Symmetric States Requiring System Asymmetry

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TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016) YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907

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Symmetry

Complex networks

MacArthur, Sánchez-García, & Anderson, Discrete Appl. Math. 156, 3525 (2008)

Symmetry

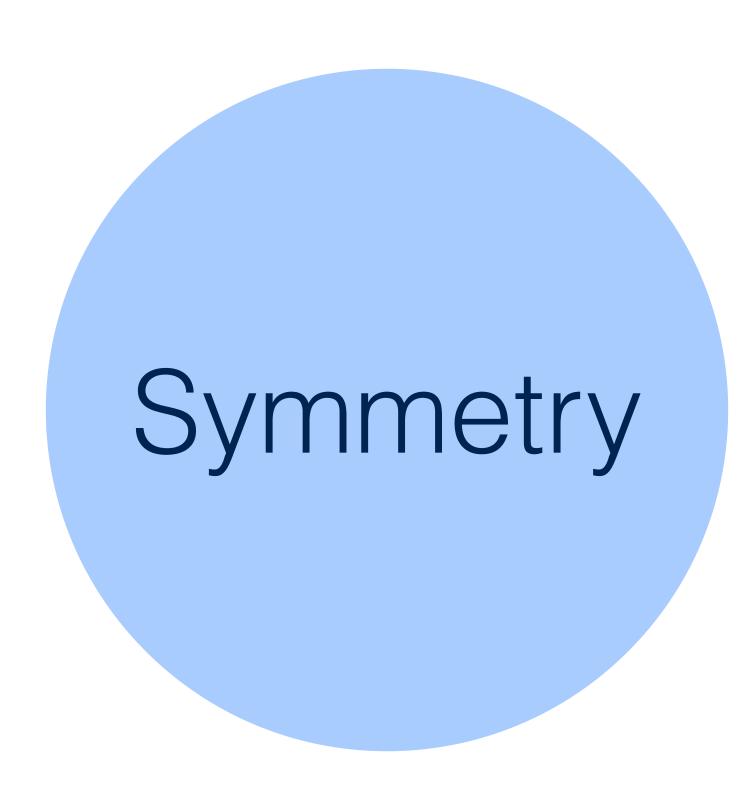
Synchronization

Strogatz, *Sync: The Emerging Science of Spontaneous Order* (2003)
Pikovsky, Rosenblum, & Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (2003)

Symmetry

Network symmetry \iff dynamical symmetry

Golubitsky & Stewart, *The Symmetry Perspective* (2002) Pecora, Sorrentino, Hagerstrom, Murphy, & Roy, Nat. Commun. **5**, 4079 (2014)





Weyl, *Symmetry* (1952)
Golubitsky & Stewart, *The Symmetry Perspective* (2002)

Chimera states

Kuramoto & Battogtokh, Nonlinear Phenom. Complex Syst. **5**, 380 (2002)
Abrams & Strogatz, Phys. Rev. Lett. **93**,174102 (2004)
Tinsley, Nkomo, & Showalter, Nat. Phys. **8**, 662 (2012)
Hagerstrom, Murphy, Roy, Hovel, Omelchenko, & Schöll, Nat. Phys. **8**, 658 (2012)

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^n \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i)$$

Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

$$\dot{\theta}_i = \omega + r_i - 1$$

$$\dot{r}_i = b_i r_i (1 - r_i)$$

Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

Limit cycle
$$\theta_i(t) \equiv \theta_0 + \omega t, \quad r_i(t) \equiv 1$$

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^n \sin(\theta_j - \theta_i)$$

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Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

Synchronous state

$$\theta_1(t) = \dots = \theta_n(t) \equiv \theta_0 + \omega t, \quad r_1(t) = \dots = r_n(t) \equiv 1$$
 uniform and symmetric

TN & AEM, Symmetric states requiring system asymmetry, Phys. Rev. Lett. 117, 114101 (2016)

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^n \sin(\theta_j - \theta_i)$$

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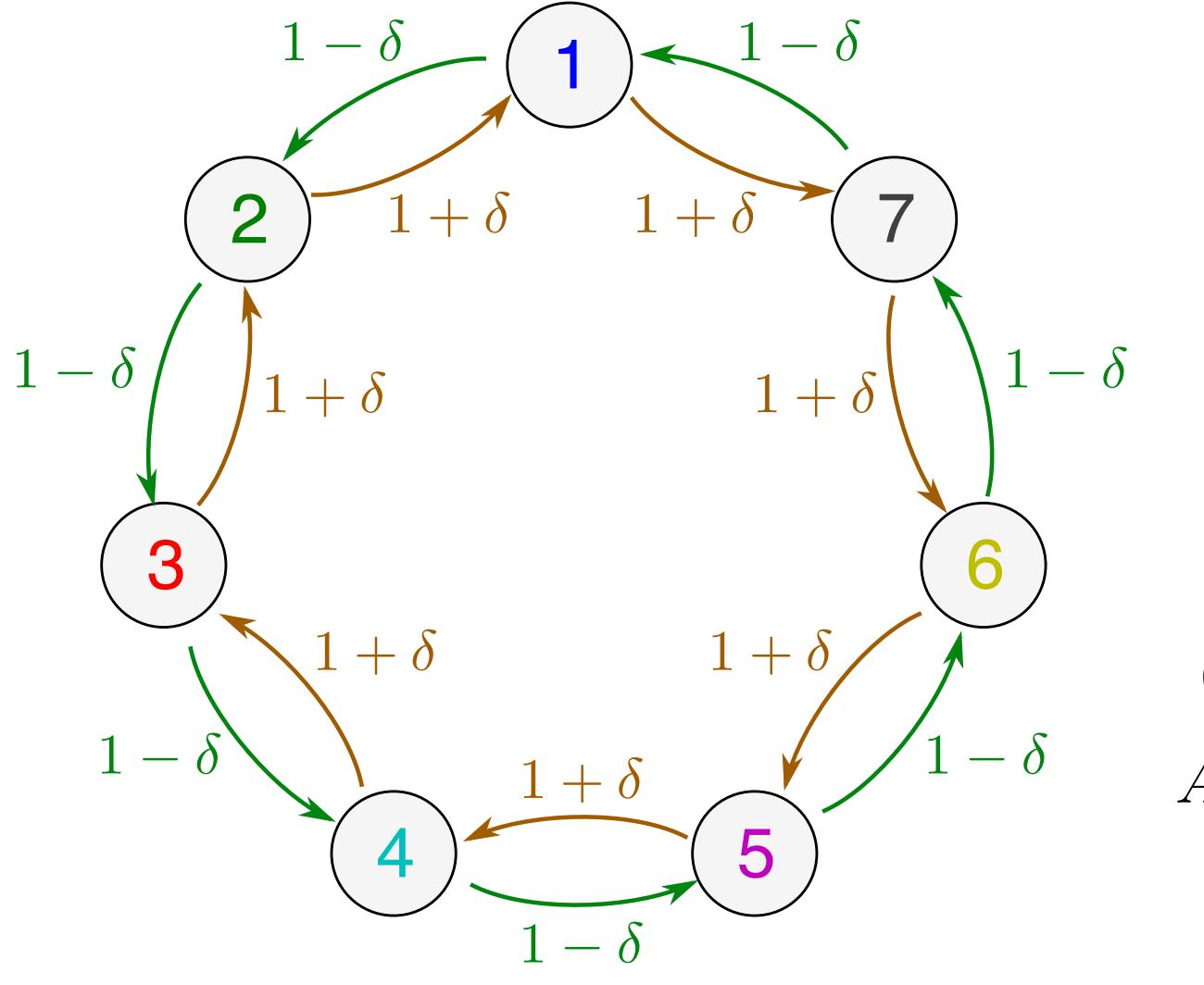
Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

tunable oscillator parameters

Synchronous state

$$\theta_1(t) = \dots = \theta_n(t) \equiv \theta_0 + \omega t, \quad r_1(t) = \dots = r_n(t) \equiv 1$$
uniform and symmetric

Symmetric network structure



Coupling strength

$$A_{ij} = 1 + \delta \text{ or } 1 - \delta$$

$$1.3 \text{ or } 0.7$$

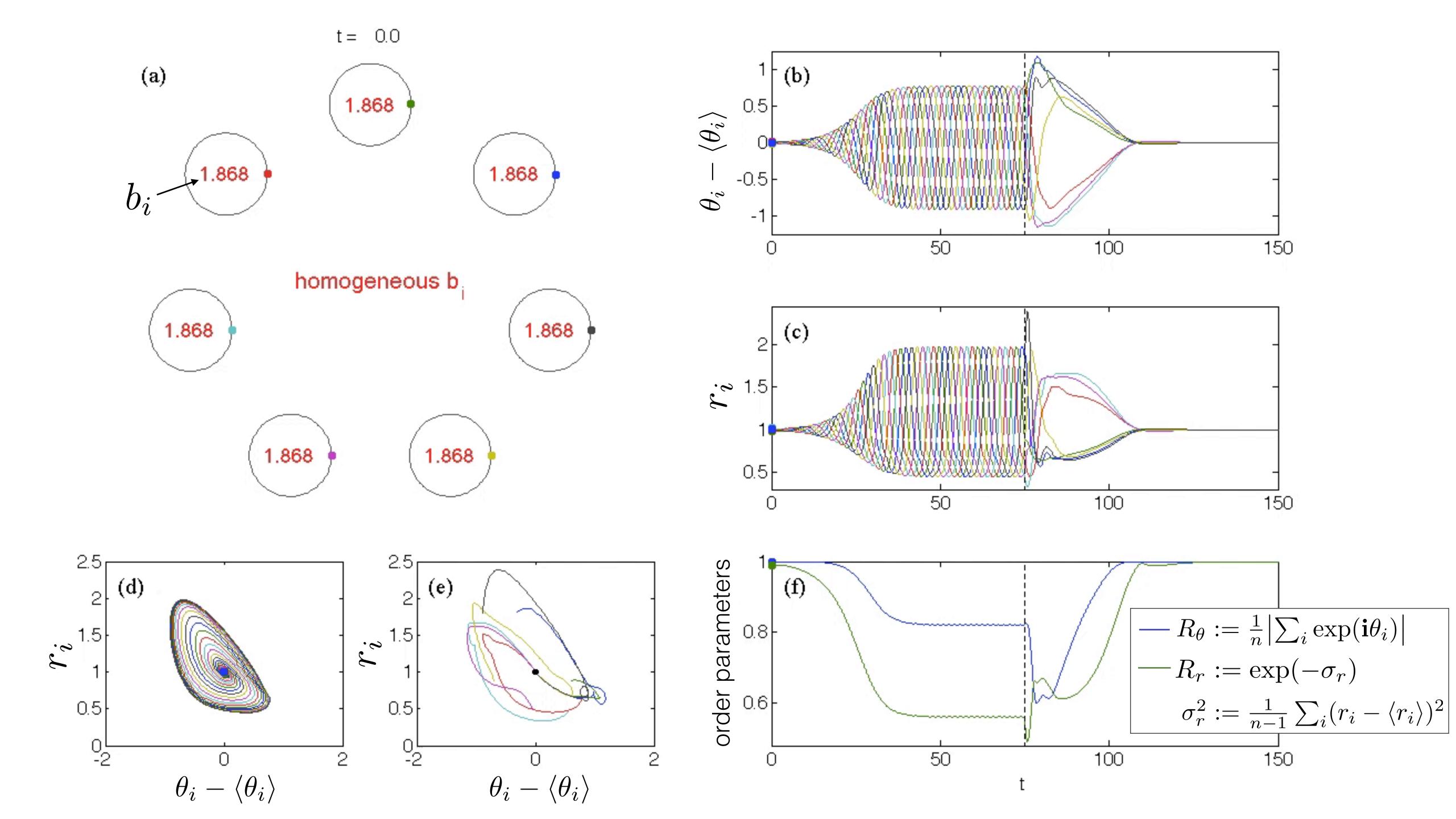
$$(\delta = 0.3)$$

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^n \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i)$$

$$\begin{array}{c} \text{Best} \\ \text{homogeneous} \\ b_i \text{ value} \end{array}$$

$$\begin{array}{c} \text{1.868} \\ \text{homogeneous b}_i \\ \text{1.868} \end{array}$$



Synchronization dynamics

Complete synchronization with nonidentical oscillators Complete synchronization only with nonidentical oscillators

symmetric state

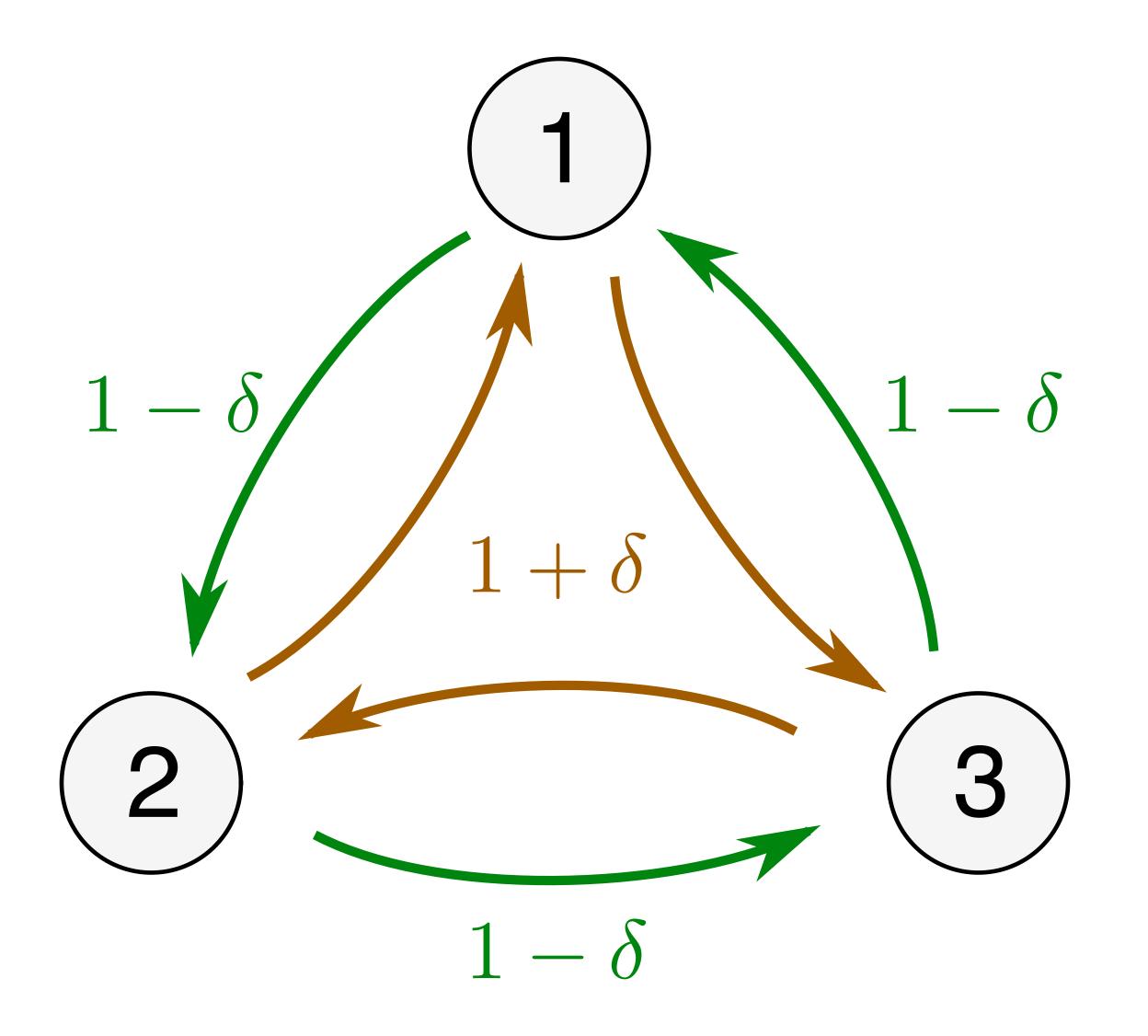
system asymmetry

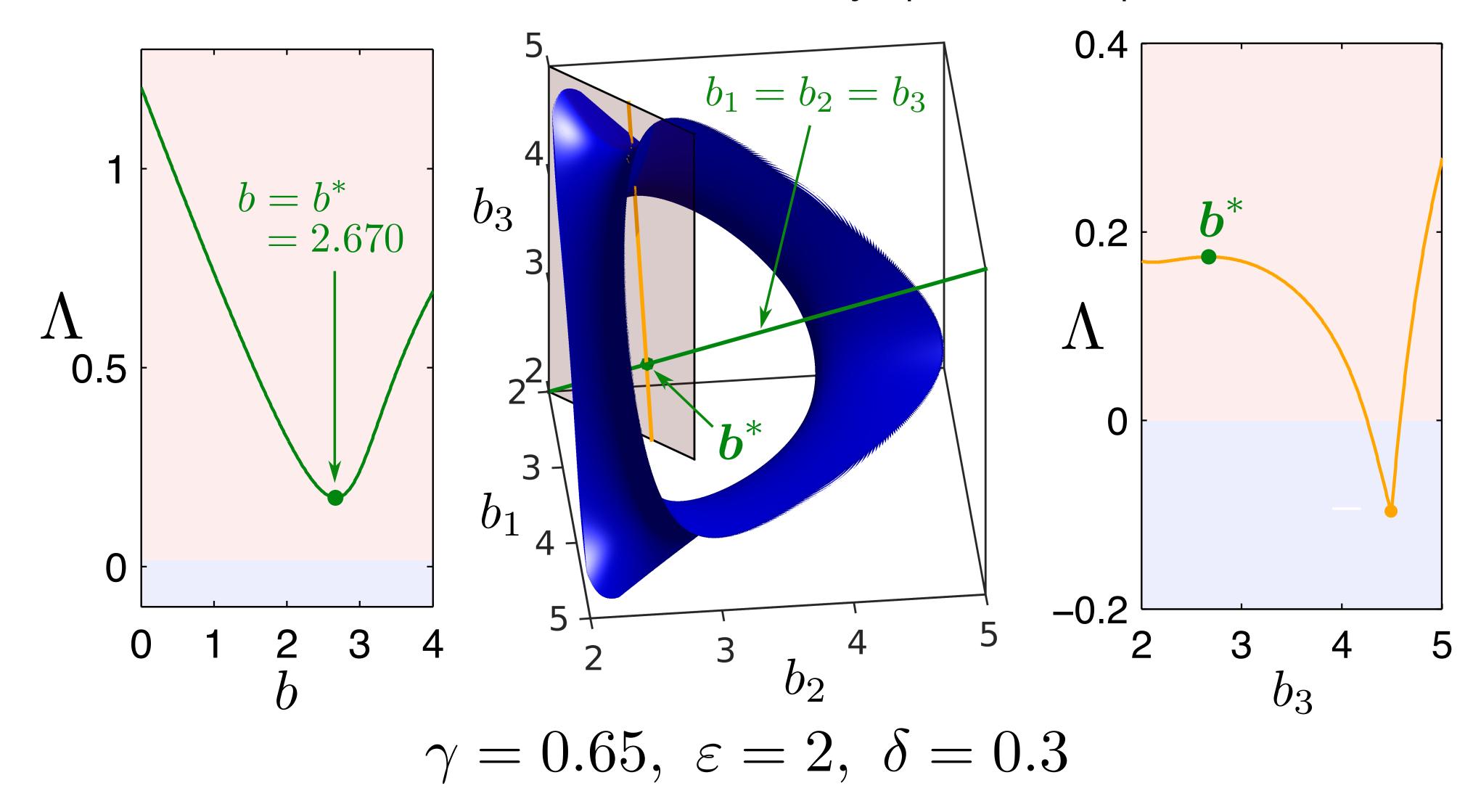
"Symmetric states requiring system asymmetry"

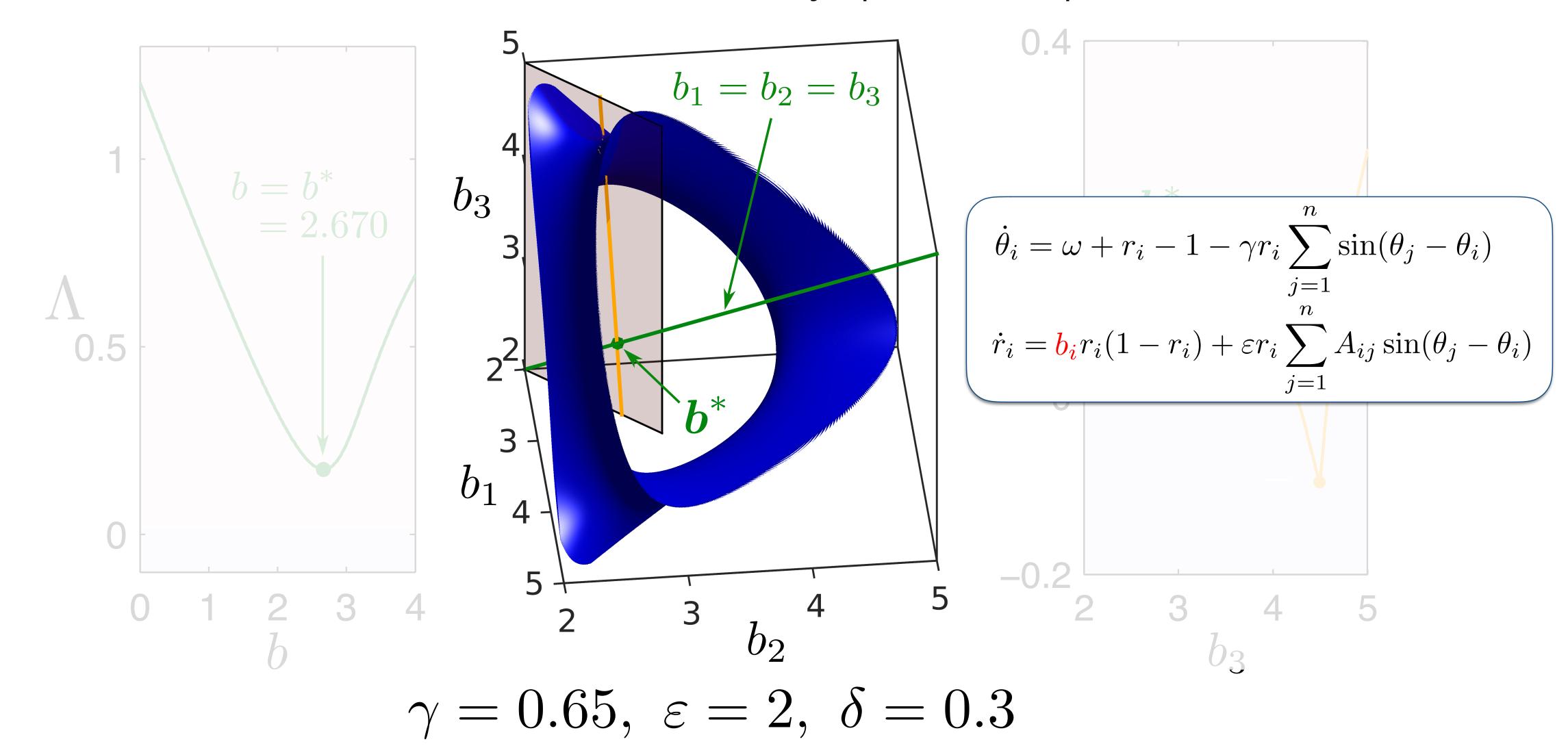
We have a converse of symmetry breaking

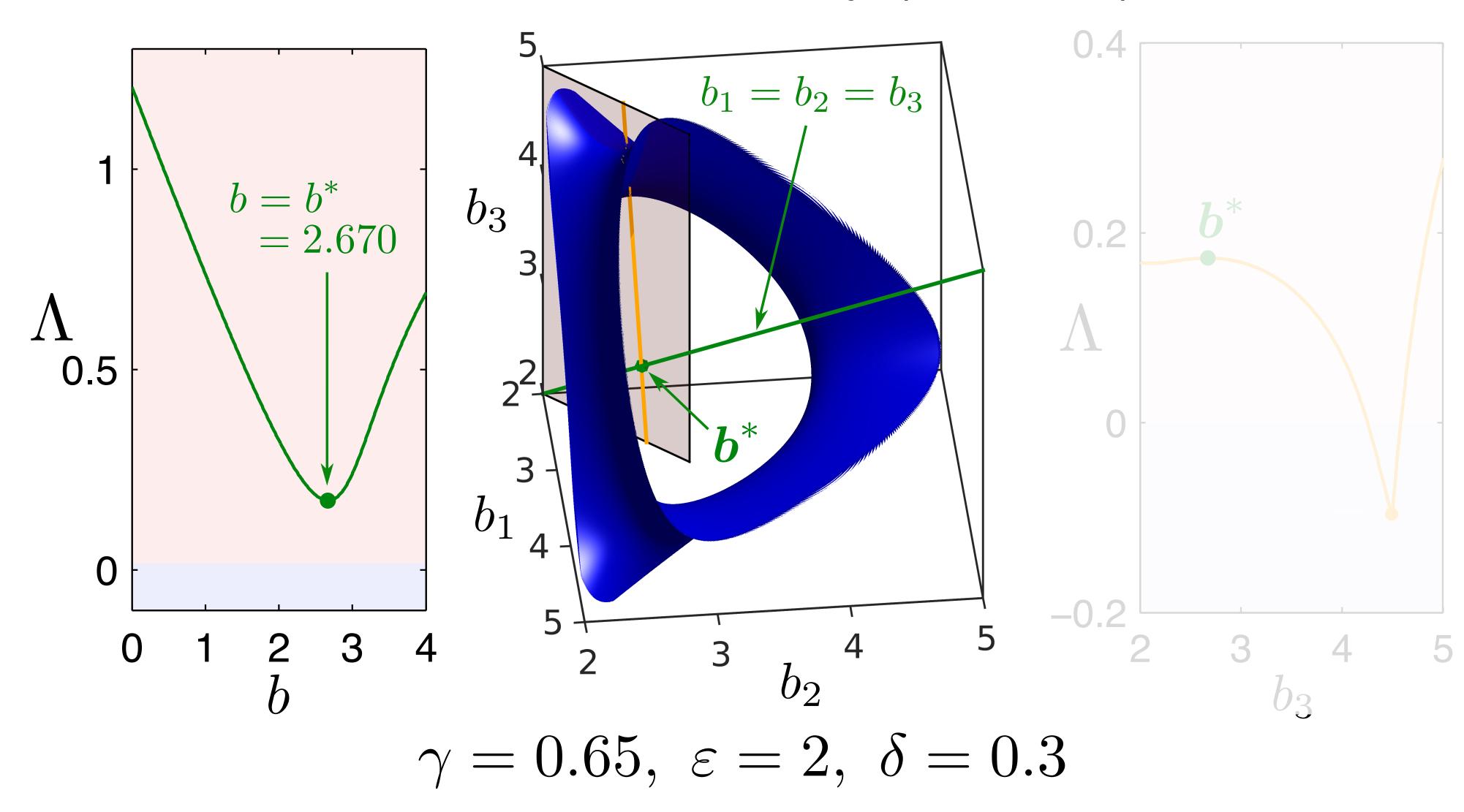
Symmetric stable state requiring system to be asymmetric

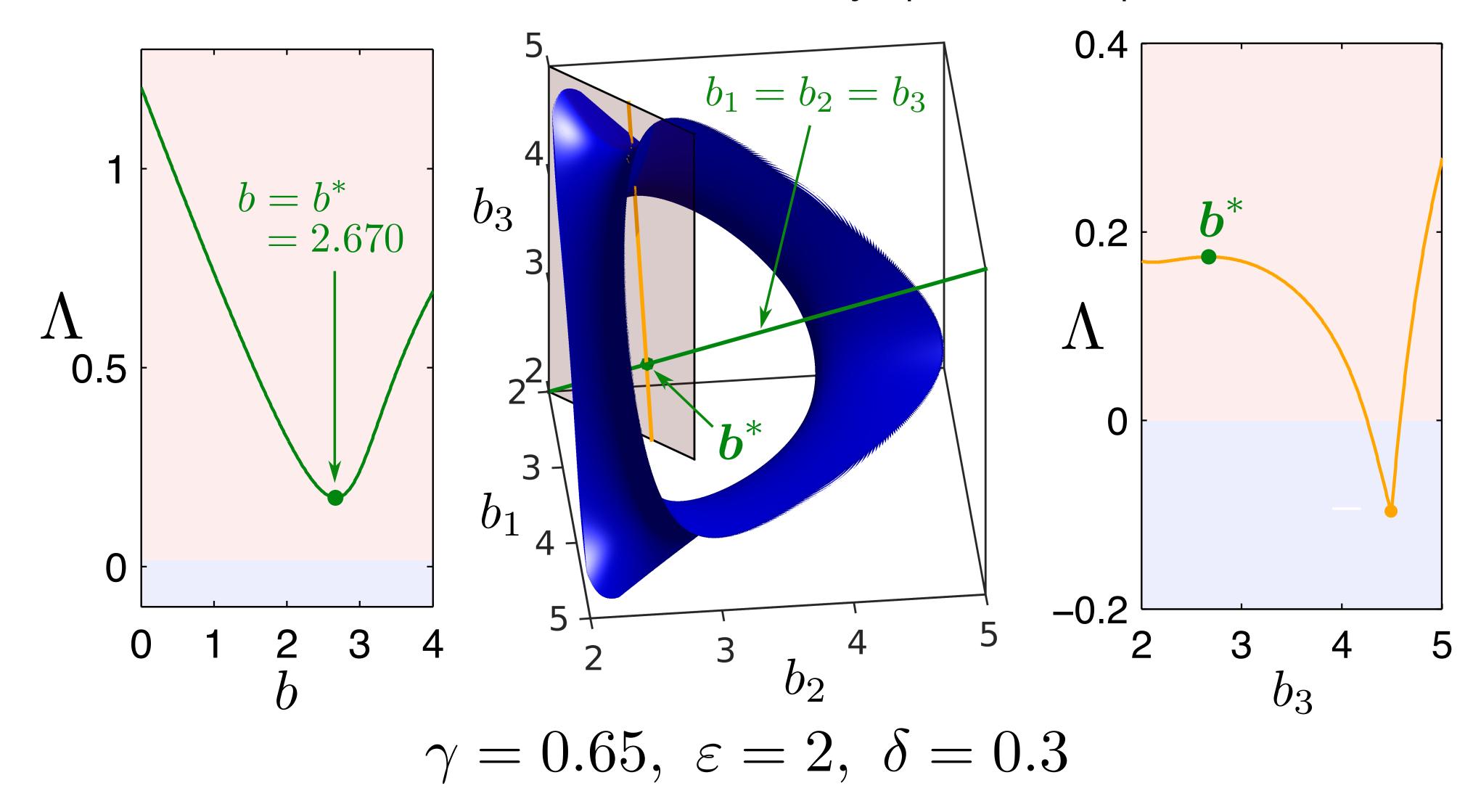
Symmetric **system** *requiring* **stable state** to be asymmetric (symmetry breaking)



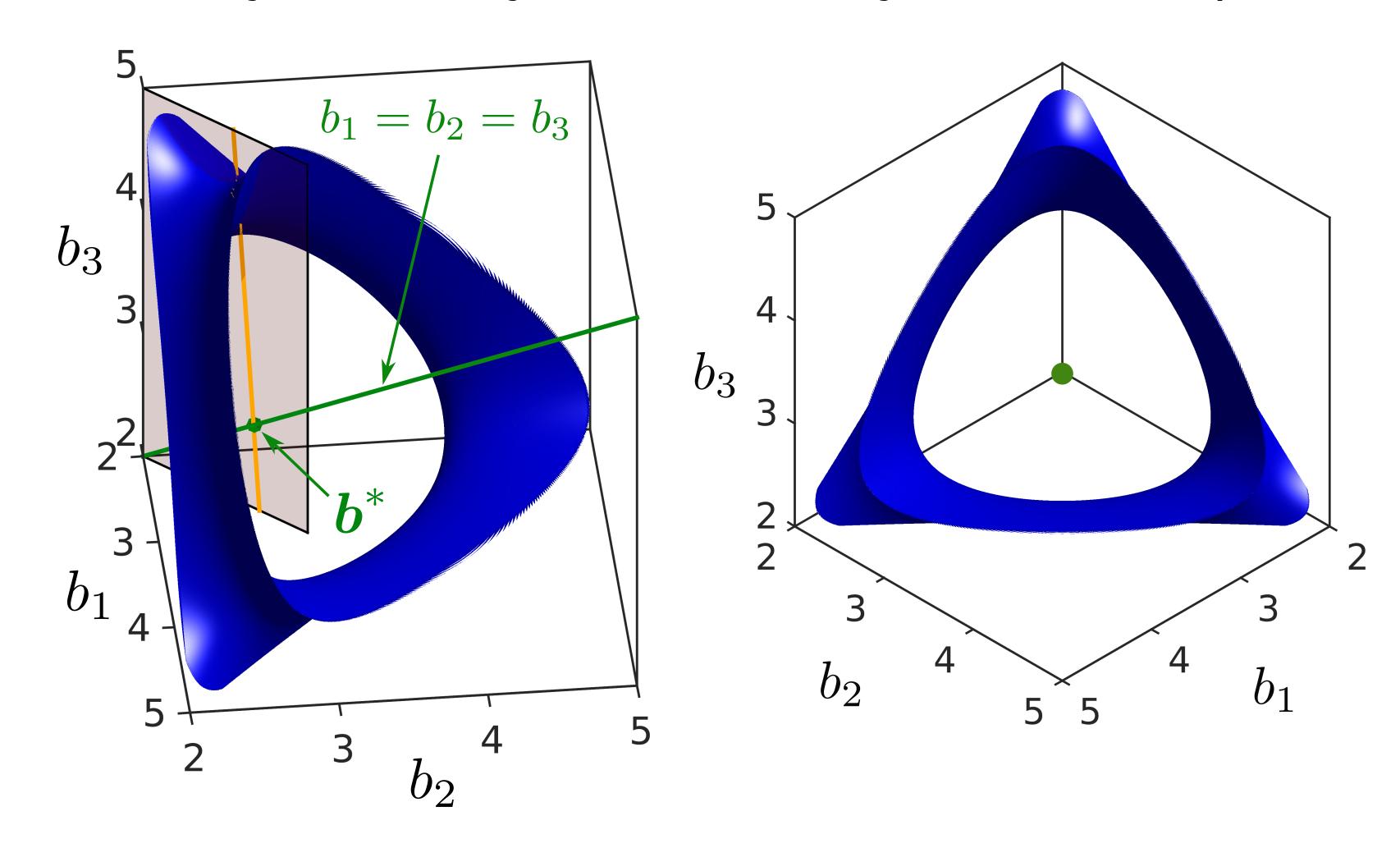






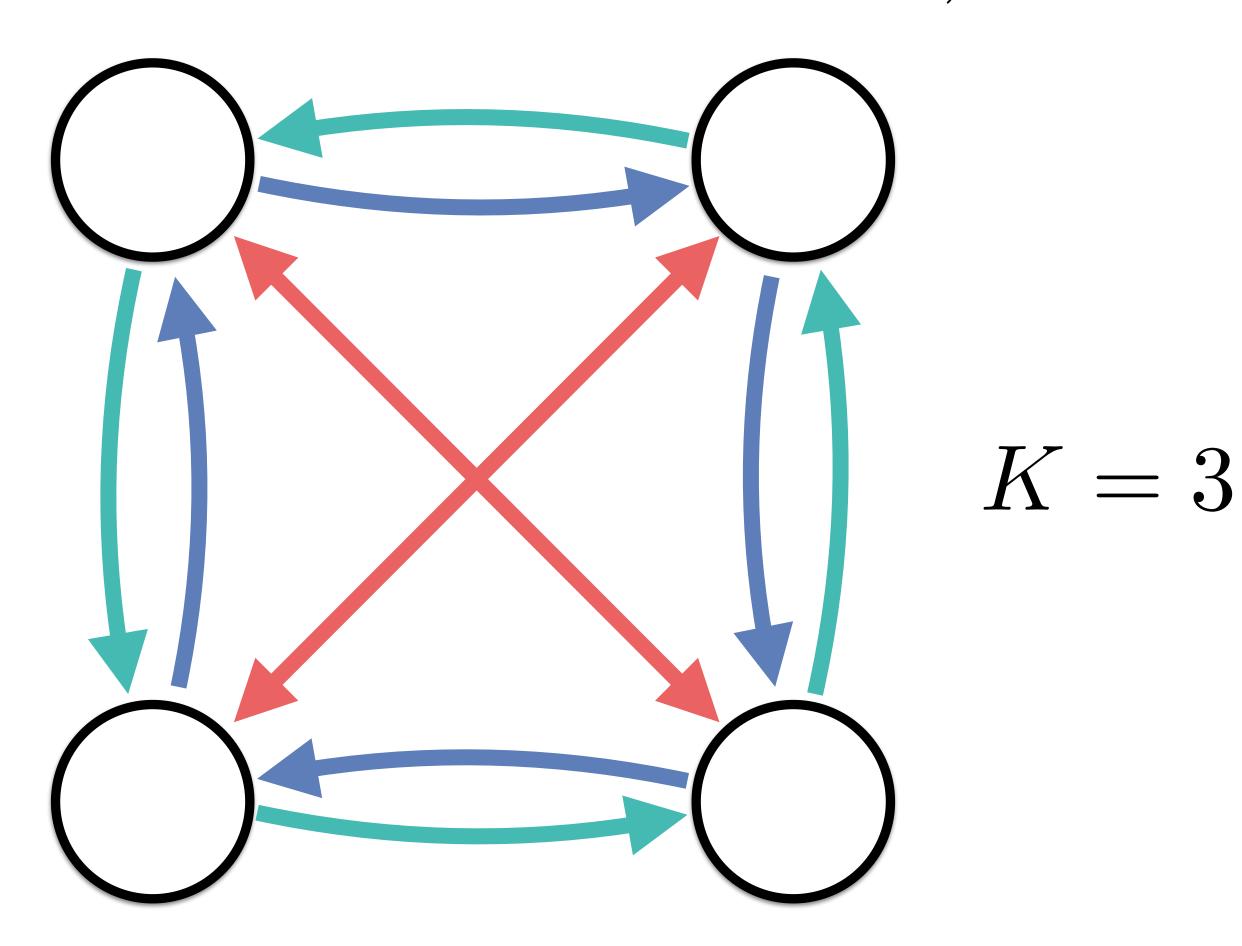


Symmetry of stability landscape

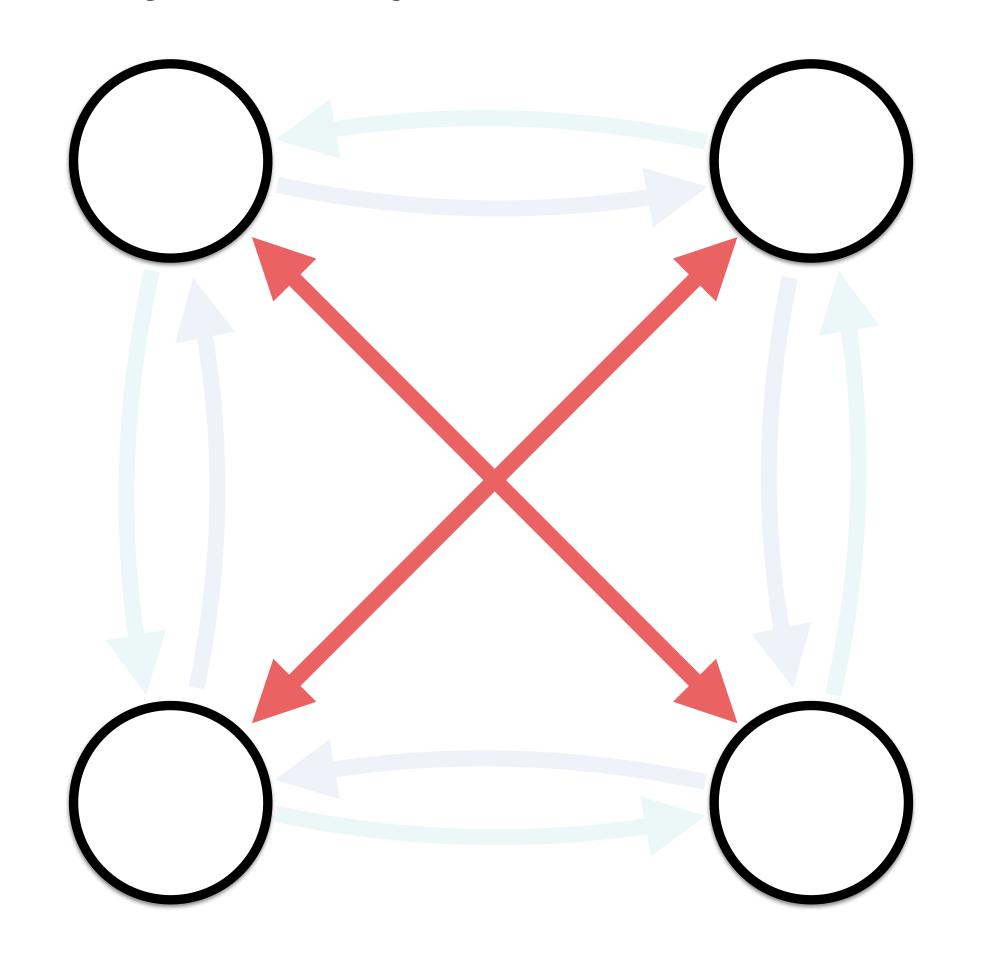


How often does this occur?

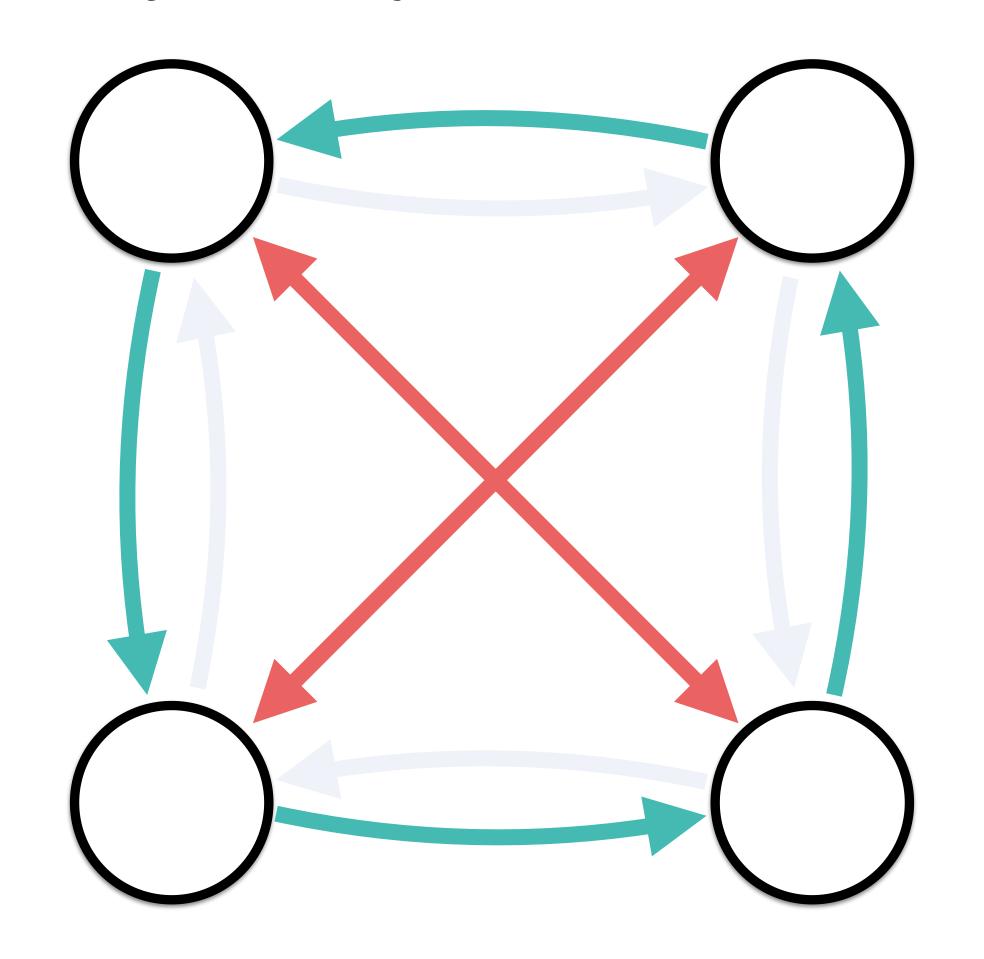
Adjacency matrices $A^{(\alpha)}$, $\alpha = 1, \ldots, K$



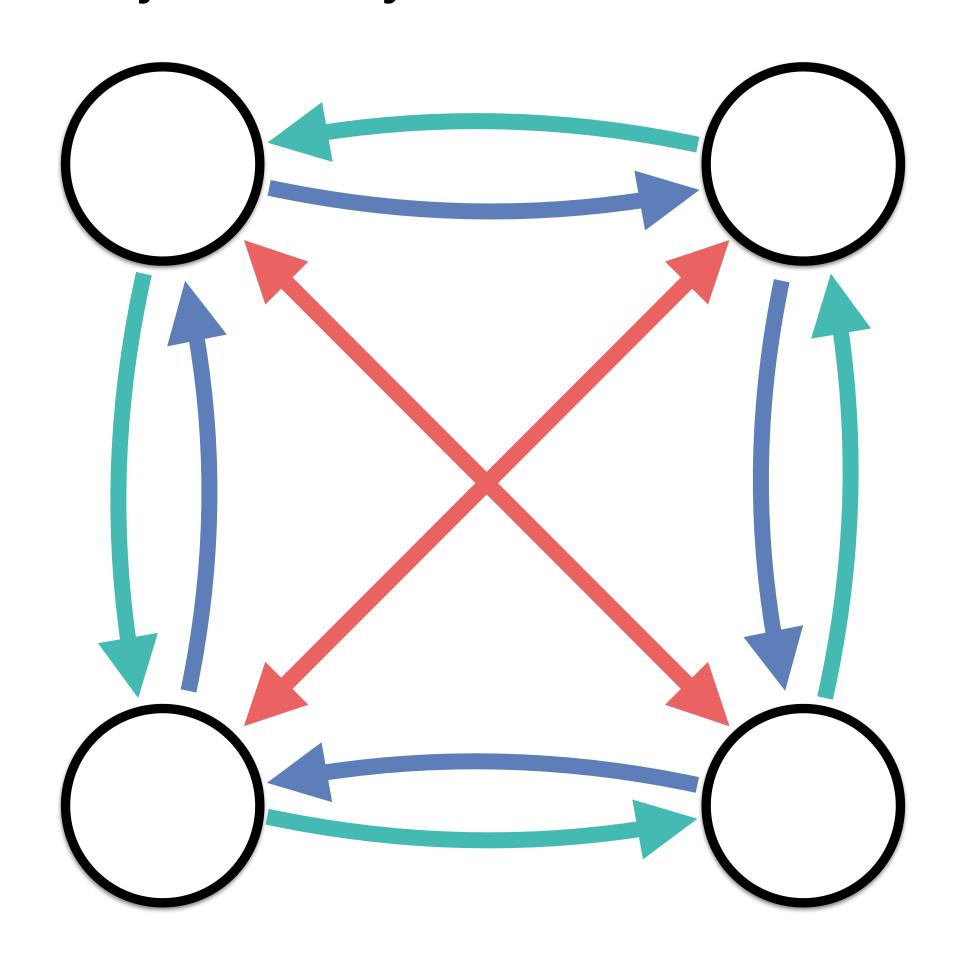
Adjacency matrices $A^{(1)}$



Adjacency matrices $A^{(1)}$, $A^{(2)}$

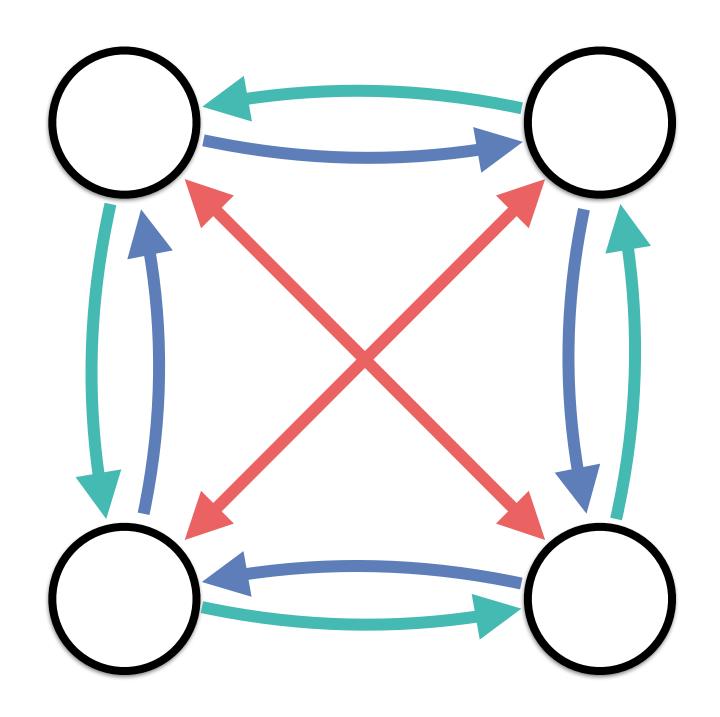


Adjacency matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$



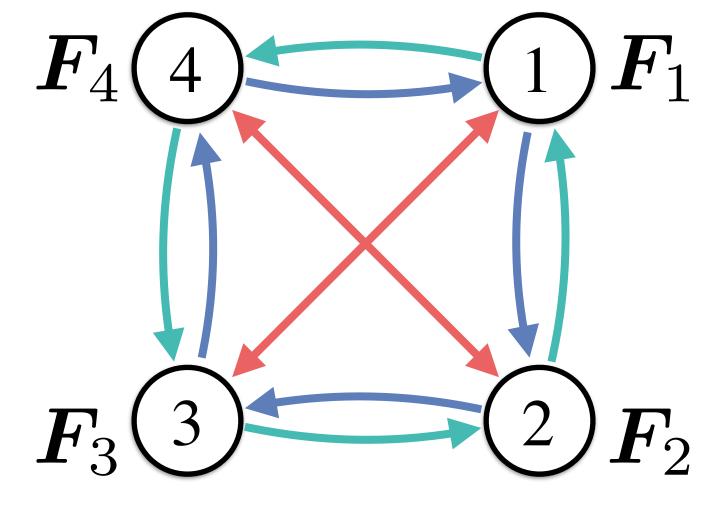
Symmetric network structures

- *Symmetric network*: every node can be mapped to any other node by some permutation of nodes without changing any $A^{(\alpha)}$.
- For undirected networks with a single link type, they are called *vertex-transitive graphs*.
- Includes *circulant graphs*, defined as a network whose nodes can be arranged in a ring so that the network is invariant under rotations.



Example of symmetric network (circulant graph)

 $oldsymbol{X}_i = F_i(oldsymbol{X}_i)$: dynamics of isolated node i

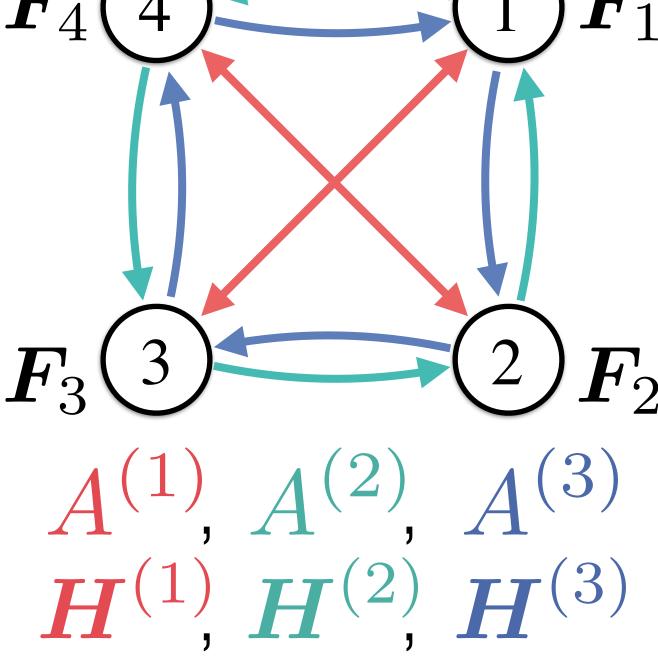


Network of non-identical oscillators

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{lpha=1}^K \sum_{\substack{i'=1 \ i'
eq i}}^N A_{ii'}^{(lpha)} \boldsymbol{H}^{(lpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$

 $A_{ii'}^{(\alpha)}$: directed link of type lpha from node i' to node i

 $m{H}^{(lpha)}(m{X}_i,m{X}_{i'})$: coupling function

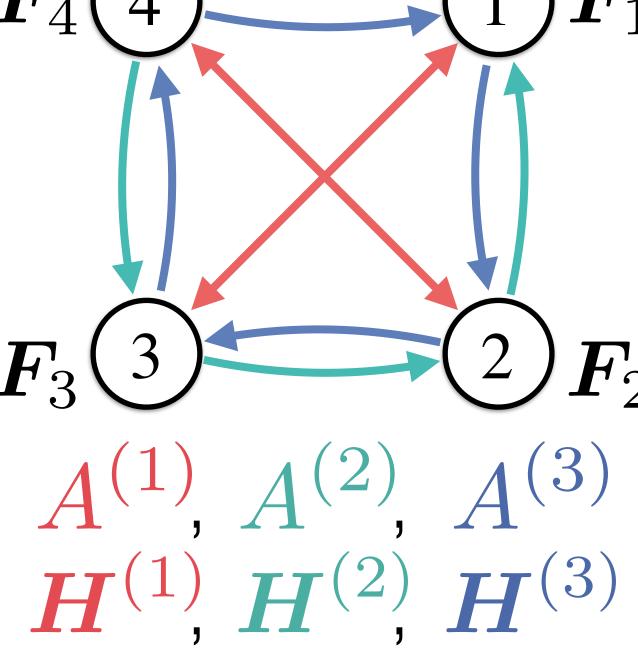


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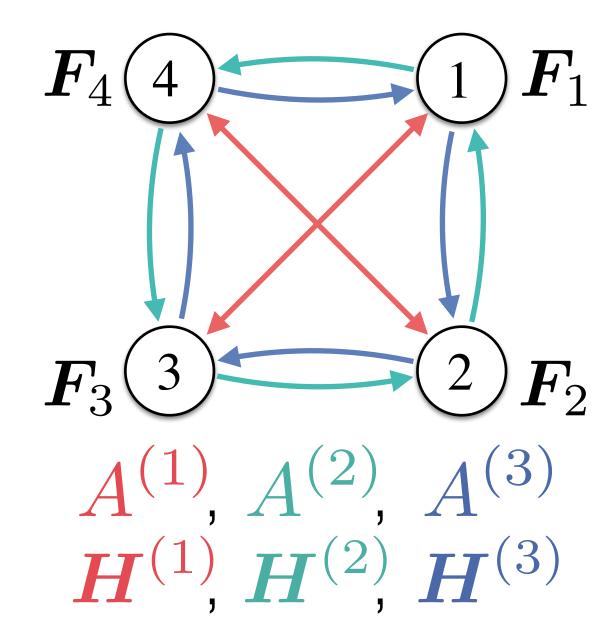


Defining asymmetry-induced synchronization

For a symmetric network

$$\dot{\boldsymbol{X}}_{i} = \boldsymbol{F}_{i}(\boldsymbol{X}_{i}) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1\\i'\neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_{i}, \boldsymbol{X}_{i'})$$

with completely synchronous state,



Conditions

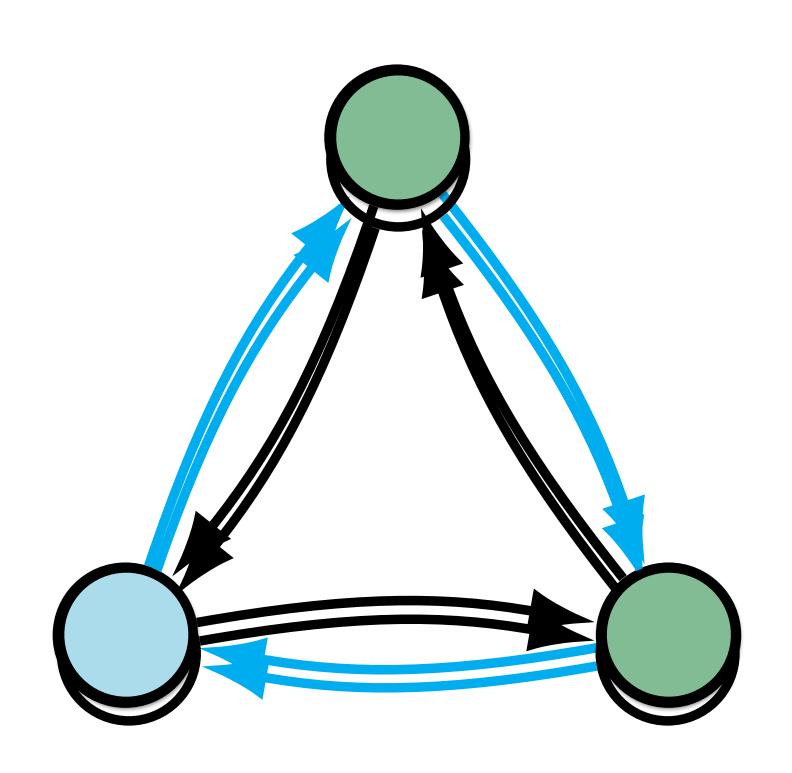
1. Synchronous state is unstable for any homogeneous system.

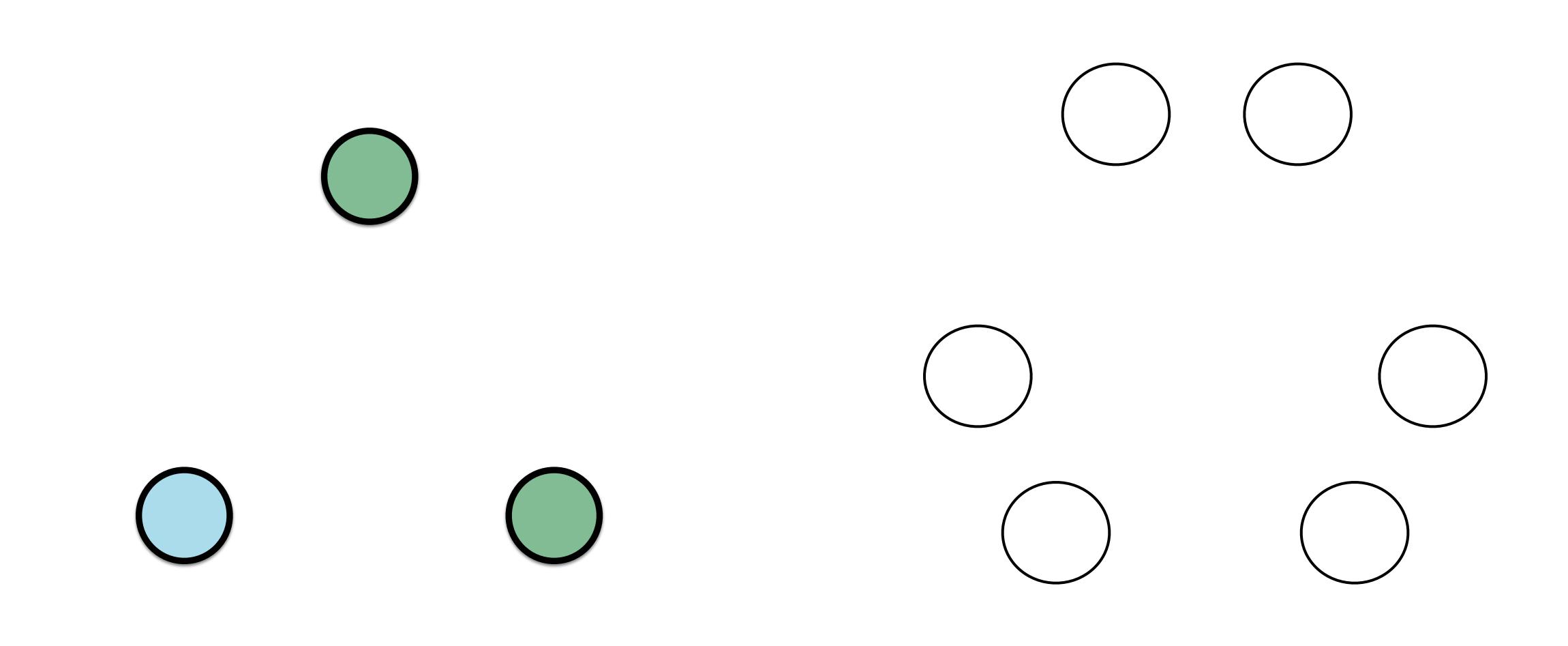
$$F_1 = \cdot \cdot \cdot = F_N$$

2. Synchronous state is stable for some heterogeneous system. $F_i \neq F_{i'}$ for some $i \neq i'$

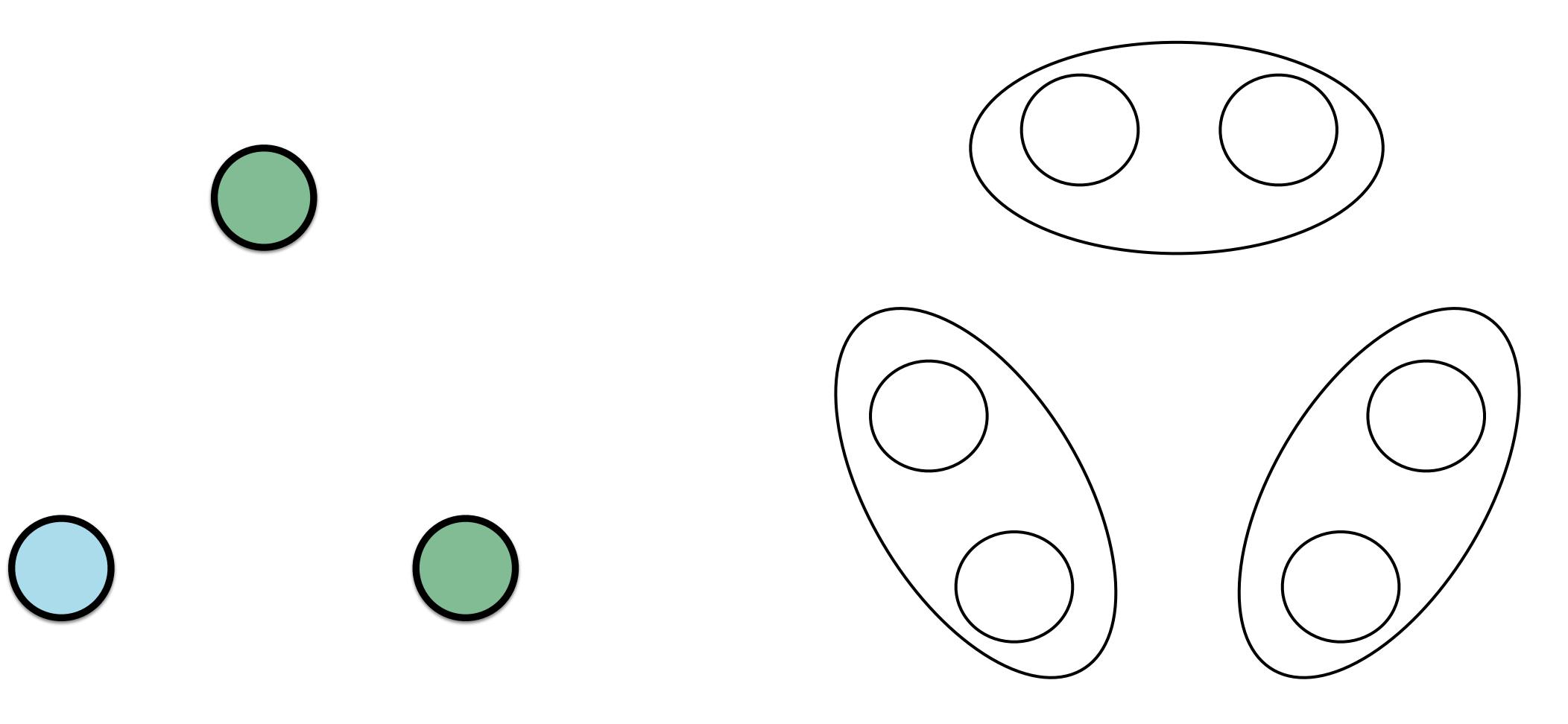
$$\dot{\boldsymbol{X}}_{i} = \boldsymbol{F}_{i}(\boldsymbol{X}_{i}) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1\\i'\neq i}}^{K} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_{i}, \boldsymbol{X}_{i'})$$

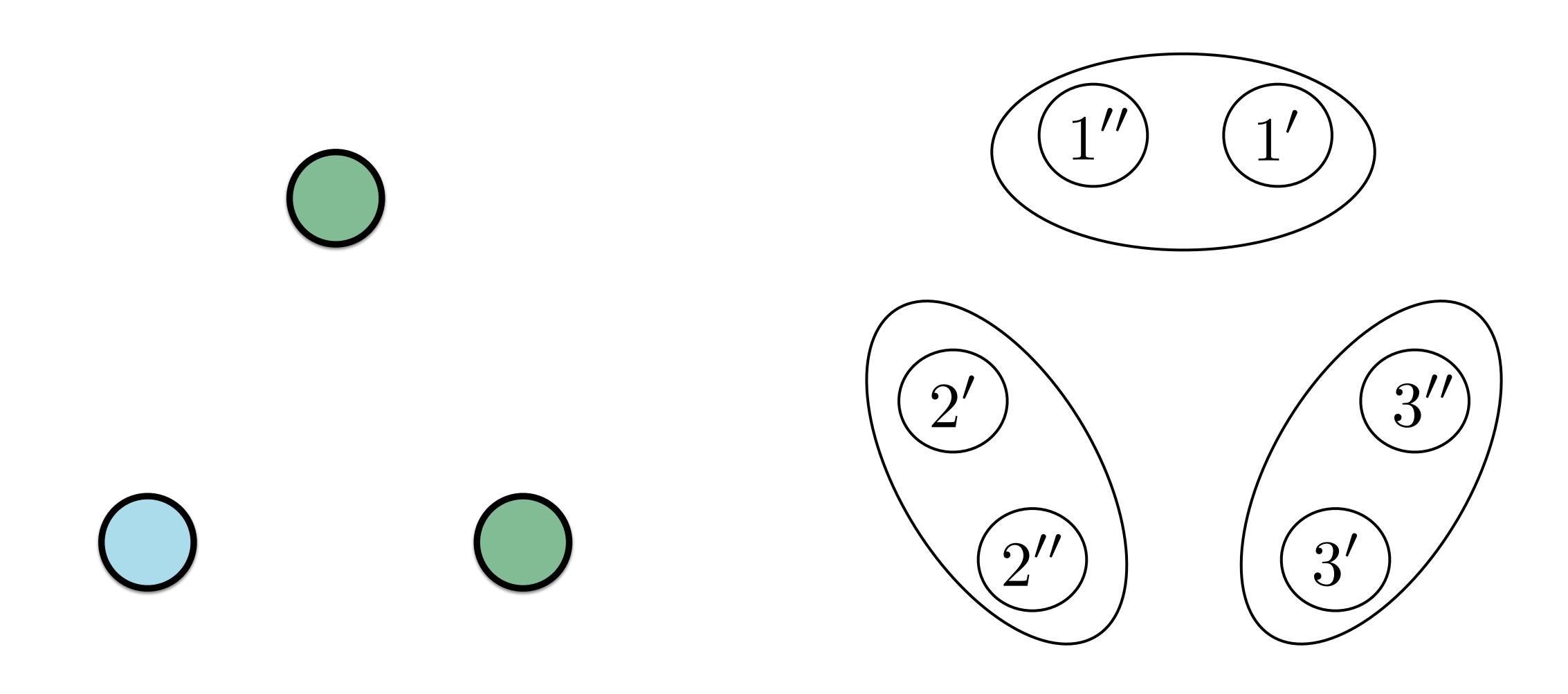
Class of multilayer systems



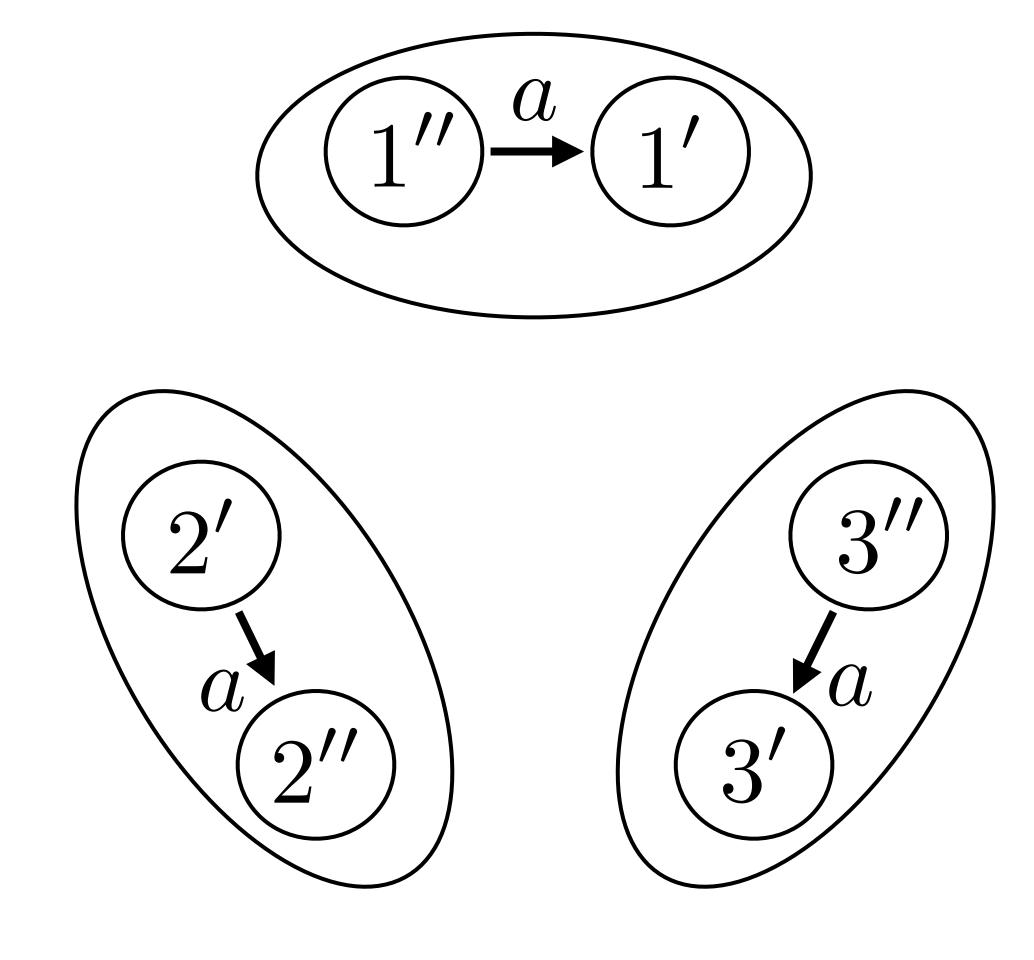


2 subnodes for each node

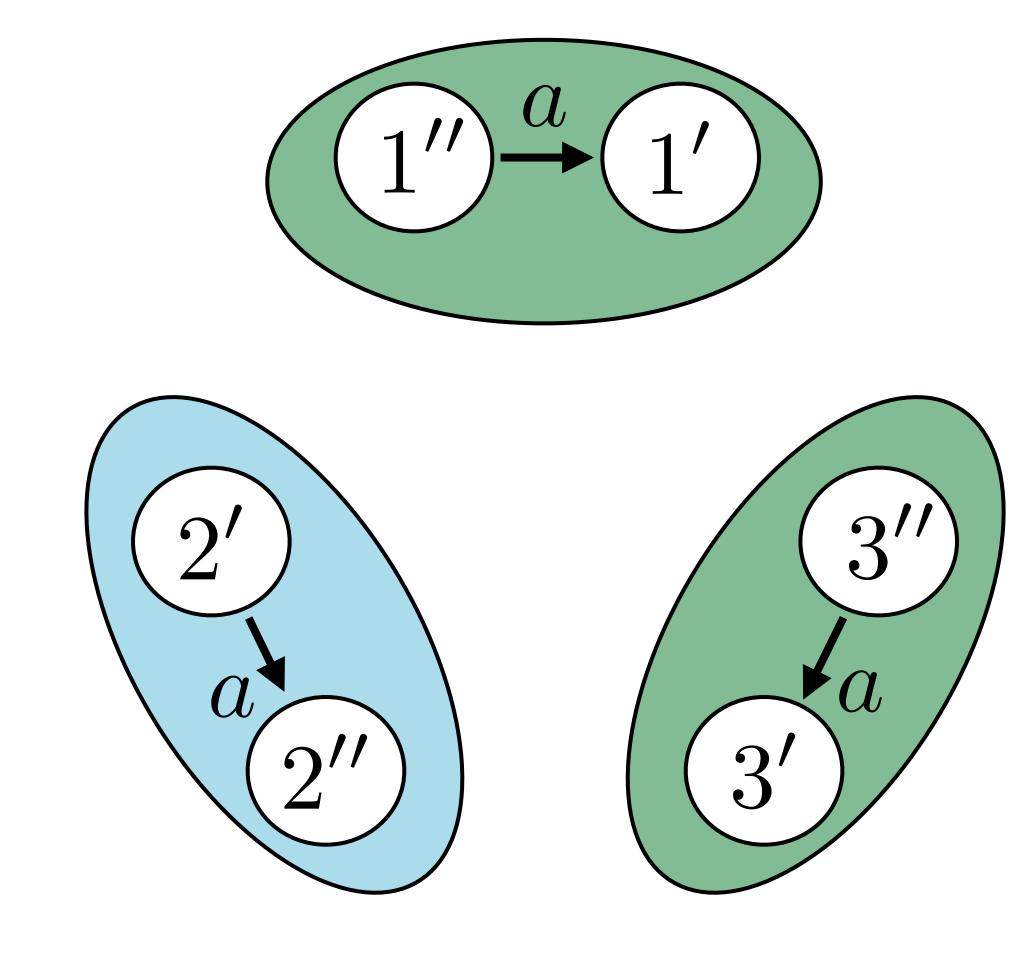


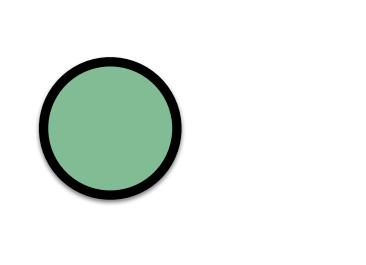


Internal links with strength a



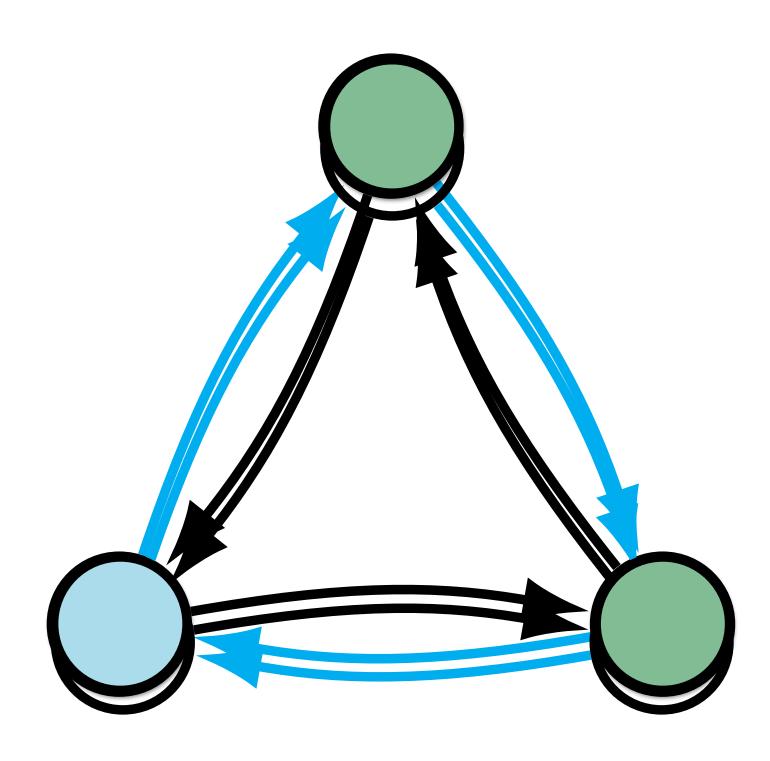
Internal link pattern defines node type

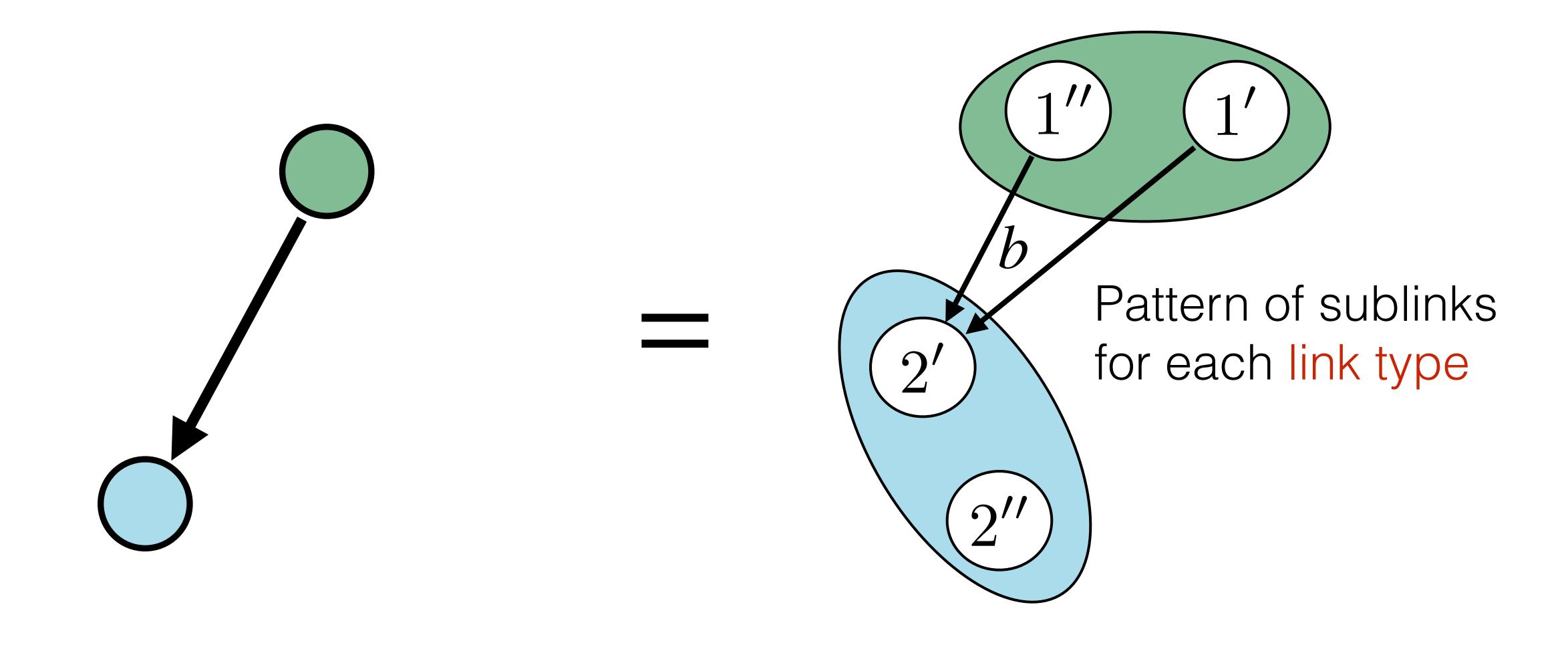


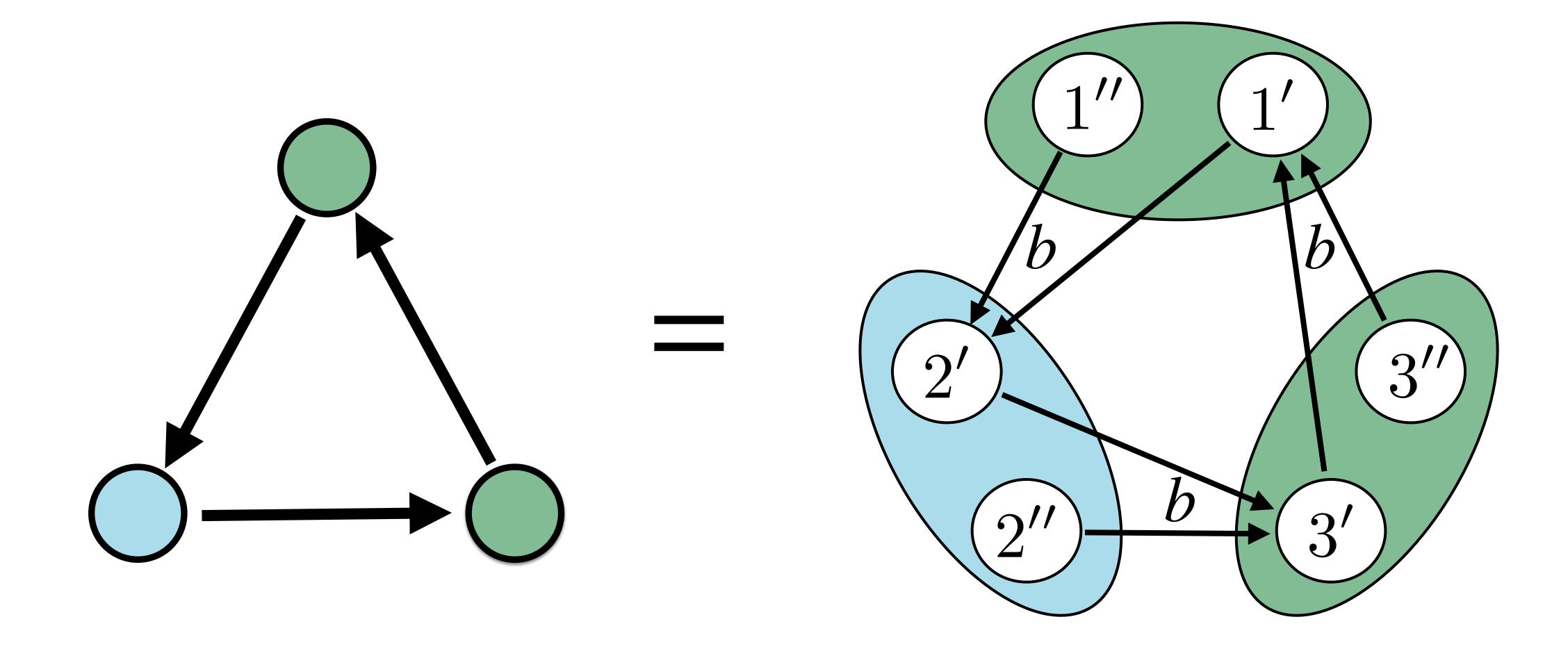


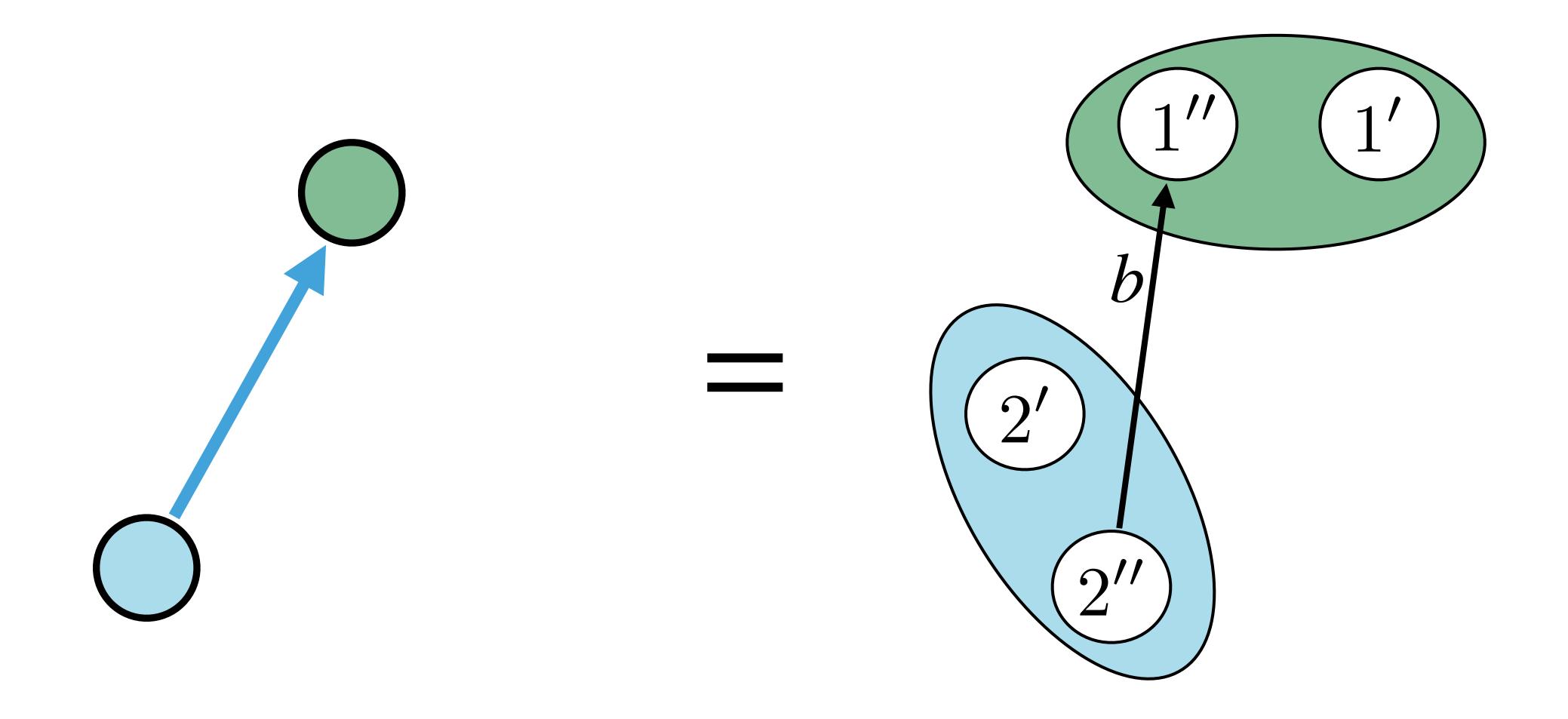


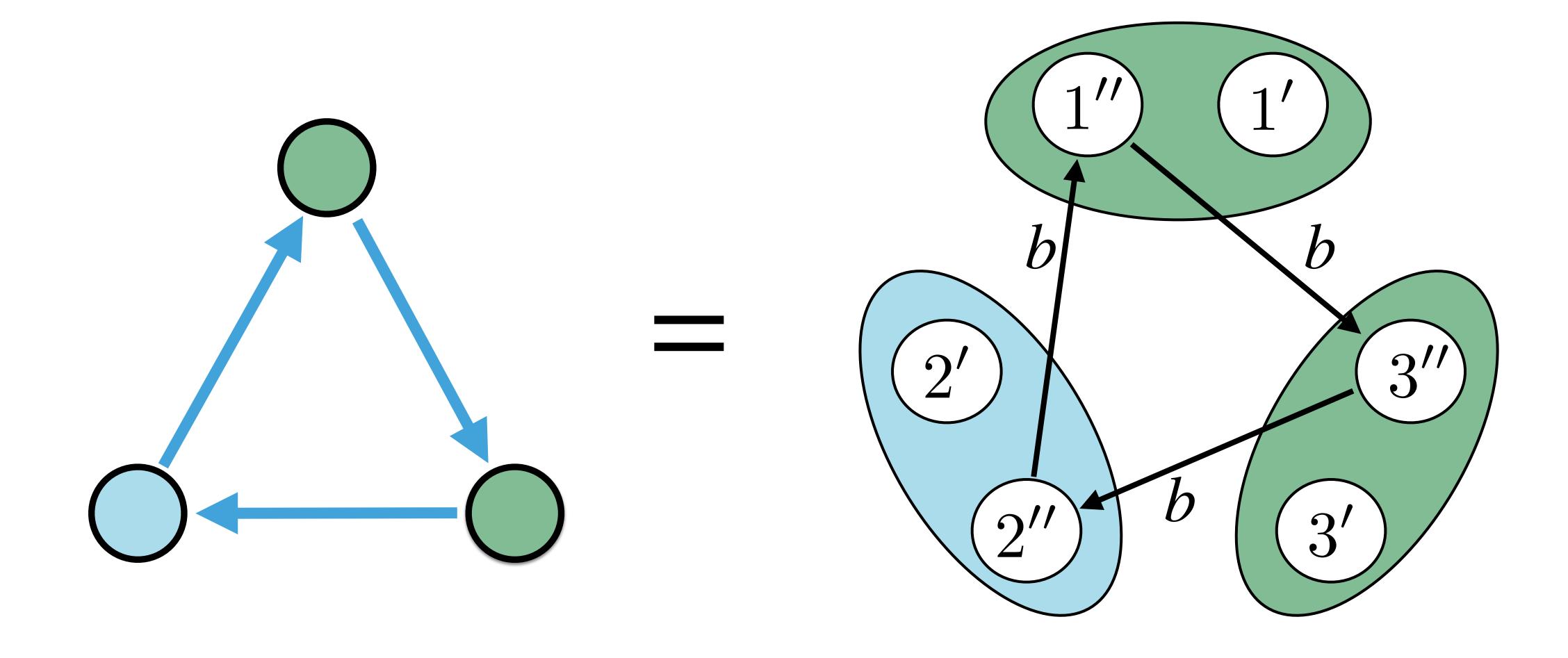
External links?

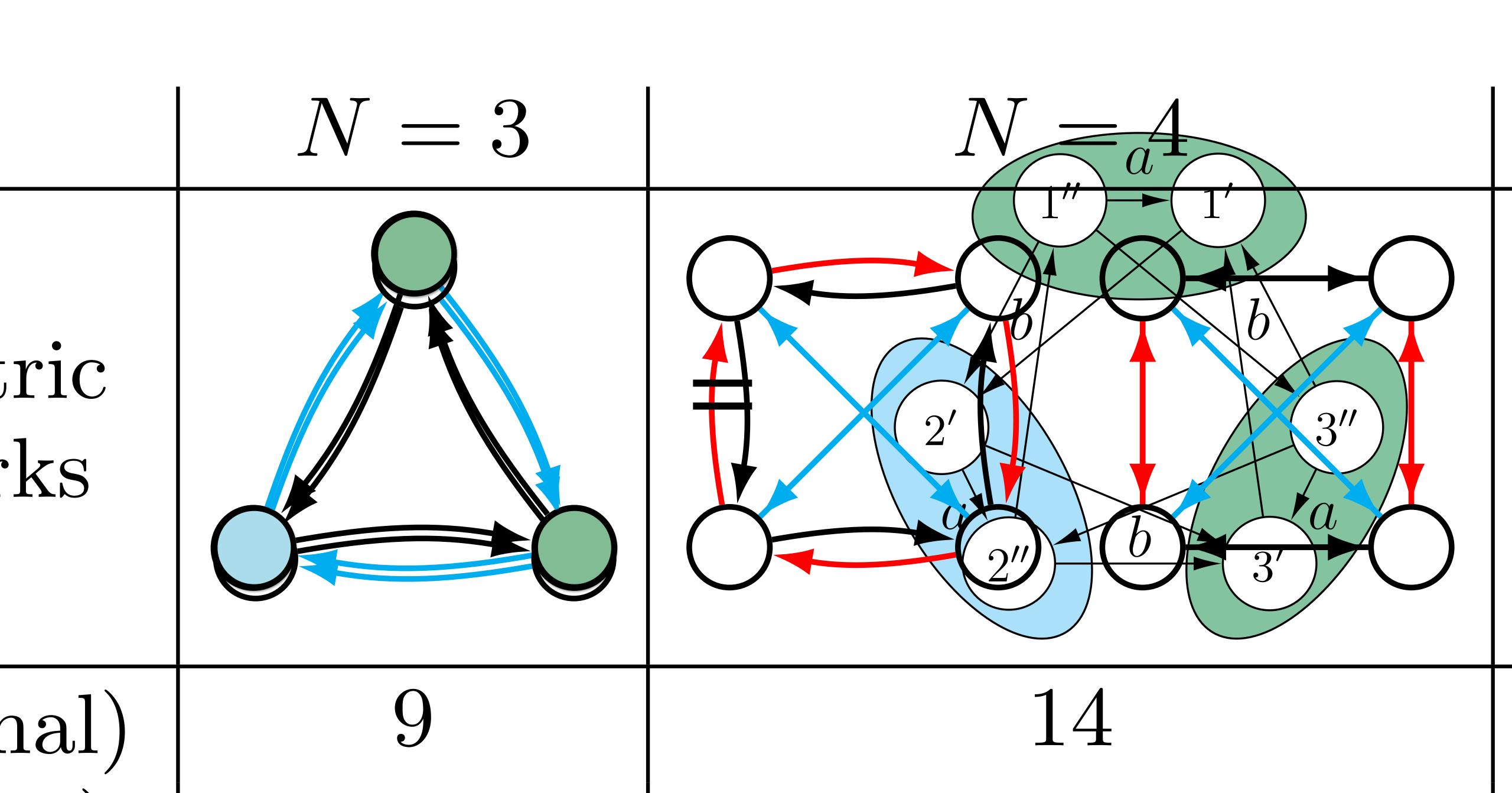


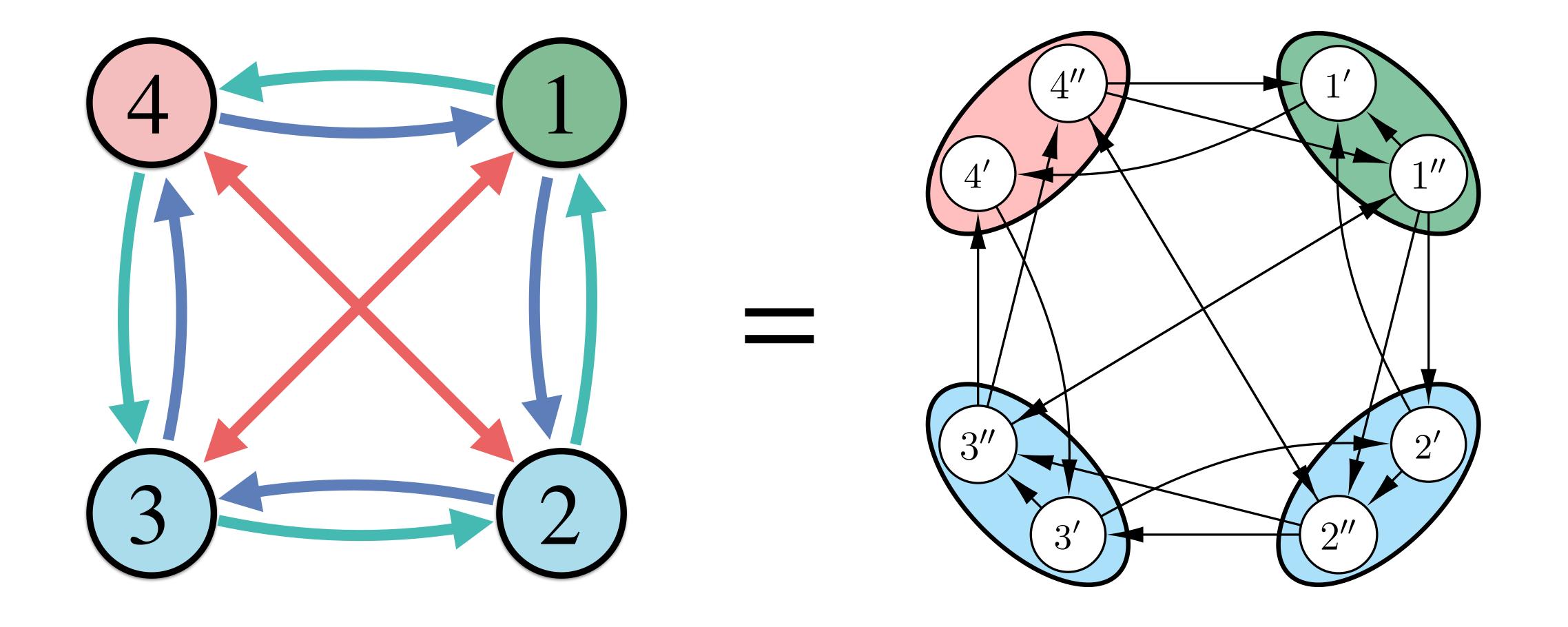


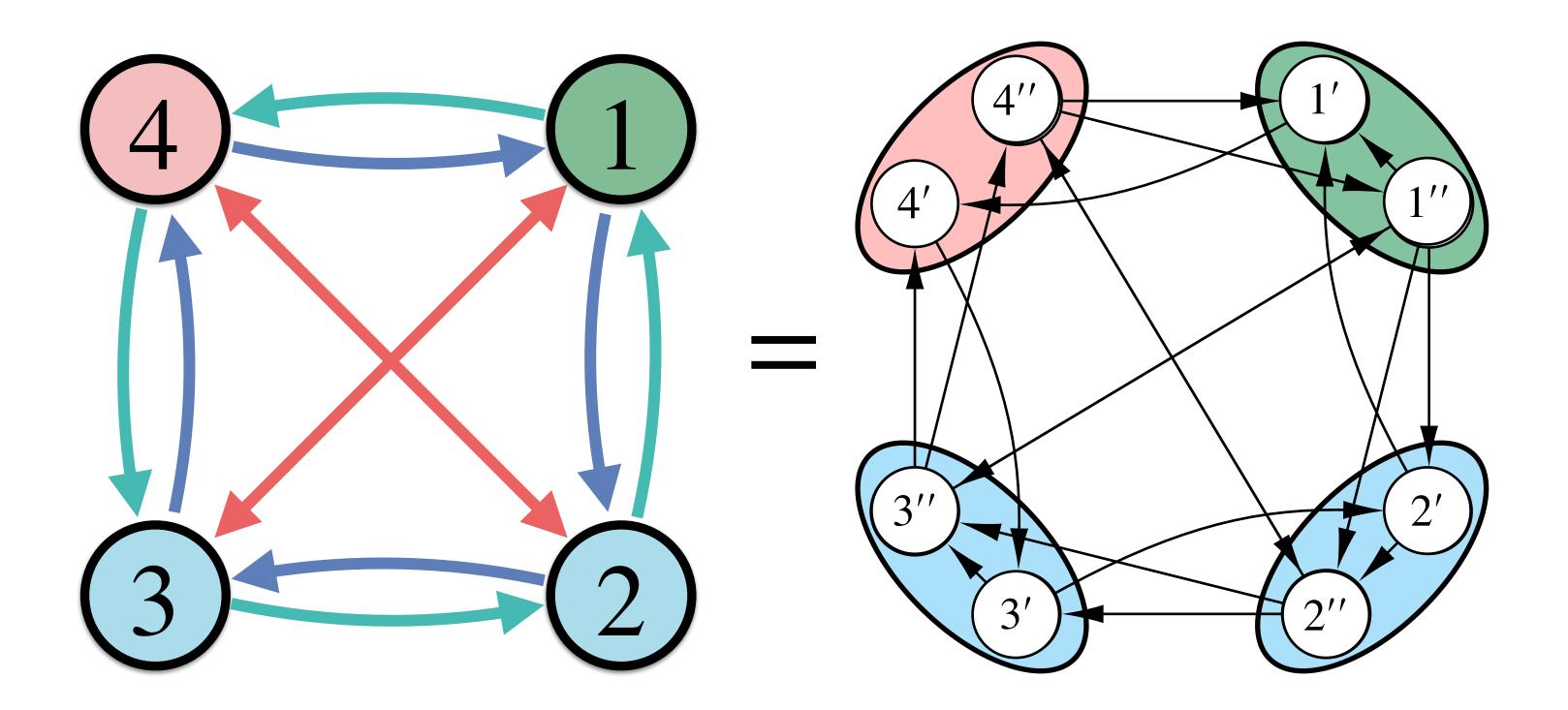




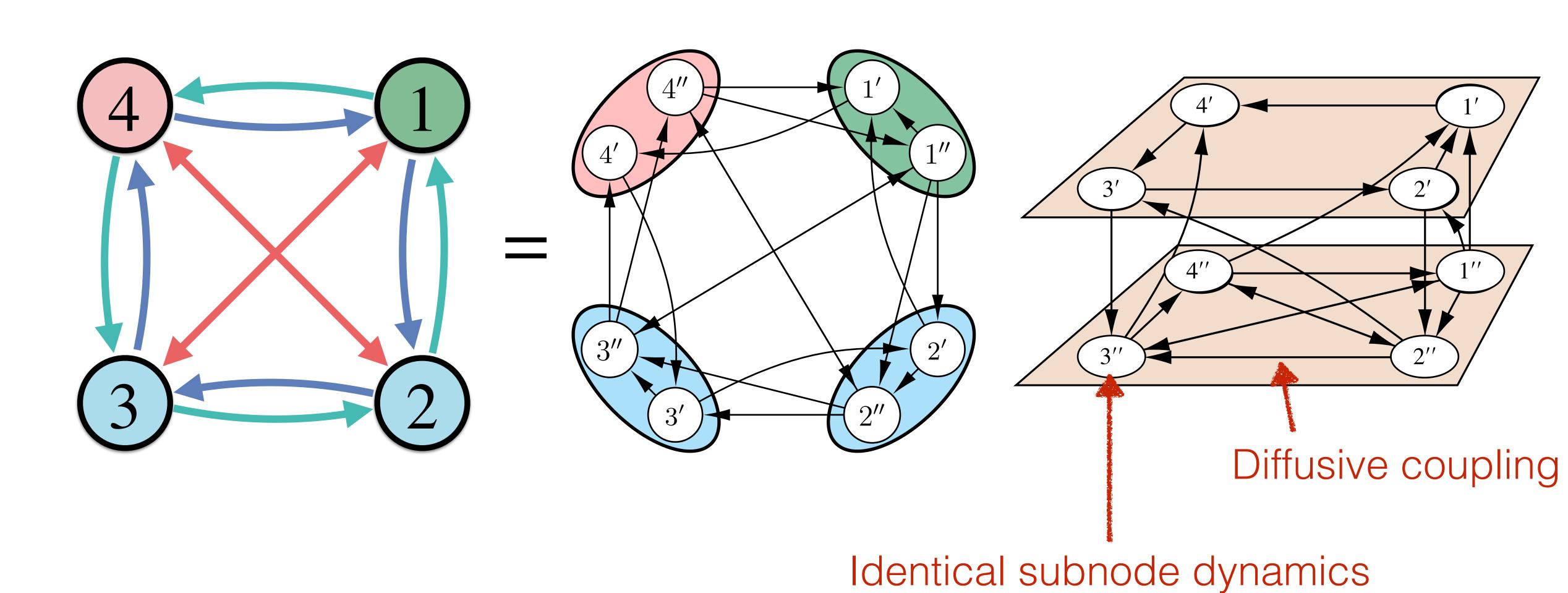








Multilayer network



termines the dynamics of every isolated subnode, and \boldsymbol{h} Multilaxeful fixed function common to all sublinks. Here Multilaxeful fixed sublinks the interaction function common to all sublinks. Here Multilaxeful fixed sublinks is the interaction function common to all sublinks. Here Multilaxeful fixed sublinks is the interaction function common to all sublinks. Here is the interaction function common to all sublinks. Here is the interaction function common to all sublinks. Here is the interaction function common to all sublinks. Here is the interaction function common to all sublinks. Here is the interaction function common to all sublinks. Here is the interaction function common to all sublinks. Here is the interaction function common to all sublinks. corresponding coupling matrix $\widetilde{A}^{(ii')} := (\widetilde{A}^{(ii')}_{\ell\ell'}), i \neq i',$ is the same and encodes the subnode connection pattern for that link type In contrast, the subnode connection pattern within each hode is encoded in the matrix Tote that the node-to-mode interactions are not necessarily diffusive, but the submode-to-subnode interactions are diffusive. This guarantees the existence of a^i synchropious state of $\widetilde{E}_{\ell'}^{(ii')}(2)$ hein by $a^{(i)}(t)$ $\Rightarrow s(t)$, $\forall i, \ell \text{ with } \dot{s} = f(s)$, which corresponds to a synchronous state of Eq. (1). Thus, we have a general class of multi-open electric networks that admit complete • Stahility beardily on that estimation, Master Stability Function ils). To facilitate the stability Caralysis, we featen 1996 1996 1-• Vallay for arbitrary fapor sentation into a single layer (see Fig. 1(c) for an example). By defining a single index

function which to a n nizatio totical MSF . for ve our m An

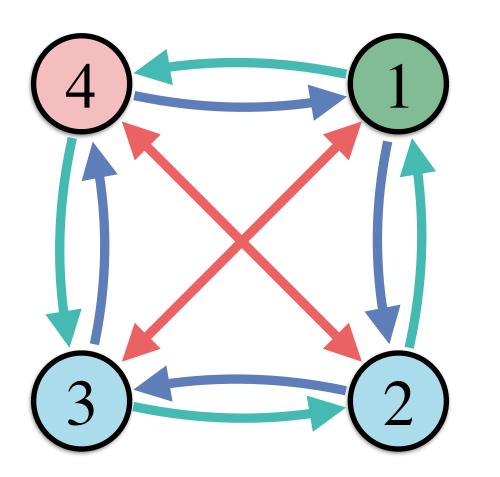
An Fig. 2

to link in Fig

of whi

 $\begin{array}{c} \text{mines} \\ \text{and } b \end{array}$

Network of nodes and links

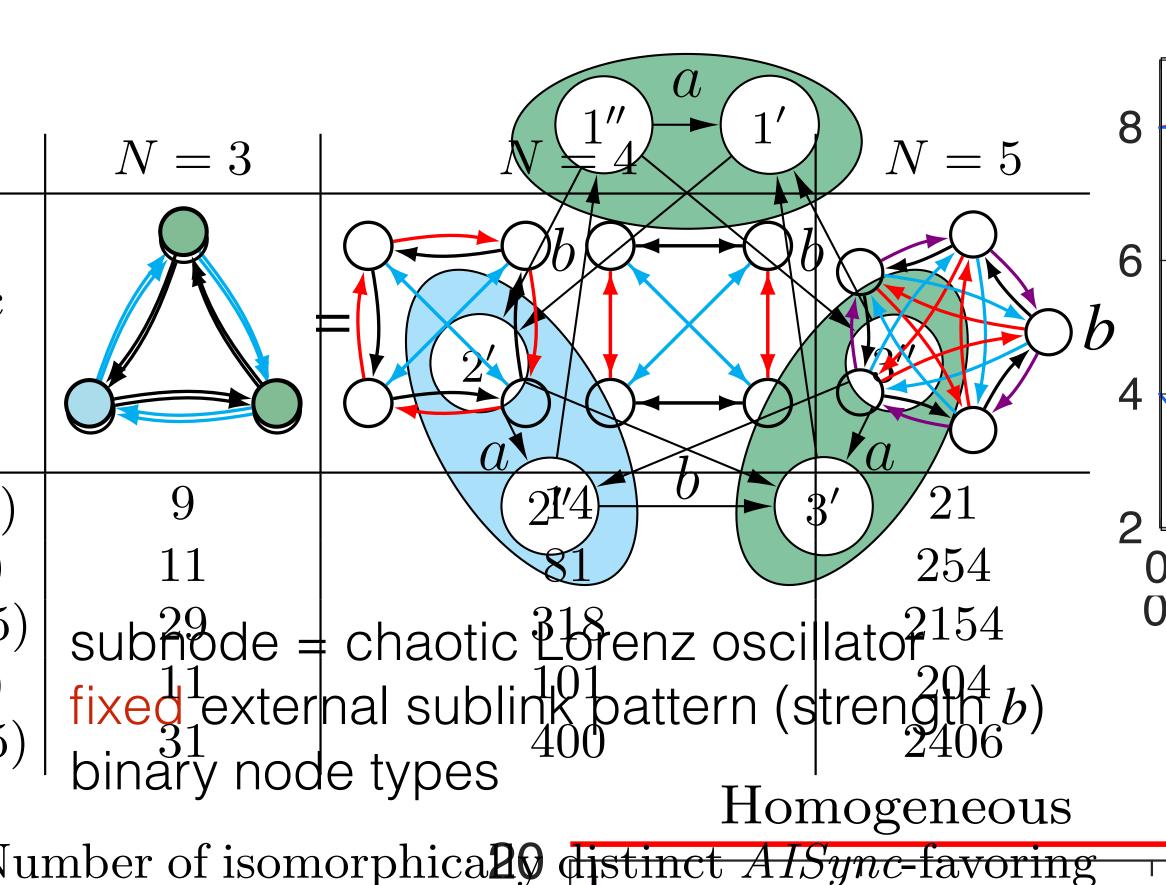


node dynamics is different if

internal link pattern is different

$$\dot{\boldsymbol{X}}_{i} = \boldsymbol{F}_{i}(\boldsymbol{X}_{i}) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1\\i'\neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_{i}, \boldsymbol{X}_{i'})$$

- Synchronous state is guaranteed
- Stability readily computed using Master Stability Function
 L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. 80, 2109 (1998)
- Valid for arbitrary f and h



th N=3,4,5 nodes and L=2 layers 2 (with 2" to enable counting). The numbers are given for (B) and quaternary (Q) choices of internal subtractions, as well as for different (M) constrained in the text. The network diagrams cossible symmetric network of a given size. See the constraints and network diagrams for larger N.

works if N is prime). Sampling uniformly emithins this class (see SM [26], Sec. ??, for details [30]), we observe that significant fraction of external sublink structures are Alsync-favoring over a range of sublink densities [Fig. 3(a)] and network sizes [Fig. 3(b)]. We also observe that sparse and dense structures favor Alsync more often than 5 nedim-1-density ones, despite the expectation that the effect (a internal sublink heterogeneity would be smaller washing mer than 5 nedim-1-density ones, despite the expectation that the effect (a internal sublink heterogeneity would be smaller washing mer than 5 nedim-1-density ones, despite the expectation that the effect (b internal sublink heterogeneity would be smaller washing mer than 5 nedim-1-density ones, despite the expectation that the effect (b internal sublink heterogeneity would be smaller washing mer than 5 nedim-1-density ones.)

Given a symmetric network of identical oscillators, it heterogeneous is instructive to compare our results above in which the symmetry is broken by making the oscillators nonidentical with the alternative scenario in which the symmetry is broken by making the network structure asymmetric. When the symmetry is broken by making the network structure asymmetric didentical oscillators, it can be shown that: 1) with the exception of the complete graphs, all topologies that optimize symmetric being the graphs, all topologies that optimize symmetric symmetry (i.e., those with $\sigma = 0$) are asymmetric asymmetric symmetry.

metric [31]; 2) any network topology that can be spanned

What about other symmetric networks?

AlSync strength *r* quantifies the degree to which a network structure favors AlSync.

- $r = 0 \Rightarrow \text{No AlSync}$
- Larger $r \Rightarrow$ Favors AlSync more strongly
- $r=1 \Rightarrow$ There is an optimal heterogenous system.

	N=3	N=4	N=5
symmetric networks			
All 4 types (optimal)	9	14	21
All 4 types $(r \gg 0.2)$	11	81	254
All 4 types $(r \gg 0.05)$	29	318	2154
Binary $(r > 0.2)$	11	101	204
Binary $(r > 0.05)$	31	400	2406
	•		

Node types I. Number of isomorphically distinct AISync-favoring systems with N=3,4,5 nodes and L=2 layers (with

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that

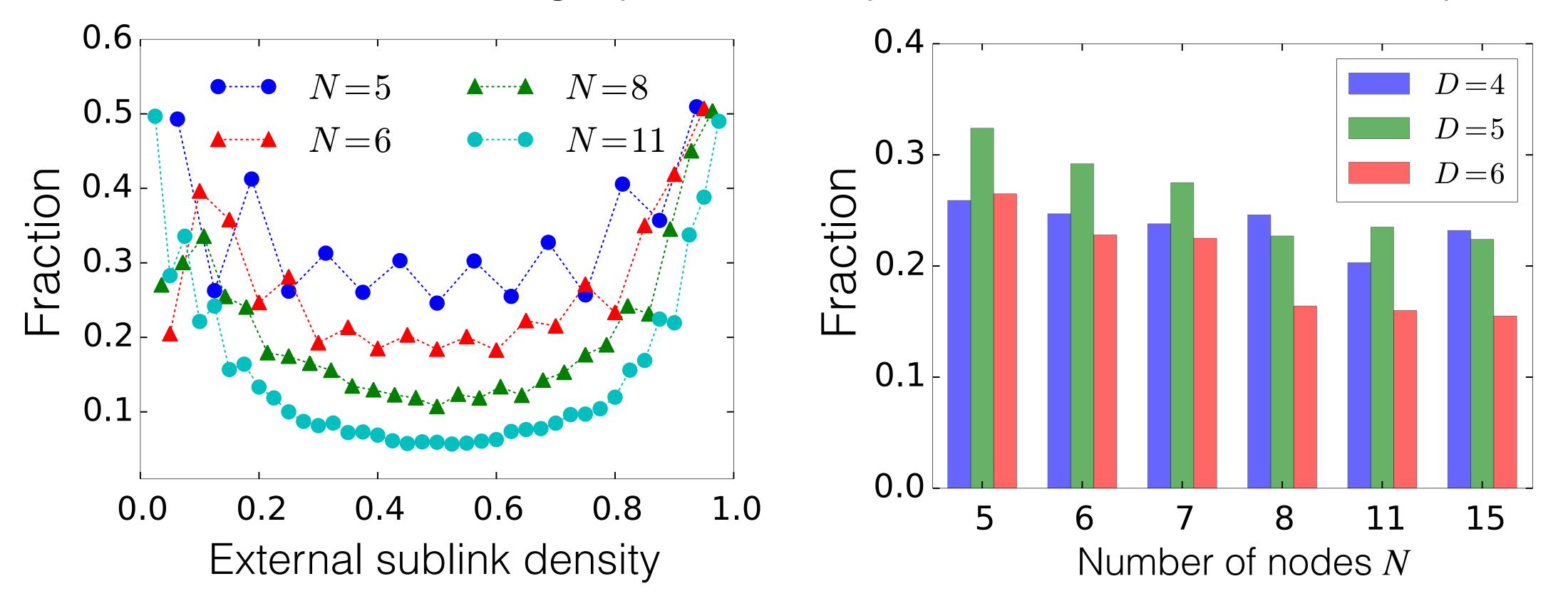
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is in

Sym cal s

Fraction of networks with r > 0.05

Within class of circulant-graphs (= all symmetric networks, if N is prime)



Significant fraction of systems are AlSync-favoring for a range of system parameters

Summary

Symmetric states requiring system asymmetry (converse of symmetry breaking)

- In network synchronization: fully synchronous state stable only when the oscillators are non-identical
- Observed quite often in the class of multilayer networks we considered