The Parameterization Method for Computing Periodic and Quasi-Periodic Orbits in Symplectic Maps without using Symmetries

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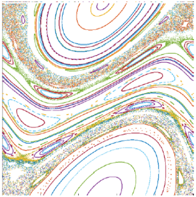
Hamiltonian Systems and Symplectic maps

Let Ω be a symplectic form on the symplectic manifold $\mathcal{M} = \mathbb{R}^n \times \mathbb{T}^n$
A Hamiltonian vector field X_H satisfies that, $\mathcal{L}_{X_H}\Omega = 0$, and its flow f_t is a symplectic map satisfying

$$f_t^*\Omega = \Omega$$

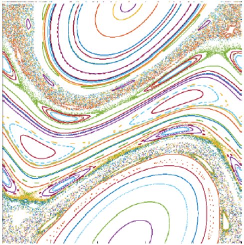


Invariant Tori

They are
prominent as invariant sets in symplectic dynamical systems since they play a fundamental role in chaotic transport

The dynamics in
an invariant torus are conjugate to a rotation by an irrational rotation number or vector

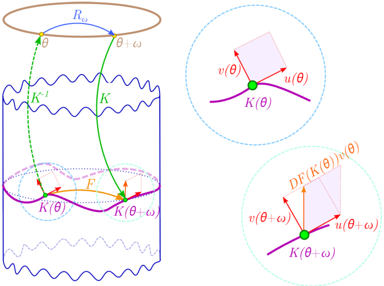
They are landmarks of regular dynamics which are persistent



The parameterization method

For a symplectic map F, let $K : \mathbb{T} \to \mathcal{M}$ so that

$$F(K(\theta)) = K(\theta + \omega)$$



The parameterization method provides an embedding that conjugates the dynamics to a rigid rotation number

de la Llave-Gonzalez-Jorba-Villanueva, Haro-Canadell-Figueras-Luque-Mondelo

Parameterization algorithm

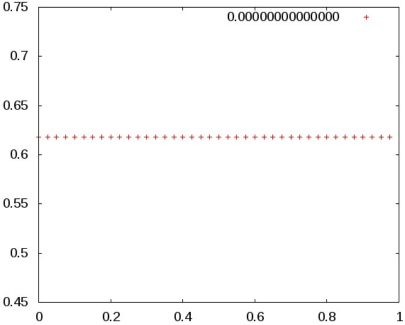
Fix a Diophantine frequency ω
$|\omega \cdot k - n| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{N}.$
Set $E(\theta) = F(K(\theta)) - K(\theta + \omega)$
A Newton method

$$DF(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) = -E(\theta)$$

Make a change of variables $\Delta(\theta) = M(\theta)W(\theta)$, the Quasi Newton step is,

$$\begin{pmatrix} 1 & S_0(\theta) \\ 0 & 1 \end{pmatrix} W(\theta) - W(\theta + \omega) = -M^{-1}(\theta + \omega)E(\theta)$$

Constant coefficient cohomology equations that can be solved with $O(N \log N)$ operations (N is the number of points that we use to discretize the torus)


0.00000000000000
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0
0.2
0.4
0.6
0.8
1

Fix a Diophantine frequency ω
$|\omega \cdot k - n| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{N}.$

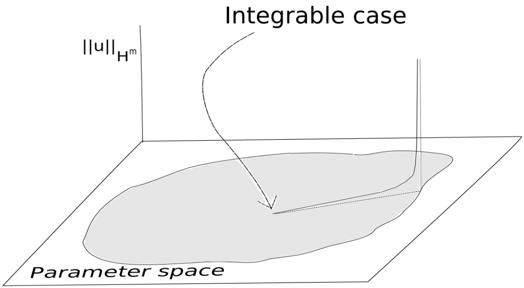
1. KAM theory ensures the existence of smooth quasi-periodic solutions for "quasi-integrable" system
2. There are examples with no smooth equilibria

- *Where is the boundary of existence of smooth solutions?*
- *What happens near the boundary?*

Sobolev method (C-de la Llave)

Local uniqueness and bootstrap of regularity are given. In practice, the functionals we need to check are:

- Non-degeneracy of the problem
- That the approximate solution is regular enough



Periodic orbits

To determine the breakup of invariant tori one can find a sequence of rotational numbers $\frac{p}{q}$ that limit on a given Diophantine number

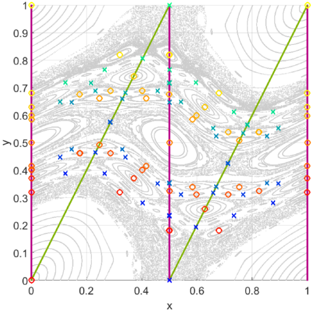
Look for periodic orbits with the rotation numbers approximating a Diophantine number or vector

One could implement a 2 dimensional Newton method but it is better to reduce the dimensions as much as possible

Periodic orbits and symmetry lines

F is called *reversible* if it can be written as the composition, $F = I_2 \circ I_1$, of two functions I_1 and I_2 with the property,

$$I_k \circ I_k = Id \,, \qquad k = 1, 2 \,,$$



Reichl, Shenker-Kadanoff, Piña-Jiminez, Kook-Meiss, Tompaidis, Olvera-Petrov, ...

Greene's method

One can ascertain the existence or not of a KAM torus (i.e. a smooth invariant torus whose dynamics is smoothly conjugated to a rotation) by examining the linearization of periodic orbits of rotation numbers close to it

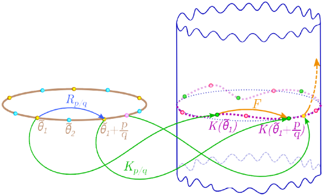
$$R_q = \frac{1}{2}[2 - \mathrm{Tr}(DF^q)]\,.$$

A rotational invariant circle of an area-preserving twist map with a given irrational rotation number ω exists if and only if the residues of its convergent Birkhoff orbits, R_q, remain bounded as $\frac{p}{q} \to \omega$.

Parameterization method for periodic orbits

Another way to reduce the dimension is to implement a parameterization method for periodic orbits

$$F(K(\theta_q)) = K(\theta_q + \frac{p}{q})$$



Parameterization method for periodic orbits

Set $E(\theta_q) = F(K(\theta_q)) - K(\theta_q + \frac{p}{q})$
A Newton method

$$DF(K(\theta))\Delta(\theta_q) - \Delta(\theta_q + \frac{p}{q}) = -E(\theta)$$

Make a change of variables $\Delta(\theta_q) = M(\theta_q)W(\theta_q)$, the Quasi Newton step is,

$$\begin{pmatrix} 1 & S_0(\theta_q) \\ 0 & 1 \end{pmatrix} W(\theta_q) - W(\theta_q + \frac{p}{q}) = -M^{-1}(\theta_q + \frac{p}{q})E(\theta_q)$$

Constant coefficient cohomology equations that can be solved with $O(q \log q)$ operations (q is the number of points that we use to discretize the torus in the periodic orbit)

Cohomology equations

$$\varphi(\theta_q) - \varphi(\theta_q + \frac{p}{q}) = \eta(\theta_q)$$

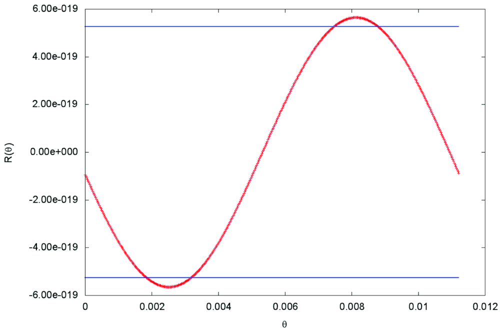
In Fourier coefficients

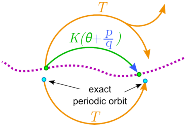
$$(1 - e^{2\pi ik\frac{p}{q}})\varphi_k = \eta_k.$$

For $k \neq \ell \cdot q$, we can divide
In all the other cases we can set $\varphi_{\ell \cdot q} = 0$
This works since η is in the same space of trigonometric polynomials with $\eta_{\ell \cdot q} = 0$

Residue and phase

$$R(\theta) = \frac{1}{2}\left[2 - \mathrm{Tr}\Big(DF^{q}(K(\theta))\Big)\right]$$





Newton-Gauss method

An alternative and more efficient method to improve the accuracy of a periodic orbit seed, is a collocation or multi-shooting approach

$$T^Q(z_0) - z_0 = E_0$$

$$\left\{ \begin{array}{ll} T(z_{Q-1}) - z_0 & = e_0 \ , \\ T(z_0) - z_1 & = e_1 \ , \\ \quad \vdots & \\ T(z_{q-1}) - z_{Q-1} & = e_{Q_1} \ , \end{array} \right.$$

$$\begin{pmatrix} -I & 0 & 0 & \dots & 0 & DT(z_{q-1}) \\ DT(z_0) & -I & 0 & \dots & 0 & 0 \\ 0 & DT(z_1) & -I & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & DT(z_{q-1}) & -I \end{pmatrix} \begin{pmatrix} \delta z_0 \\ \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_{q-1} \end{pmatrix} = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{q-1} \end{pmatrix},$$

Examples

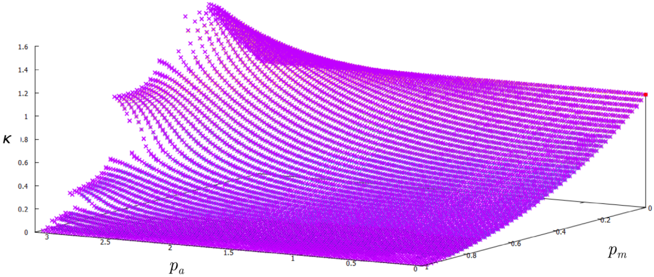
$$x_{n+1} = x_n + y_{n+1}$$
$$y_{n+1} = y_n + \frac{\kappa}{2\pi} V'(x_n)$$

with the perturbation function,

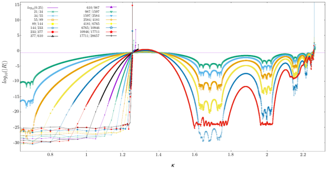
$$V'(x) = f(x) - \int_0^1 f(s) ds \,,$$

where,

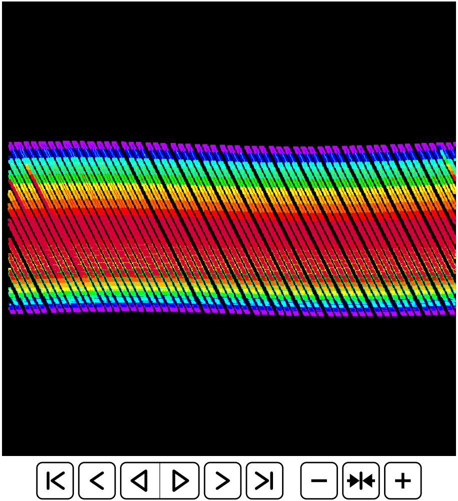
$$f(x) = \frac{\sin(2\pi x + p_a)}{1 - p_m \cos(2\pi x)}$$

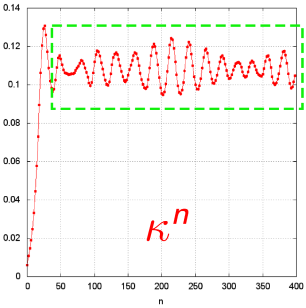


1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
K
3
2.5
2
1.5
1
0.5
0
p_a
0
0.2
0.4
0.6
0.8
1
p_m



$log_{10}(0.25)$
21/34
34/55
55/89
89/144
144/233
233/377
377/610
610/987
987/1597
1597/2584
2584/4181
4181/6765
6765/10946
10946/17711
17711/28657
$log_{10}(|R|)$
15
10
5
0
-5
-10
-15
-20
-25
-30
0.8
1
1.2
1.4
1.6
1.8
2
2.2
κ



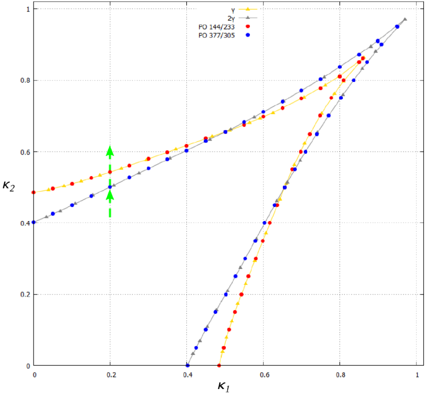

0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
50
100
150
200
250
300
350
400
n
κ^n

Examples

$$\hat{x}_{n+1} = \hat{x}_n + \hat{y}_{n+1} \mod (1)$$
$$\hat{y}_{n+1} = \hat{y}_n + \frac{\kappa_n}{2\pi}\sin(2\pi\hat{x}_n)$$

with

$$\kappa_n = \begin{cases} \kappa_1 & \text{if } n \text{ is odd,} \\ \kappa_2 & \text{if } n \text{ is even.} \end{cases}$$



Thank you

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- **R.C.** and de la Llave, R., **Fast numerical computation of quasi-periodic equilibrium states in 1-D statistical mechanics, including twist maps**, Nonlinearity 22, (2009)
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