Minitutorial MT2

Directions for Graduate and Undergraduate Modeling Courses

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There are many approaches to mathematical modeling pedagogy.

We'll discuss them today and provide some activities to help you make choices about what will work best for you and your students.

Outline of minitutorial

<u>Part I: Motivation and Background</u>

- Why teach mathematical modeling?
- What is the difference between teaching mathematical models and mathematical modeling?

<u>Part II: Design and Objectives</u>

- Course design
- Tools/algorithms
- Career preparation

Part I: Motivation and Background

Why should you teach mathematical modeling?

Your students have likely heard of mathematical modeling before they come to college.

K-12 Common Core State Standards
8 Standards for Mathematical Practice

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments and critique reasoning of others.

Model with Mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

CCSS.Math.Practice.MP4

Model with mathematics.

Mathematically proficient students

- **apply the mathematics** they know to solve problems
- are comfortable making **assumptions and approximations** to simplify a complicated situation, realizing that these may need **revision** later.
- **identify important quantities** in a practical situation and **map their relationships** using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
- **analyze** those relationships mathematically **to draw conclusions**.
- **interpret** their mathematical results in the **context** of the situation and **reflect** on whether the results make sense, possibly **improving** the model if it has not served its purpose.

Mathematical Modeling Competitions

Moody's Mega Math Challenge®

Participation in high school mathematical modeling competitions is steadily increasing.

The Mathematical Contest in Modeling (MCM)
The Interdisciplinary Contest in Modeling (ICM)

International Mathematical Modeling Challenge (IM²C)

Undergraduate competitions are also growing, especially in participation by Chinese teams.

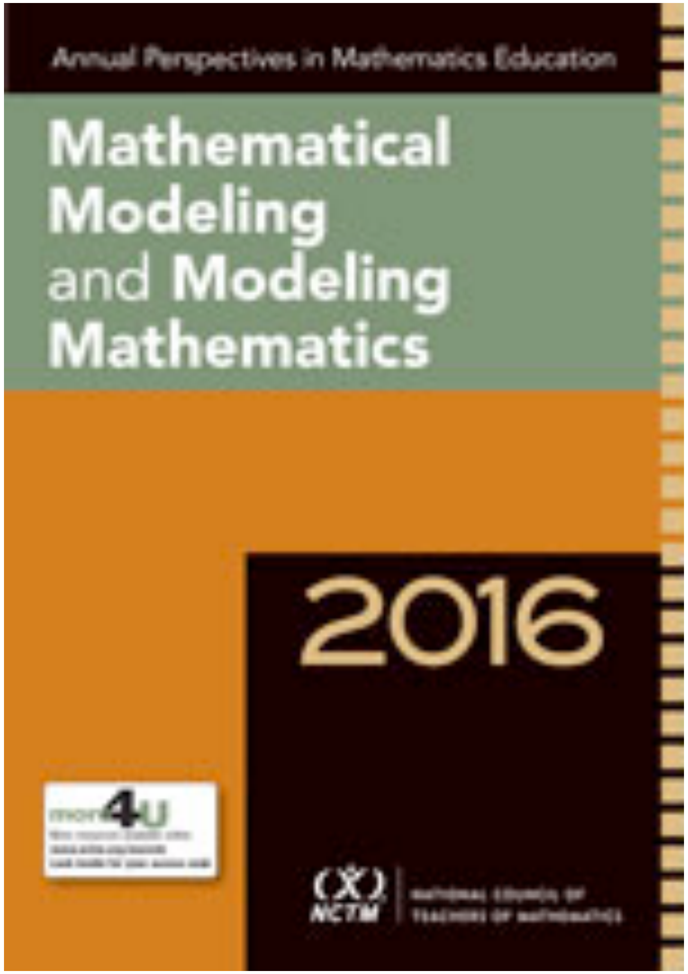
GAIMME Report (SIAM, COMAP 2016)

Guidelines for Assessment and Instruction in Mathematical Modeling Education

Contents

- What is Mathematical Modeling?
- Early Grades (K–8)
- High School (9–12)
- Undergraduate
- Resources

http://www.siam.org/reports/gaimme.php


Annual Perspectives in Mathematics Education
Mathematical Modeling and Modeling Mathematics
2016
NCTM

MATH MODELING

GETTING STARTED & GETTING SOLUTIONS

K. M. BLISS
K. R. FOWLER
B. J. GALLUZZO

As we discuss mathematical modeling today, please keep in mind these questions:

1) Who are your learners?
2) What do you want your students to accomplish in a mathematical modeling course?
3) Will you teach a stand alone course or infuse modeling into an existing course?

Take a few minutes to discuss these questions with a neighbor or two.

Who is teaching…

- Undergraduates?
- Graduate Students?
- Math majors?
- Other majors? (Name them.)
- Modeling as part of an existing course?
- As a stand alone course?

We'll discuss more details about goals later…

What is the difference between teaching mathematical models and mathematical modeling?

three perspectives

What is a model?

In [Field], a model is .....

- Biology: ... a hypothesis explaining experiments (or a mouse)
- Engineering: … a simple design/prototype of a more complex system
- Comp Sci: … is description of the complete operations of a system
- Numerics: … is a simulation that gives the same quantitative results
- Differential eqns: … is a simpler eqn derivable from the full problem

What is modeling?

Many definitions of the scope and intrinsic elements of “math modeling” – it's applied math, but beyond that, many views on aspects:

On Starting points – classes of problems:

- Broad questions for un-formulated real world problems vs.
- Specific questions for well-defined but intractable direct problems

On Ending points – desired form of solutions:

- Formulating new frameworks/theories
- Finding efficient computing/analysis approaches
- Developing a proof-of-concept approximation
- Using teamwork and interdisciplinary collaboration to present communicate progress

Focus of teaching and priorities:

- Emphasis on formulation or analysis/solution techniques
- Experience with open problems vs. specific problem classes
- The modeling process vs. studies of model types

Continuum of Problems

Solve a 2^{nd} order constant coeff linear forced DE

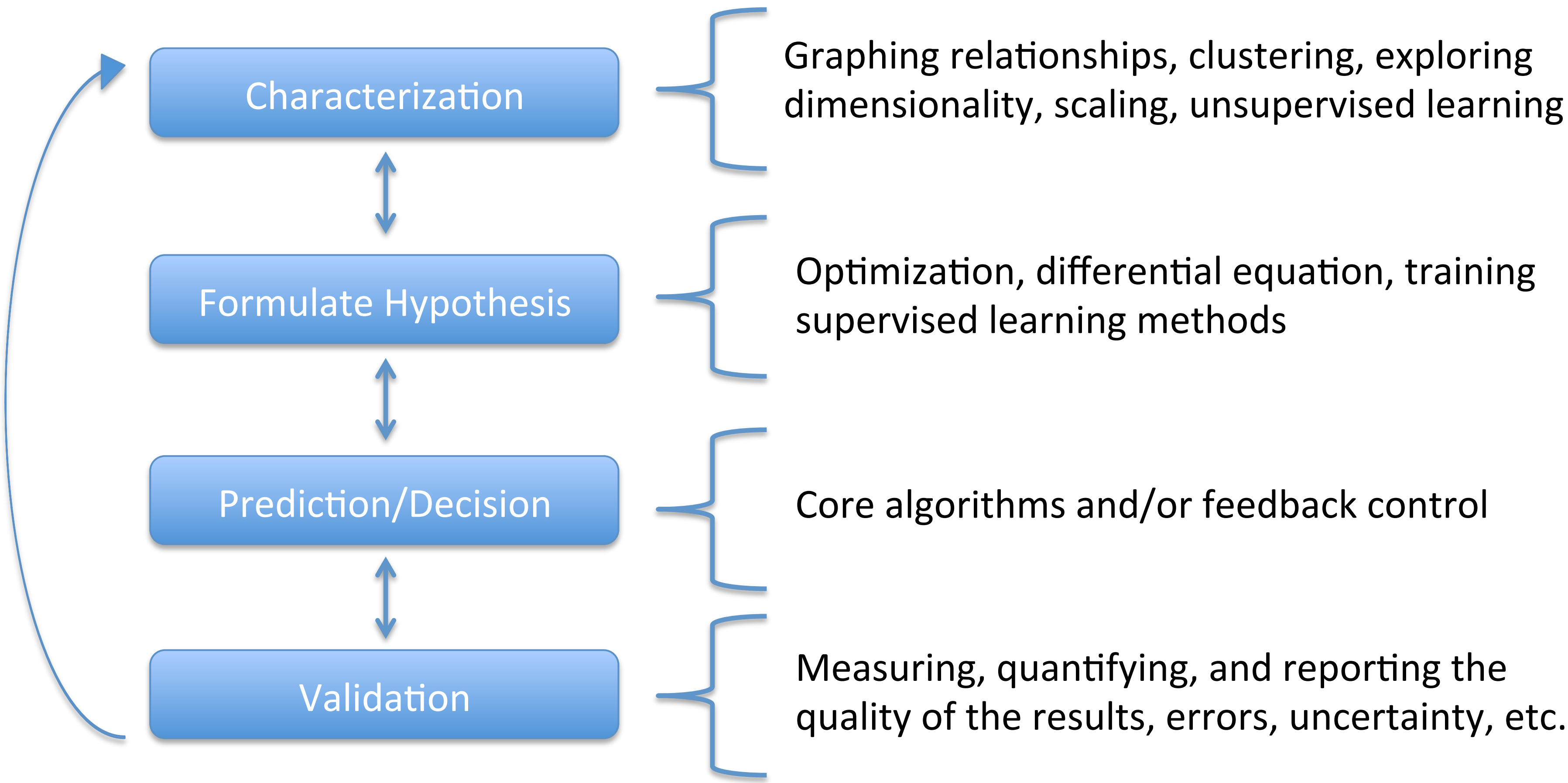
Mass Spring Equation

How can I model a building in an earthquake?

Math — **Model** — **Modeling**

Mathematical modeling is the Scientific Method

(where your hypothesis is a mathematical relationship)



Classifying models: black boxes, white boxes and shades of gray

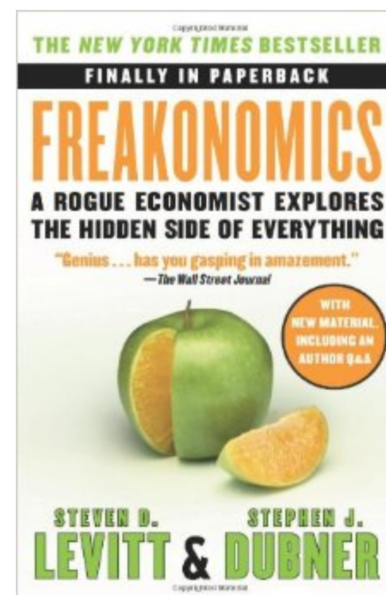
Shades of Model Uncertainty

Black Box Models	Gray Box Models	White Box Models
Fitting Data	Combination of data and first principles	First Principles
$$\min \sum_{i=1}^{n} \ell(y_i - f(x_i))$$		$$u_t + \nabla \cdot f(u) = 0$$
Purely data driven		Conservation Laws

Black Box Methods: Mapping inputs to outputs

- Regression
  - Curve Fitting
  - Time-Series Analysis
  - Econometrics
  - Credit Scoring

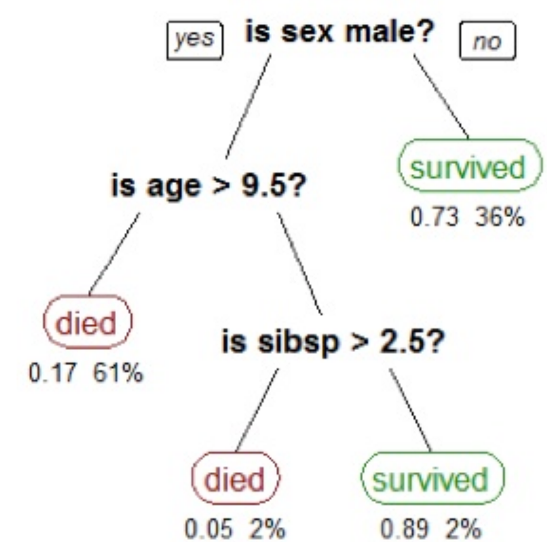
- Classification
  - Credit Card Fraud
  - Speech Recognition
  - Image Recognition
- Freakonomics: In black box disciplines, you are are a good researcher if you ask clever questions and find good data sets.
- Not very good at prediction unless the future looks like the past



Black-Box Example: Surviving the Titanic

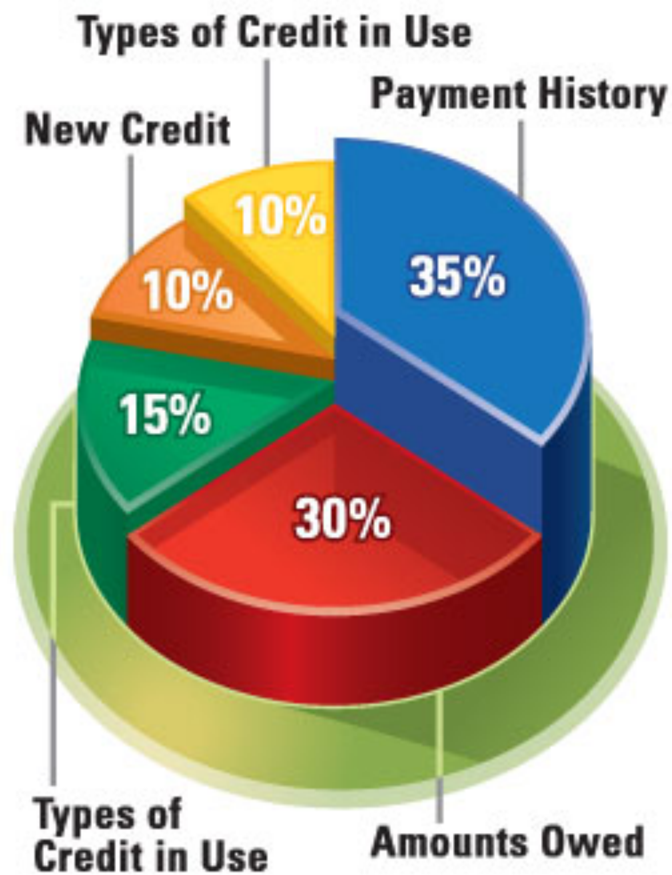
- Use classification to predict who lived and who died
  - Class of ticket (first class, etc)
  - Gender
  - Age
  - With spouse or children
- Can get around 94% accuracy in predicting who lived and died.

	A	B	C	D	E	F	G	
1	pclass	survived	name	sex	age	sibsp	parch	
2	1	1	Allen, Miss. Elisabeth Walton	female	29	0	0	24
3	1	1	Allison, Master. Hudson Trevor	male	0.917	1	2	11
4	1	0	Allison, Miss. Helen Loraine	female	2	1	2	11
5	1	0	Allison, Mr. Hudson Joshua Creighton	male	30	1	2	11
6	1	0	Allison, Mrs. Hudson J C (Bessie Waldo Daniels)	female	25	1	2	11
7	1	1	Anderson, Mr. Harry	male	48	0	0	1
8	1	1	Andrews, Miss. Kornelia Theodosia	female	63	1	0	1
9	1	0	Andrews, Mr. Thomas Jr	male	39	0	0	11
10	1	1	Appleton, Mrs. Edward Dale (Charlotte Lamson)	female	53	2	0	11
11	1	0	Artagaveytia, Mr. Ramon	male	71	0	0	P
12	1	0	Astor, Col. John Jacob	male	47	1	0	P
13	1	1	Astor, Mrs. John Jacob (Madeleine Talmadge Force)	female	18	1	0	P
14	1	1	Aubart, Mme. Leontine Pauline	female	24	0	0	P
15	1	1	Barber, Miss. Ellen "Nellie"	female	26	0	0	1
16	1	1	Barkworth, Mr. Algernon Henry Wilson	male	80	0	0	2



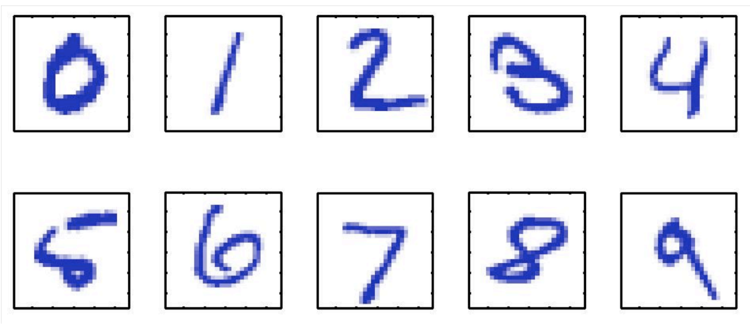
Black-Box Example: Consumer Credit Rating

- Logistic regression is the old school way to do analytics.
- Credit ratings use this technique to give you scores that have some meaning in weighing the fidelity of a borrower.
- Has reasonable track record when predictions are “in sample”
- Have little to no fidelity when predictions are “out of sample”
- Feature selection is the name of the game



Black-Box Example: Handwriting Recognition

- Classify zip codes for USPS (famous MNIST dataset)
- Each image is 28x28 = 784 pixels with 256 shades of grey
- Training data (60k samples) is much smaller than the set of all possible vectors
- Methods:
  - Best in Show: Deep convolutional neural networks. Now up to 99.79% accuracy (as of 2013)
  - Entry level via nearest neighbors
  - with Logistic regression: 92.5% accuracy
  - Average Humans: 98% accuracy



Gray-Box Methods: Uncovering the Structure

- Dynamical models (cause and effect)
- Managing uncertainty, peeling back the layers to understand what is in the box
- Understanding feedback effects
- Understanding first principles: "Why?", not just "How?".
- Modeling the thing that generate data, not the data.
- Able to make predictions when the future is not a repeat of the past.
- Examples:
  - Natural Language Processing (underlying semantics)
  - Math Biology (individual processes and reactions)
  - Economic Modeling (individual decisions)

Gray-Box Example: Handwriting Recognition

- Adding human motor control to the model
- Understanding context with the surrounding letters

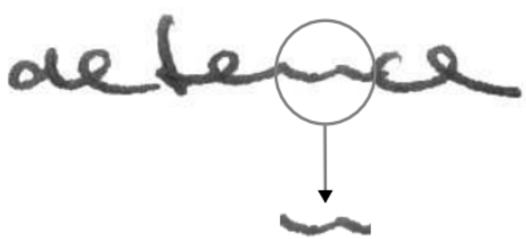
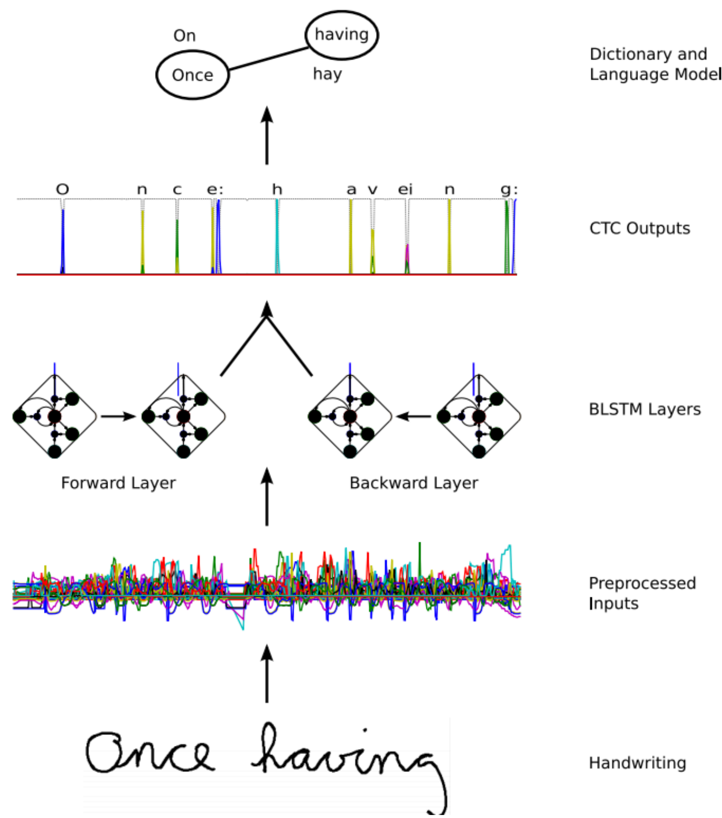


Fig. 9. Importance of context in handwriting recognition. The word 'defence' is clearly legible, but the letter 'n' in isolation is ambiguous.



Gray-Box Example: Weather Prediction

- Use compressible fluid dynamical models, thermodynamics, oceanic models, climate models, etc., combined with statistical models.
- National Weather Service uses two clusters each with 10k processors and 210TF peak performance (210 Trillion floating point operations per second).
- Still fairly accurate to 8 days.
- European Center has 10 times that power and forecasts better.

Gray-Box Example: Weight Management

$$EE = \underbrace{\delta BW}_{\text{physical activity}} + \underbrace{\beta_{tef} EI}_{\text{thermic effect of eating}} + \underbrace{\beta_{at} EI + \gamma_F F + \gamma_L L + \eta_F \frac{dF}{dt} + \eta_L \frac{dL}{dt} + K}_{\text{resting metabolic rate (RMR)}},$$

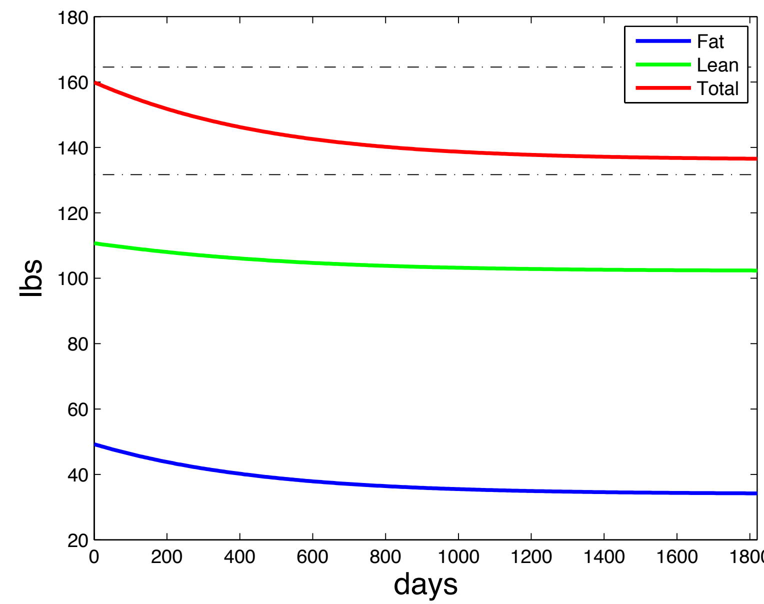
Conservation of Energy

$$\rho_F \frac{dF(t)}{dt} = (1 - p(t))EB(t),$$
$$\rho_L \frac{dL(t)}{dt} = p(t)EB(t)$$

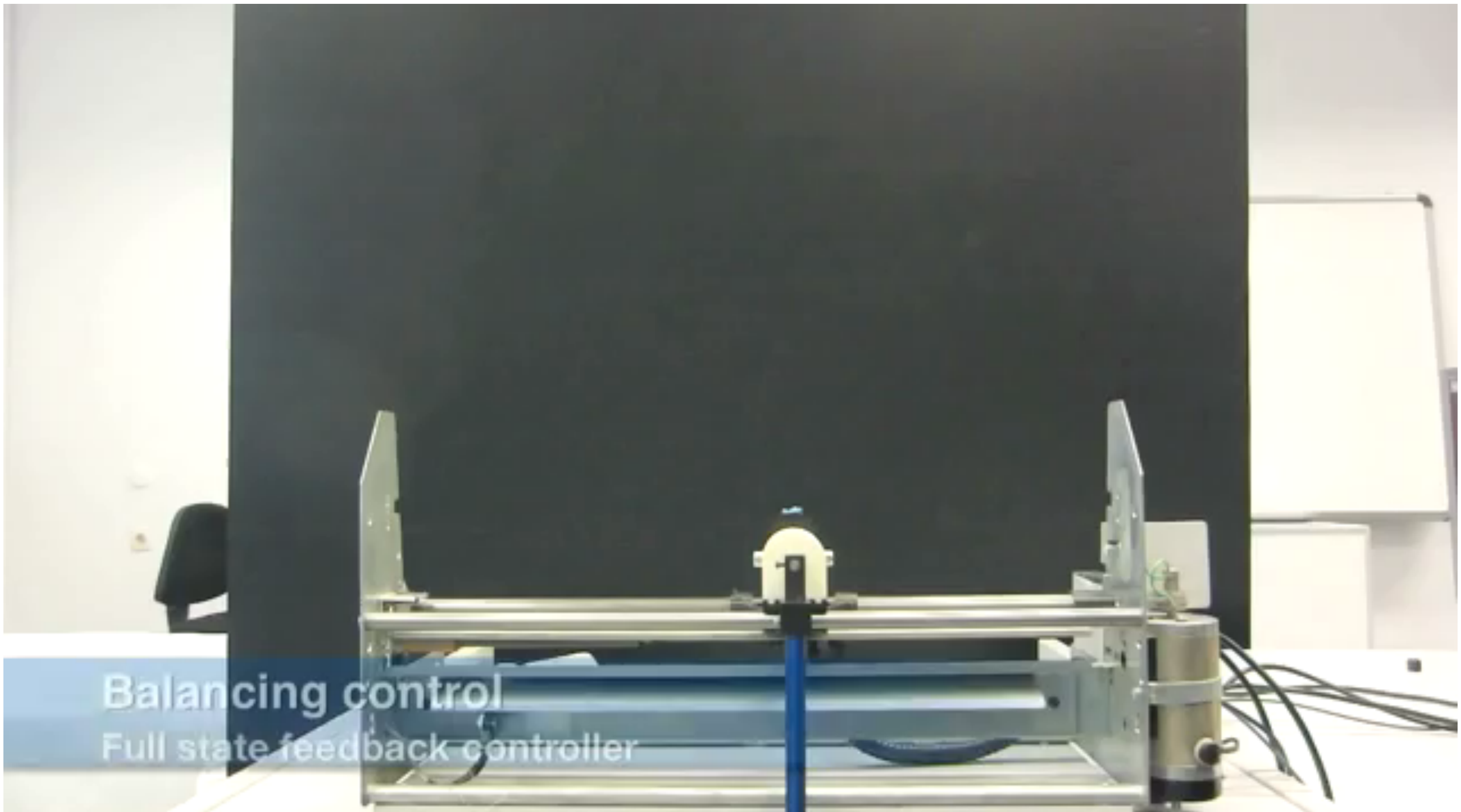
Compartmental Model

$$\frac{dF}{dL} = \frac{F}{10.4}$$

Forbes' Law (empirical)



Gray Box Example: Inverted Pendulum



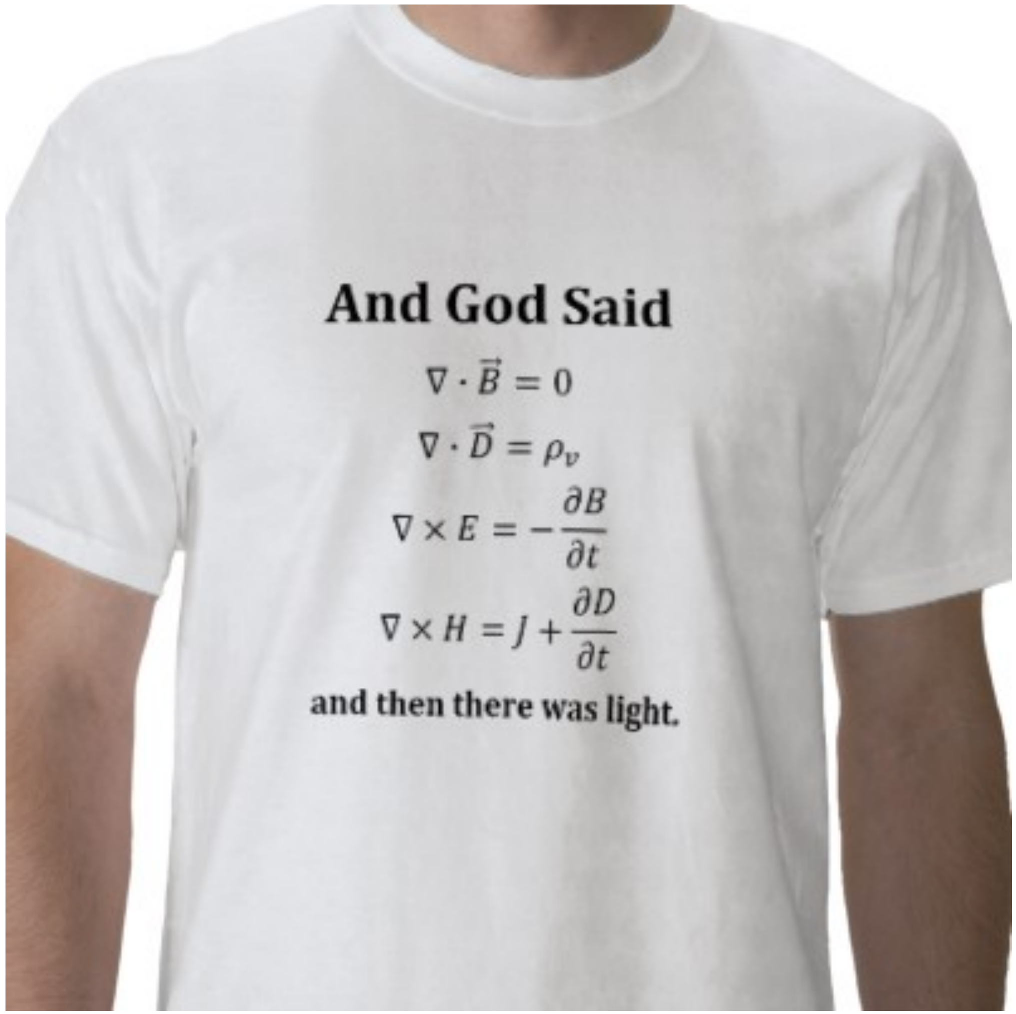
Gray-Box Example: Jet Control

- X-29 forward swept wing, DARPA funded (1980s)
- Dynamically unstable: A flying inverted pendulum
- Computer stabilized: Makes 40 corrections per second
- Inspired many fighter jet designs in use today

White-Box Methods: Uncovering the Truth

- Models are “essentially” right
- Intimate connection with symmetries and conservation laws
- Examples:
  - Classical Mechanics
  - Electromagnetism
  - Quantum Theory
  - Statistical Physics
  - General Relativity
  - Fluid Dynamics (mostly)

White Box Example: Maxwell's Equations



White Box Example: Fluid Dynamics

(with a sliver of gray)

The Navier-Stokes Equation:

Mass → $$\rho_t + \nabla \cdot (\rho \vec{v}) = 0$$

Momentum → $$(\rho \vec{v})_t + \nabla \cdot (\rho \vec{v} \otimes \vec{v}) = \nabla \cdot \mathbb{P}$$

Energy → $$\left[\rho\left(e + \tfrac{1}{2}|\vec{v}|^2\right)\right]_t + \nabla \cdot \left[\rho\left(e + \tfrac{1}{2}|\vec{v}|^2\right)\vec{v}\right] = \nabla \cdot (\mathbb{P}\vec{v} + \kappa \nabla T),$$

where

- ρ is the density, v the velocity, and e the specific energy
- $\mathbb{P} = \left[-p + \eta \nabla \cdot \vec{v}\right]\mathbb{I} + 2\mu\, \mathrm{Sym}(\nabla \vec{v})$ is the stress tensor,
- μ, η are the viscosity coefficients ($\mu > 0$, $2\mu + \eta > 0$), and
- $\kappa > 0$ is the heat conductivity.

Part II: Design and Objectives

Facilitating Modeling

Modeling Teaching Principles (from GAIMME report)

1. Modeling (like real life) is open-ended and messy.

2. When students are modeling, they must be making genuine choices.

3. Start big, start small, just start

4. Assessment should focus on the process, not the product.

5. Modeling does not happen in isolation

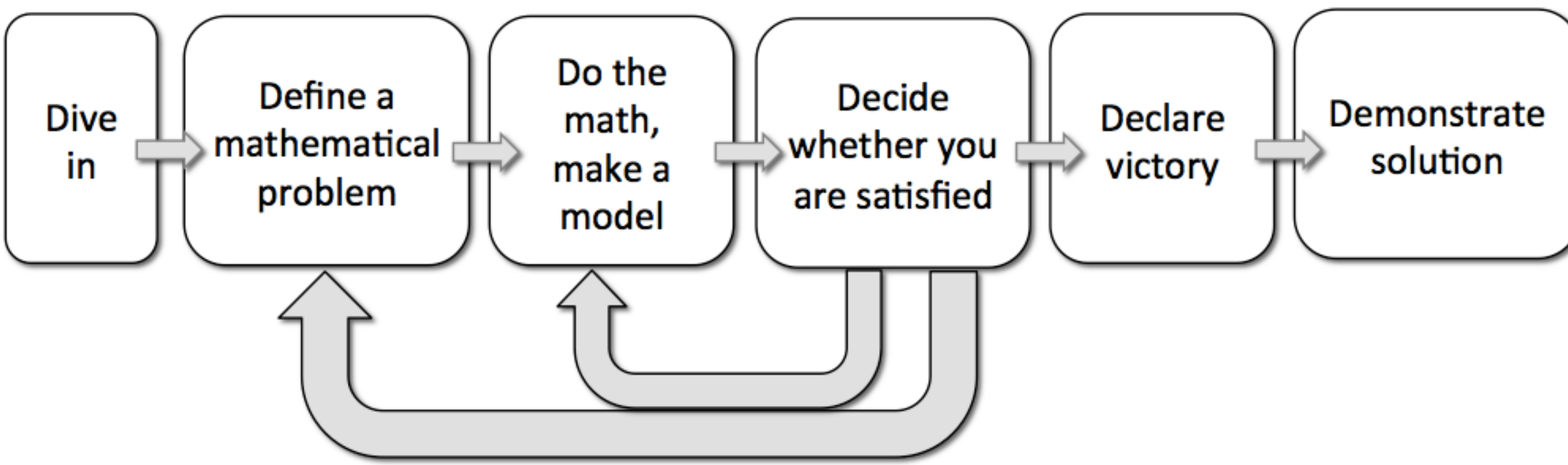
What makes a good modeling problem?

- Modelers can relate to the context
- Openness
  - Beginning: multiple entry points
  - Middle: multiple mathematical approaches
  - End: multiple solutions
- Inspires questions right away to focus problem through assumptions
- Information needed to inform and test solutions
- Someone cares about solutions; solutions are useful

Questions that arise when facilitating mathematical modeling

- How should I form student groups?
- Should I anticipate/encourage particular mathematical approaches?
- What do I do when student work reveals a unrecognized assumption or mathematical misconception?
- How many days should I stay on this project?
- What if students focus on an aspect of the problem other than mathematics?

Student modeling process



Are My Students Modeling?

DID THEY START WITH A BIG, MESSY, REAL WORLD PROBLEM?

DID THEY ASK QUESTIONS AND THEN MAKE ASSUMPTIONS TO DEFINE THE PROBLEM?

DID THEY IDENTIFY WHAT CHANGES AND WHAT STAYS THE SAME?

ARE THEY USING MATHEMATICAL TOOLS TO SOLVE THE PROBLEM?

ARE THEY COMMUNICATING WITH SOMEONE WHO CARES ABOUT THE SOLUTION?

HAVE THEY EXPLAINED IF/WHEN THEIR ANSWERS MAKE SENSE?

HAVE THEY TESTED THEIR MODEL/ SOLUTION AND REVISED IF NECESSARY?

Course design

Who are you planning to teach?

- What are their majors? Career plans? Interests?
- What prerequisite knowledge do they bring? What do they lack?
  - Mathematics
  - Programming/Algorithms
  - Modeling
  - Familiarity with software
  - Teamwork
  - Application background/interest
- Do they have to take your course or is it an elective?

Syllabus organizational principles

- **Mathematical topics**– usually ordered from simple to more complex. Model and context follows introduction of math as illustration of its use. ("Applied math class")

- **Types of models or algorithms** – here the mathematics appears within a particular established modeling approach. Context follows model as illustration of its use. ("Greatest hits models")

- **Context** – start with something like traffic/climate/health and introduce mathematics to apply to that context. ("Topics course")

- **Process-based modeling project** – emphasize steps of the process applied to open-ended problems. Choice of mathematical tools usually up to the student, though instructors could make suggestions. ("Modeling class")

- **Current research/practice** – Topics or themes based on current journal articles or industrial projects ("Journal club")

Syllabus Organized by Model Type

Mathematical Modeling (Math 5740)

- Growth & interaction models (population dynamics, epidemics)
- Wave models (traffic)
- Conflict models (games)
- Graphs (shortest path, max flow, transport)
- Shapes, smooth/space-fitting, fractal curves (gerrymandering)
- Lattice models (structures and materials)

Andrej Cherkaev, Utah

Syllabus organized by Context

Mathematical Modeling (Applied Math 115)

- **Population Dynamics:** ODEs, linear stability of fixed points, numerical solutions of ODEs, Markov processes, stochastic equations, stochastic simulations, equations for probability of distribution function, phase planes, oscillations vs limit cycles, model validation, maps as dynamical systems, their fixed points, oscillation and chaos.
- **Climate:** multiple equilibria, bifurcation, hysteresis, catastrophes, stochastic DEs, white noise, red noise, Fourier transform, spectrum, AR(n), Markov processes, variance, autocorrelation
- **Renewable and Exhaustible Resources:** maximum solution in optimal control, singular solutions and bang-bang control, shadow prices
- **Traffic flow:** discrete modeling, delayed ODEs and PDEs, waves, characteristics, shocks
- **Interacting agents:** Markov chains, ergodic distribution, recurrent classes

Tziperman and Fudenberg, Harvard

First Year BYU ACME (4 Courses)

Mathematical Analysis

- Vector Spaces
- Linear Transformations
- Inner Product Spaces
- Spectral Theory
- Metric Topology
- Differentiation
- Contraction Mappings
- Integration
- Integration on Manifolds
- Complex Analysis
- Adv. Spectral Theory
- Psudospectrum

Algorithm Design & Optimization

- Intro Algorithms
- Graph Algorithms
- Discrete Probability
- Fourier Theory
- Wavelets
- Interpolation
- Unconstrained Optimization
- Convex Analysis
- Linear Optimization
- Nonlinear Optimization
- Dynamic Optimization
- Markov Decision Processes

First Year BYU ACME (Labs)

Mathematical Analysis

- Intro Python
- NumPy
- MatPlotLib
- Complexity/Sparse Matrices
- Linear Systems
- QR (householder)
- QR (givens)
- Markov Chain Lab
- Image Segmentation
- Facial Recognition (SVD)
- Finite Differences
- Conditioning
- Newton Cotes vs. Monte Carlo
- Sparse Grid Approximation
- Variance Reduction Methods
- Complex Analysis
- Profiling and Wrapping
- PageRank on Tournaments
- Arnoldi Iteration and GMRES
- The Pseudospectrum

Algorithm Design & Optimization

- Standard Library
- Object Oriented Programming
- Data Structures
- Depth/Breadth First
- Nearest Neighbor Search
- Scientific Visualization
- Maximum Likelihood Estimation
- FFT and Applications
- Wavelets
- Chebychev Polynomials
- Gaussian Quadrature
- Polynomial Interpolation
- Optimization Packages
- Line Search Methods
- Conjugate Gradient Methods
- Simplex Method
- Compressed Sensing Lab
- Interior Point Methods
- Dynamic Optimization
- Multi-Armed Bandits

Second Year BYU ACME (4 Courses)

Modeling with Uncertainty & Data

- Random Spaces & Variables
- Distributions & Expectation
- Limit Theorems
- Markov Processes
- Poisson, Queuing, Renewal
- Information Theory
- Martingales, Diffusion
- Kalman Filtering & Time-Series
- Principal Components
- Clustering
- Bayesian Statistics (MCMC)
- Logistic Regression
- Random Forests
- Support Vector Machines

Modeling with Dynamics & Control

- ODE Existence & Uniqueness
- Linear ODE
- Nonlinear Stability
- Boundary-Value Problems
- Hyperbolic PDE
- Parabolic PDE
- Elliptic PDE
- Calculus of Variations
- Optimal Control
- Stochastic Control

Second Year BYU ACME (Labs)

Modeling with Uncertainty & Data

- Relational Databases and SQL
- Regular Expressions
- Web Technologies
- Scraping with BeautifulSoup
- MPI and OpenMP
- Pandas and Hadoop
- MongoDB/noSQL
- Kalman Filtering
- Time Series
- Naïve Bayes
- Discrete HMMs
- Continuous HMMs (speech recognition)
- Gibbs Sampling and LDA
- Metropolis Hastings
- PCA and LSI
- Clustering with k-means
- Logistic Regression
- Random Forests
- SVM on Handwriting Recognition

Modeling with Dynamics & Control

- Harmonic Oscillators and Resonance
- Weightloss Models
- Predator-Prey Models
- Shooting Methods and Applications
- Compartmental Models (SIR)
- Pseudospectral methods for BVP
- Lyapunov Exponents and Lorenz Attractors
- Hysteresis in population models
- Conservation Laws and Heat Flow
- Anisotropic diffusion
- Poisson equation, finite difference
- Nonlinear Waves
- Finite Volume Methods
- Finite Element Methods
- Scattering Problems
- PID Control
- LQR and LQG Control
- Guided Missiles
- Merton Model in Finance

Let's discuss Pros and Cons… Discussion: Which approach(es) will best serve your students?

- **Mathematical topics**
- **Types of models or algorithms**
- **Context**
- **Process-based modeling project**
- **Current research/practice**

Assignment types – again, what will best serve your students?

Doing the Math (more closed)

- Solving problems to practice a mathematical skill (little or no context)
- Solving given problem using particular tool (tool practice is the focus)
- Solving mathematical problems where the context matters

Modeling projects (more open)

- Teacher-proposed
- Student-proposed
- Industrial / community client

Presentations

- Problem presentation in class
- Data visualization
- Oral presentations
- Written presentations

Tools / algorithms

Jupyter Notebook Demonstration

Career preparation

<u>Beyond the classroom</u>: math modeling courses as intro **guided research experiences**

What are the short/long-term take-aways for students? (GAIMME 2016 report)

- Improved skills with applying specific **mathematical tools and techniques**
- Independently **formulating** research goals
- Doing searches to find sources for background
- Experience with **teamwork, project management, or leadership**
- **Communications skills**: intra-team and external reports and presentations

All are valuable ***<u>transferable skills</u>*** that students can carry-over to future experiences

Math modeling courses are **gateways** to many opportunities:

- Independent research projects (in math or application areas)
- **Summer schools** and **workshops on modeling**
  - **MPI** (Mathematical Problems in Industry) workshops (US Univs)
  - **GSMMC** (Graduate Student Mathematical Modeling Camp) (RPI)
  - **IMSM** (Industrial Mathematical & Statistical Modeling Workshop) (NCSU/SAMSI)
  - **RIPS** (Research in Industrial Projects for Students) (UCLA/IPAM)
  - **ESGI** (European Study Groups in Industry, Medicine, etc) (UK, EU, worldwide)
  - Other programs in Canada (Fields, PIMS, MITACS) and other sources…
  - **PIC Math**
- Internships with companies and research labs...

<u>Beyond the classroom</u>: preparation for **broader career paths**

- Many thriving areas of **high technical careers** involving research and design via mathematical sciences **outside academia**!
- Essential for **sustainability** – academia does not offer enough jobs to balance the yearly production of graduates

What **industrial areas**? (SIAM Mathematics in Industry 1996, 2012 reports)

- Engineering, aerospace, defense
- Biotech, medicine, pharmaceuticals
- Information science, communications, computers, security
- Manufacturing and materials
- Finance, business, management

<u>What mathematical backgrounds</u>?

- Statistics
- Applied mathematics
- Probability
- Discrete math, algebra, geometry, ...
- Numerical analysis, optimization, operations research, …

<u>Beyond the classroom</u>: modeling in modernized advanced math curriculum

<u>NSF-IPAM Workshop on Internships in Math Sciences</u> (UCLA/IPAM, Fall 2015)
Recommendations for preparing math students for current real-world jobs:

- Encouraging **internship** experiences
- Industry-academic **collaborations**
- Multi-level National **infrastructure** to provide support and information: via Professional Associations (SIAM, AMS, ASA, ACM, etc), Mathematical Institutes (IPAM, ICERM, SAMSI, Fields, etc), …
  see BIG (**<u>B</u>**usiness **<u>I</u>**ndustry **<u>G</u>**overnment) Math Network (https://bigmathnetwork.wordpress.com)
- Key elements of **modern training in applied math**
  - Programming, Data science, **Modeling and simulation**
  - Professional development experience

Very consistent with earlier Math-in-Industry reports, also:

- **Depth** of knowledge **in a current active application** area is important
- Some **"cultural" tensions** exist in academia regarding **valuing academic vs. industrial-track jobs** for graduates, but core training **shares fundamental principles** of problem solving and modeling via applied math

SIAM Conference on Applied Mathematics Education

September 30–October 2, 2016

DoubleTree by Hilton Hotel, Philadelphia Center City

Philadelphia, Pennsylvania, USA

Find a blend
that works for your learners.

models + modeling
mathematics + algorithms
theory + implementation
formulation + solutions

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Mathematical Modeling Motivates

- Mathematical Vocabulary
- Interdisciplinary Connections
- Quantitative Literacy
- Creativity
- Collaboration
- Critical Thinking
- Communication
- Formalizing & Mathematizing
- Problems Solving
- Iteration & Revision
- Computation
- Multiple representations