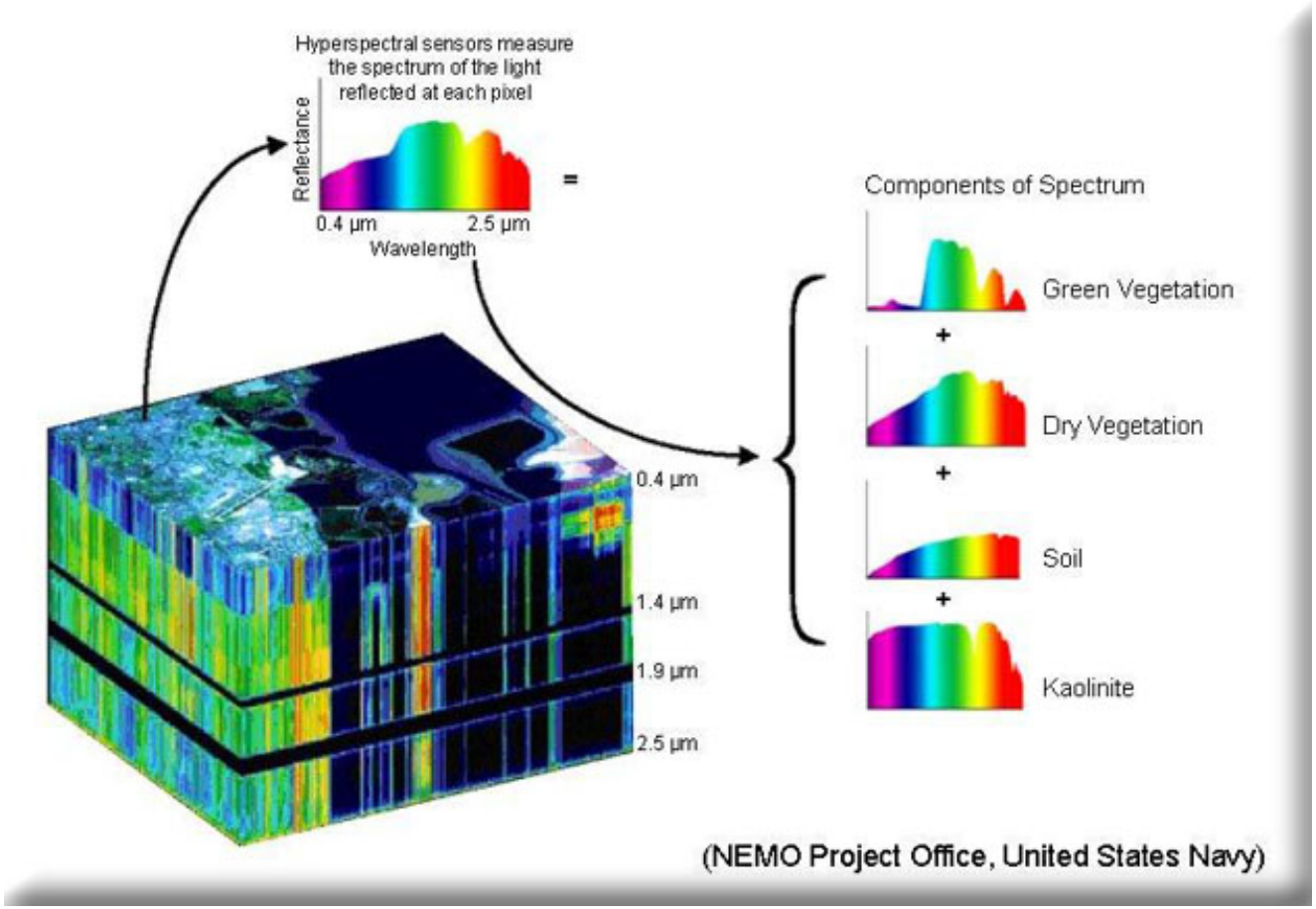


J. Lee, CBS, D. Coomes, et al. '15– Illustration courtesy of Simonson

Tree segmentation for forest conservation

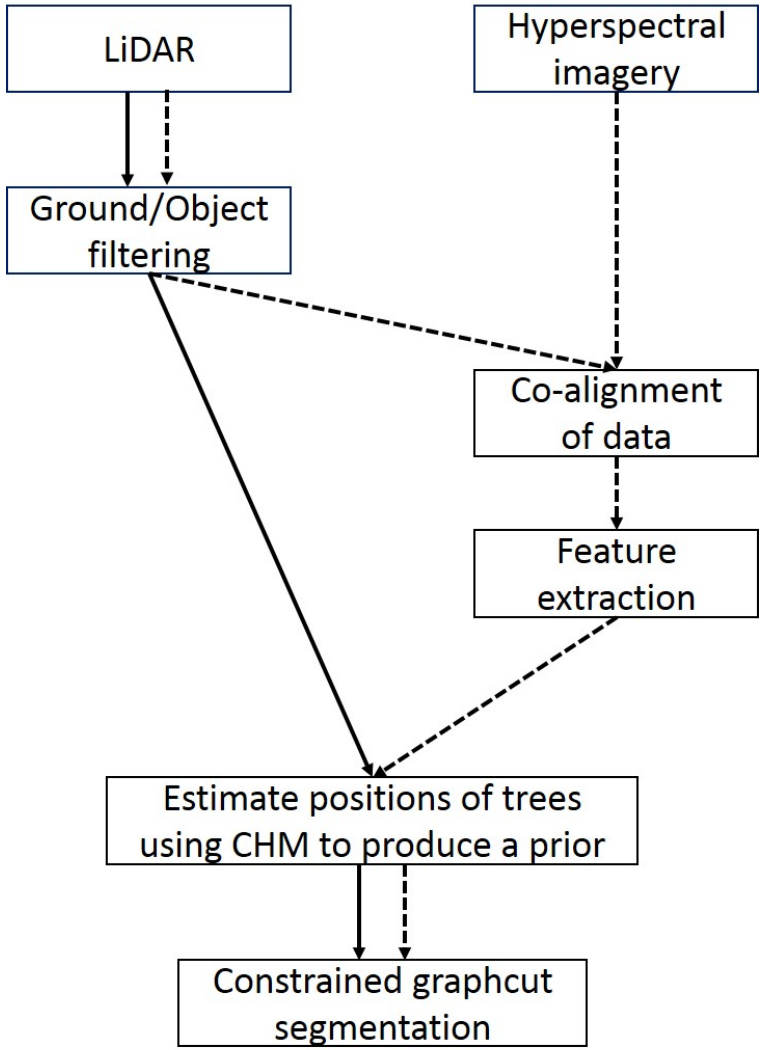
CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Multi-sensor data: hyperspectral images



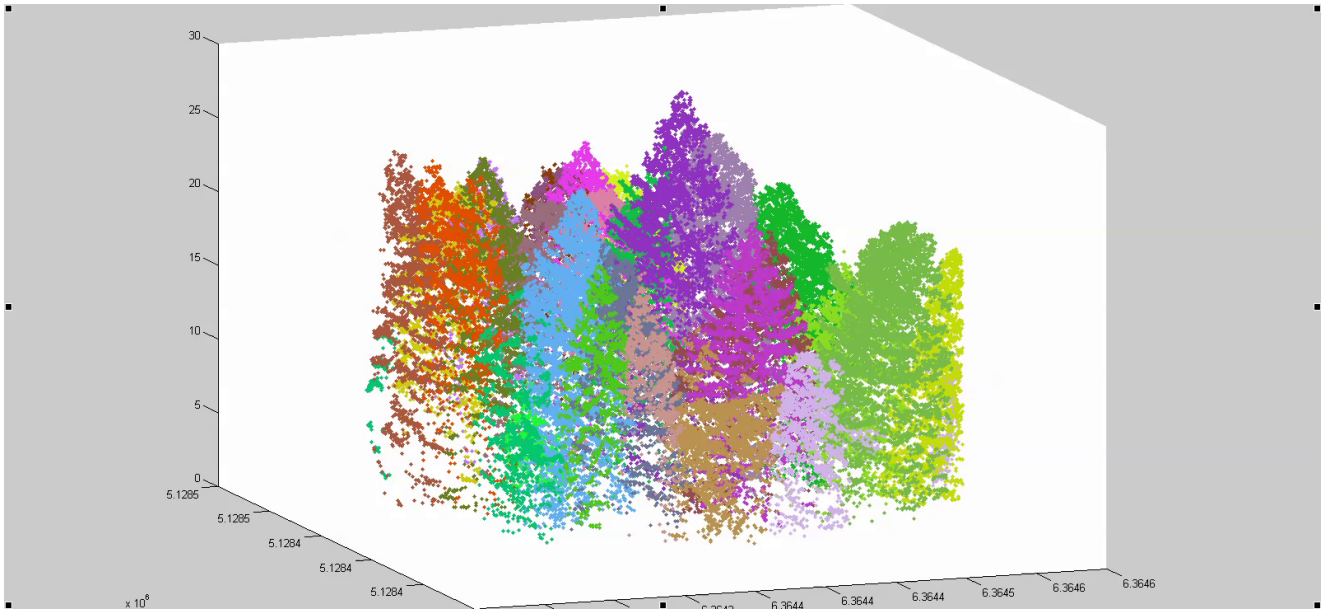
J. Lee, CBS, D. Coomes, et al. ’15–

Tree segmentation for forest conservation



J. Lee, CBS, D. Coomes, et al. '15–

Tree segmentation on the graph



J. Lee, CBS, D. Coomes, et al. ’15–

Evaluating sexual attractiveness . . .

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Joint work with Luca Calatroni, Yves van Gennip and Hannah Rowland

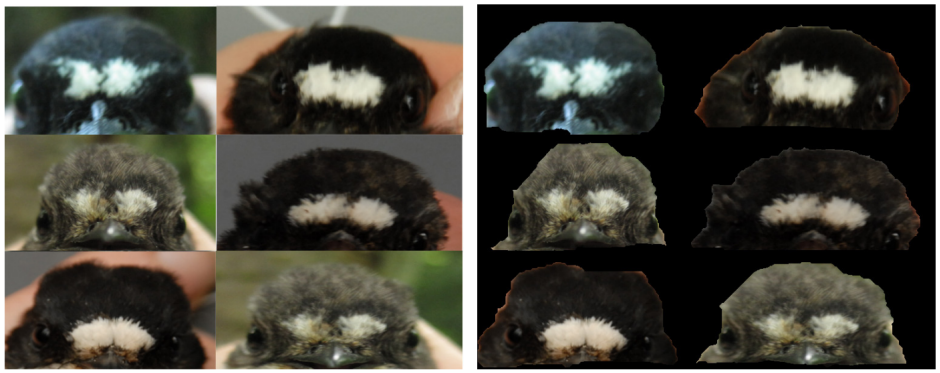




Evaluating sexual attractiveness . . .

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

with graph cut!



Bertozzi, Flenner, 2012; L. Calatroni, Y. van Gennip, H. Rowland, CBS, A. Flenner, 2016 forward

Mathematical imaging by examples

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- Arts restoration

Neidthart frescoes



. . . and a fun example if there is time . . .

Mathematical analysis can make you fly!

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



Mathematics can make you fly! J. Grah, K. Papafitsoros, CBS, EPSRC Science Photo Award '14, Burger, He, CBS '09; CBS, Bertozzi '11; CBS, CUP '15; Chan, Shen '01; Bertalmio et al. '00; Masnou, Morel '98.

Forensics



Low quality fingerprint



Novel enhancement method.

$$\begin{cases} u_t = \operatorname{div}\left(D\left(\mathcal{J}_\rho(\nabla u_\sigma), \mathrm{OF}\right)\nabla u\right) & \text{on } \Omega \times (0,\infty) \\ u(x,0) = f(x) & \text{on } \Omega \\ \langle D\left(\mathcal{J}_\rho(\nabla u_\sigma), \mathrm{OF}\right)\nabla u, \vec{n}\rangle = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

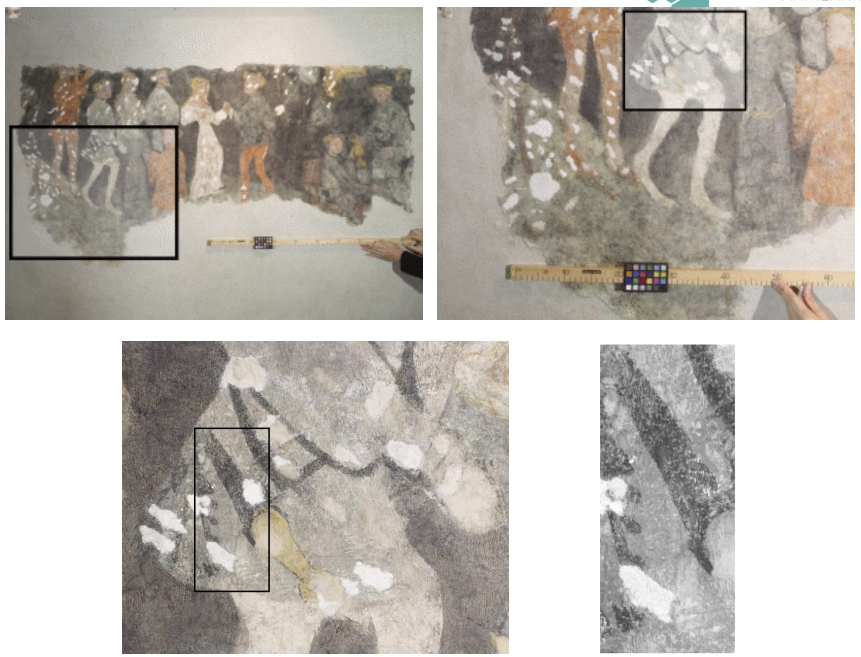
where diffusion tensor D is dependent on image structure modelled by tensor $\mathcal{J}_\rho$, and precomputed direction of orientation OF.

Gottschlich, CBS '12; Weickert '98

Restoring profanity



Detail of the Neidhart frescoes (13th century Viennese artwork).



$$u_t = \alpha\Delta\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) + \chi_{\Omega\setminus D}(g-u)$$

TV-H^{-1} image inpainting.

Baatz, Fornasier, Markowich, CBS, '10; Burger, He, CBS '09; CBS, Bertozzi '11; Efros, Leung '99; Bertalmio et al. '03.

Restoring paintings

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Joint work with Marie D'Autume, Spike Bucklow and Stella Panayotova





Restoring paintings

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



Figure: Adoration of the Shepherds, Sebastiano Del Piombo (1519)

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

Restoring paintings

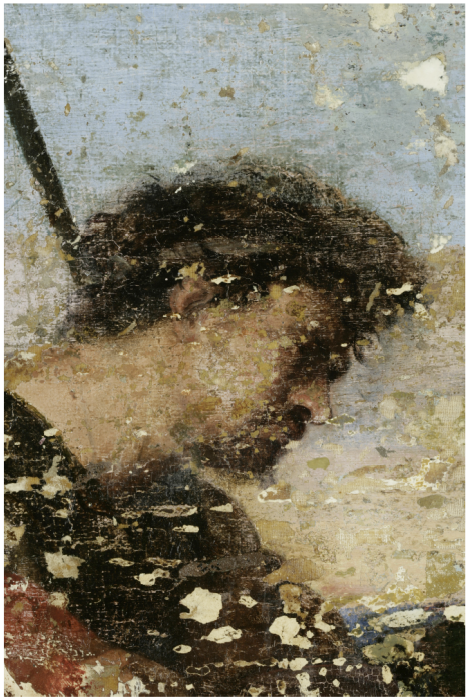


Figure: A detail

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

Restoring paintings

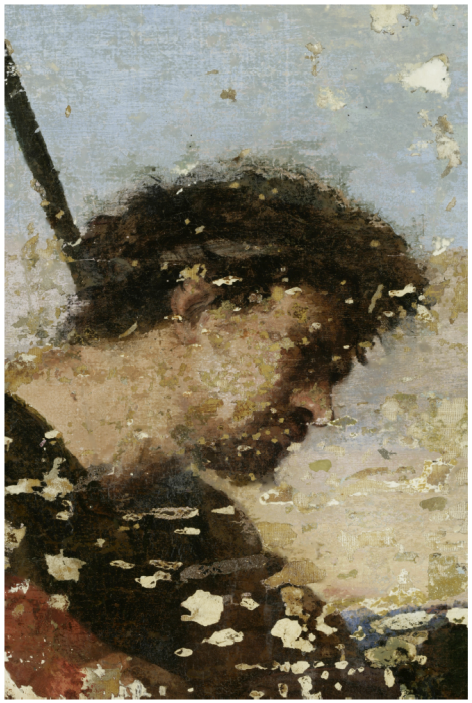


Figure: Small damages restored with structure (TV) inpainting[1]

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

[1] Chan, Shen '00

Restoring paintings

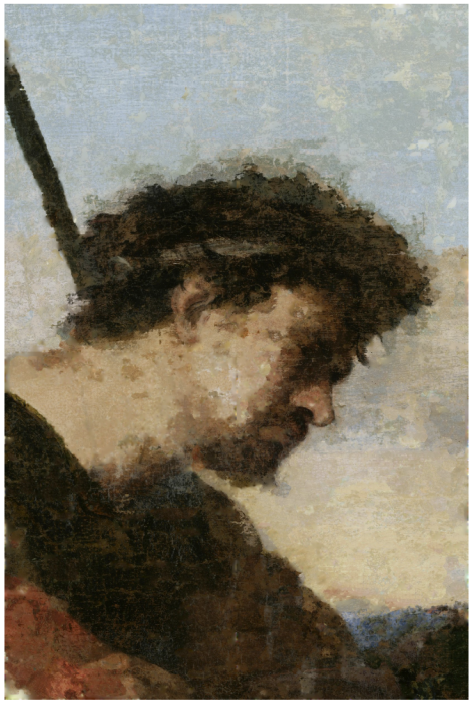


Figure: Large damages restored with structure & texture inpainting[1]

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Spike Bucklow (Hamilton Kerr Institute)

[1] Arias, Facciolo, Caselles, Sapiro '11; CBS, Bertozzi '11; Burger, He, CBS '09; Bertalmio et al. '03; Criminisi et al. '04; Efros, Leung '99.

Unveiling the invisible



Figure: Manuscript by Claude of Franceś Primer

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Unveiling the invisible

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



Figure: Detail

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Unveiling the invisible



Figure: Infrared image of detail

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Unveiling the invisible

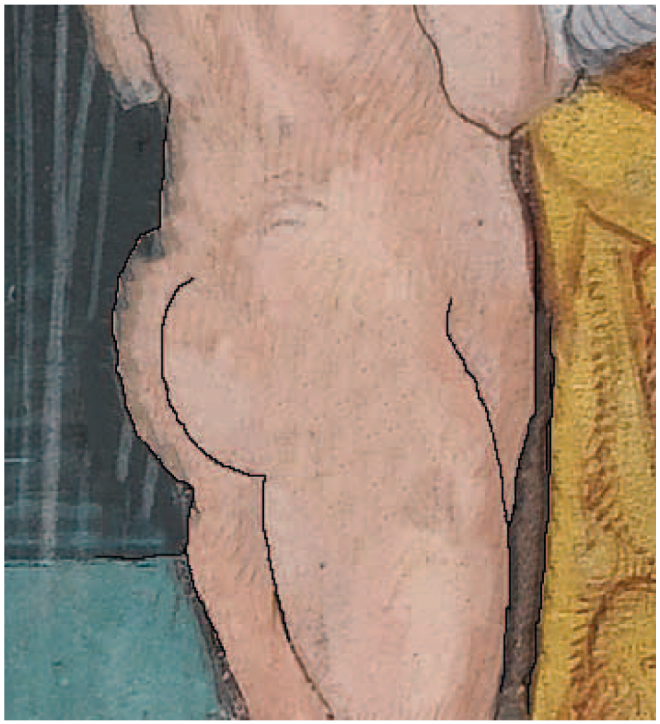


Figure: Unveiled Adam[2]

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

[2] Weickert '13

Unveiling the invisible



Figure: Unveiling the invisible

References: Bachelor thesis Marie Autume (intern from ENS Cachan) in collaboration with Stella Panayotova and Paola Ricciardi (Fitzwilliam Museum)

Adam and Eve naked again!

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

sign in | become a supporter | subscribe | search | jobs | dating | more | UK edition

theguardian

UK | election | world | sport | football | opinion | culture | business | lifestyle | fashion | environment | tech | travel | browse all sections

home › culture › art & design | stage | classical | film | tv & radio | music | games | books

Exhibitions

Unveiled: Adam and Eve naked again after centuries-old cover-up

Exhibition at Fitzwilliam Museum in Cambridge shows biblical pair in original state after onetime owner painted over nudity

This article is 10 months old

546 | 44

Maev Kennedy

Monday 25 July 2016
10.31 BST



Adam and Eve are seen as they were painted 500 years ago. Photograph: Andrew Morris/The Fitzwilliam Museum

Adam and Eve are once again as naked as the day they were created, centuries after some prudish hand wrapped his loins in a grass skirt and draped a veil

Advertisement



Development of methodology, taylor-made for particular application . . . mathematical concepts?

Key mathematical concepts for modern image analysis

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- Large data – small information $\Rightarrow$ sparsity, structured data processing
- Multiple data types – one question $\Rightarrow$ registration, fusion
- Noise in data – robustness $\Rightarrow$ stable processing
- Enormous data sizes to process $\Rightarrow$ scalable and parallelisable optimisation

My take on some of the above . . .

Non-smooth and non-convex optimisation

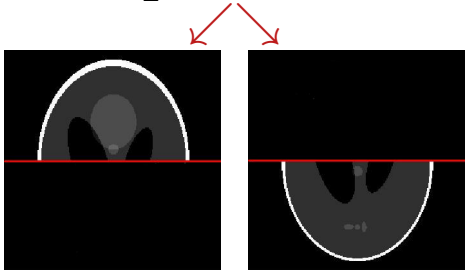
CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

$$\min_u \begin{cases} \|\nabla u\|_1 \\ \|u\|_1 \end{cases}$$

under linear or nonlinear constraints or penalisation, e.g. for problems arising in computational harmonic analysis (sparse basis recovery).

- M. Benning, M. Betcke, M. Ehrhardt, and CBS, *Choose your path wisely: gradient descent in a Bregman distance framework*, in preparation.
- A. Chambolle, M. Ehrhardt, P. Richtarik, and CBS, *Stochastic Primal-Dual Hybrid Gradient Algorithm with Arbitrary Sampling and Imaging Application*, arxiv 2017.
- M. Benning, F. Knoll, CBS and T. Valkonen, *Preconditioned ADMM with nonlinear operator constraint*, IFIP Advances in Information and Communication Technology TC7-2015 Proceedings, Springer, 2016.
- M. Fornasier, A. Langer, CBS: *A convergent overlapping domain decomposition method for total variation minimization*, Numerische Mathematik, Vol. 116, Nr. 4, pp. 645 - 685 (2010).

$$\alpha\|\nabla u\|_1 + \tfrac{1}{2}\|Tu - g\|^2 \to \min_u$$



Structured image processing

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- Modelling and analysis of prior / regularisation theory
  - M. Benning, M. Foged-Schmidt, CBS, *Inverse scale space decomposition*, 2016.
  - M. Burger, H. Dirks and CBS, *A variational model for joint motion estimation and image reconstruction*, arXiv:1607.03255, 2016.
  - J. Lellmann, K. Papafitsoros, CBS, and D. Spector, *Analysis and Application of a Non-Local Hessian*, SIAM Journal on Imaging Sciences, 8(4), 2161-2202, 2015.
  - J. Lellmann, D. Lorenz, CBS, and T. Valkonen, Imaging with Kantorovich-Rubinstein discrepancy, SIAM Journal on Imaging Sciences, 7(4), 2833-2859, 2014.

TV - sparse gradient regularisation click





Learning with structure

- Deep variational models – bilevel learning
  - M. Benning, M. Ehrhardt, G. Maierhofer, J. C. De Los Reyes, and CBS, *Bilevel learning for optimal sampling in MRI*, in preparation.
  - M. Benning, CBS, T.Valkonen, and V. Vlacic, *Explorations on anisotropic regularisation of dynamic inverse problems by bilevel optimisation*, arXiv:1602.01278
  - J. C. De Los Reyes, CBS, and T. Valkonen, *The structure of optimal parameters for image restoration problems*, Journal of Mathematical Analysis and Applications 434 (2016), 464–500.
  - J. C. De Los Reyes, and CBS, *Image denoising: Learning noise distribution via PDE-constrained optimisation*, Inverse Problems and Imaging, Vol. 7, 1183-1214, 2013.



The End

$$\min_{\lambda} \text{loss}(u(\lambda))$$
$$\text{s.t.} \quad u(\lambda) = \text{argmin}_v \mathcal{J}(v, \lambda)$$

Outline

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

1 Variational regularisation and PDEs for imaging

2 Learning of variational models in imaging

3 Conclusions and outlook . . . and maybe a fun example

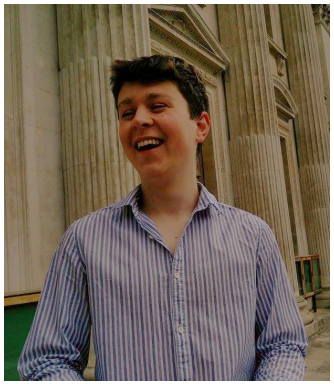
Learning with structure

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Joint work with Juan Carlos De Los Reyes, Martin Benning, Luca Calatroni, Matthias Ehrhardt, Georg Maierhofer, Ferdia Sherry, Tuomo Valkonen …








Learning variational models - one idea

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Assumptions

Training set of pairs $(f_k, u_k),\ k = 1, \dots, N$ with

- f_k imperfect data
- u_k represent the ground truth

Determine optimal regulariser R, data model ϕ, and α in admissible set $\mathcal{A}$

$$\min_{(R,\phi,\alpha,T)\in\mathcal{A}} \sum_k \text{Cost}(\bar{u}_k, u_k)$$

subject to

$$\bar{u}_k = \text{argmin}_u \left\{ \alpha\ R(u) + \int_\Omega \phi(Tu, f_k)\ dx \right\}$$

Learning by optimisation in imaging

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Some contributions

- Odone '05–, Tappen et al. '07, '09; Domke '11–: Markov Random Field models; stochastic descent method
- Lui, Lin, Zhang and Su '09: optimal control approach, no analytical justification; promising numerical results.
- Horesh, Tenorio, Haber et al. '03–: optimal design; ℓ_1 minimisation.
- Kunisch and Pock '13, Pock 13' –: results for finite dimensional case; optimal image filters; optimal SVM; optimal reaction-diffusion ...
- De Los Reyes, CBS '13 –: results on bilevel learning in function space and development of numerical optimisation.
- Hintermüller et al. '14 – : bilevel optimisation for blind deconvolution, and for adaptive TV denoising.
- Nikolova, Steidl, Weiss '15
- Fonseca, Liu et al. '16 –: bilevel model for higher-order TV type regularisation and Mumford-Shah; analysis in function space ...

Learning by optimisation in imaging

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Some contributions

- Odone '05–, Tappen et al. '07, '09; Domke '11–: Markov Random Field models; stochastic descent method
- Lui, Lin, Zhang and Su '09: optimal control approach, no analytical justification; promising numerical results.
- Horesh, Tenorio, Haber et al. '03–: optimal design; ℓ_1 minimisation.
- Kunisch and Pock '13, Pock 13' –: results for finite dimensional case; optimal image filters; optimal SVM; optimal reaction-diffusion . . .
- De Los Reyes, CBS '13 –: results on bilevel learning in function space and development of numerical optimisation.
- Hintermüller et al. '14 – : bilevel optimisation for blind deconvolution, and for adaptive TV denoising.
- Nikolova, Steidl, Weiss '15
- Fonseca, Liu et al. '16 –: bilevel model for higher-order TV type regularisation and Mumford-Shah; analysis in function space . . .

Analysis in function space & resolution independent optimisation.

Learning a parametrised model

Look for $\lambda = (\lambda_1, \ldots, \lambda_M)$ and $\alpha = (\alpha_1, \ldots, \alpha_N)$ solving

$$\min_{(\lambda,\alpha)\in[0,\infty]^{M+N}} F(u_{\lambda,\alpha})$$

subject to

$$u_{\lambda,\alpha} \in \operatorname{argmin}_{u\in X} \sum_{i=1}^{M} \int_\Omega \lambda_i(x)\phi_i([Tu](x))\, dx + \sum_{j=1}^{N} \int_\Omega \alpha_j(x)\, d|A_j u|(x).$$

Here $T : X \to Y \subset L^1(\Omega;\mathbb{R}^d)$ with X, Y Banach spaces, $A_j : X \to \mathcal{M}(\Omega;\mathbb{R}^{m_j})$, $(j = 1, \ldots, N)$ are appropriate linear operators, $|A_j u|$ total variation measure, F is cost function.

Cost function: how to measure optimality?

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Cost functions: For noise free data $\tilde{u}$ we take either PSNR

$$F_{L^2}(v) = \frac{1}{2}\|\tilde{u} - v\|_2^2,$$

or Huberised TV cost

$$F_{L^1 \nabla_\gamma}(v) = |D(\tilde{u} - v)|_\gamma(\Omega).$$

For a training set, take average, e.g.

$$F_{L^2}(v_1, \ldots, v_K) = \sum_{i=1}^{K} \|v_i - \tilde{u}_i\|_2^2.$$

Analytic results

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- Theorem 1: Existence of an optimal solution for non-smooth problem (under appropriate assumptions optimal parameter lies in the interior!) ✓
- Theorem 2: Existence and continuity of optimal solution for smoothed problem with Huber-regularisation & elliptic term (Hilbert space) ✓
- Theorem 3: Derivation of optimality system and gradient of solution map for smoothed problem ✓
- Theorem 4: Convergence of solutions of regularised problem to non-smooth problem in Hilbert space ✓

De Los Reyes 2012; CBS, De Los Reyes 2013; Calatroni, CBS, De Los Reyes 2014; **De Los Reyes, CBS, Valkonen 2015**

Numerical strategy

Solve

$$\min_{(\lambda,\alpha)\in[0,\infty]^{M+N}} F(u_{\lambda,\alpha})$$

subject to

$$u_{\lambda,\alpha} = \operatorname{argmin}_u \frac{\mu}{2}\sum_{j=1}^{N}\|A_j u\|_2^2 + \sum_{i=1}^{M}\int_\Omega \lambda_i(x)\phi_i(x,[Tu](x))\,dx + \sum_{j=1}^{N}\int_\Omega \alpha_j(x)\,d|A_j u|_\gamma(x).$$

by quasi-Newton method (BFGS)

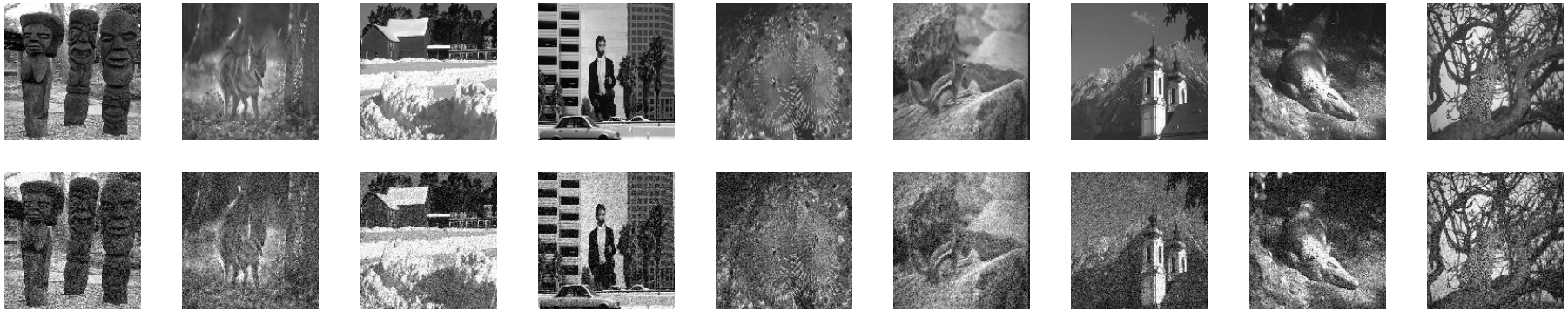
- state equation is solved by Newton type algorithm (varies with ϕ)
- evaluation of the gradient of the cost functional with adjoint information
- Armijo line search with curvature verification.
- For large training set we use dynamic sampling technique for constraints á la Byrd et al.

Parameters: we typically choose $10^{-10} \approx \mu \ll 1$, $100 \approx \gamma \gg 1$.

TV regularisation

Cross-validated computations on the Berkeley database split into two halves (100 images each):

Total variation regularisation with L^2 cost and fidelity. Noise variance $\sigma = 10$.



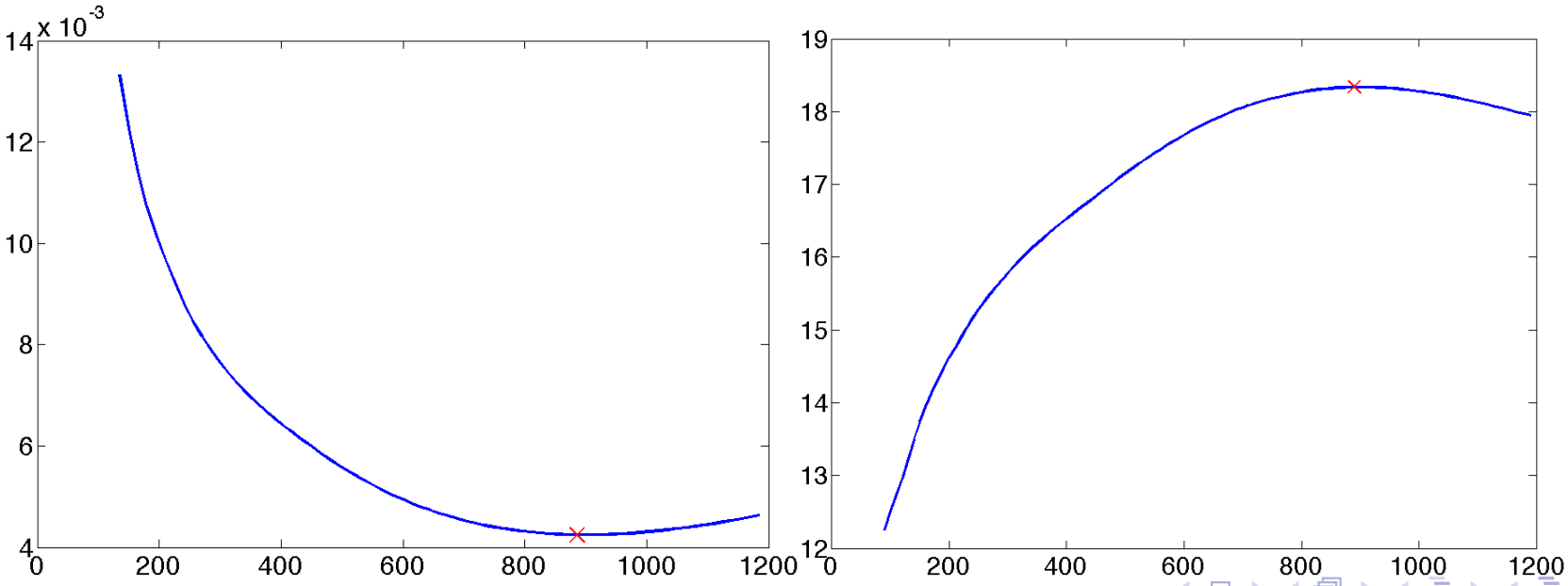
Validation	Learning	α	Average PSNR	Average SSIM
1	1	0.0190	31.3679	0.8885
1	2	0.0190	31.3672	0.8884
2	1	0.0190	31.2619	0.8851
2	2	0.0190	31.2612	0.8850

Parameter optimality?

Quality measure

- Original cost functional (left figure) $\|u - u_k\|_{L^2}^2$
- Signal to noise ratio (right figure)

$$SNR = 20 \times \log_{10}\left(\frac{\|u_k\|_{L^2}}{\|u - u_k\|_{L^2}}\right),$$



TGV - training set

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Cross-validated computations on the Berkeley database split into two halves (100 images each): TGV^2 regularisation with L^2 cost and fidelity. Noise variance $\sigma = 10$.

Validation	Learning	(β, α)	Average PSNR	Average SSIM
1	1	(0.0187, 0.0198)	31.4325	0.8901
1	2	(0.0186, 0.0191)	31.4303	0.8899
2	1	(0.0186, 0.0191)	31.3281	0.8869
2	2	(0.0187, 0.0198)	31.3301	0.8870

Learning (β, α) in $\mathrm{TGV}^2_{\beta,\alpha}$

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



(a) Original image



(b) Noisy image

Optimal $\mathrm{TGV}^2_{\beta,\alpha}$



(c) TGV^2 denoising, $L_1\nabla_\gamma$ cost functional



(d) TGV^2 denoising, L_2^2 cost functional

$L^1\nabla_\gamma$ cost

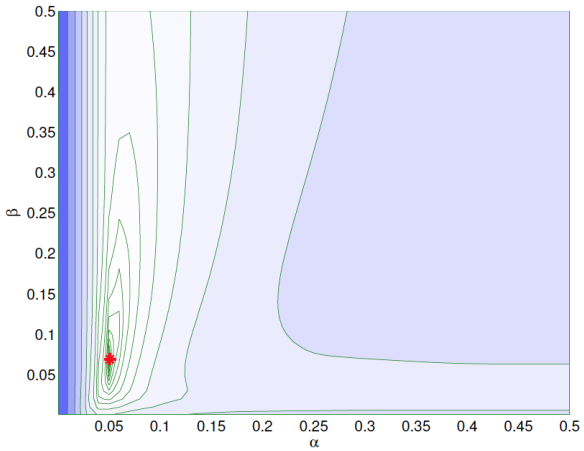
$(\alpha, \beta) = (0.069/n^2, 0.051/n)$

L^2 cost

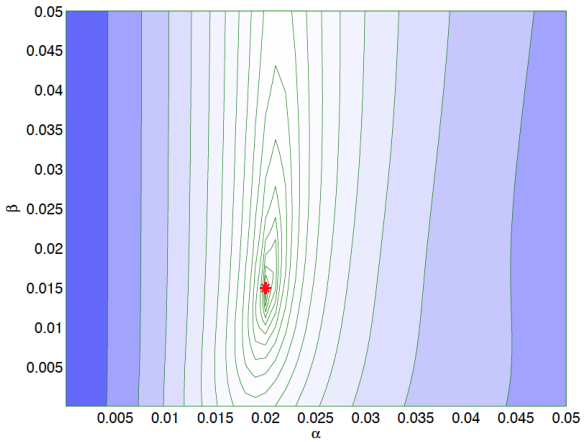
$(\alpha, \beta) = (0.058/n^2, 0.041/n)$

Optimal (β, α) in $\mathrm{TGV}^2_{\beta,\alpha}$?

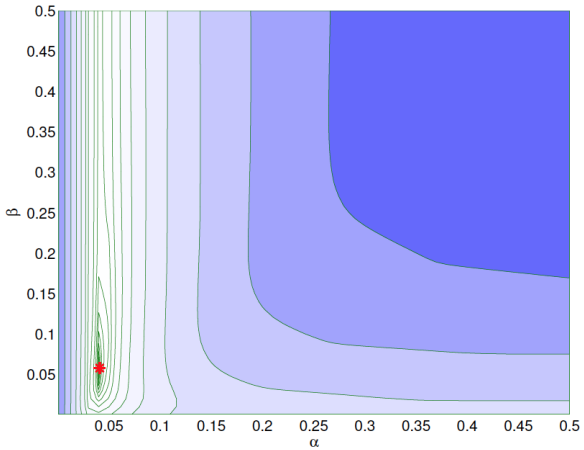
CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



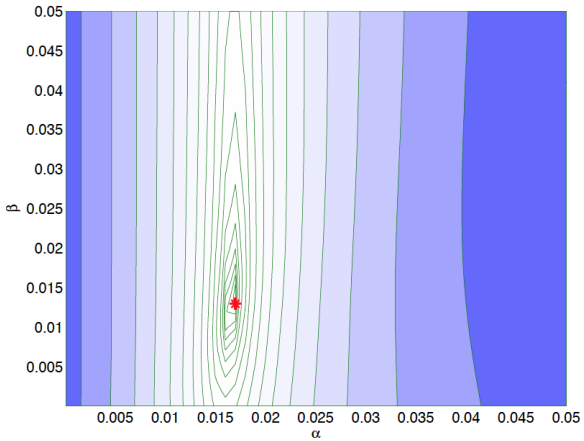
(a) Parrot, TGV^2, $L_1\nabla_\gamma$ cost functional



(b) Uplands, TGV^2, $L_1\nabla_\gamma$ cost functional



(c) Parrot, TGV^2, L_2^2 cost functional



(d) Uplands, TGV^2, L_2^2 cost functional

For TGV a good initialisation is important!

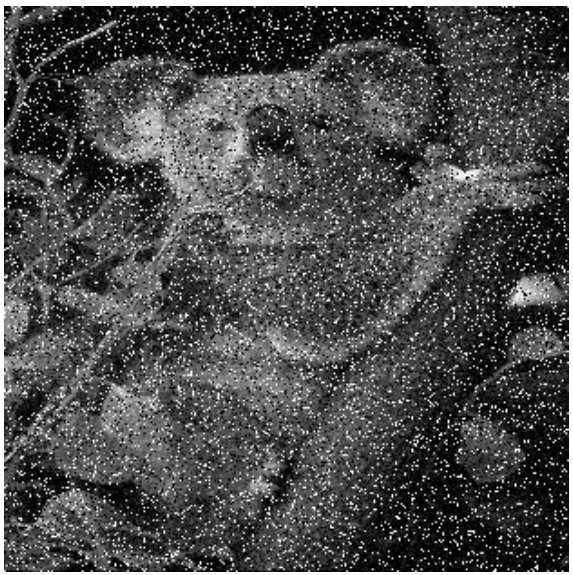
Mixed Impulse & Gaussian noise

$$\min_{\lambda_1,\lambda_2\geq 0} \frac{1}{2}\|u - u_{org}\|_{L^2}^2$$

where u is the solution of the optimisation problem:

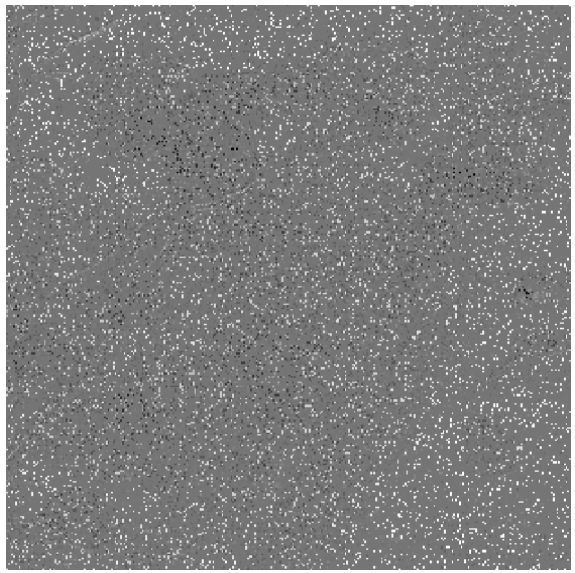
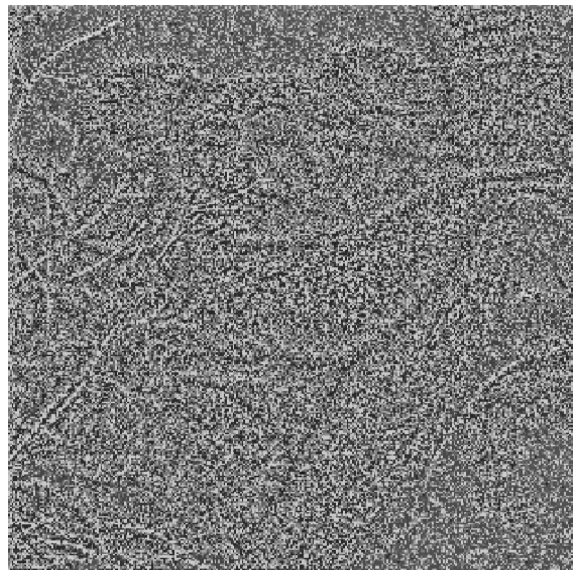
$$\min_{\substack{v\in BV\\ n\in L^2}} \left\{ \frac{\mu}{2}\|\nabla v\|_{L^2}^2 + \|Dv\|_\gamma + \lambda_1\|n\|_\gamma + \lambda_2\|f - v - n\|_{L^2}^2 \right\},$$





Original image (left) and noisy image (right) corrupted by impulse

Mixed Impulse & Gaussian noise

From left to right: Denoised image, impulse noise residuum and Gaussian noise residuum. Optimal parameters: $\lambda_1^* = 351.23$ and $\lambda_2^* = 5200.1$.

And that is not all . . .

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

A few more recent examples of bringing together model-based imaging and learning . . .

Thomas Pock et al.

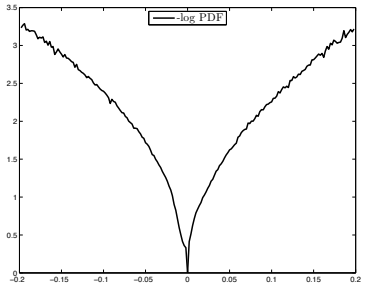
CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

The nonconvex fields of experts model

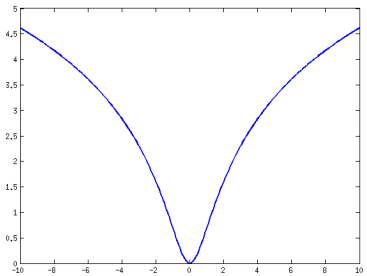
- Let us consider the following nonconvex model [Roth, Black '09], [Samuel, Tappen '09], called the "Fields of Experts" model:

$$\mathcal{R}(u) = \sum_{k=1}^{q} \sum_{i,j=1}^{m,n} \rho_k((K_k u)_{i,j})$$

- $\{K_k\}$ are arbitrary filter kernels, and $\{\rho_k\}$ are potential functions
- Has much more parameters compared to the ℓ_1 model (several thousands)
- Allows only to compute a stationary point (local minimum)
- Suitable potential functions ρ_l are derived from statistics of natural images [Huang and Mumford '99]:



$$\rho_k(t) = \alpha_k \log(1 + \beta_k t^2)$$

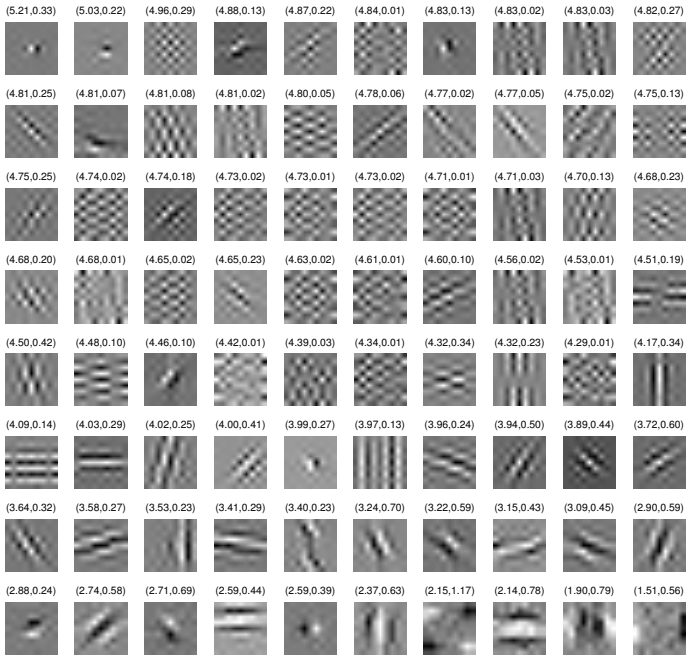


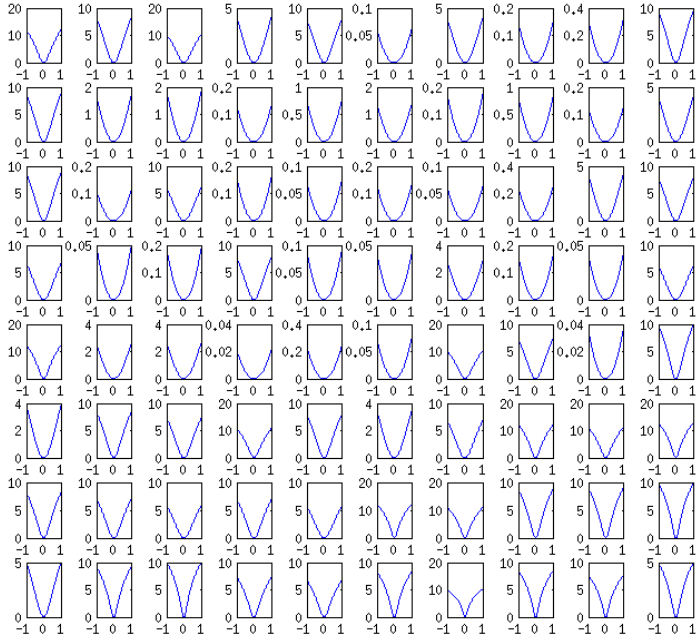
Thomas Pock et al.

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

The learned filters and functions

- In [Chen, Ranftl, P. '14] we learned 80 filters of size 9×9 plus function parameters $\rightarrow$ 6480 parameters on a database of $\sim$ 200 images
- ... two weeks later ...





Thomas Pock et al.

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Evaluation

- Comparison with five state-of-the-art approaches: K-SVD [Elad and Aharon '06], FoE [Q. Gao and Roth '12], BM3D [Dabov et al. '07], GMM [D. Zoran et al. '12], LSSC [Mairal et al. '09]
- We report the average PSNR on 68 images of the Berkeley image data base [Chen, P. 14]

σ	KSVD	FoE	BM3D	GMM	LSSC	BL7x7	BL9x9
15	30.87	30.99	31.08	31.19	**31.27**	31.18	31.22
25	28.28	28.40	28.56	28.68	**28.70**	28.66	**28.70**
50	25.17	25.35	25.62	25.67	25.72	25.70	**25.76**

- Performs equally or better as the state-of-the-art

Variational networks

- Inspired by the conditional shrinkage fields (CSF) [Schmidt, Roth '14], we allow to change the parameters during the iterations:

$$\begin{cases} u^0 = f \\ u^{t+1} = u^t - \lambda^t \left(\sum_{k=1}^{q} (K_k^t)^\top (\rho_k^t)'(K_k^t u^t) + (u^t - f) \right), \ t = 0 \dots T-1 \end{cases}$$

- In each step we perform one gradient descent on a learned variational energy
- Can be interpreted as one cycle of a block incremental gradient descent
- Can also be interpreted as learned non-linear diffusion, trying to "invert" the convolution $\int p(f|u)p(u)\mathrm{d}u$
- And it can be interpreted as a convolutional neural network with T layers

Thomas Pock et al.

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Quantitative evaluation

- We evaluated our learned models on a standard database of 68 images

Method	σ		St.	$\sigma = 15$	
	15	25		$\text{TRD}_{5\times5}$	$\text{TRD}_{7\times7}$
BM3D	31.08	28.56	2	31.14	31.30
LSSC	31.27	28.70	5	31.30	**31.42**
EPLL	31.19	28.68	8	31.34	**31.43**
opt-MRF	31.18	28.66		$\sigma = 25$	
RTF_5	–	28.75		$\text{TRD}_{5\times5}$	$\text{TRD}_{7\times7}$
WNNM	31.37	28.83	2	28.58	28.77
$\text{CSF}^5_{5\times5}$	31.14	28.60	5	28.78	**28.91**
$\text{CSF}^5_{7\times7}$	31.24	28.72	8	28.83	**28.95**

Ozan Öktem & Jonas Adler

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

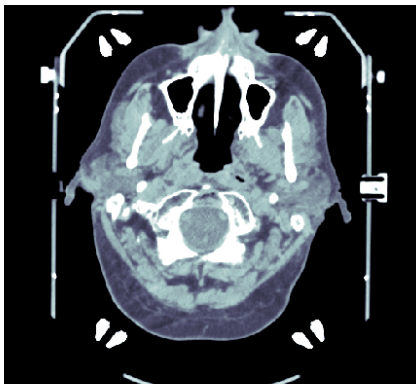
Learning to reconstruct

- Variational regularization: Iterative schemes
- Learned operators
- Data in $\rightarrow$ reconstruction out

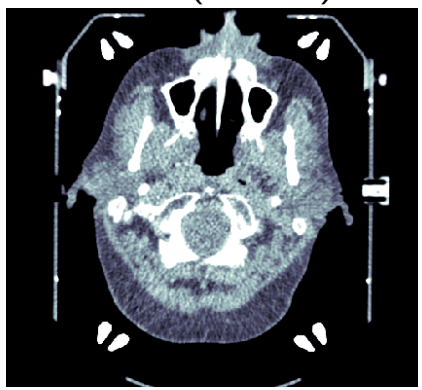
Algorithm 1 Learned Gradient

1: **for** $i = 1, \ldots$ **do**
2: $\quad \Delta f_i \leftarrow \Lambda_\Theta\big(f_i, \nabla\big[\mathcal{L}(\mathcal{T}(\cdot), g)\big](f_{i-1})\big)$
3: $\quad f_i \leftarrow f_{i-1} + \Delta f_i$

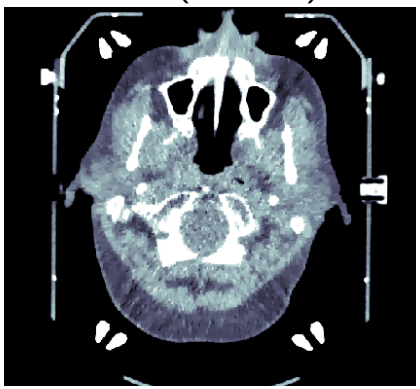
Ground truth



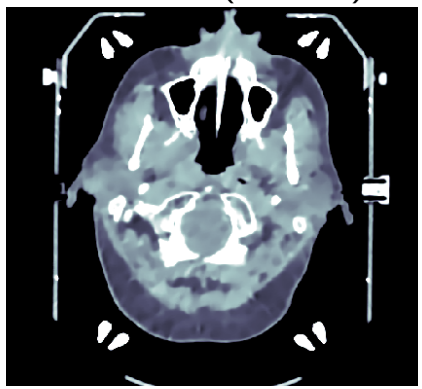
FBP (36 dB)



TV (38 dB)



Learned (44 dB)



See also M. Unser et al. 2017

Ozan Öktem & Jonas Adler

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

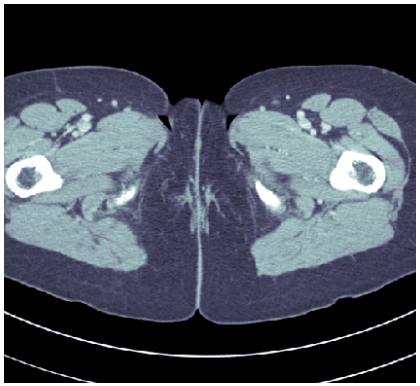
Learning to reconstruct

- Variational regularization: Iterative schemes
- Learned operators
- Data in $\rightarrow$ reconstruction out

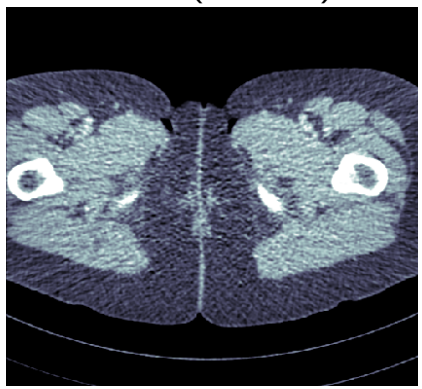
Algorithm 2 Learned Primal-Dual

1: **for** $i = 1, \dots, I$ **do**
2: $\quad h_i \leftarrow \Gamma_{\Theta_i^d}\left(h_{i-1}, \mathcal{T}(\bar{f}_{i-1}), g\right)$
3: $\quad f_i, \bar{f}_i \leftarrow \Lambda_{\Theta_i^p}\left(f_{i-1}, \mathcal{T}^*(h_i)\right)$
4: $\mathcal{T}_\Theta^\dagger(g) \leftarrow f_I^{(1)}$

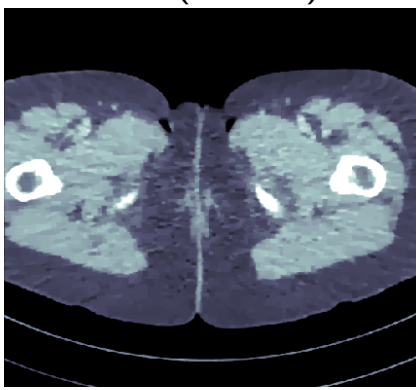
Ground truth



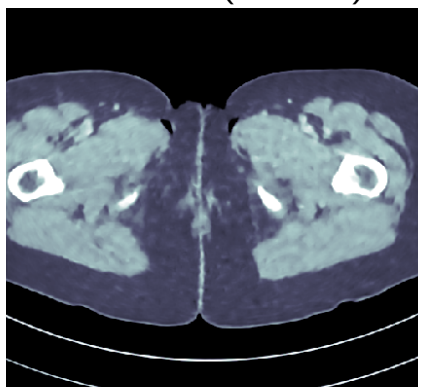
FBP (34 dB)



TV (37 dB)



Learned (44 dB)



Optimal MR sampling

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Question

What is an ideal (sparse) sampling pattern for a given (2d)-object?

Mathematical formulation

For $\mathbf{p} = (\mathbf{s}, \alpha) \in [0,1]^m \times \mathbb{R}_{\geq 0}$,

$$\min_{\mathbf{p}} \frac{1}{2}\|u(\mathbf{p}) - \hat{u}\|_2^2 + \beta\|\mathbf{s}\|_{l_1} + \tilde{\epsilon}\alpha \text{ , subject to}$$

$$u(\mathbf{p}) \in \arg\min_{u \in \mathbb{R}^m_{\geq 0}} \left\{ \frac{1}{2}\sum_{j=1}^{m} s_j |(\mathcal{F}u)_j - \tilde{y}_j|^2 + \alpha \| \|\nabla u\|_2 \|_1 + \frac{\epsilon}{2}\|u\|_{H^1}^2 \right\}$$

where $\hat{u} \in \mathbb{R}^m_{\geq 0}$ is some ground truth, and $\epsilon, \tilde{\epsilon} \ll \alpha, \beta$.

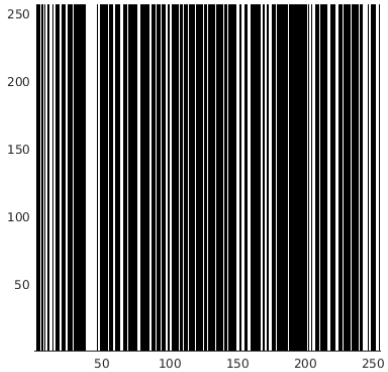
M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Optimal MR sampling

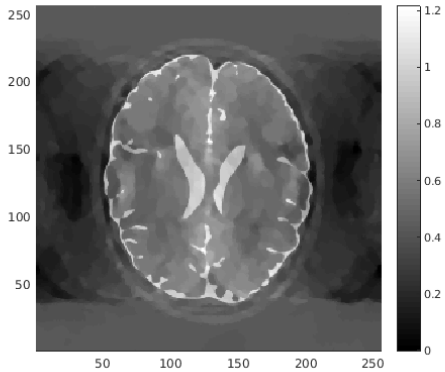
Consider $\tilde{f} = f + \epsilon$

$$u^* \in \arg\min_{u \in \mathbb{R}^m_{\geq 0}} \frac{1}{2}\|S\mathcal{F}u - S\tilde{f}\|_2^2 + TV(u)$$

where $S : \mathbb{C}^m \to \mathbb{C}^m$ is a linear sampling operator.



(a) Sampling pattern



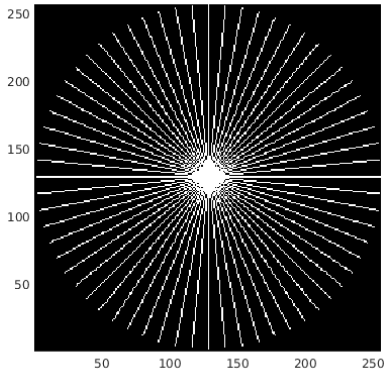
(b) Reconstructed image

Optimal MR sampling

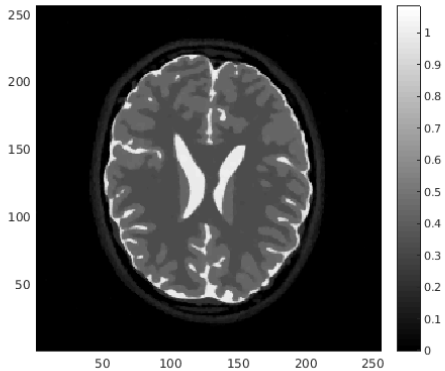
Consider $\tilde{f} = f + \epsilon$

$$u^* \in \arg\min_{u \in \mathbb{R}^m_{\geq 0}} \frac{1}{2}\|S\mathcal{F}u - S\tilde{f}\|_2^2 + TV(u)$$

where $S : \mathbb{C}^m \to \mathbb{C}^m$ is a linear sampling operator.



(c) Sampling pattern



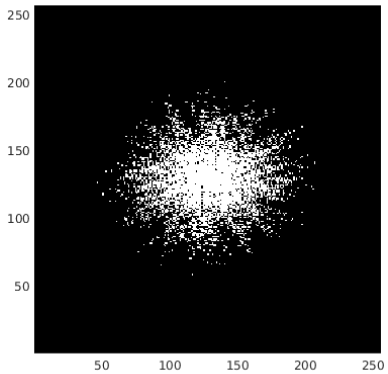
(d) Reconstructed image

Optimal MR sampling

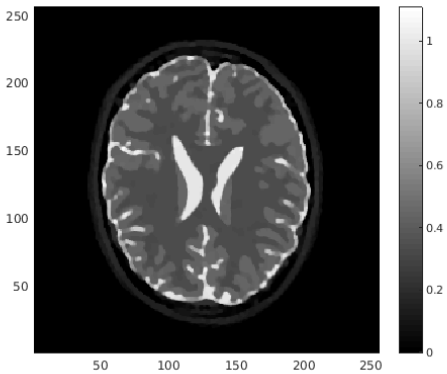
Consider $\tilde{f} = f + \epsilon$

$$u^* \in \arg\min_{u \in \mathbb{R}^m_{\geq 0}} \frac{1}{2}\|S\mathcal{F}u - S\tilde{f}\|_2^2 + TV(u)$$

where $S : \mathbb{C}^m \to \mathbb{C}^m$ is a linear sampling operator.



(e) Sampling pattern



(f) Reconstructed image

Compressed sensing theory: Adcock, Hansen et al. 11–; R. Ward et al. 12–; P. Weiss et al. 13–

Optimal MR sampling

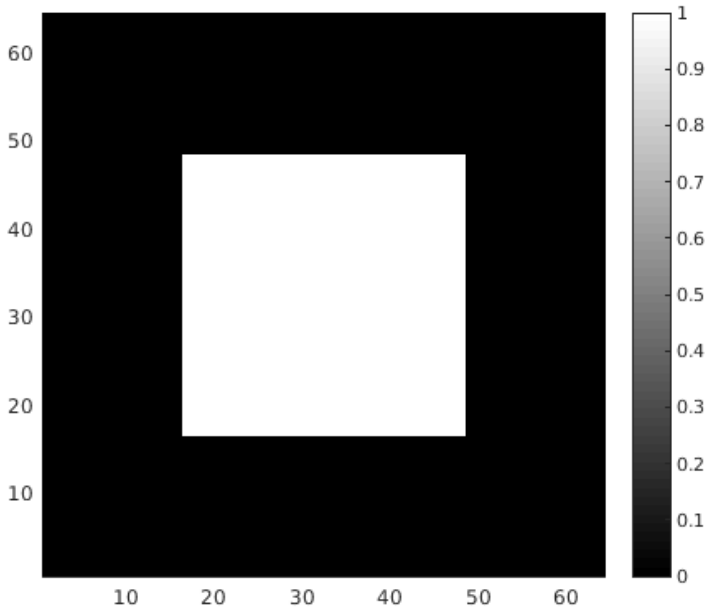
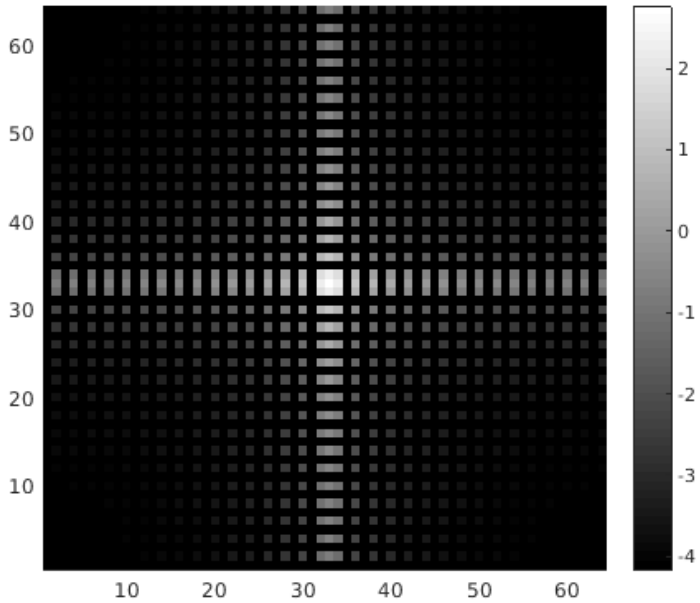


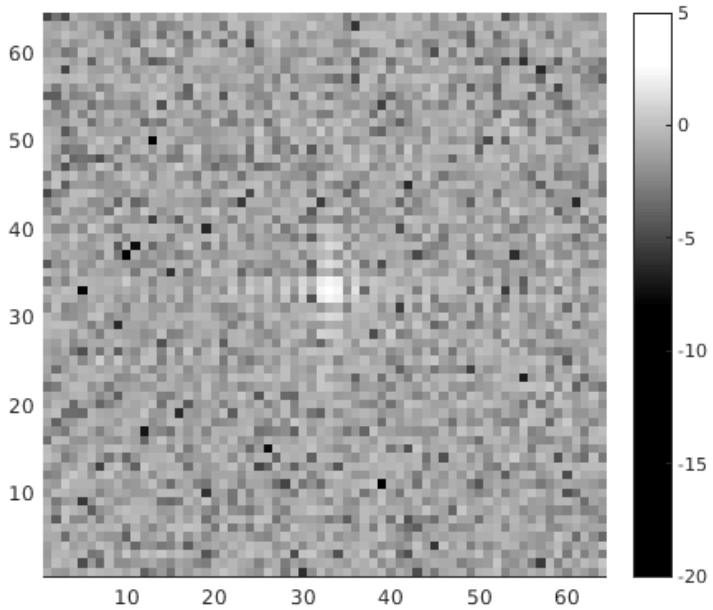
Figure: Discrete 2d bump

M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Optimal MR sampling



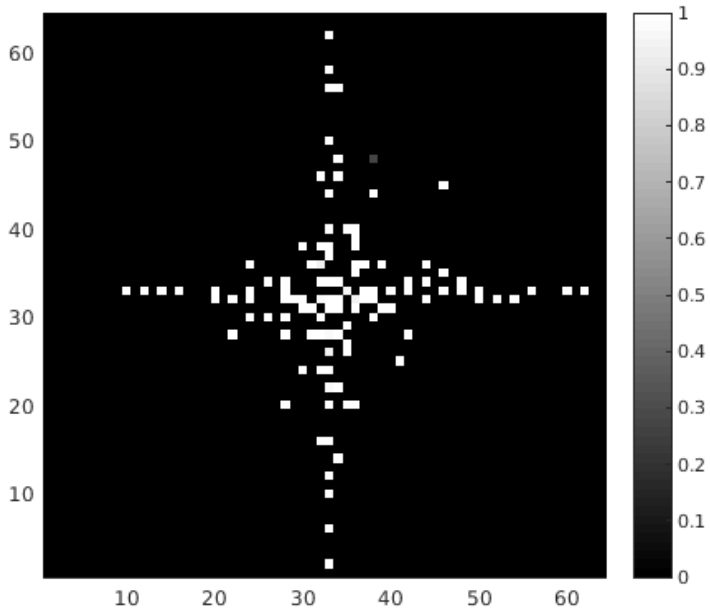
(a) Original data: $\log|y|$



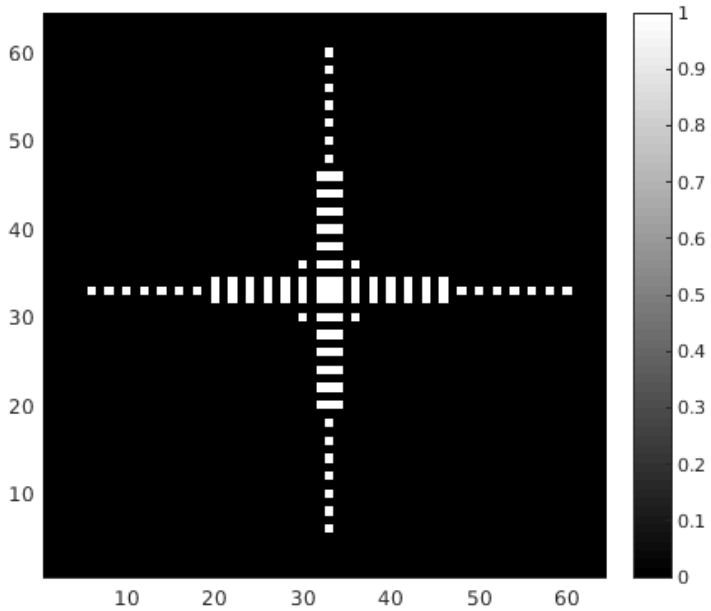
(b) Noisy data: $\log|\tilde{y}|$

M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Optimal MR sampling



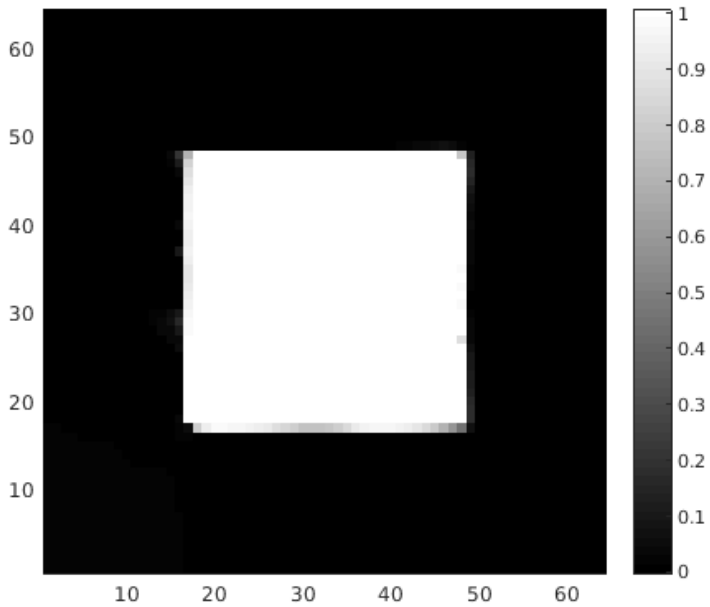
(c) Learned sampling pattern



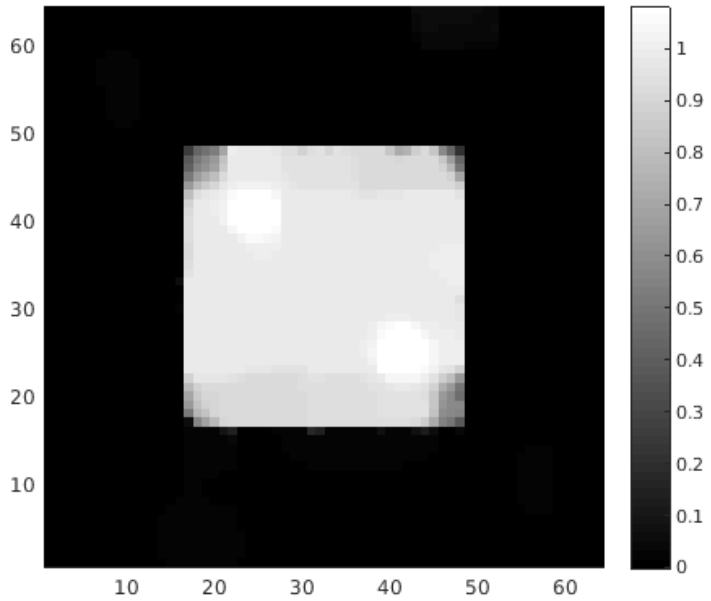
(d) Largest 2.76% Fourier Coefficients

M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Optimal MR sampling



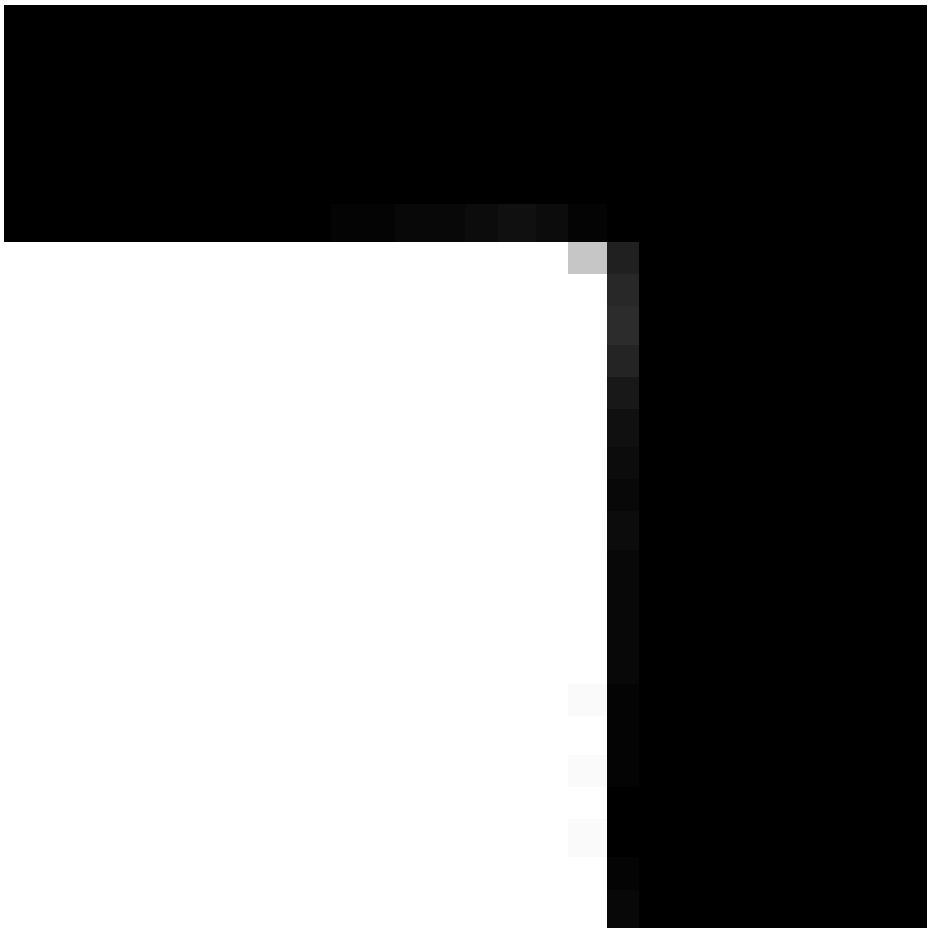
(e) Learned sampling pattern



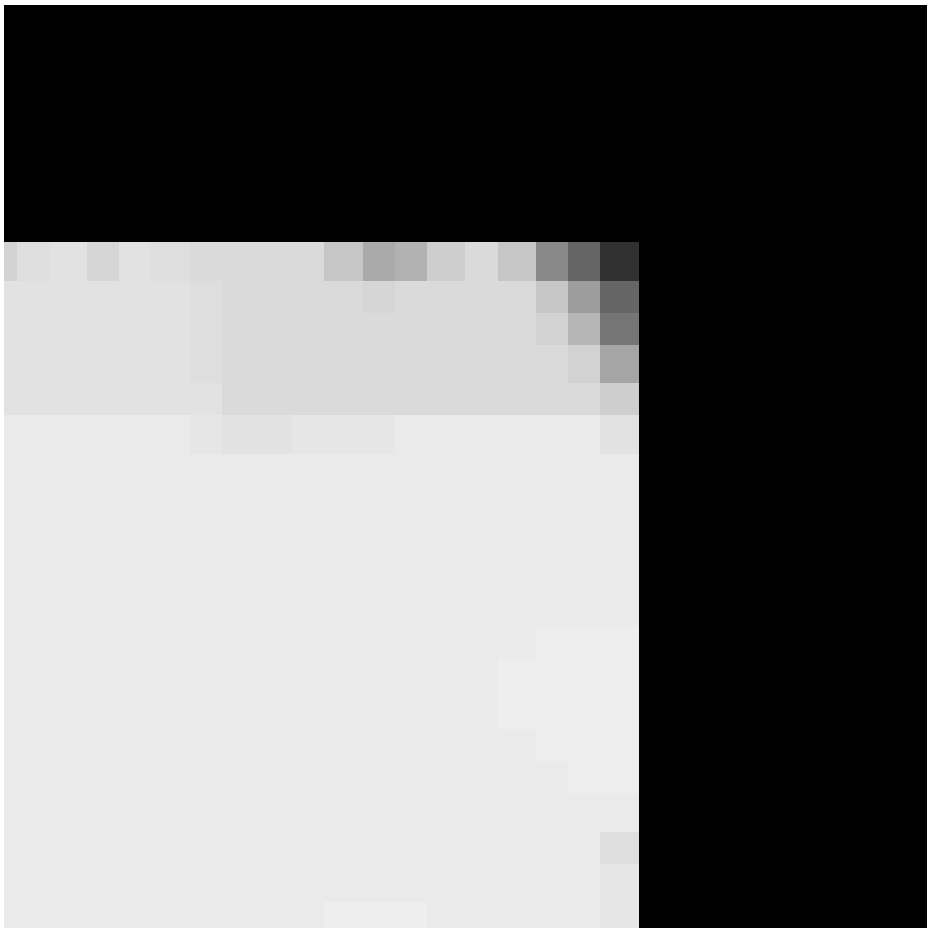
(f) Largest 2.76% Fourier Coefficients

M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Optimal MR sampling



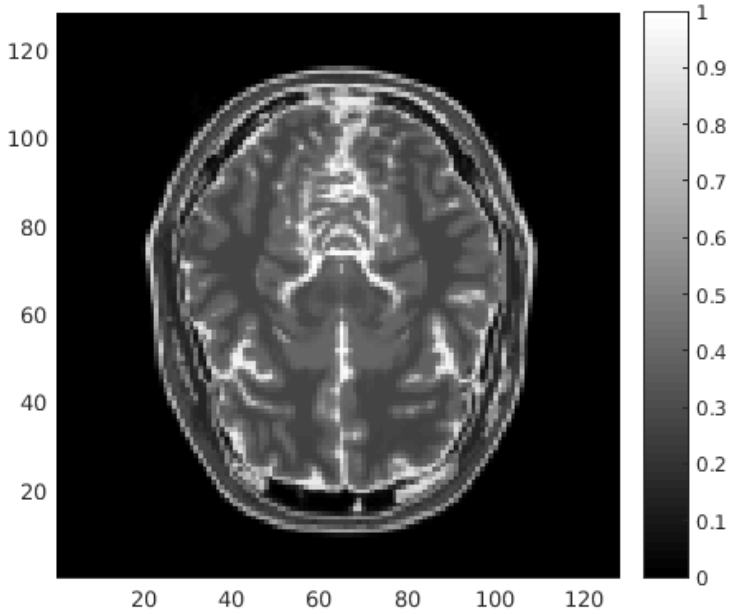
(g) Learned sampling pattern



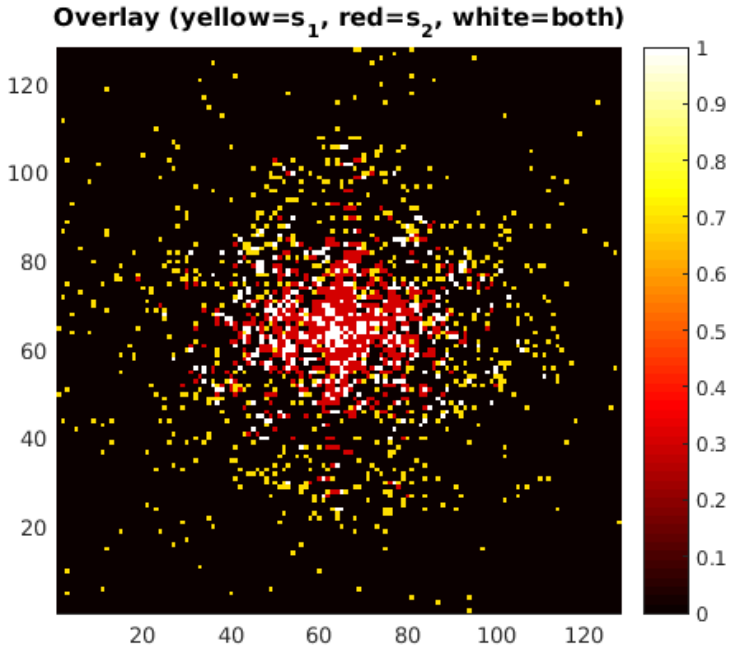
(h) Largest 2.76% Fourier Coefficients

M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Optimal MR sampling



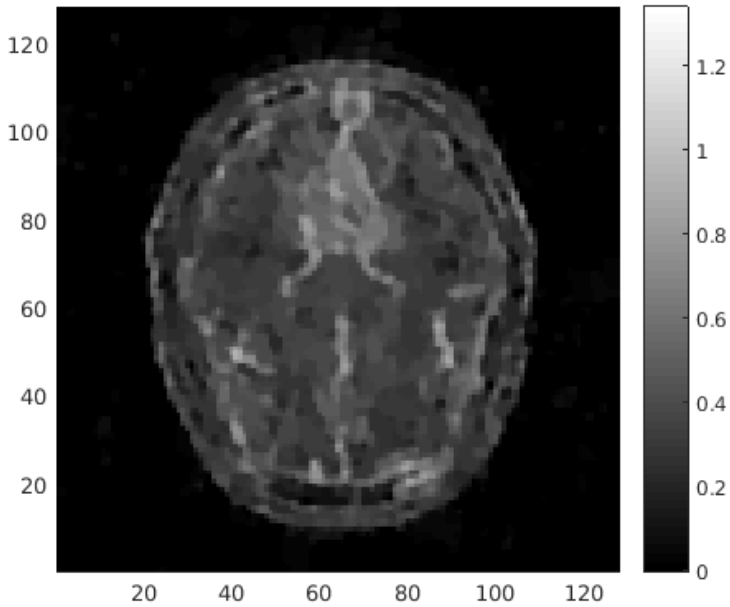
(i) Original image



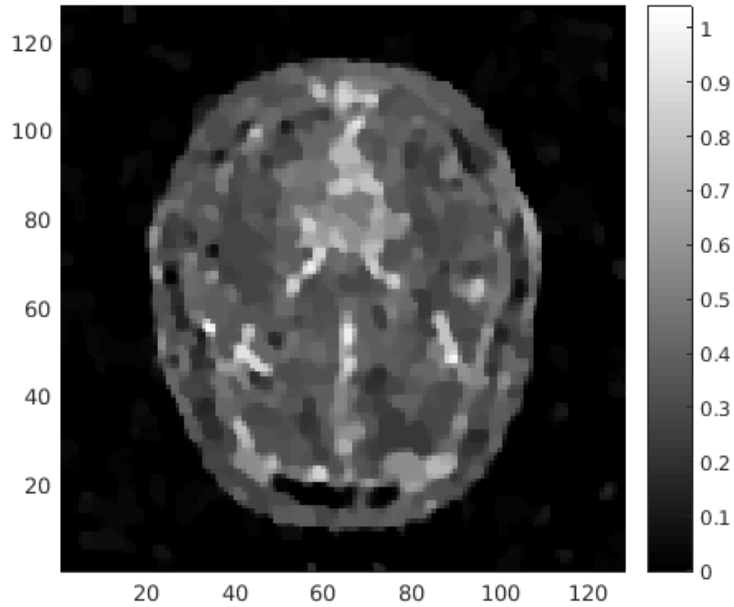
(j) Learned sampling pattern vs. FC

M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Optimal MR sampling



(k) Learned sampling pattern



(l) Largest 6.5% Fourier Coefficients

$$Similarity_{(a)} = 57.8690$$

$$Similarity_{(b)} = 58.5188$$

M. Benning, M. Ehrhardt, G. Maierhofer, F. Sherry, CBS, 2017

Outline

1 Variational regularisation and PDEs for imaging

2 Learning of variational models in imaging

3 Conclusions and outlook . . . and maybe a fun example

Learning versus variational modelling?

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Variational modelling	Recent data learning
physical model	non-physical
high generalizability	very limited generalizability
limited adaptivity to data	adaptive to data
insight in structure of problem	blackbox
reconstruction guarantees	often no guarantees
stability, error analysis, . . .	guarantees stability?
heavily relies on a-priori model	learns the model from the data

Learning versus modelling?

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Happy marriage of variational modelling and data learning?

Hope: adaptive, physical models with theoretical guarantees.

Take away messages

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- PDEs and variational models in imaging – rigorous and generalisable tools in imaging
- Learning of variational models – towards rigorous, generalisable and adaptive imaging models
- The aim: learning structured but adaptive models with guarantees.

- There are still many things to do: richer parametrisation, multi-level & multi-scale; better optimisation schemes for non-convex and high-dimensional learning problems with stronger guarantees and more efficiency.

Boosting and learning in . . .

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

. . . in mathematical imaging algorithms.

Organisers: Marcelo Bertalmio and Stacey Levine

What: MS67 & MS75

When: Thursday, 13 July 2017, 10:30 am -12:30 pm & 4-6 pm

Where: Spirit of Pittsburgh A.

siam®
Society for Industrial and Applied Mathematics

PostDoc in Cambridge

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Research Associate / Senior Research Associate in the Mathematics of Measurement



Tenure: 3 years

Deadline for applications: 15 August 2017

More details: `http://www.jobs.cam.ac.uk/job/14148/` or email me `cbs31@cam.ac.uk` The End

National Physical Laboratory NPL

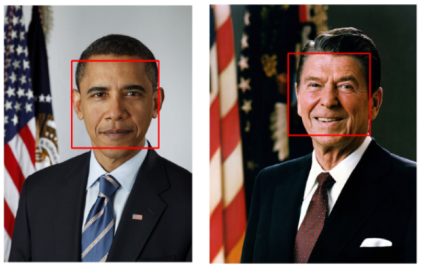
UNIVERSITY OF CAMBRIDGE

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

And the fun example . . .

Image fusion at a glance

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



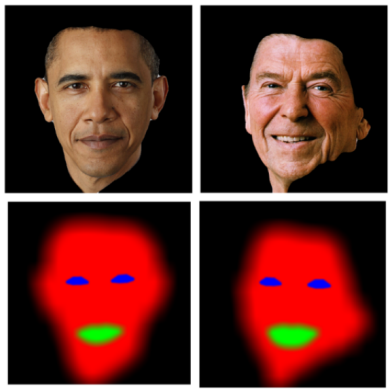
1. Face detection



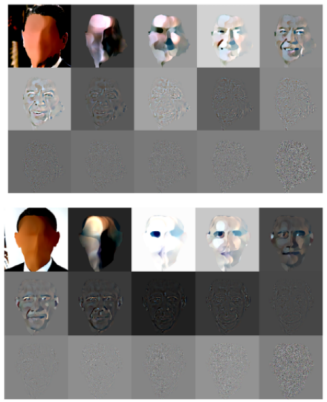
2. Landmark detection



3. Registration



4. Face segmentation



5. Spectral decomposition



6. Image fusion

$$\hat{f}(x) := \int_0^\infty H_1(x,t)\varphi_1(x,t) + R\big(H_2(x,t)\varphi_2(x,t)\big)dt + (P_0 f_1)(x)$$

Fusion image

Spectral filters & segmentation for spectrum of first image

Non-rigid registration operator

Spectral filters & segmentation for spectrum of second image

M. Moeller, M. Benning, R. Nossek, M. Burger, D. Cremers, G. Gilboa, CBS 2016.

Scale decomposition

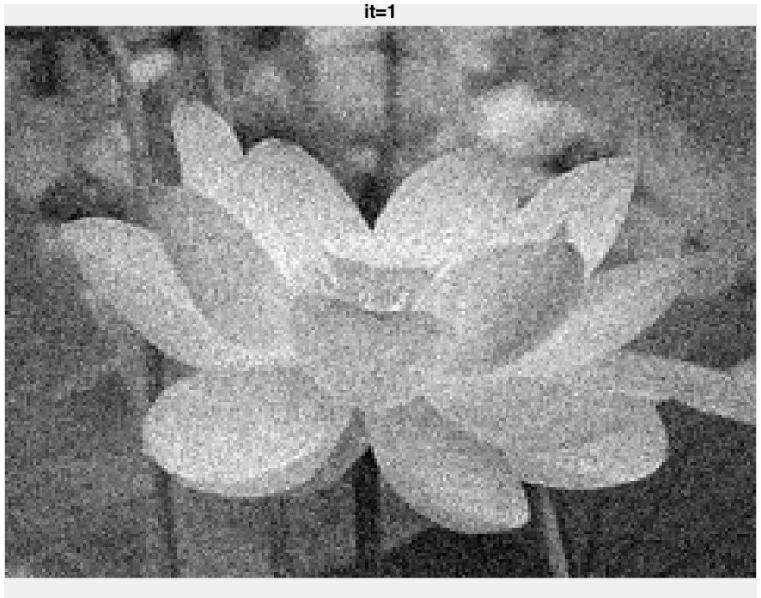


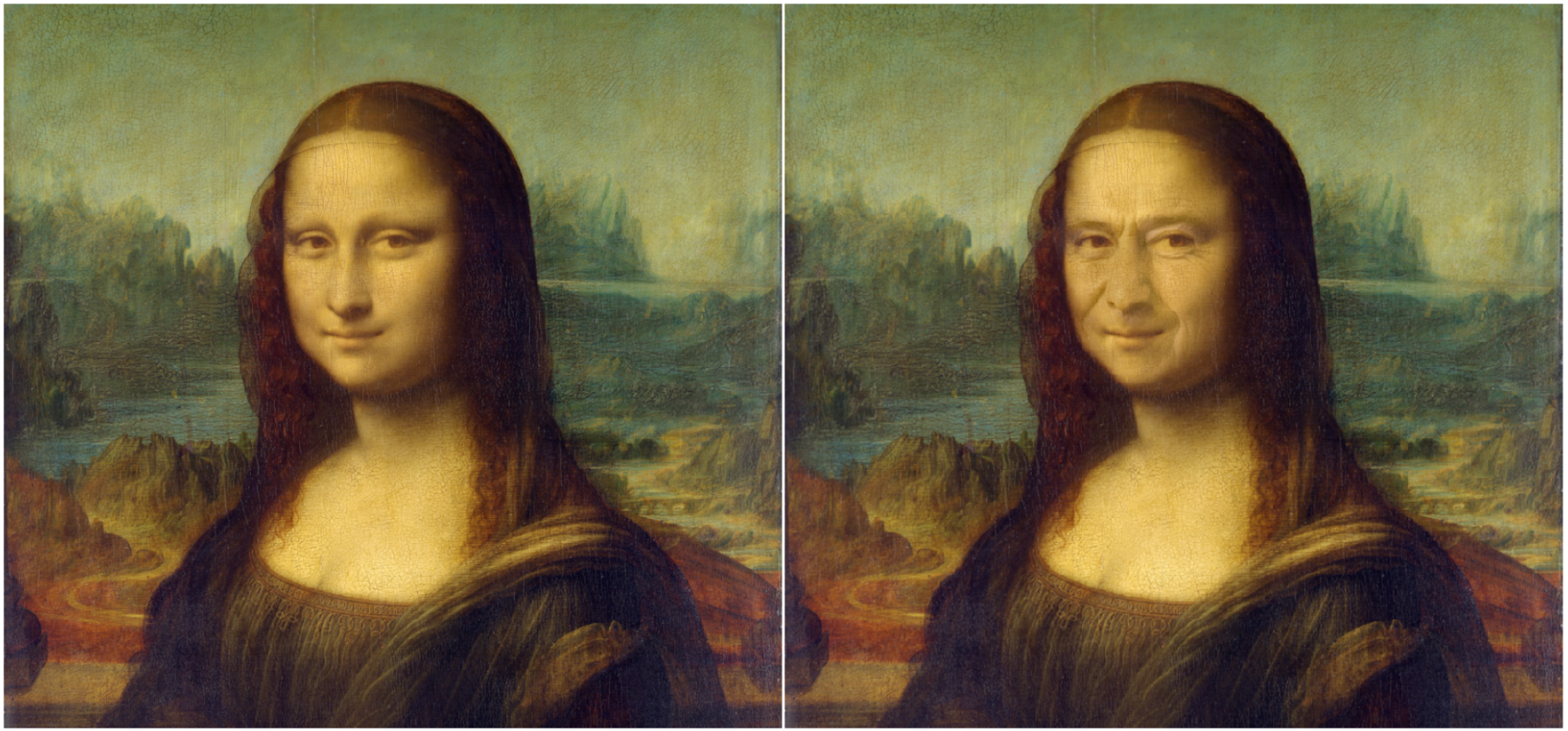
Image fusion some examples



M. Moeller, M. Benning, R. Nossek, M. Burger, D. Cremers, G. Gilboa, CBS 2016.

Image fusion some examples

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



M. Moeller, M. Benning, R. Nossek, M. Burger, D. Cremers, G. Gilboa, CBS 2016.

Image fusion some examples



M. Moeller, M. Benning, R. Nossek, M. Burger, D. Cremers, G. Gilboa, CBS 2016.

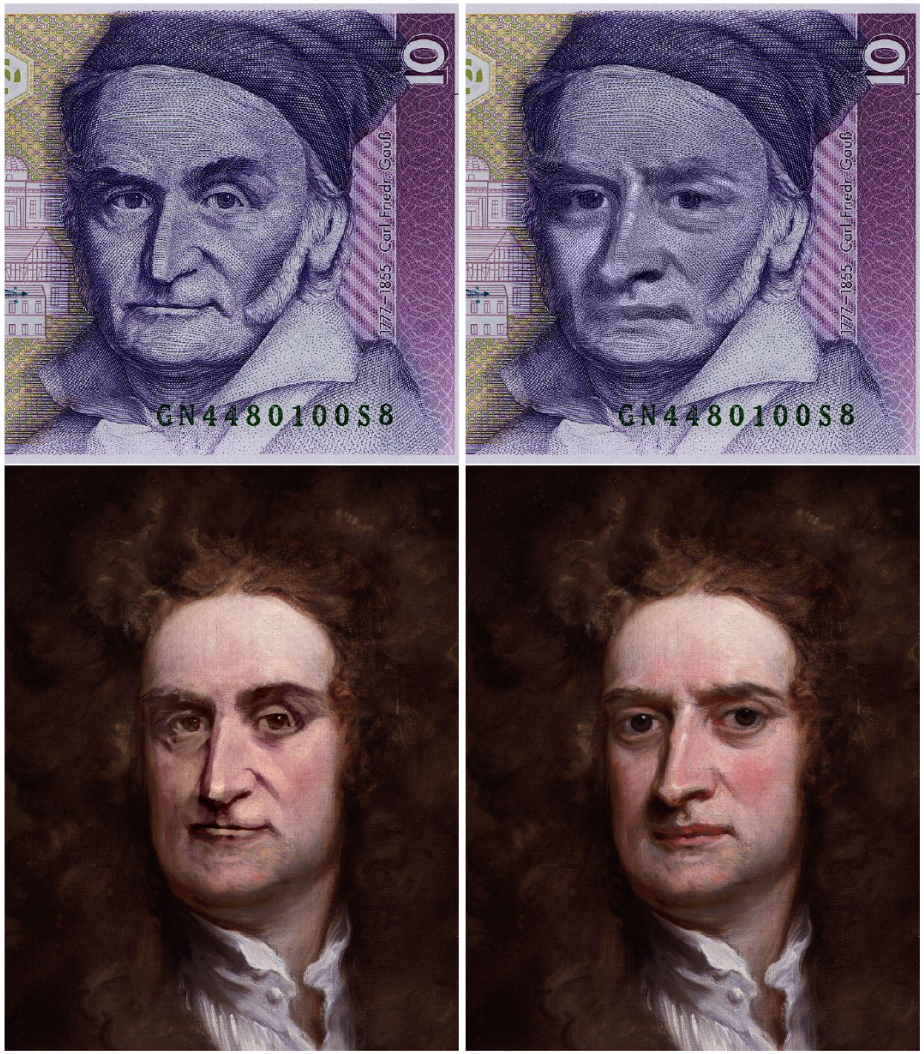
Image fusion some examples



M. Moeller, M. Benning, R. Nossek, M. Burger, D. Cremers, G. Gilboa, CBS 2016.

Image fusion some examples

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



M. Moeller, M. Benning, R. Nossek, M. Burger, D. Cremers, G. Gilboa, CBS 2016. The End

Cambridge Image Analysis

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

- Dr Martin Benning
- Dr Matthias Ehrhardt
- Dr Lukas Lang
- Dr Jingwei Liang
- Veronica Corona
- Joana Grah
- Rob Hocking
- Lisa Kreusser
- Simone Parisotto
- Erlend Riis
- Ferdia Sherry
- Rob Tovey
- Jon Williams

Visit us at `http://www.damtp.cam.ac.uk/research/cia/`
and check out our IMAGES network at
`http://www.images.group.cam.ac.uk`



Sponsored by EPSRC, London Mathematical Society, The Leverhulme Trust, Alan Turing Institute, Horizon2020, and The Royal Society.

Thank you very much for your attention!



UNIVERSITY OF CAMBRIDGE



CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION



cmih
centre for mathematical imaging in healthcare

EPSRC
Pioneering research and skills

The Alan Turing Institute

More information see:

`http://www.ccimi.maths.cam.ac.uk`

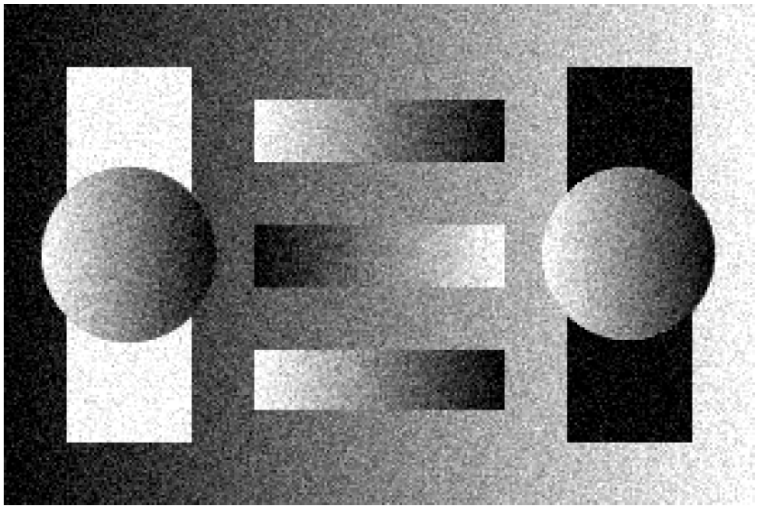
`http://www.cmih.maths.cam.ac.uk`

`http://www.damtp.cam.ac.uk/research/cia/`

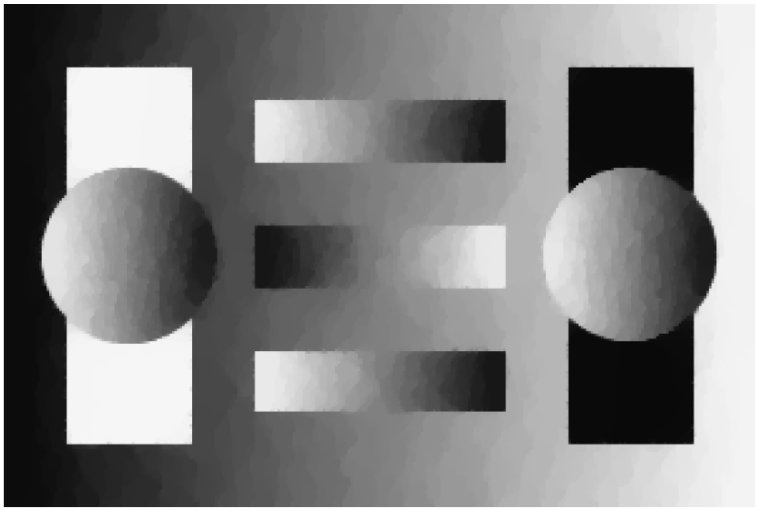
Email: `cbs31@cam.ac.uk`

The regularisation effect

$\min_u \{\alpha\|\nabla u\|_1 + \|u - f\|_2^2\}$



Noisy image



TV denoised image

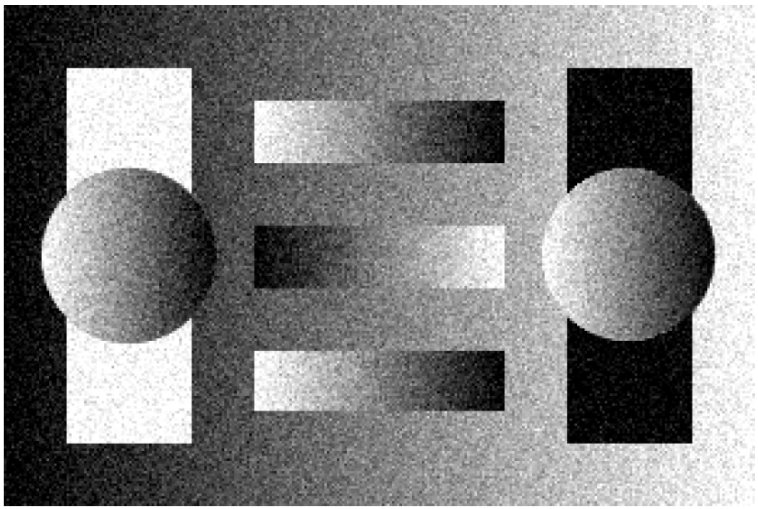
Image courtesy of K. Papafitsoros

References: Rudin, Osher, Fatemi '92; Hinterberger, Scherzer, Computing '06; Bredies, Kunisch, Pock, SIAM Imaging '10; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 . . .

back

The regularisation effect

$\min_u \{\alpha\|\nabla^2 u\|_1 + \|u - f\|_2^2\}$



Noisy image



TV^2 denoised image

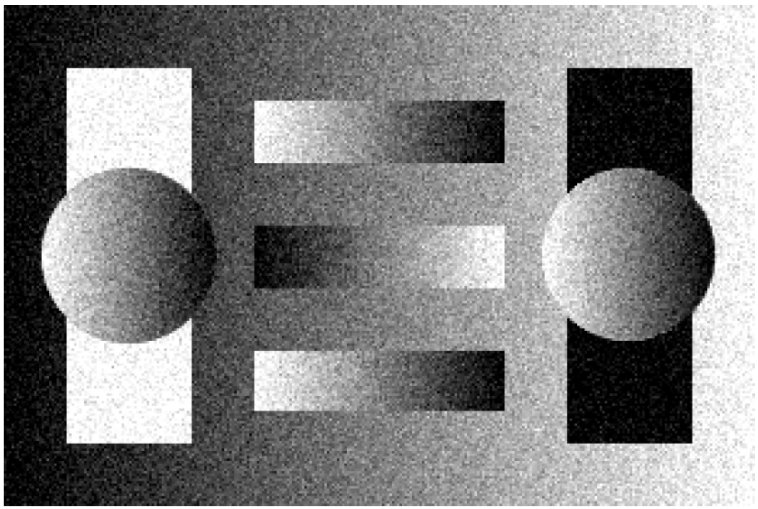
Image courtesy of K. Papafitsoros

References: Rudin, Osher, Fatemi '92; Hinterberger, Scherzer, Computing '06; Bredies, Kunisch, Pock, SIAM Imaging '10; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 . . .

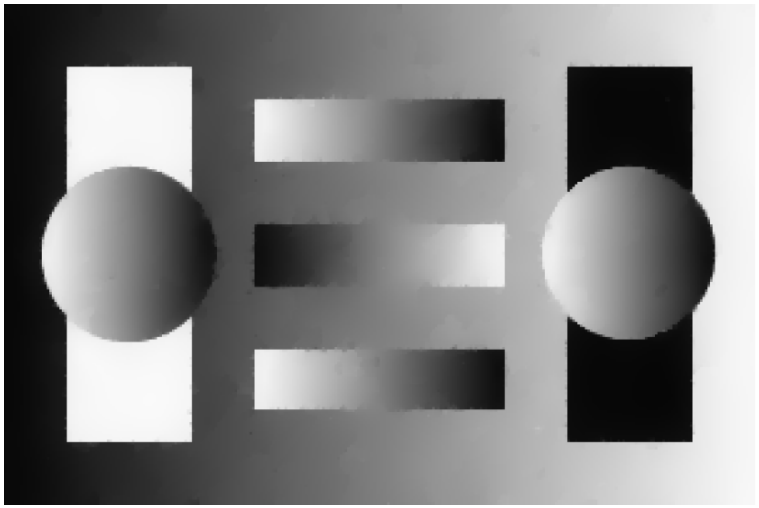
back

The regularisation effect

$$\min_u \left\{ \min_w \{\alpha_1 \|\nabla u - w\|_1 + \alpha_2 \|Ew\|\} + \|u - f\|_2^2 \right\}$$



Noisy image



TGV^2 denoised image

Image courtesy of K. Papafitsoros

References: Rudin, Osher, Fatemi '92; Hinterberger, Scherzer, Computing '06; Bredies, Kunisch, Pock, SIAM Imaging '10; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 . . .

back