KAM theory and Quasi-periodic solutions for fully non-linear PDEs on the circle

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Quasi-periodic solutions for PDEs

Let us start with classes of dispersive PDEs on the circle

$$-\mathrm{i}\partial_t u - \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0 \tag{1}$$

- $\mathfrak{L}(-\mathrm{i}\partial_x)$ is a real-on-real linear (pseudo-)differential operator of order ν
- f is a non-linear operator (of order $q \leq \nu$).

Example: non-linear Schrödinger equation

$$-\mathrm{i}u_t + u_{xx} + f(x, u, u_x, u_{xx}) = 0\,, \qquad \mathfrak{L}(k) = k^2$$

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KdV equation

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DP equation

$$-u_t + u_{xxt} + u_{xxx} - 4u_x + uu_{xxx} + 4uu_x - 3u_x u_{xx} = 0 \qquad \mathfrak{L}(k) = \frac{k^3 + k}{1 + k^2}$$

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Wave equation

$$y_{tt} - y_{xx} + \mathtt{m}y + f(x, y, y_x, y_{xx}) = 0 \quad \rightsquigarrow u = (-\partial_{xx} + \mathtt{m})^{-1}(y + \mathrm{i}y_t)$$

Quasi-periodic solutions for PDEs

$$(-\mathrm{i}\partial_t - \mathfrak{L}(-\mathrm{i}\partial_x))u + f(u) = 0 \tag{2}$$

This is a rather wide class of PDEs,
the main point is that if we ignore the non-linearity In the linear system all the solutions are of the form

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k e^{ikx + i\mathfrak{L}(k)t} \qquad (\text{recall that } \mathfrak{L}(k)\in\mathbb{R})$$

Namely periodic, quasi-periodic or almost periodic.
When we study the equation close to u=0 it is natural to look for quasi-periodic solutions

Definition

Quasi-periodic solution of frequency $\omega\in\mathbb{R}^d$: a torus embedding $\mathbb{T}^d \ni \varphi \to u(\varphi, x)$ such that $u(\omega t, x)$ solves the equation

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Namely periodic, quasi-periodic or almost periodic.
This depends on the support of the solution and on the dispersion law $\mathfrak{L}(k)$
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Quasi-periodic solutions for PDEs

Definition

Quasi-periodic solution of frequency ω: a torus embedding $\mathbb{T}^d \ni \varphi \to u(\varphi, x)$ such that $u(\omega t, x)$ solves the equation

the embedding $\mathbb{T}^d \ni \varphi \to u(\varphi, x)$ solves

Equation for the torus embedding

$$-\mathrm{i}\omega \cdot \partial_\varphi u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$$

The unknowns are ω, u.

We need to be more specific on the regularity:
say f has C^q regularity
look for small solutions in the Sobolev space $H^s(\mathbb{T}^d \times \mathbb{T}; \mathbb{C})$ for some $s \le q$
if f is an analytic function
look for small analytic solutions $H^s(\mathbb{T}^{d+1}_a; \mathbb{C})$

$$\mathbb{T}^{d+1}_a := \{x + \mathrm{i}y \ : x \in \mathbb{T}^{d+1}, \quad y \in \mathbb{R}^{d+1}, |y| \le a\}$$

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Example: Forced fully non-linear NLS on the circle

Consider a forced fully-non linear NLS of the circle

$$-\mathrm{i}u_t + u_{xx} + \varepsilon f(\omega t, x, u, u_x, u_{xx}) = 0$$

with diophantine forcing $\omega \in \Lambda \subset \mathbb{R}^d$

Theorem (Feola-M.P. 15)

for every nonlinearity $f \in C^q$ such that the PDE is either reversible or Hamiltonian + some technical conditions then for all $\varepsilon \in (0, \varepsilon_0)$ small enough, there exists a Cantor set $\mathcal{C}_\varepsilon \subset \Lambda$ of asymptotically full Lebesgue measure, i.e.

$$|\mathcal{C}_\varepsilon| \to 1 \quad as \quad \varepsilon \to 0, \tag{3}$$

such that for all $\omega \in \mathcal{C}_\varepsilon$ there exists a solution $u(\varepsilon, \omega) \in H^s$ to the NLS equation with $||u(\varepsilon, \omega)||_s \to 0$ as $\varepsilon \to 0$. In addition, $u(\varepsilon, \omega)$ is reducible and linearly stable.

A result on the reversible autonomous NLS

Consider a reversible NLS equation

$$-\mathrm{i}u_t + u_{xx} + f(u, u_x, u_{xx}) = 0 \tag{4}$$

where

$$f(u, u_x, u_{xx}) = \mathtt{a}_1|u|^2u + \mathtt{a}_2|u|^2u_{xx} + \mathtt{a}_3|u_x|^2u + \mathtt{a}_4|u_x|^2u_{xx} + \mathtt{a}_5|u_{xx}|^2u + \mathtt{a}_6|u_{xx}|^2u_{xx} + h.o.t. \tag{5}$$

with $\mathtt{a}_i \in \mathbb{R}$ for $i = 1, \dots, 6$. Suppose that

$$(\mathtt{a}_1, \mathtt{a}_2, \mathtt{a}_3, \mathtt{a}_4, \mathtt{a}_5, \mathtt{a}_6) \neq (0, a, a, b, b, 0)$$

Theorem (Corsi,Feola,P.)

For any generic choice of tangential sites $\mathtt{j}_1, \dots, \mathtt{j}_d \in \mathbb{N}$ and for all $\varepsilon \in (0, \varepsilon_0)$, there exists a Cantor set

$$\mathcal{C}_\varepsilon \subset \varepsilon\left[\frac{1}{2}, \frac{3}{2}\right]^d, \quad |\mathcal{C}_\varepsilon| \to 1 \quad as \quad \varepsilon \to 0, \tag{6}$$

such that for all $\xi \in \mathcal{C}_\varepsilon$ the NLS has a quasi- periodic solution with frequency ω^∞;

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For any generic choice of tangential sites $\mathtt{j}_1, \ldots, \mathtt{j}_d \in \mathbb{N}$ *and for all* $\varepsilon \in (0, \varepsilon_0)$, *there exists a Cantor set*

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such that for all $\xi \in \mathcal{C}_\varepsilon$ *the NLS has a quasi- periodic solution with frequency* ω^∞*:*

$$v = \sum_{i=1}^{d} \sqrt{\xi_i} e^{\mathrm{i}\omega_i^{(\infty)}} \sin(\mathtt{j}_i x) + o(\sqrt{\xi}), \quad \omega_i^\infty(\xi) = \mathtt{j}_i^2 + \sum_j \mathcal{M}_i^j \xi_j + o(\xi)$$

The solutions analytic and linearly stable.

Preliminaries $-\mathrm{i}\partial_t u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$

The solutions we are looking for are very special
one does not expect typical initial data to evolve quasi-periodically)
First Idea: extend KAM theory to the context of infinite dimensional dynamical systems.

Think of the equation as a vector field with u in some Banach space.

$$u_t = F(u)$$

For instance if we pass to Fourier coefficients in x:
$u(x,t) = \sum_{k\in\mathbb{Z}} u_k(y)e^{\mathrm{i}kx}$
we get

$$\dot{u}_k = \mathrm{i}\mathfrak{L}(k)u_k + f_k(\{u_j\}).$$

In a finite dimensional system:

- Suppose that the $\mathfrak{L}(k)$ satisfy some non-resonance conditions
- Suppose that your system has a Hamiltonian or a Reversible structure

KAM theory (Moser counterterm theorem) implies:

- Existence of a positive measure set of maximal tori.

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This is a chain of harmonic oscillators coupled by a non-linearity.
If u is small f is a perturbation...
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- Existence of families of lower dimensional tori.

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Classic KAM results

The first results were on model Hamiltonian PDEs such as the semilinear NLS with Dirichlet boundary conditions

$$-\mathrm{i}u_t + u_{xx} + |u|^2 u + g(x,u) = 0, \quad u(t,0) = u(t,\pi) = 0$$

- *KAM theory(Semilinear PDEs with Dirichlet b.c. :* Kuksin, Wayne, Pöschel, Kuksin-Pöschel, Chierchia-You (Wave equation with periodic b.c.)

The reason for requiring Dirichlet b.c. is that one needs the linear frequencies $\mathfrak{L}(k)$ to be distinct

$$\dot{u}_k = \mathrm{i}\mathfrak{L}(k)u_k + f_k(u).$$

Classic results by Nash-Moser

A more flexible approach which was proposed to handle Periodic boundary conditions is to consider the equation of the torus embedding as a functional equation

$$\mathcal{F}(\omega, \varepsilon, u) = -\mathrm{i}\omega \cdot \partial_\varphi u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$$

with unknowns ω, u and apply a Newton algorithm.

- Craig-Wayne '93 (periodic solutions), Bourgain '94 (quasi periodic solutions), Berti-Bolle.

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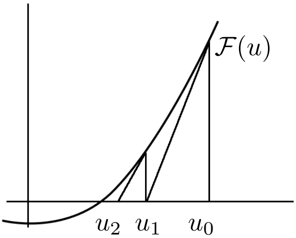


Figura: Three steps of the Newton algorithm
$u_{n+1} := u_n - (d_u\mathcal{F}(\omega, \varepsilon, u_n))^{-1}[\mathcal{F}(\omega, \varepsilon, u_n)]$

Some literature: unbounded non linearities

Recall

$$-\mathrm{i}\partial_t u - \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0, \quad \mathfrak{L} : \mathcal{H}^s \to \mathcal{H}^{s-\nu}, f : \mathcal{H}^s \to \mathcal{H}^{s-q}$$

- **semi-linear Pde's,** $q \leq \nu - 1$
  Kuksin '98, Kappeler-Pöeschel '03 KdV ($q < \nu - 1$), Liu-Yuan '10, Zhang-Gao-Yuan '11 Hamiltonian and Reversible DNLS ($q = p - 1$) Berti, Biasco, Procesi , Hamiltonian and Reversible DNLW
- **Fully Non-linear Pde's,** $q = p$
  periodic solutions
  Ioss-Plotnikov-Toland '05, water waves, Baldi Kirckhoff , Benjamin-Ono,Alazard,Baldi capillary water waves
  quasi-periodic solutions
  Baldi, Berti, Montalto, '12-'15 quasi-periodic solutions for KdV, capillary water waves

Small Divisors

The first problems come form small divisors.
The linearized equation is NOT invertible form H_s to H_s. Even in the best possible scenario it loses regularity As an example consider the NLS operaotr linearized at $u = 0$

$$\mathrm{i}\omega \cdot \partial_\varphi u - \partial_{xx} u + \varepsilon f(u) \rightsquigarrow L_\omega = \mathrm{i}\omega \cdot \partial_\varphi - \partial_{xx}$$

Eigenvalues of L_ω: $(\omega \cdot \ell + j^2)$, $\qquad (\ell, j) \in \mathbb{Z}^d \times \mathbb{Z}$.

Small divisors

$$|\omega \cdot \ell - \sigma j^2| \geq \frac{\gamma}{1 + |\ell|^\tau}, \quad \forall (\ell, j) \in \mathbb{Z}^{d+1}, \sigma = \pm 1, \tau > d,$$

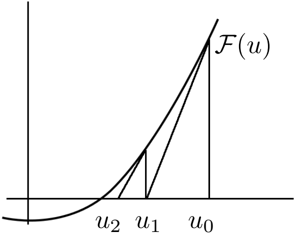
Then L_ω is invertible, but

$$L_\omega^{-1} : \mathcal{H}^s \to \mathcal{H}^{s-\tau}.$$

NO classical Implicit function theorem.

Main Ideas: Quadratic algorithms

In a Newton algorithm we need to control the inverse of the Linearized operator in a neighborhood of $u = 0$



$$\text{Newton method :} \quad u_{n+1} = u_n - (d_u\mathcal{F}(u_n))^{-1}\mathcal{F}(u_n) \tag{6}$$

$d_u\mathcal{F}(u_n)$ is an infinite matrix and in order to have convergence we need estimates on the inverse in high Sobolev norm We would like to control the loss of regularity

$$(d_u\mathcal{F})^{-1} : H_s \to H_{s-\mu}$$

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- 1st Melnikov conditions: lower bounds on the eigenvalues of $d_u\mathcal{F}$ (necessary in order to invert $d_u\mathcal{F}$)
  This is necessary but NOT sufficient in order to deduce the control on the loss of regularity
- 2nd Melnikov conditions: lower bounds on the differences of the eigenvalues $d_u\mathcal{F}$ This implies that $d_u\mathcal{F}$ is diagonable by a map $H_s \to H_s$ this is sufficient in order to deduce the control on the loss of regularity.

Main Ideas

- **Nash-Moser**: Uses *1st Melnikov conditions* (bounds on the L^2 norm of $(d_u\mathcal{F}(\lambda,\varepsilon,u_n))^{-1}$)+ multiscale analysis (used to pass form L^2 norm to H_s norm)
- **KAM theory**: Uses *1st Melnikov conditions* + *2nd Melnikov conditions* at each step perform a traslation so that $u_n \to 0$, and then a diagonalization of the linearized operator, so that in this basis it is simple to invert it and compute u_{n+1}.

  Here we also have information on the linear stability of the solutions.

Due to the presence of small divisors in order to get results one needs some parameters to modulate Consider the example of the linearized NLS operator:

$$-\mathrm{i}\omega \cdot \partial_\varphi u + \partial_{xx} u + \varepsilon |u|^2 u \quad \Longrightarrow \quad -\mathrm{i}\omega \cdot \partial_\varphi + \partial_{xx} + \varepsilon V(\varphi, x)$$

for some values of ω and of the potential $\lambda = 0$ can be an eigenvalue! but if you have parameters

$$-\mathrm{i}\omega(\xi) \cdot \partial_\varphi + \partial_{xx} + \varepsilon V(\xi, \varphi, x)$$

then "for most values of the parameters" the spectrum does not touch zero

$$|\lambda_{\ell,j}(\xi)| > \gamma |\ell|^{-\tau}$$

- No natural parameter

$$-\mathrm{i}u_t - \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0 \quad \rightsquigarrow -\mathrm{i}u_t + u_{xx} + |u|^2 u + G(x, u) = 0$$

Use as parameters the initial data.... this might be very hard

- One natural parameter

$$u_{tt} - u_{xx} + \mathtt{m}u + f(u)$$

- Add external parameters

$$-\mathrm{i}u_t + u_{xx} + M(x)u + G(x, u) = 0$$

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$$u_{tt} - u_{xx} + \mathtt{m}u + f(u)$$

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$$-\mathrm{i}u_t + u_{xx} + M_\xi(x)u + G(x,u) = 0$$

  where M_ξ is a parameter family of potentials

- consider a forced equation

$$-\mathrm{i}u_t + u_{xx} + \varepsilon f(\omega t, u)$$

  use ω as parameters

Summarizing: $-\mathrm{i}\partial_t u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$

For a autonomous equation with no external parameters there are three classes of problems:

1) Parameter extraction

2) Invertibility of the linearized operator

3) Reducibility

Summarizing: $-\mathrm{i}\partial_t u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$

1) Parameter extraction

We fix some tangential sites $S = \{\mathtt{j}_1, \dots, \mathtt{j}_d\}$ and we look for approximate solutions of the form:

$$v_0(\xi, x, t) = \sum_{i=1}^{d} \sqrt{\xi_i} e^{\mathrm{i}t\mathfrak{L}(\mathtt{j}_i) + \mathrm{i}\mathtt{j}_i \cdot x}$$

Then we need to prove that for generic choices of the $\mathtt{j}_i$ the parameters ξ modulate the spectrum of

$$-\mathrm{i}\partial_t + \mathfrak{L}(-\mathrm{i}\partial_x) + f_u(v_0) = 0.$$

This is the so-called frequency-amplitude modulation.

2) Invertibility of the linearized operator

3) Reducibility

Summarizing: $-\mathrm{i}\partial_t u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$

1) Parameter extraction

2) Invertibility of the linearized operator

Given an approximate solution $v_n(\xi, x, t)$ (appropriately close to v_0) we need to invert

$$-\mathrm{i}\partial_t + \mathfrak{L}(-\mathrm{i}\partial_x) + f_u(v_n)$$

as a tame operator: $H_s \to H_{s-\mu}$.

3) Reducibility

Summarizing: $-\mathrm{i}\partial_t u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$

1) Parameter extraction

2) Invertibility of the linearized operator

3) Reducibility

Given an approximate solution $v_n(\xi, x, t)$ prove that

$$-\mathrm{i}\partial_t + \mathfrak{L}(-\mathrm{i}\partial_x) + f_u(v_n)$$

can be diagonalized by a map $H_s \to H_s$.

Summarizing: $-\mathrm{i}\partial_t u + \mathfrak{L}(-\mathrm{i}\partial_x)u + f(u) = 0$

1) Parameter extraction

2) Invertibility of the linearized operator

3) Reducibility

Parameter extraction is potentially extremely tricky...it often reduces to a combinatoric problem.

Reducibility is NOT necessary in order to prove existence of quasi-periodic solutions.

We expect that if we can prove Reducibility then the Invertibility of the linearized operator follows.

If we do not have reducibility then one has to use multi-scale analysis in order to obtain invertibility

Strategy

Perform the parameter extraction by using a weak Birkhoff normal form

Prove that for large measure sets of parameters one may diagonalize the linearized operator

$$\mathcal{L}(u) := d_u\mathcal{F}(u) = \omega(\xi)\cdot\partial_\varphi \mathbb{1} + i\begin{pmatrix} 1+a_2(\xi) & b_2(\xi) \\ -\bar{b}_2(\xi) & -1-a_2(\xi) \end{pmatrix}\partial_{xx} + i\begin{pmatrix} a_1(\xi) & b_1(\xi) \\ -\bar{b}_1(\xi) & -\bar{a}_1(\xi) \end{pmatrix}\partial_x + \begin{pmatrix} a_0(\xi) & b_0(\xi) \\ -\bar{b}_0(\xi) & -\bar{a}_0(\xi) \end{pmatrix} \tag{7}$$

Put this into a convergent Nash-Moser scheme.

Proof of Reducibility:1

- ∂_x-reduction: find invertible bounded $\mathcal{V}_1, \mathcal{V}_2 : \mathcal{H}^s \to \mathcal{H}^s$ such that

  $$\mathcal{V}_1^{-1}\mathcal{L}\mathcal{V}_2 = \mathcal{V}_1^{-1}(\omega \cdot \partial_\varphi + \mathcal{D} + \mathcal{R})\mathcal{V}_2 = \mathcal{L}_c, \quad \mathcal{L}_c = \omega \cdot \partial_\varphi + \mathcal{D}_c + \mathcal{R}_c$$

  where $\mathcal{R} = O(\varepsilon\partial_{xx})$, $\mathcal{D}_c$ constant coefficients diff. operator, $\mathcal{R}_c = O(\varepsilon)$ BUT **bounded**.
  Tools: diffeomorphism of the torus, descent method.
  Strictly based on the pseudo-differential structure of the linearized operator of a Pde, namely $\sum a_i(\varphi, x)\partial_x^i$.

Proof of Reducibility:2

- ε-reduction: find invertible bounded $\Psi_n : \mathcal{H}^s \to \mathcal{H}^s$ such that $\exists \lim_n \Psi_n \circ \ldots \circ \Psi_1 =: \lim_n \Phi_n = \Phi_\infty$ where

$$\Phi_n^{-1} \mathcal{L}_c \Phi_n = \omega \cdot \partial_\varphi + \mathcal{D}_n + \mathcal{R}_n$$

  with $|\mathcal{R}_n| = O(\varepsilon^{2^n})$.
  Classical KAM reduction scheme: works on bounded operators on scales of sequences spaces. Requires $2nd$ Melnikov condition.

Generalizations

This methods works essentially in all cases of dispersive evolution PDEs on the circle.
Analytic setting: the scheme seems to fail in the analytic context.
Consider $\mathbb{T}_a := \{x \in \mathbb{C} : |\mathrm{Im} x| < a\}$. The diffeomorphism $x \to x + \xi(\varphi, x)$ maps $\mathbb{T}_{a'}$ to $\mathbb{T}_a$, with $a' + |\xi|_\infty < a$. At each step $|\xi|_\infty \approx \varepsilon$. One cannot iterate!
use the KAM idea
At each step ONLY apply the change of variables such that

$$d_u\mathcal{F} \to \omega \cdot \partial_\varphi + \mathcal{D} + \mathcal{R}$$

with $\mathcal{R} \sim O(\varepsilon)$ bounded. then at the next step you still have the multiplication structure
but at the step n one has $|\xi^{(n)}|_\infty \approx \varepsilon_n \sim 2^{-\chi^n}$

dispersion laws: if the dispersion law is linear or sub linear when one conjugate $\mathcal{L}$ with $x + \xi(\varphi, x)$, derivatives in time interact with derivatives in space.
example: $\mathcal{T}_2^{-1} \omega \cdot \partial_\varphi \mathcal{T}_2 = \omega \cdot \partial_\varphi + (\omega \cdot \partial_\varphi \xi) \partial_x$
One need a different $\mathcal{T}_2$
Works in progress:

- Degasperis-Procesi equations: linear dispersion law, analytic case, Baldi, Feola, P.
- Water waves equations: sub linear dispersion, Berti, Baldi, Montalto.

Thanks for the attention!