Symmetric States Requiring System Asymmetry

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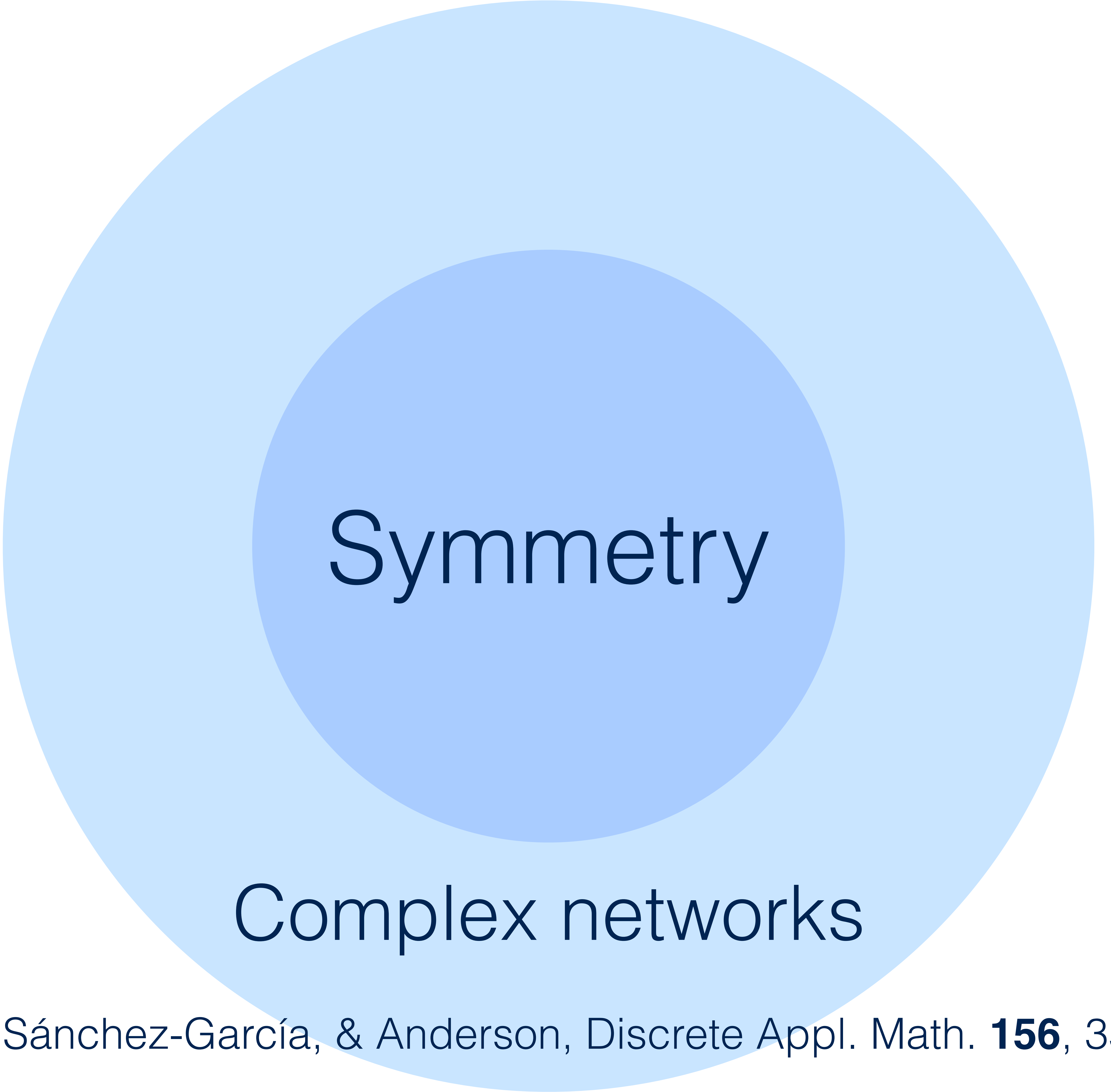
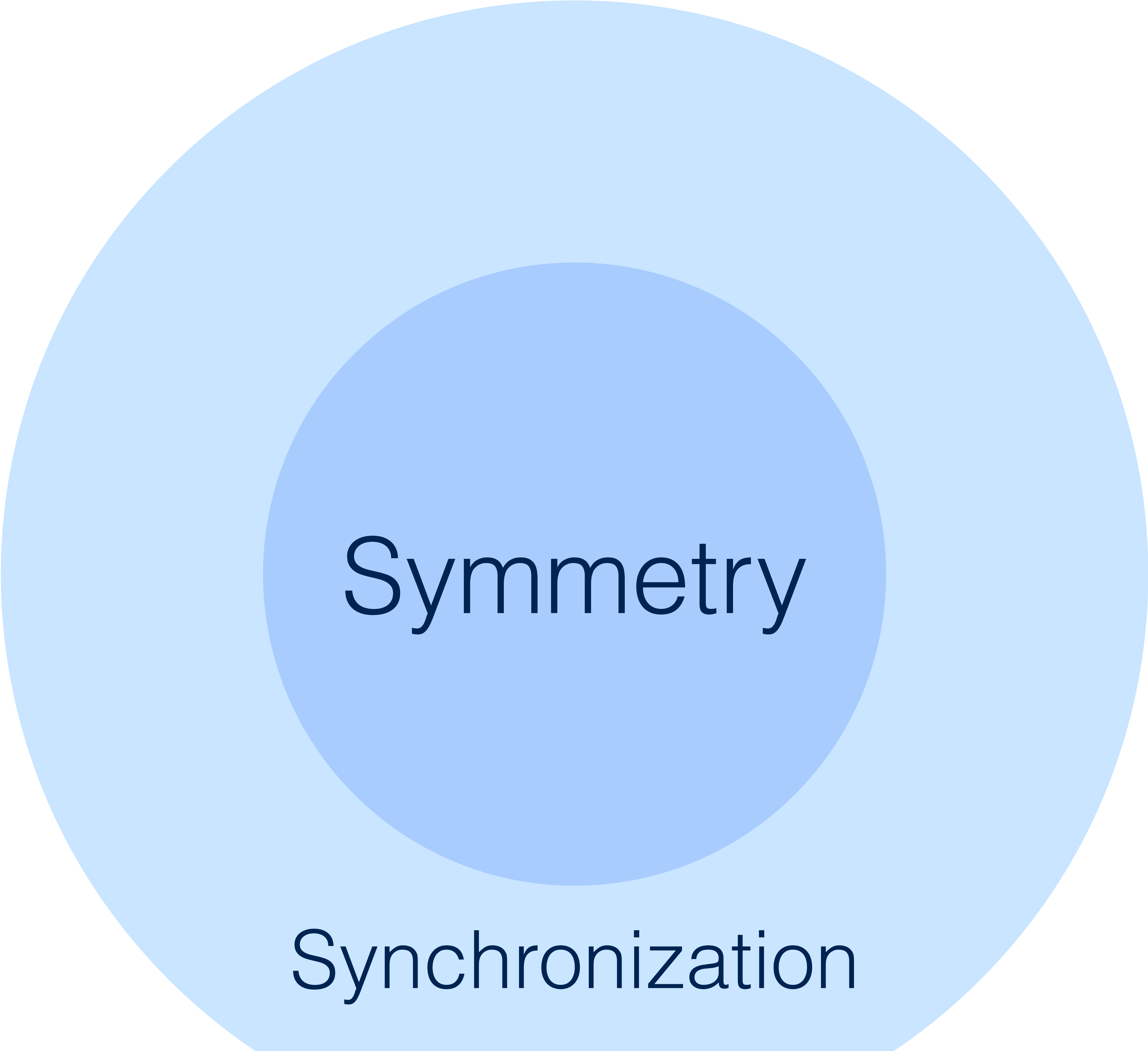
TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)
YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907

SIAM Conference on Applications of Dynamical Systems
MS142: Symmetry, Asymmetry, and Network Synchronization - May 25, 2017



MacArthur, Sánchez-García, & Anderson, Discrete Appl. Math. **156**, 3525 (2008)



Strogatz, *Sync: The Emerging Science of Spontaneous Order* (2003)
Pikovsky, Rosenblum, & Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (2003)

Symmetry

Network symmetry $\Longleftrightarrow$ dynamical symmetry

Golubitsky & Stewart, *The Symmetry Perspective* (2002)
Pecora, Sorrentino, Hagerstrom, Murphy, & Roy, Nat. Commun. **5**, 4079 (2014)

Symmetry

Symmetry breaking

Weyl, *Symmetry* (1952)
Golubitsky & Stewart, *The Symmetry Perspective* (2002)

Chimera states

Kuramoto & Battogtokh, Nonlinear Phenom. Complex Syst. **5**, 380 (2002)
Abrams & Strogatz, Phys. Rev. Lett. **93**,174102 (2004)
Tinsley, Nkomo, & Showalter, Nat. Phys. **8**, 662 (2012)
Hagerstrom, Murphy, Roy, Hovel, Omelchenko, & Schöll, Nat. Phys. **8**, 658 (2012)

Example: Network of n phase-amplitude oscillators

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i)$$

Parameters: $\omega = 1,\ \gamma = 0.1,\ \varepsilon = 2$

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)

Example: Network of n phase-amplitude oscillators

$$\dot{\theta}_i = \omega + r_i - 1$$

$$\dot{r}_i = b_i r_i (1 - r_i)$$

Parameters: $\omega = 1, \ \gamma = 0.1, \ \varepsilon = 2$

Limit cycle

$$\theta_i(t) \equiv \theta_0 + \omega t, \quad r_i(t) \equiv 1$$

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Synchronous state

$$\theta_1(t) = \cdots = \theta_n(t) \equiv \theta_0 + \omega t, \quad r_1(t) = \cdots = r_n(t) \equiv 1$$

uniform and symmetric

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)

Example: Network of n phase-amplitude oscillators

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

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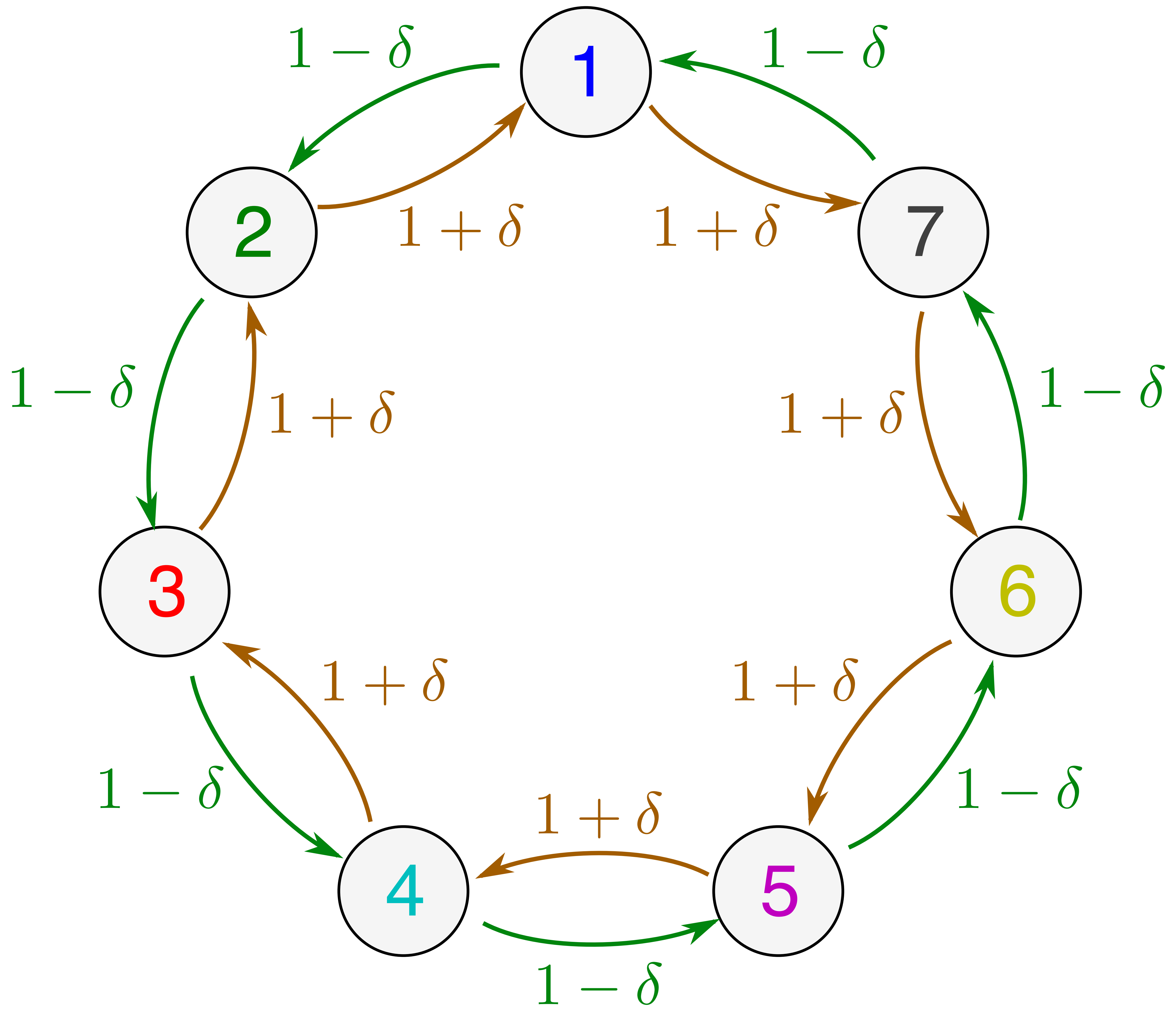
tunable oscillator parameters

Synchronous state

$$\theta_1(t) = \cdots = \theta_n(t) \equiv \theta_0 + \omega t, \quad r_1(t) = \cdots = r_n(t) \equiv 1$$

uniform and symmetric

Symmetric network structure



Coupling strength

$A_{ij} = 1+\delta$ or $1-\delta$

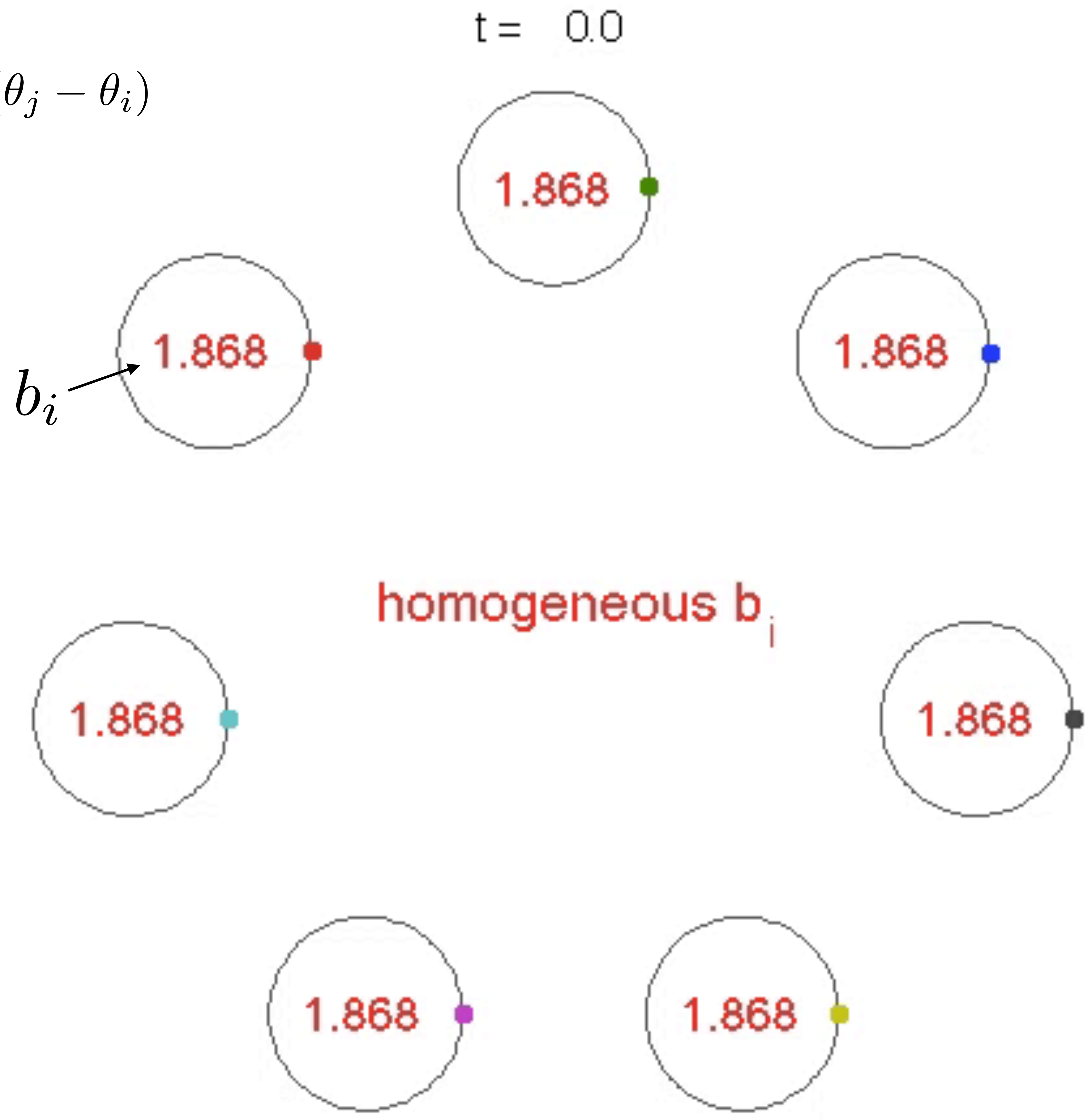
1.3 or 0.7

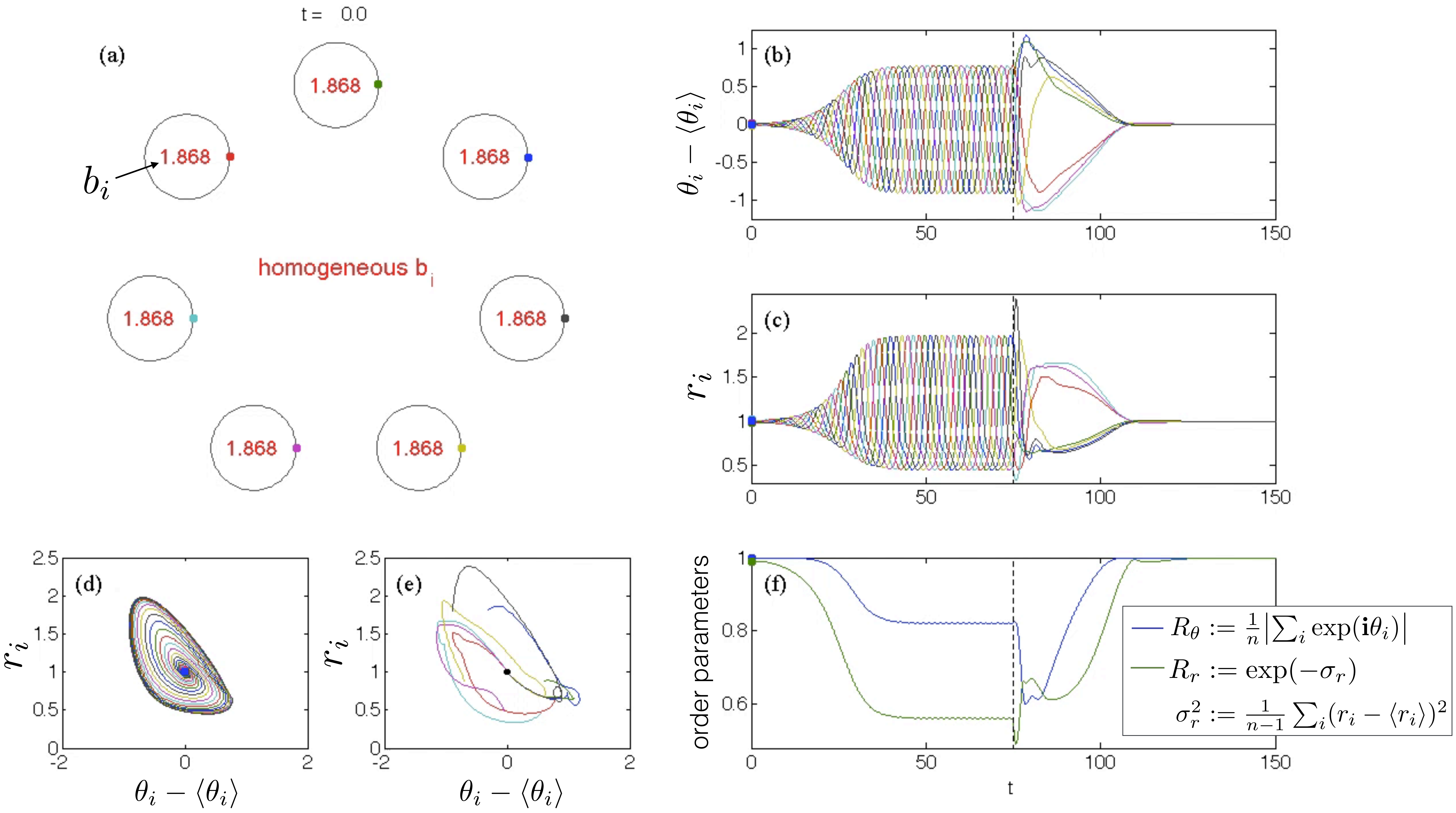
$(\delta = 0.3)$

$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i)$$

Best
homogeneous
b_i value





t = 0.0
(a)
b_i
1.868
homogeneous b_i
(b)
$\theta_i - \langle\theta_i\rangle$
(c)
r_i
(d)
(e)
(f)
order parameters
t
$R_\theta := \frac{1}{n}\left|\sum_i \exp(\mathbf{i}\theta_i)\right|$
$R_r := \exp(-\sigma_r)$
$\sigma_r^2 := \frac{1}{n-1}\sum_i (r_i - \langle r_i\rangle)^2$

Synchronization dynamics

Complete synchronization with nonidentical oscillators

Complete synchronization only with nonidentical oscillators

symmetric state

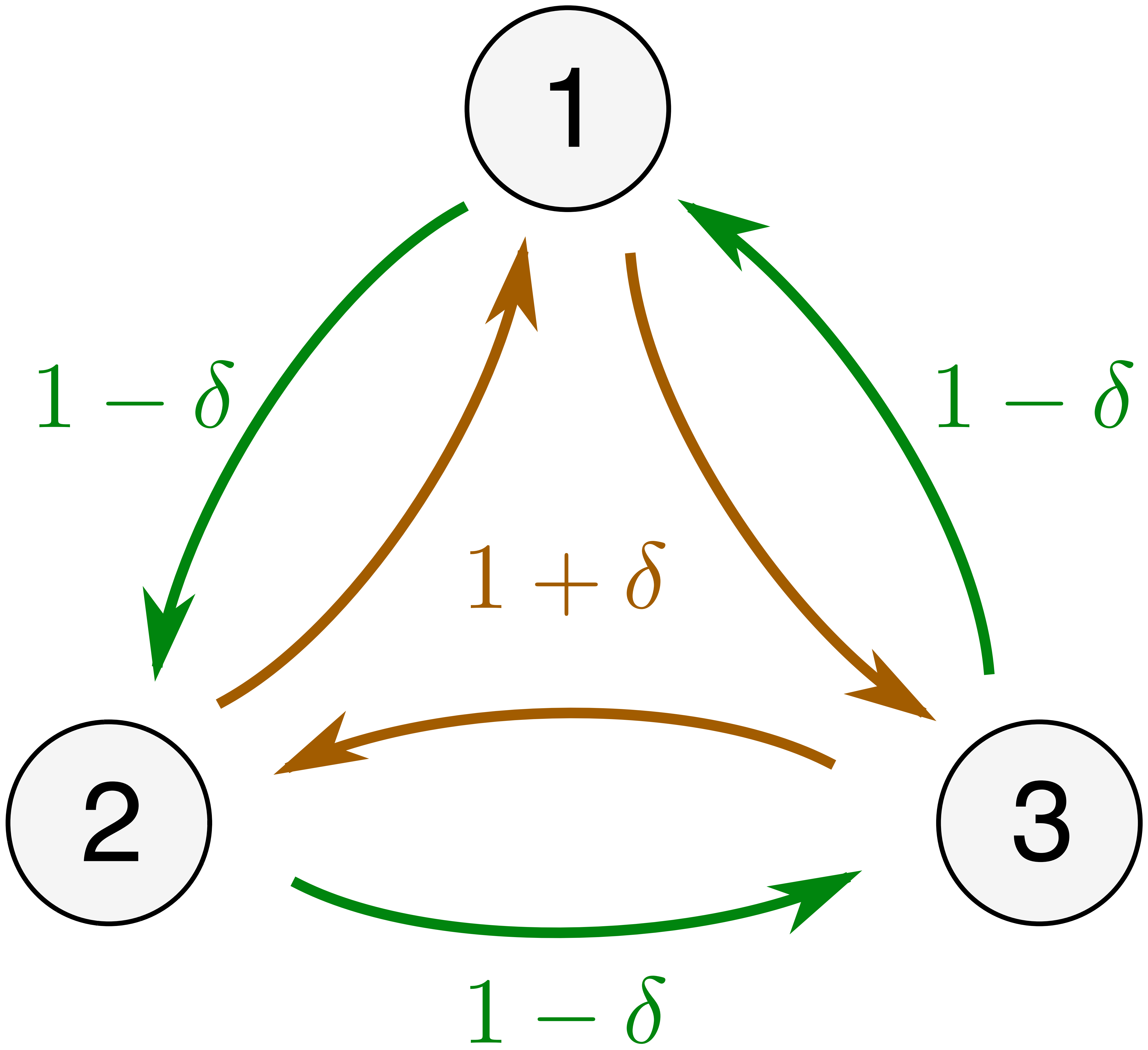
system asymmetry

"Symmetric states *requiring* system asymmetry"

We have a converse of symmetry breaking

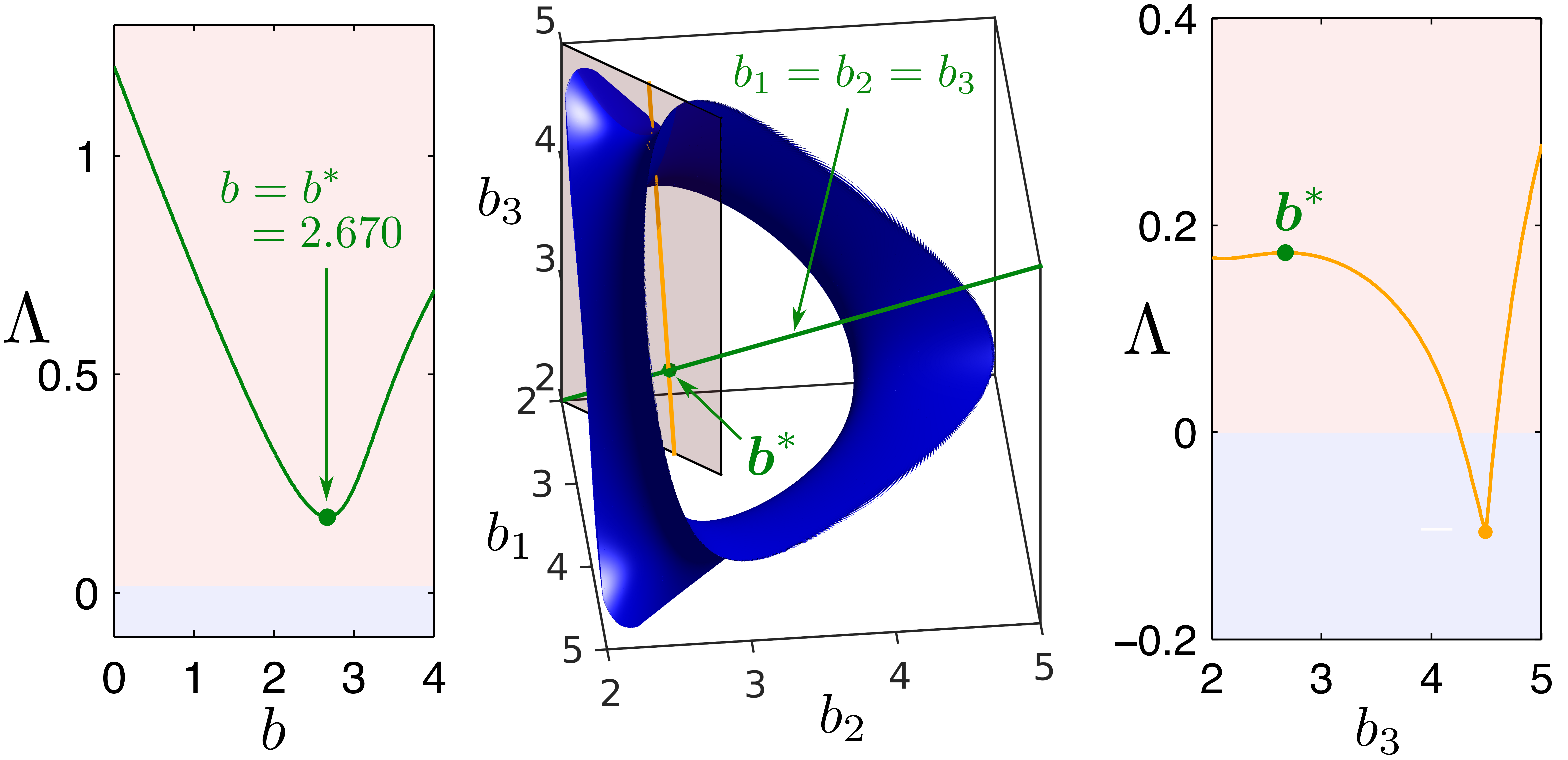
Symmetric **stable state** *requiring* **system** to be asymmetric

Symmetric **system** *requiring* **stable state** to be asymmetric
(symmetry breaking)


1
2
3
$1-\delta$
$1-\delta$
$1+\delta$
$1-\delta$

Stability landscape

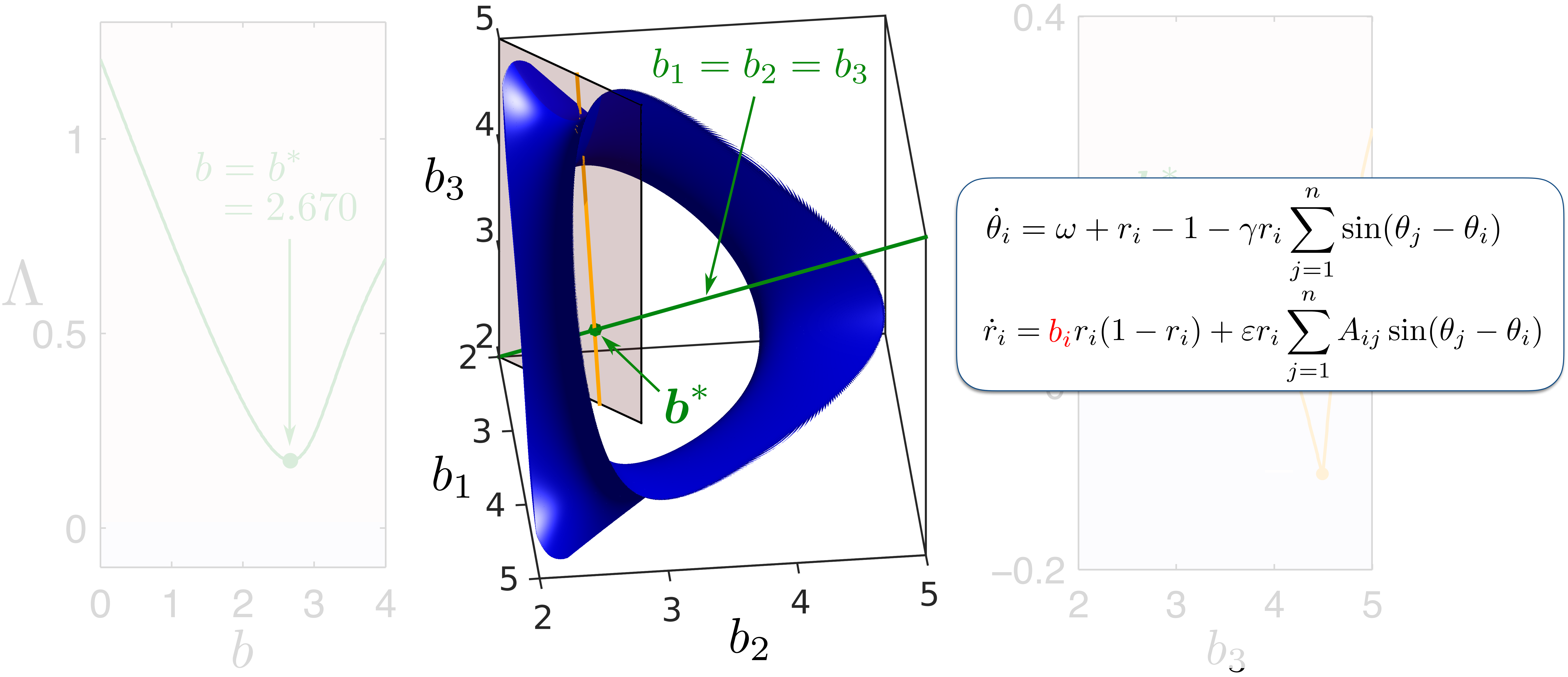
Λ = maximum transverse Lyapunov exponent



$$\gamma = 0.65,\ \varepsilon = 2,\ \delta = 0.3$$

Stability landscape

Λ = maximum transverse Lyapunov exponent



$$\dot{\theta}_i = \omega + r_i - 1 - \gamma r_i \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

$$\dot{r}_i = b_i r_i (1 - r_i) + \varepsilon r_i \sum_{j=1}^{n} A_{ij} \sin(\theta_j - \theta_i)$$

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Stability landscape

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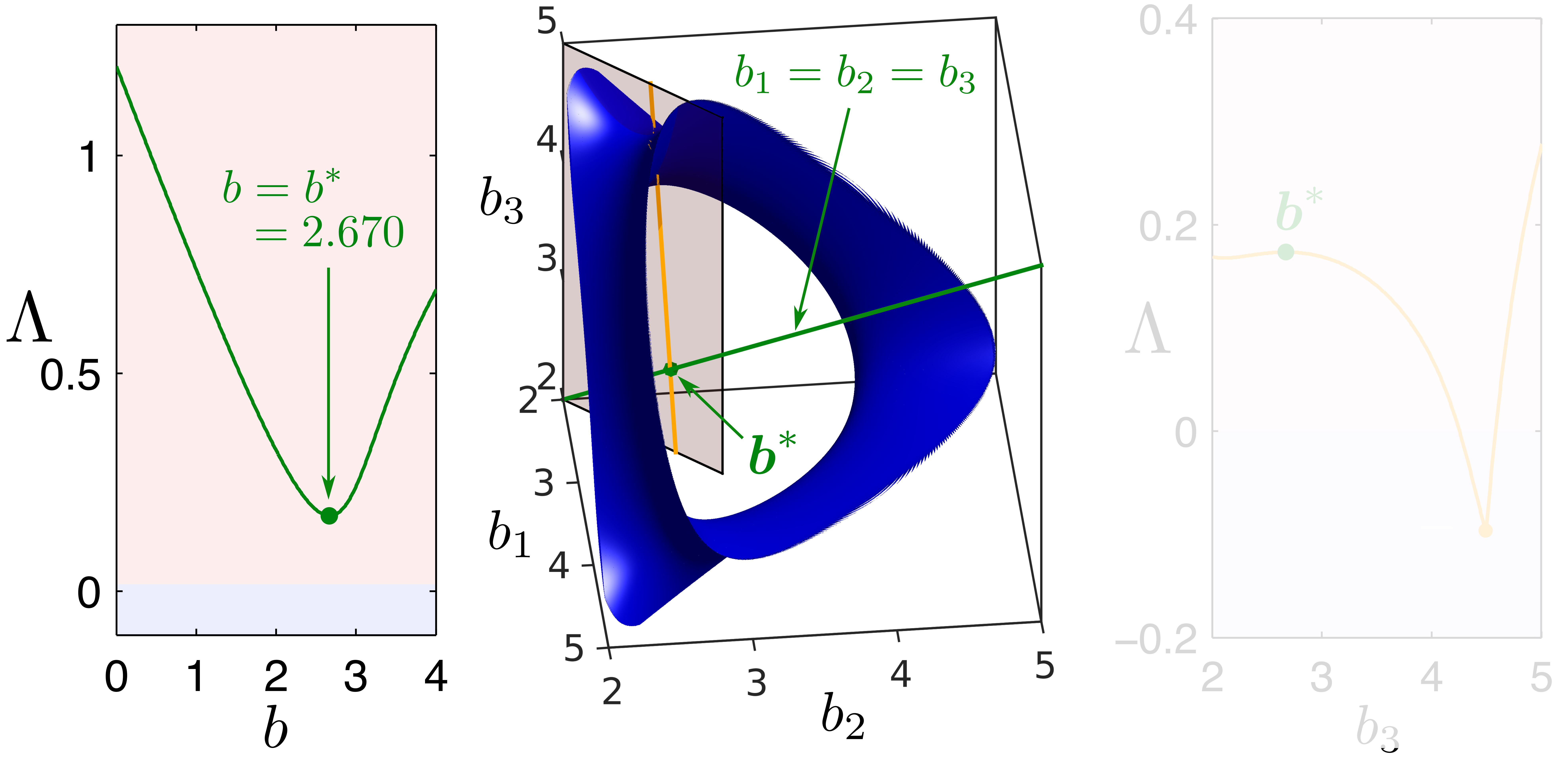
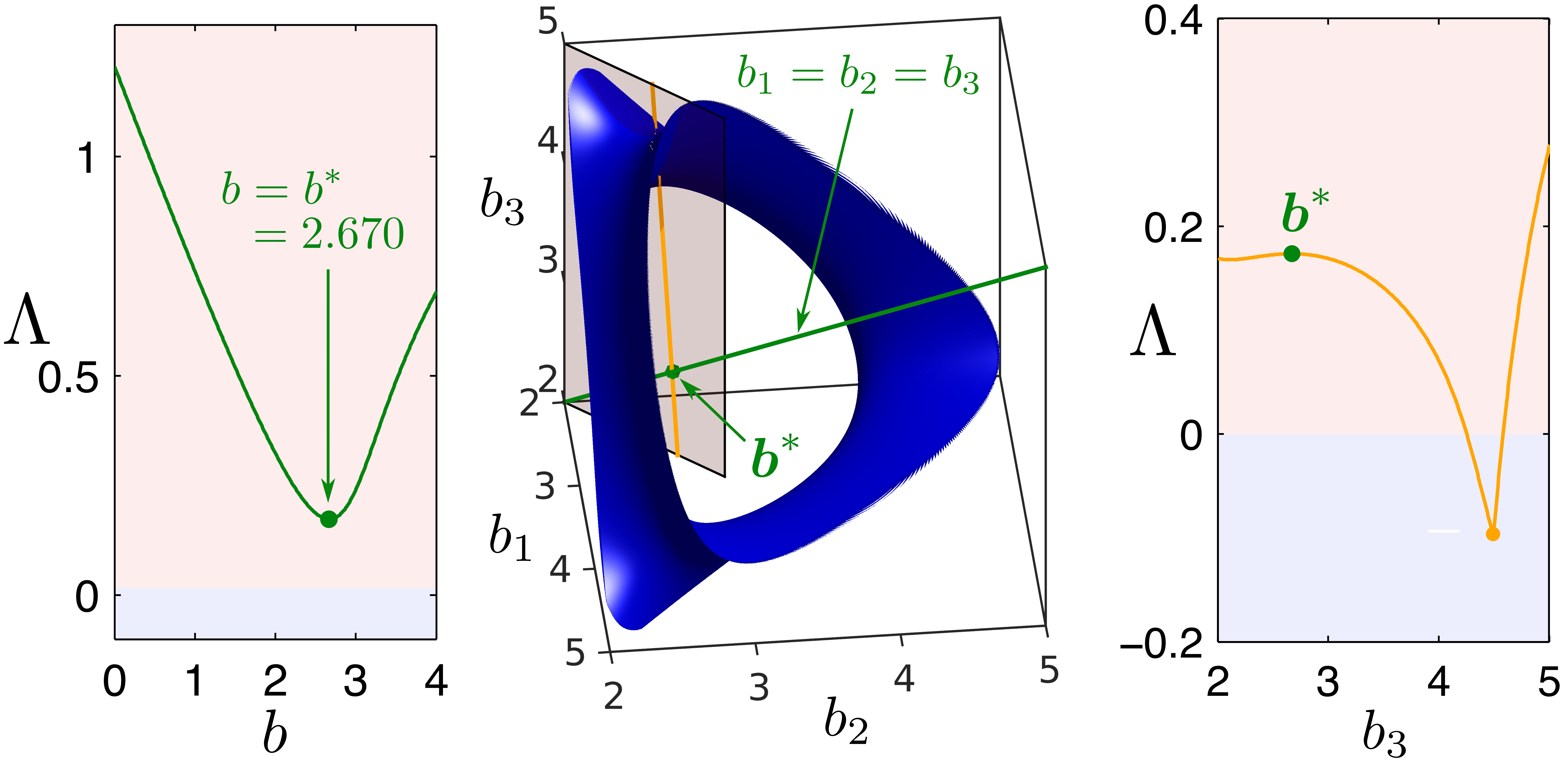


$$\gamma = 0.65,\ \varepsilon = 2,\ \delta = 0.3$$

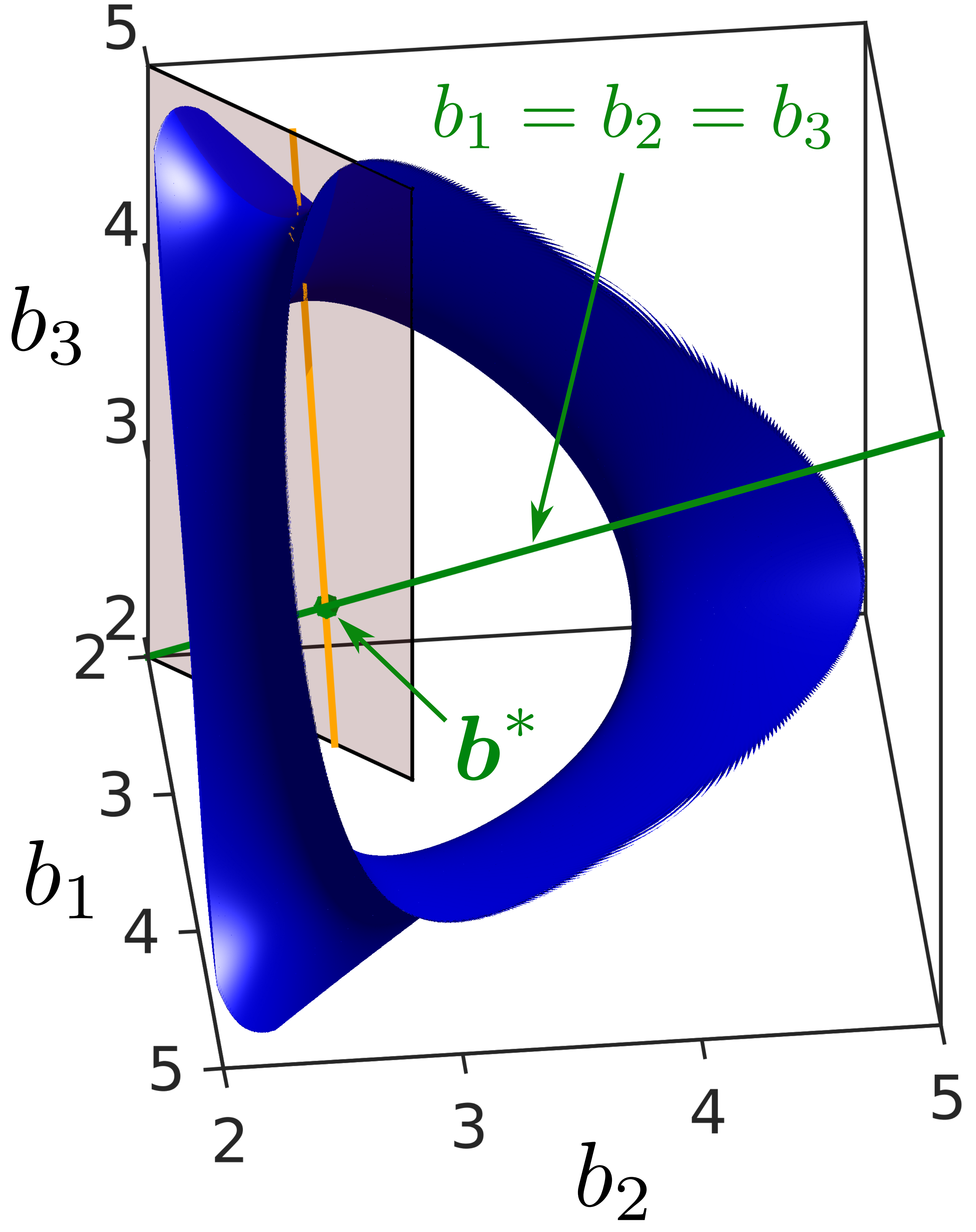
Stability landscape

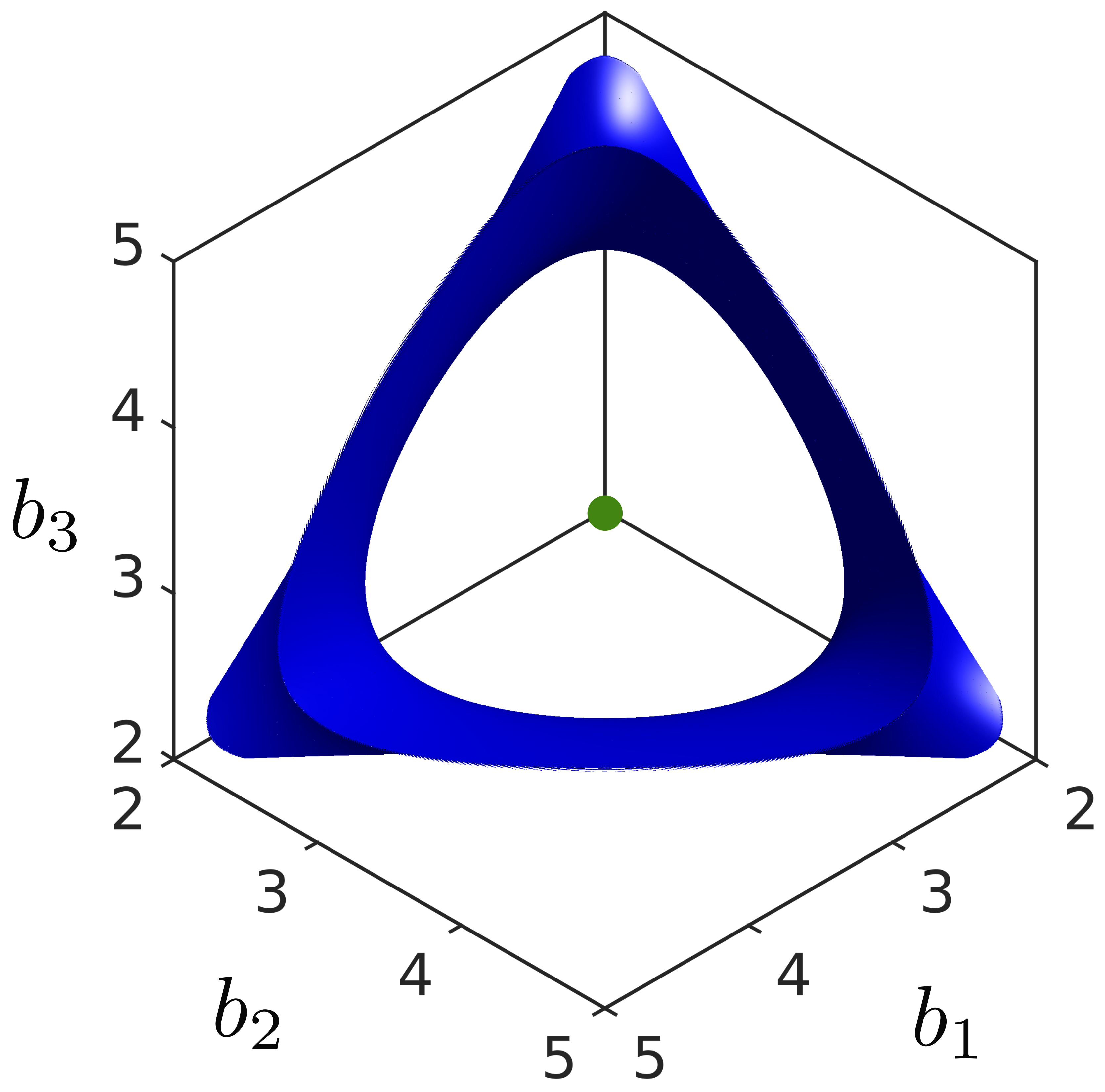
Λ = maximum transverse Lyapunov exponent



$\gamma = 0.65,\ \varepsilon = 2,\ \delta = 0.3$

Symmetry of stability landscape



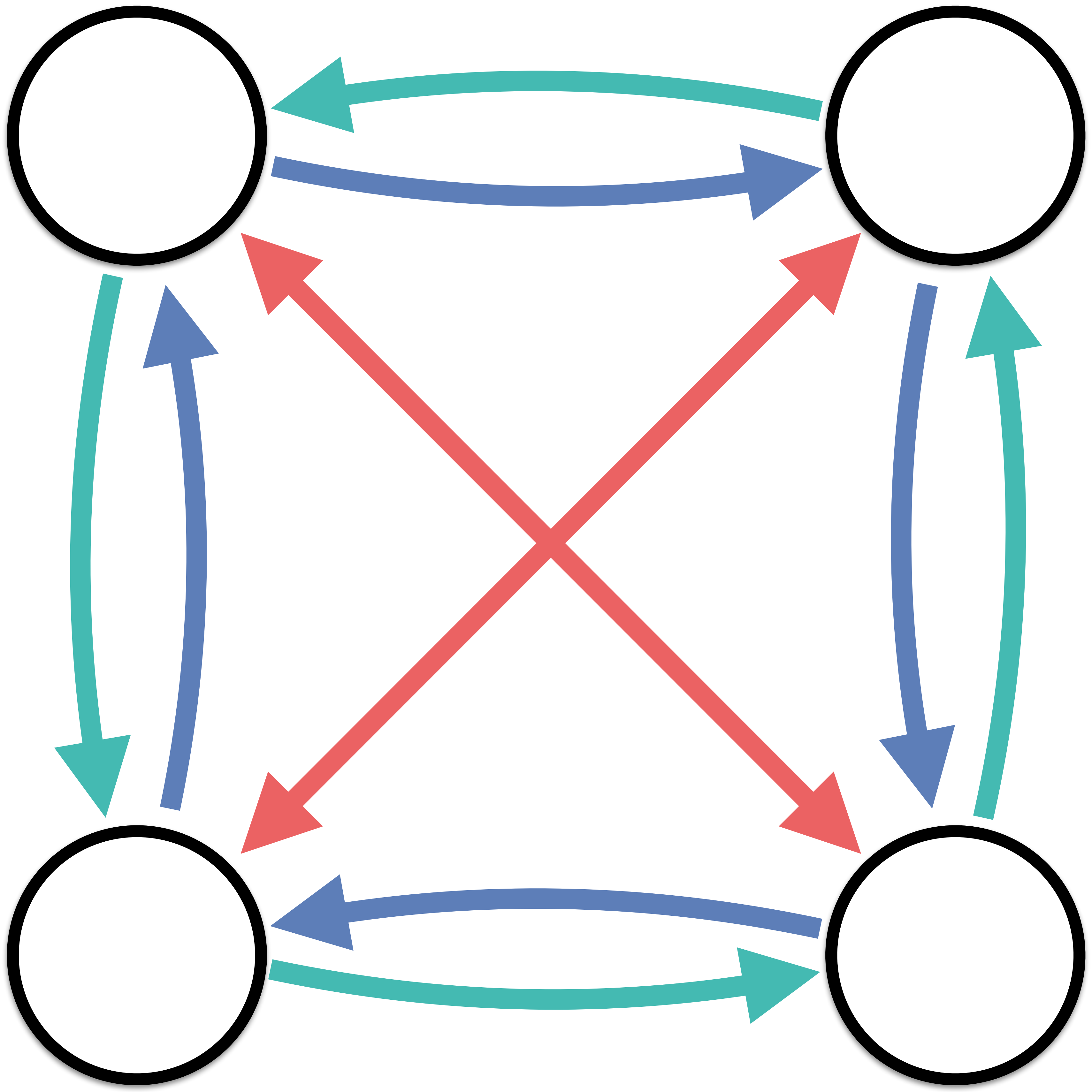


How often does this occur?

YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907

Networks with multiple link types

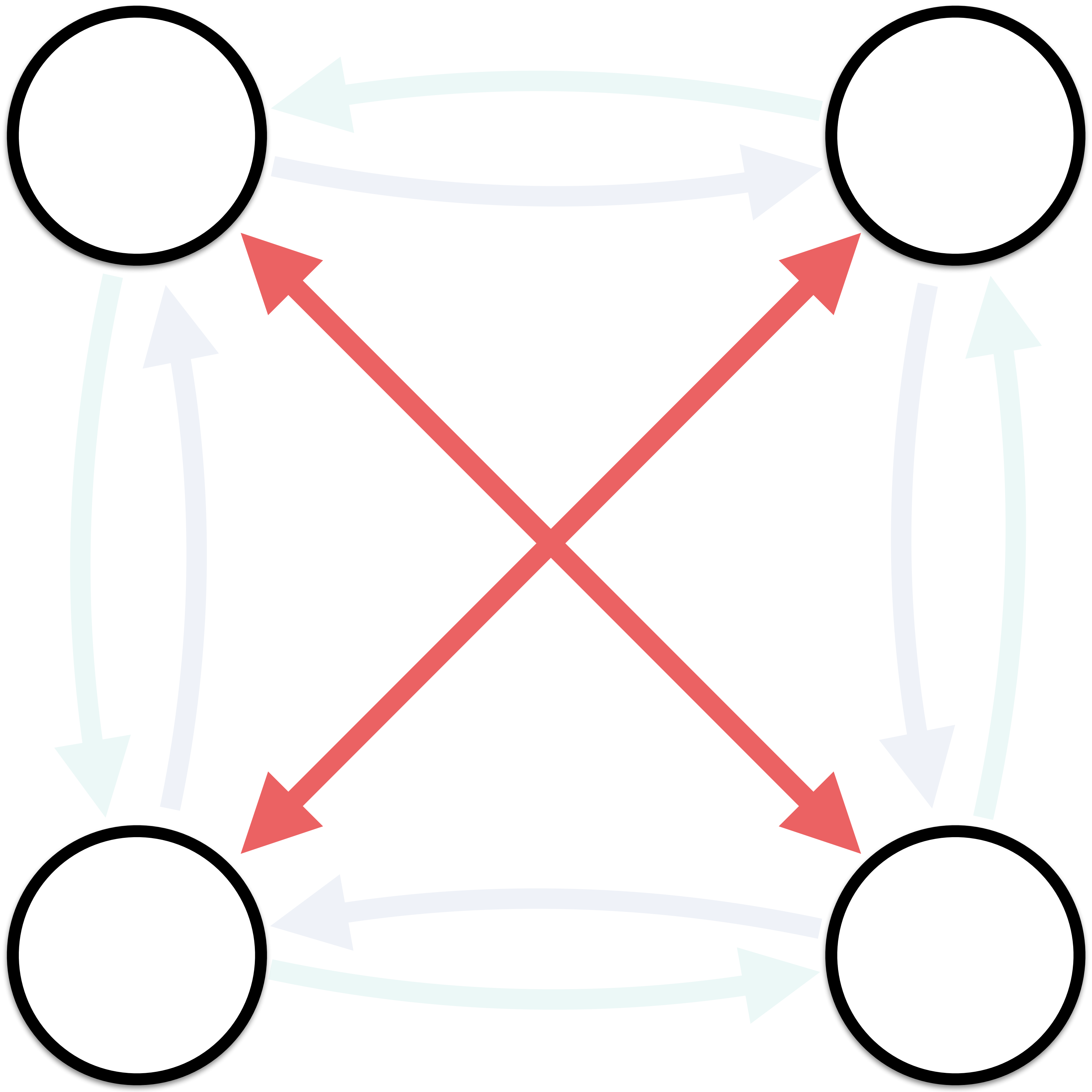
Adjacency matrices $A^{(\alpha)}, \quad \alpha = 1, \ldots, K$



$K = 3$

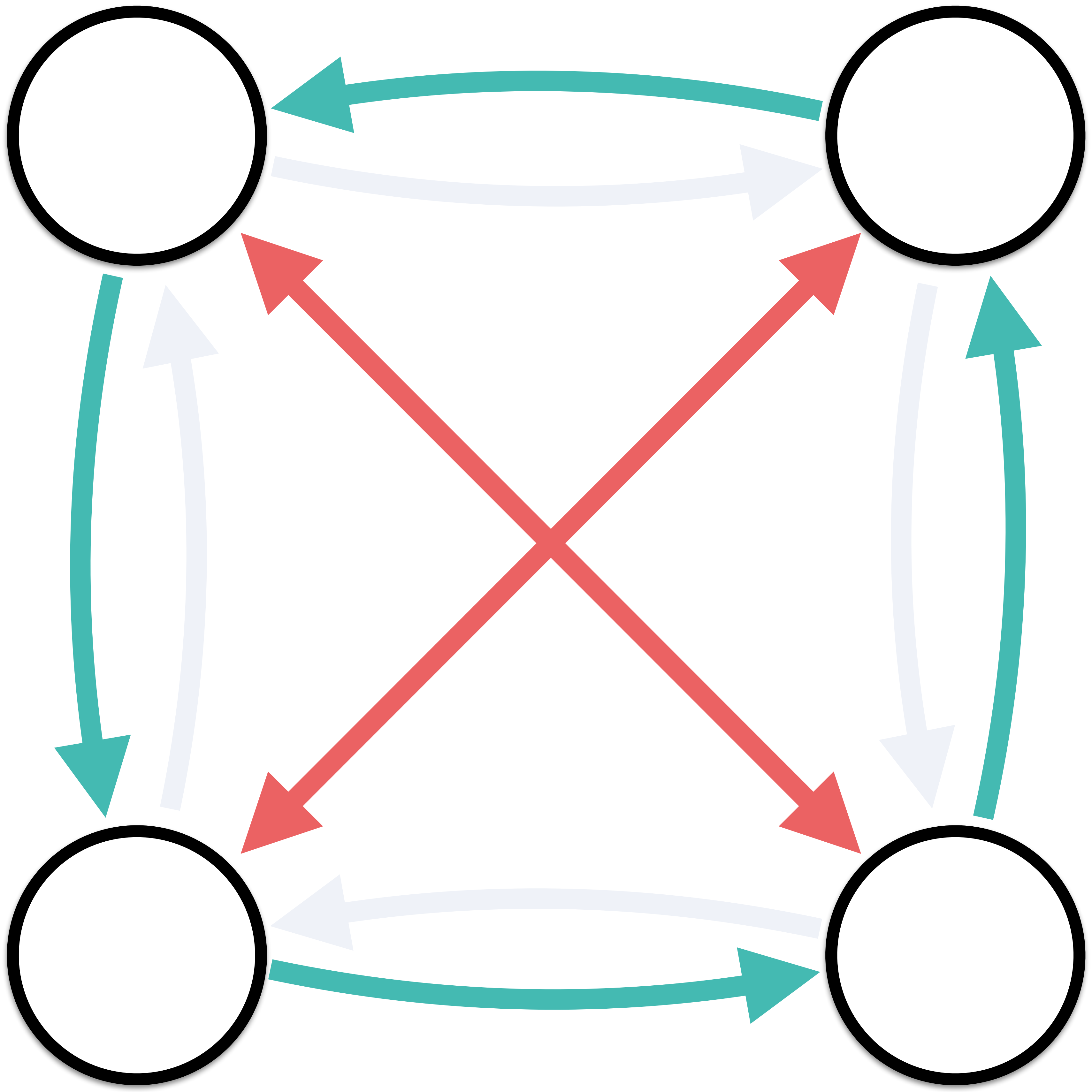
Networks with multiple link types

Adjacency matrices $A^{(1)}$



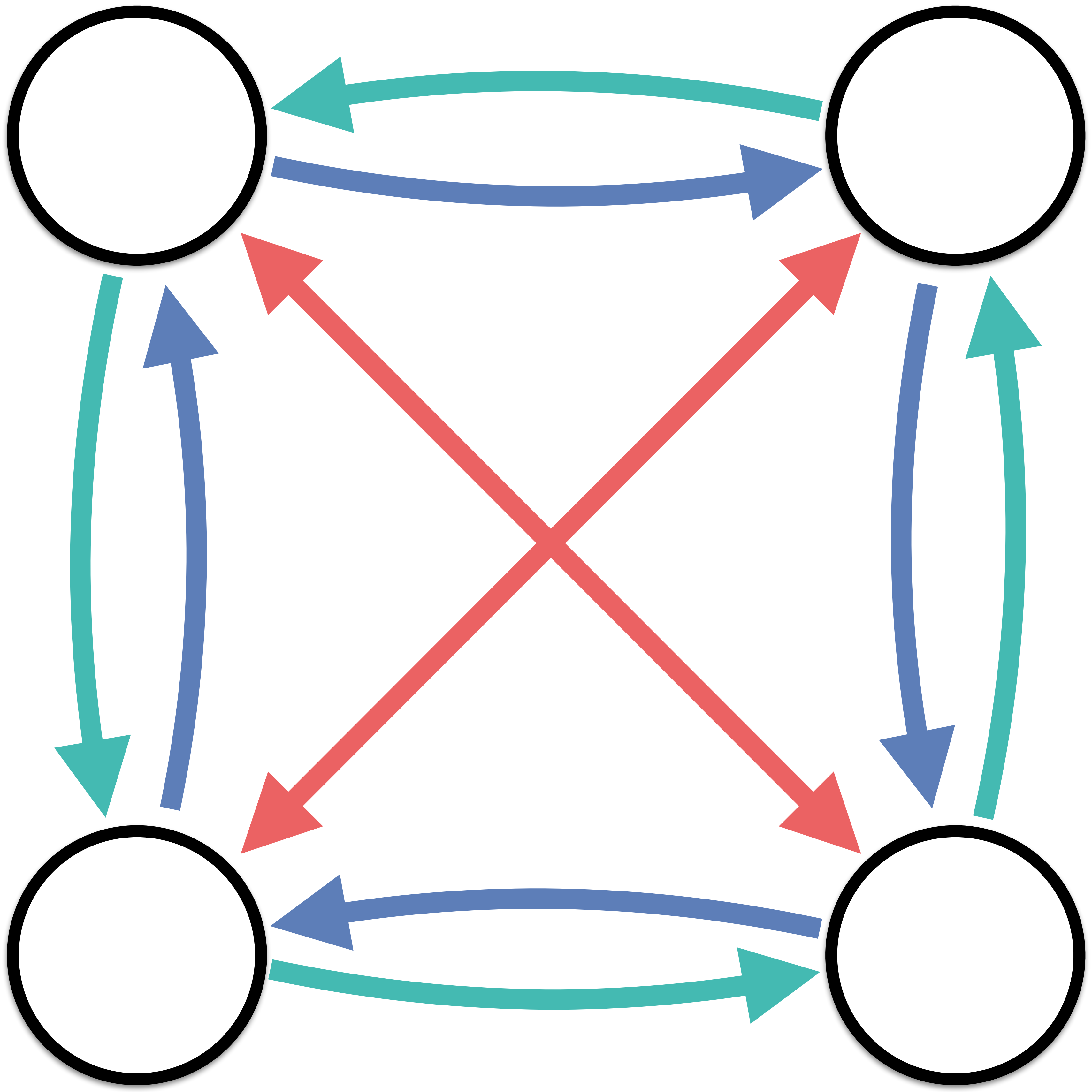
Networks with multiple link types

Adjacency matrices $A^{(1)}$, $A^{(2)}$



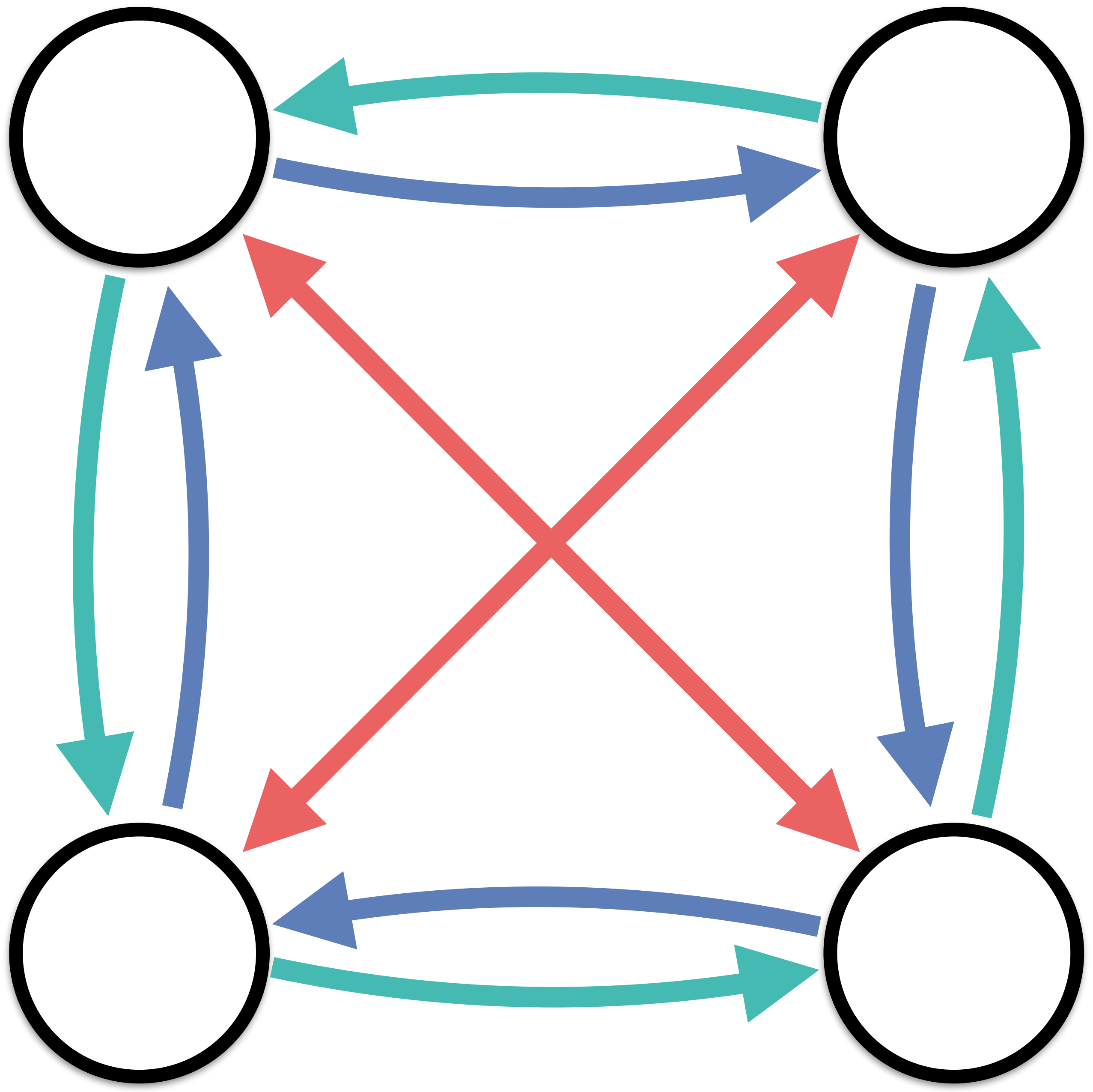
Networks with multiple link types

Adjacency matrices $A^{(1)}, A^{(2)}, A^{(3)}$



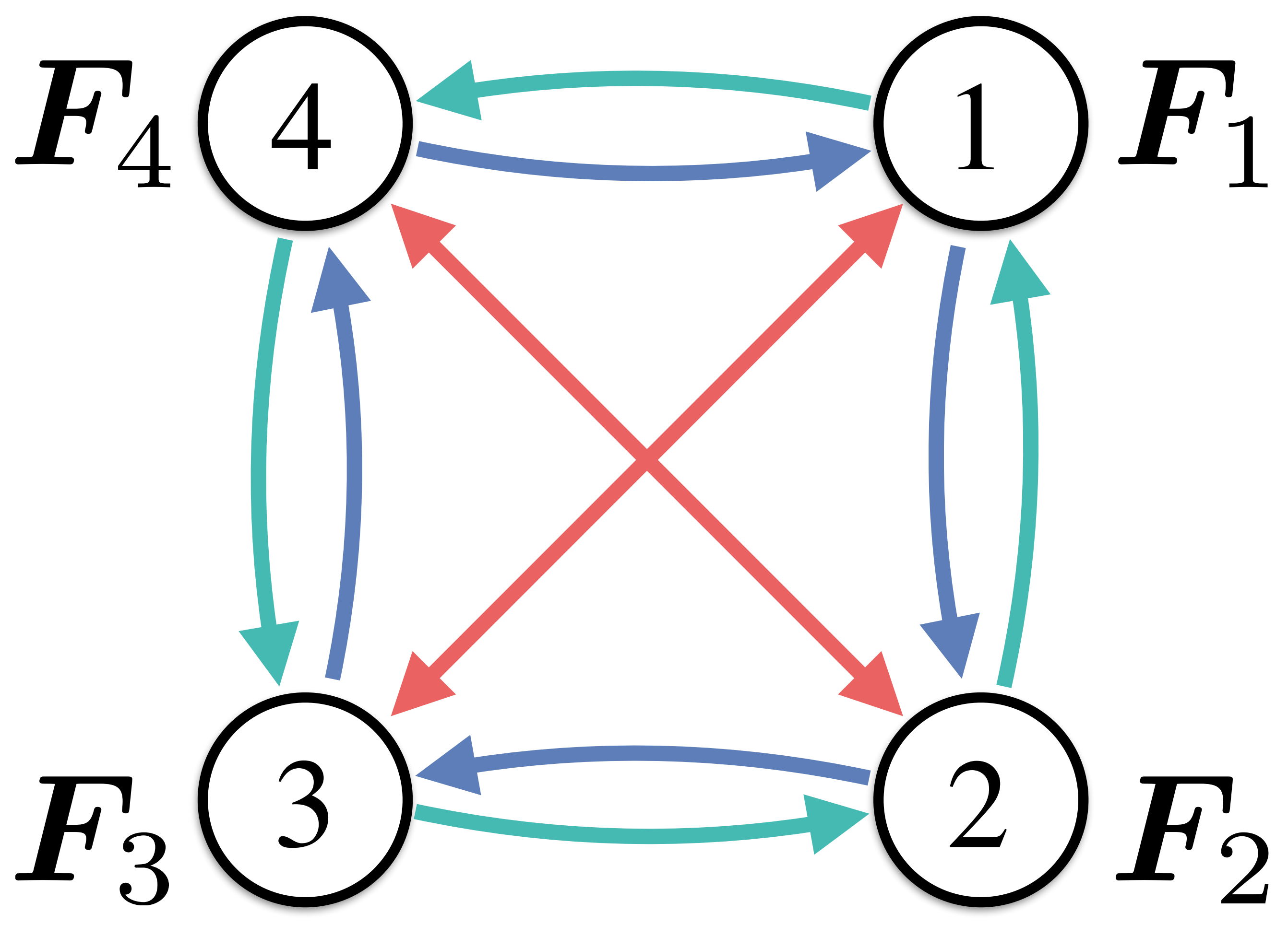
Symmetric network structures

- *Symmetric network*: every node can be mapped to any other node by some permutation of nodes without changing any $A^{(\alpha)}$.
- For undirected networks with a single link type, they are called *vertex-transitive graphs*.
- Includes *circulant graphs*, defined as a network whose nodes can be arranged in a ring so that the network is invariant under rotations.



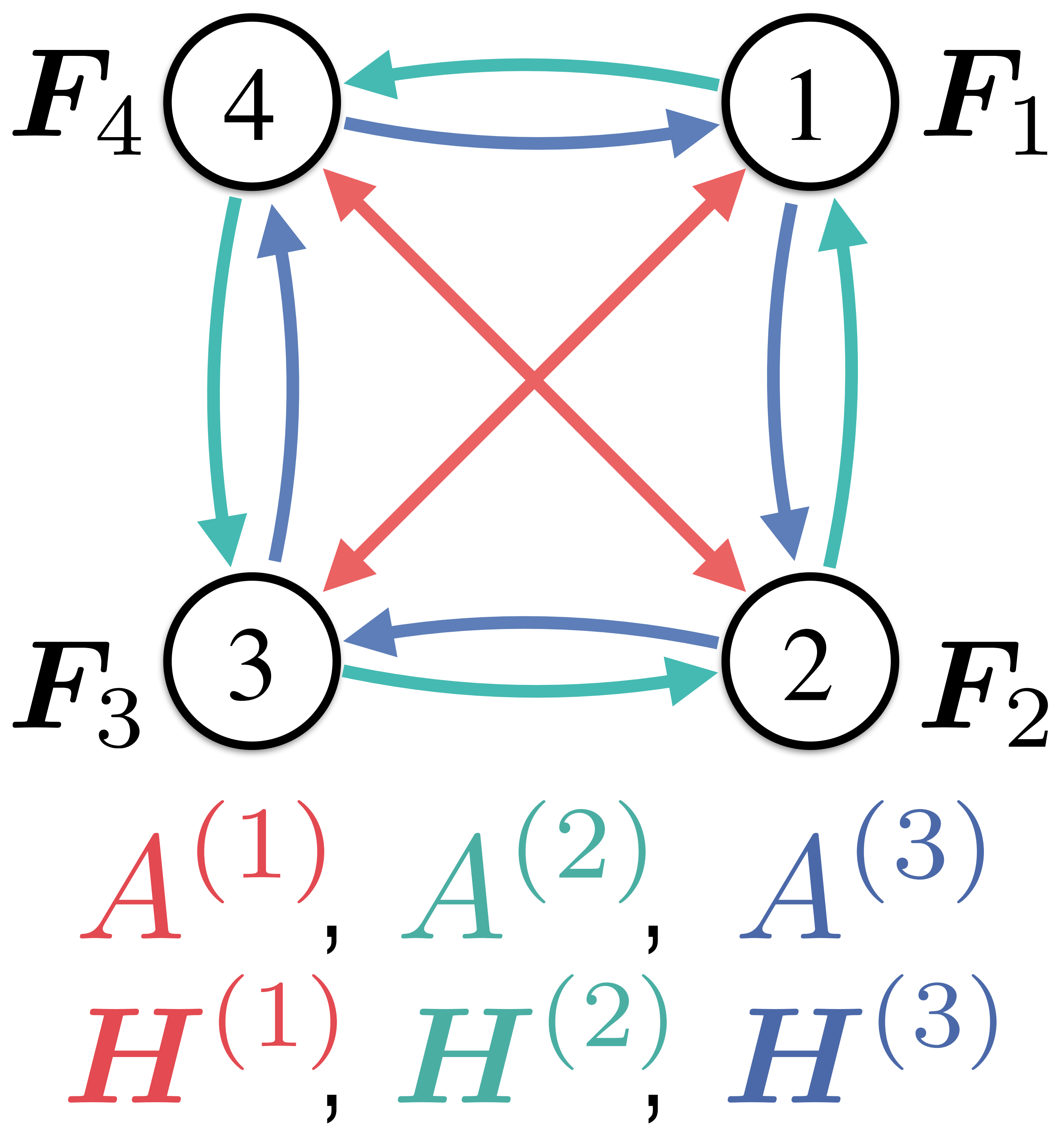
Example of symmetric network (circulant graph)

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i)$$: dynamics of isolated node i



Network of non-identical oscillators

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$



$A_{ii'}^{(\alpha)}$: directed link of type α from node i' to node i

$\boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$: coupling function

Network of non-identical oscillators

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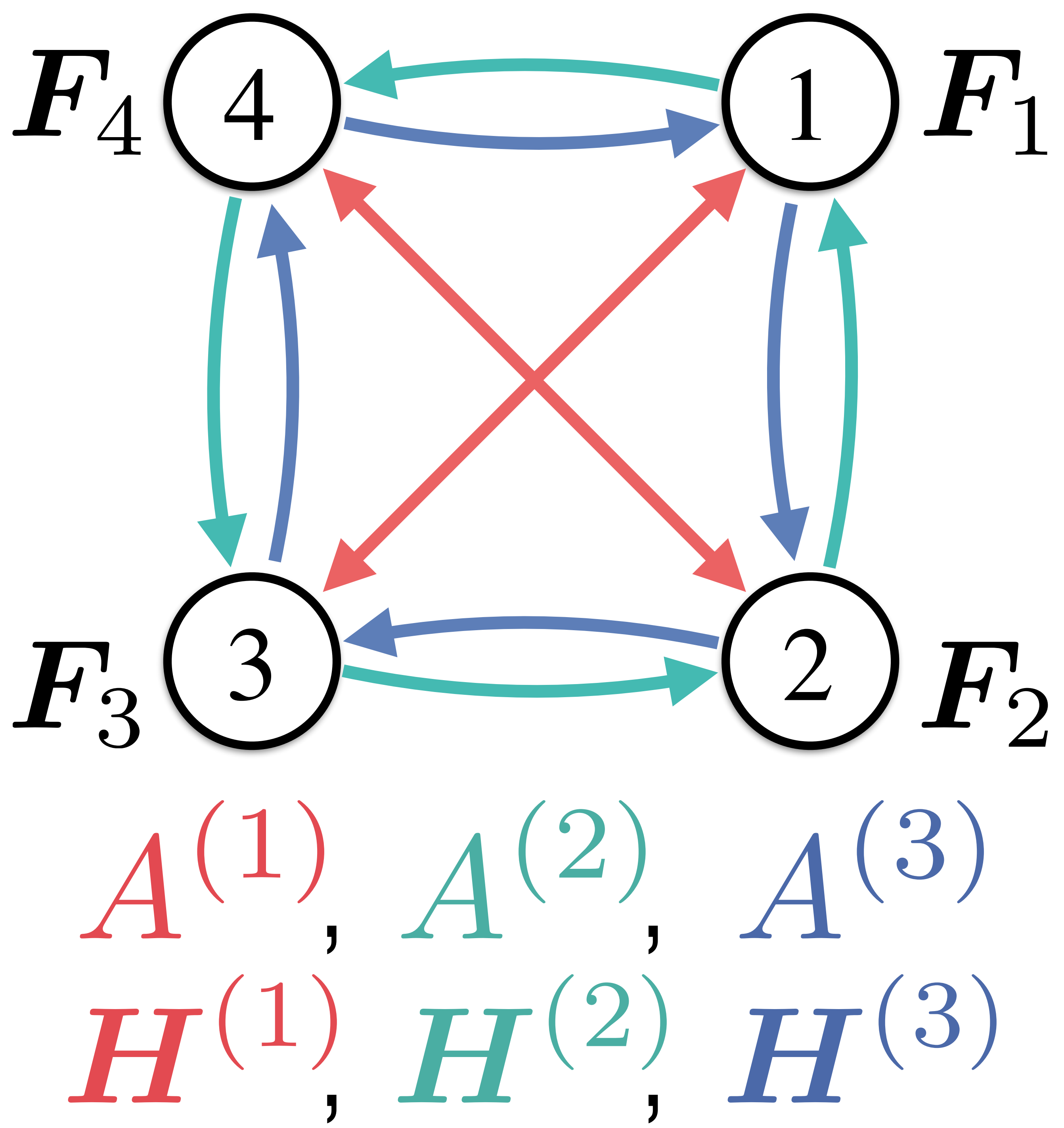
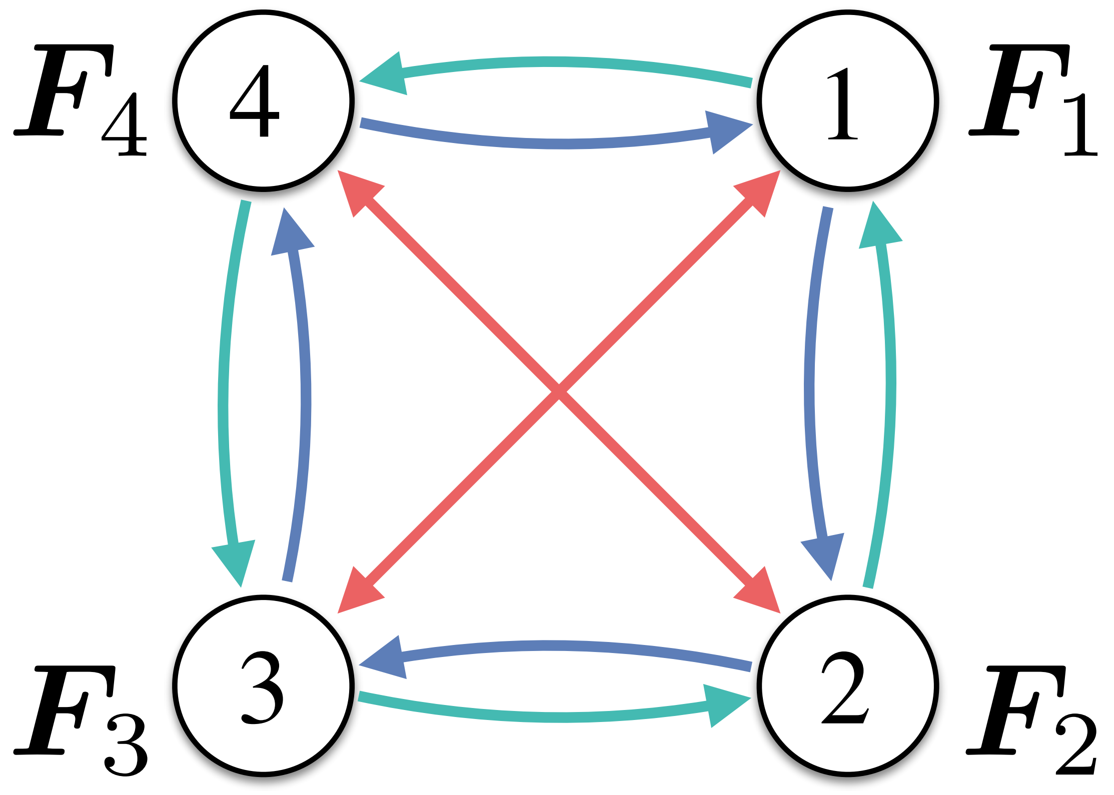


Defining asymmetry-induced synchronization

For a symmetric network

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$

with completely synchronous state,



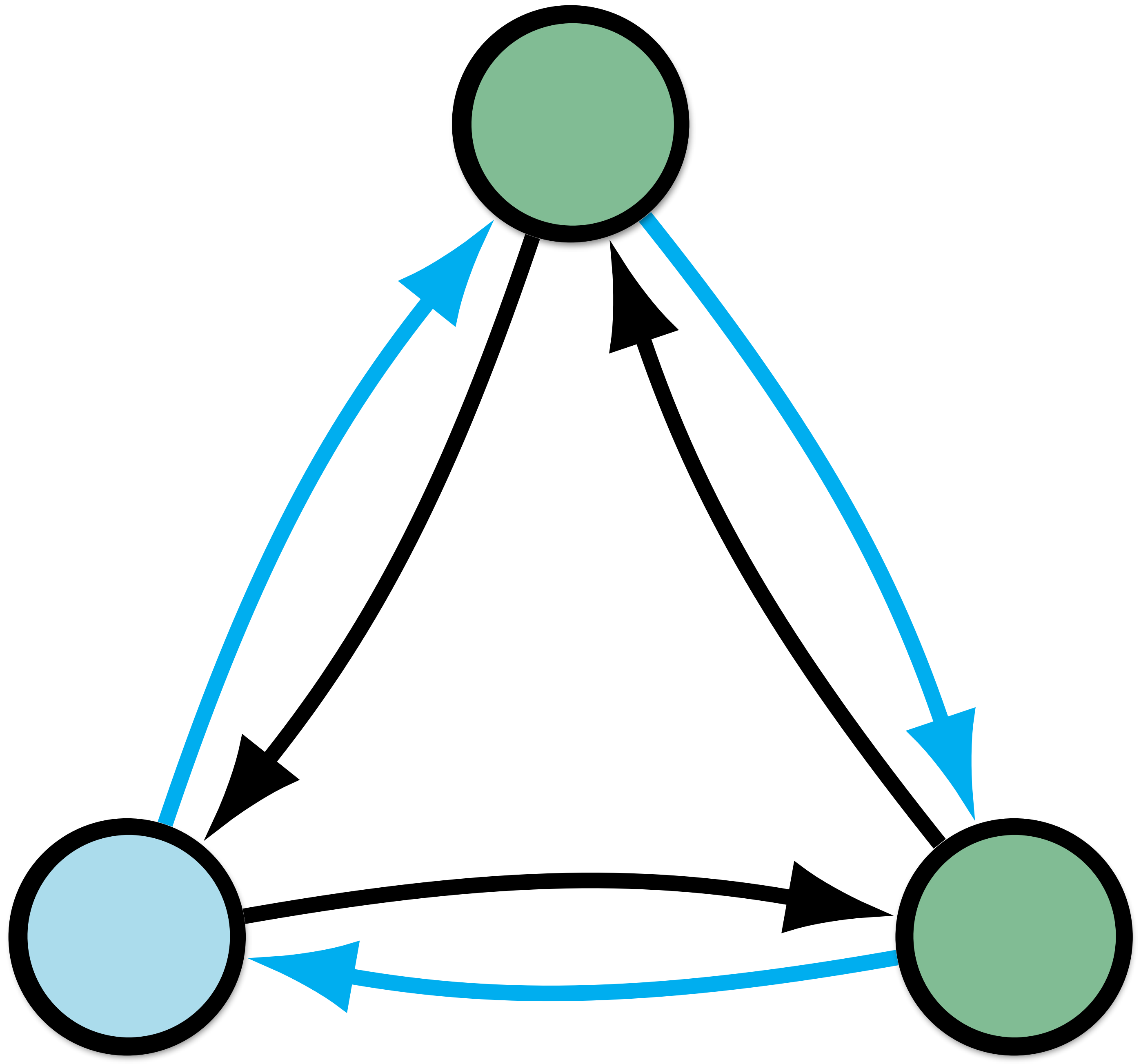
Conditions

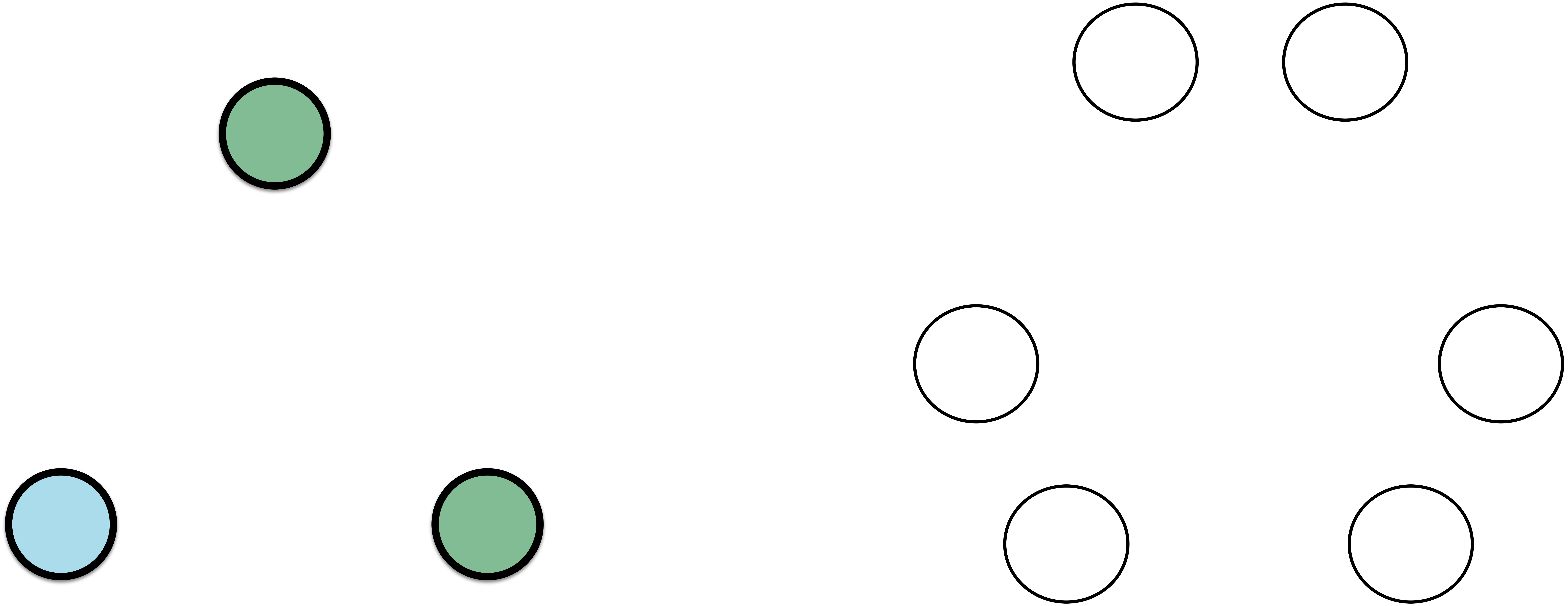
1. Synchronous state is unstable for any homogeneous system.
   $F_1 = \cdots = F_N$

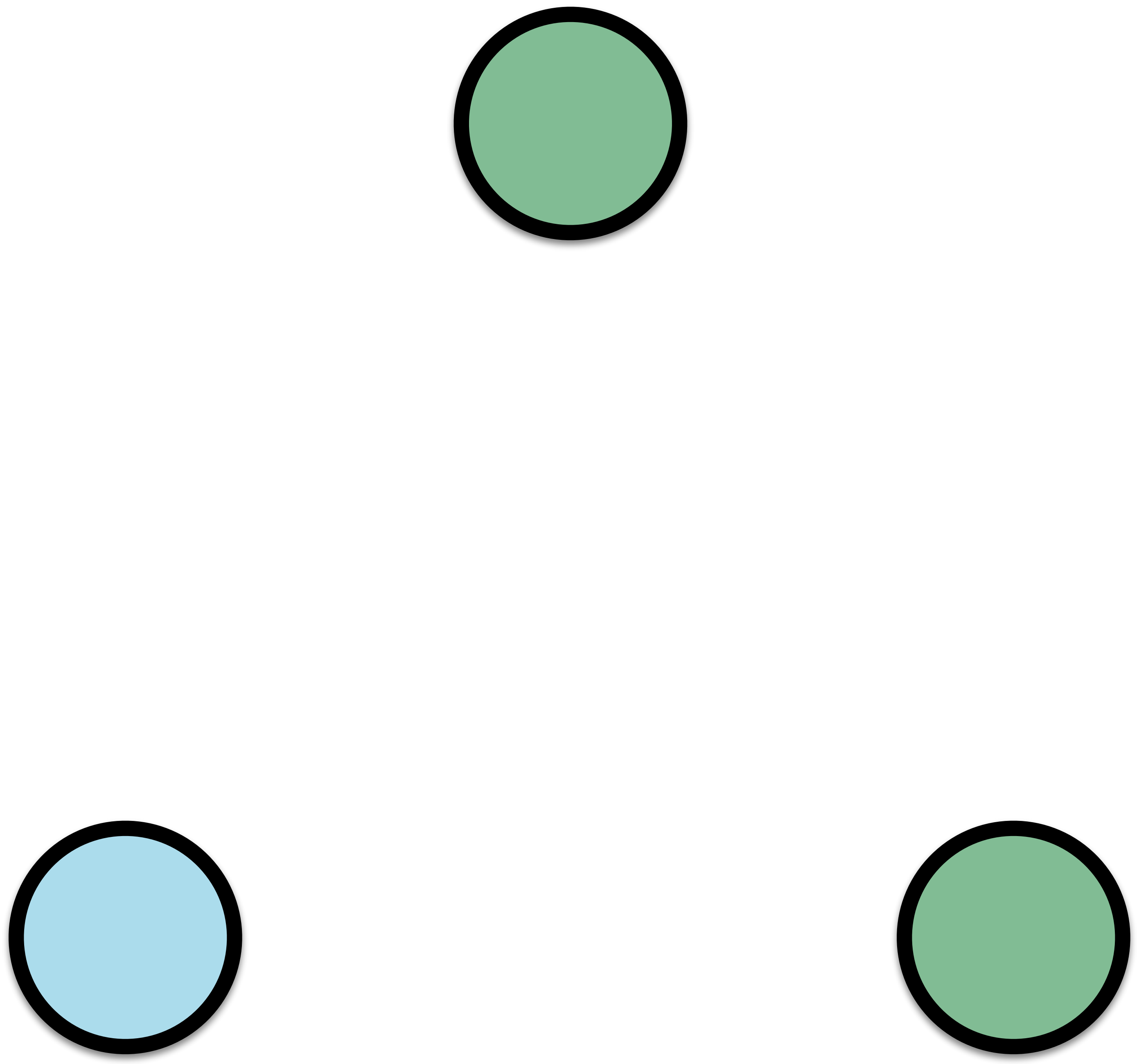
2. Synchronous state is stable for some heterogeneous system.
   $F_i \neq F_{i'}$ for some $i \neq i'$

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$

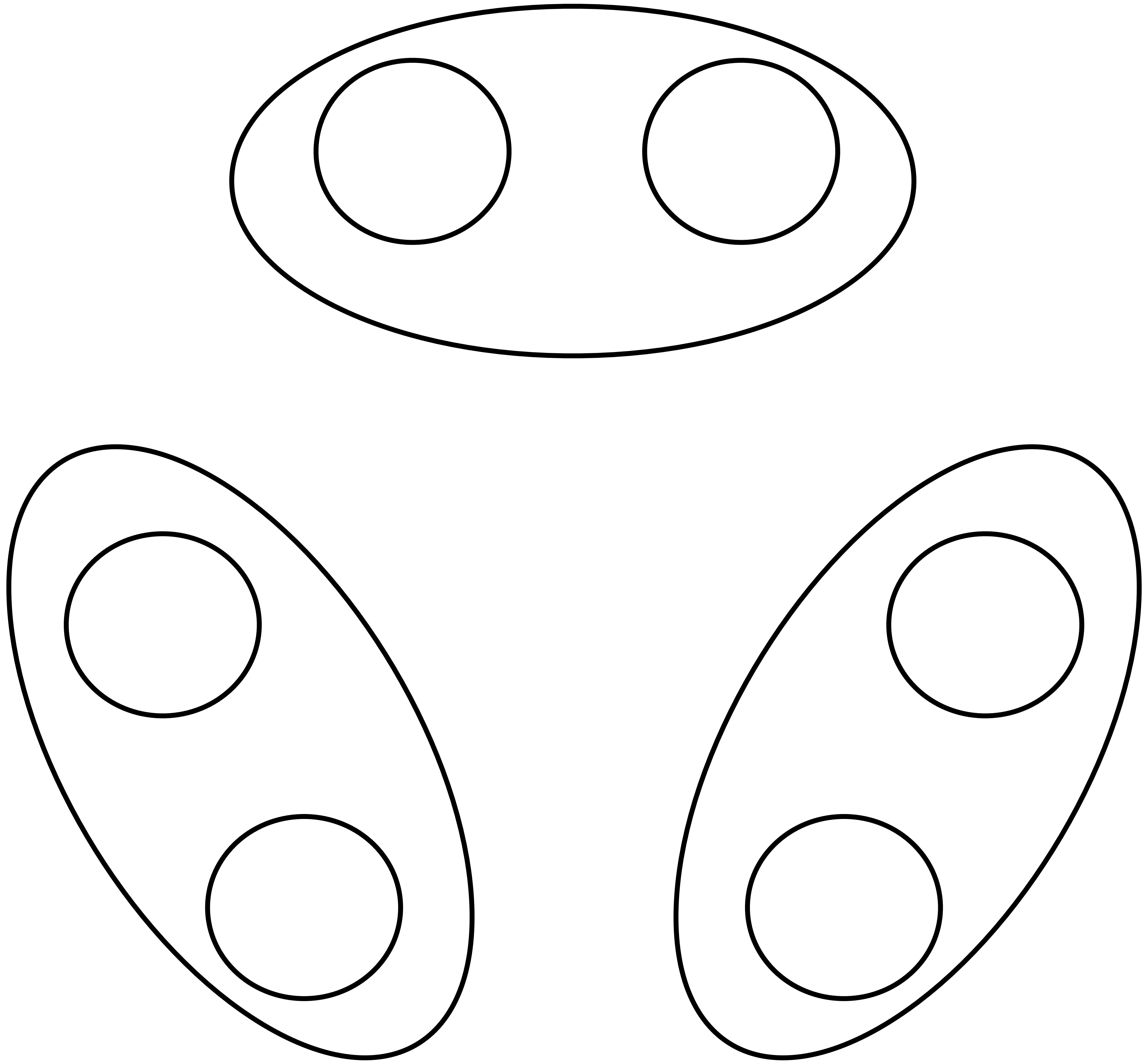
Class of multilayer systems

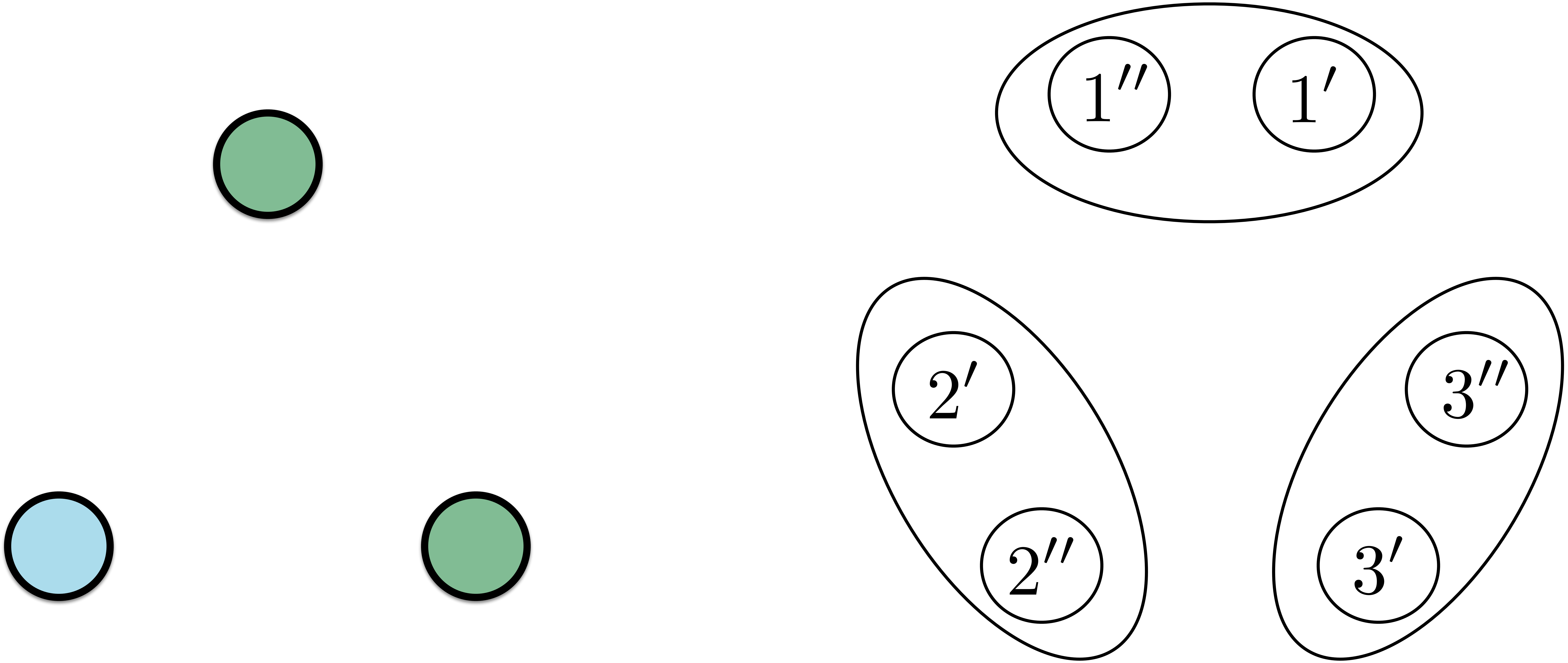




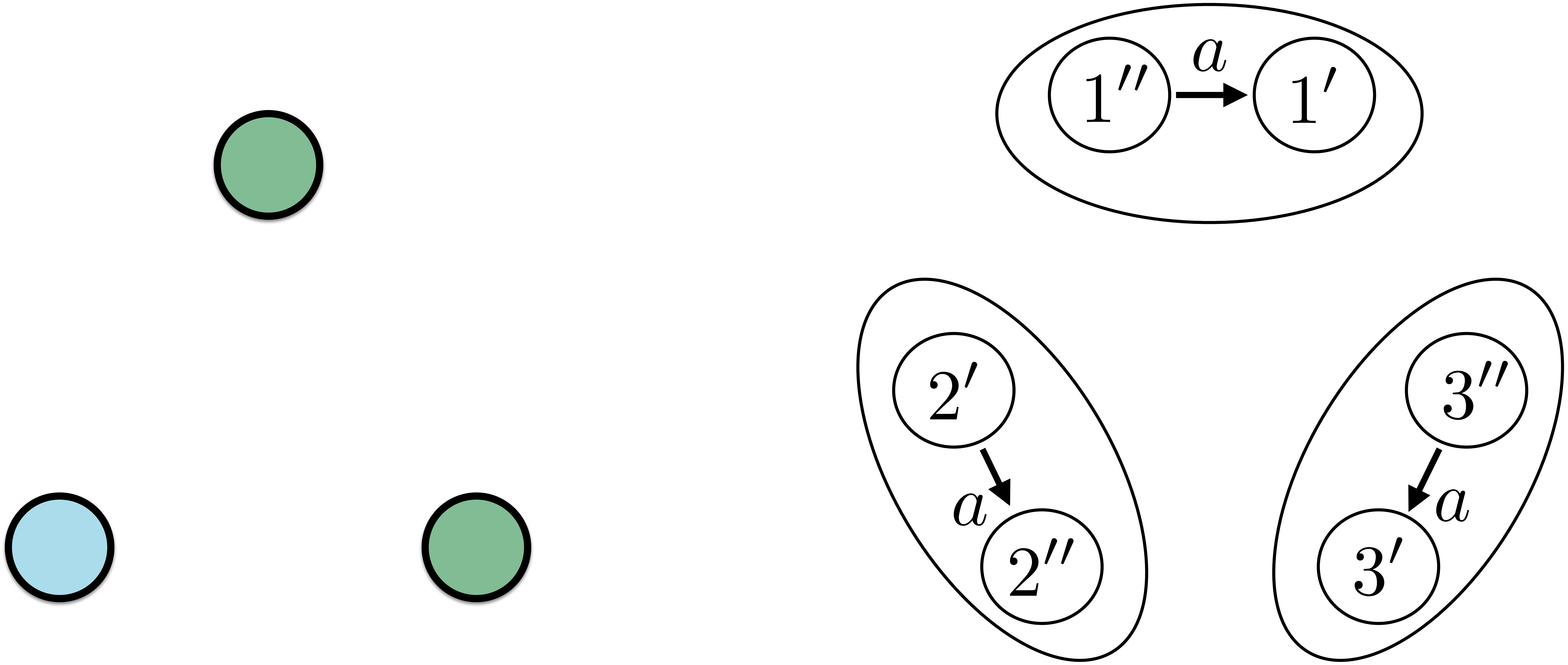


2 subnodes for each node

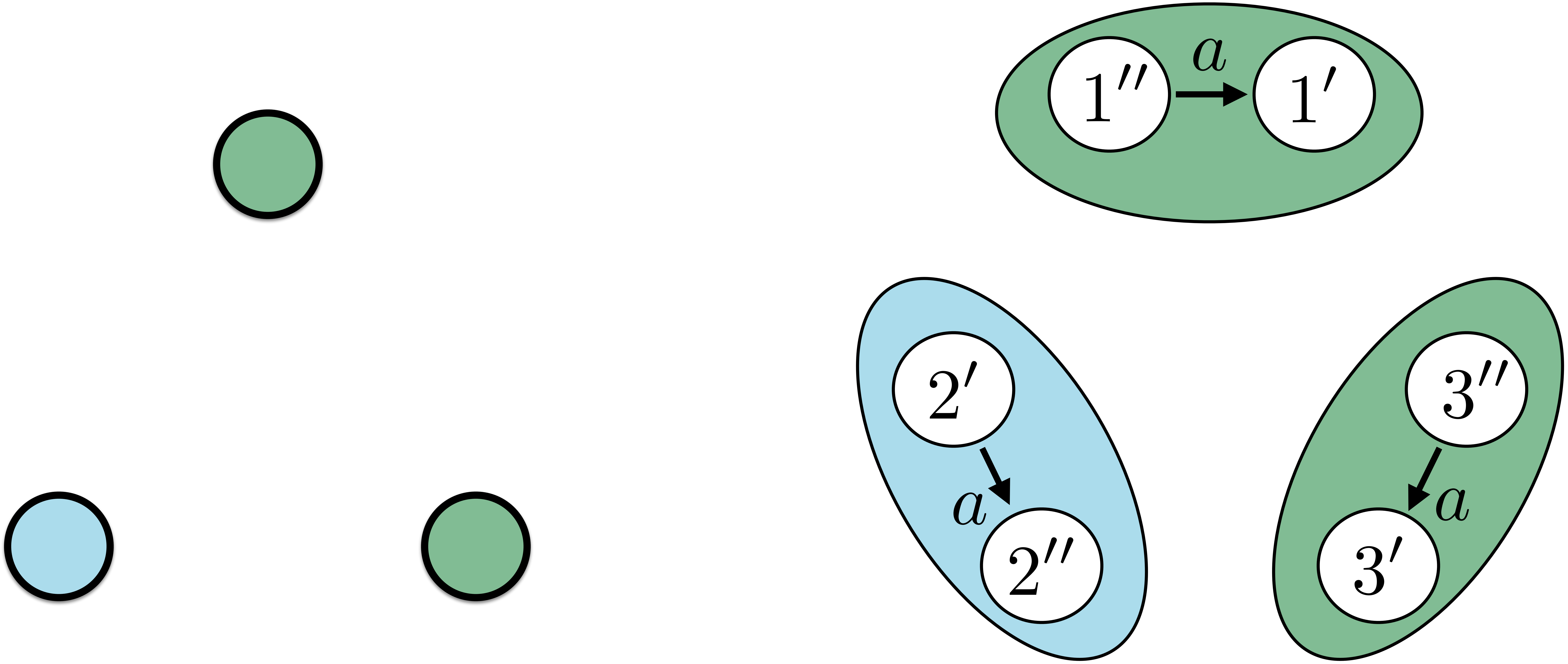



1′′
1′
2′
3′′
2′′
3′

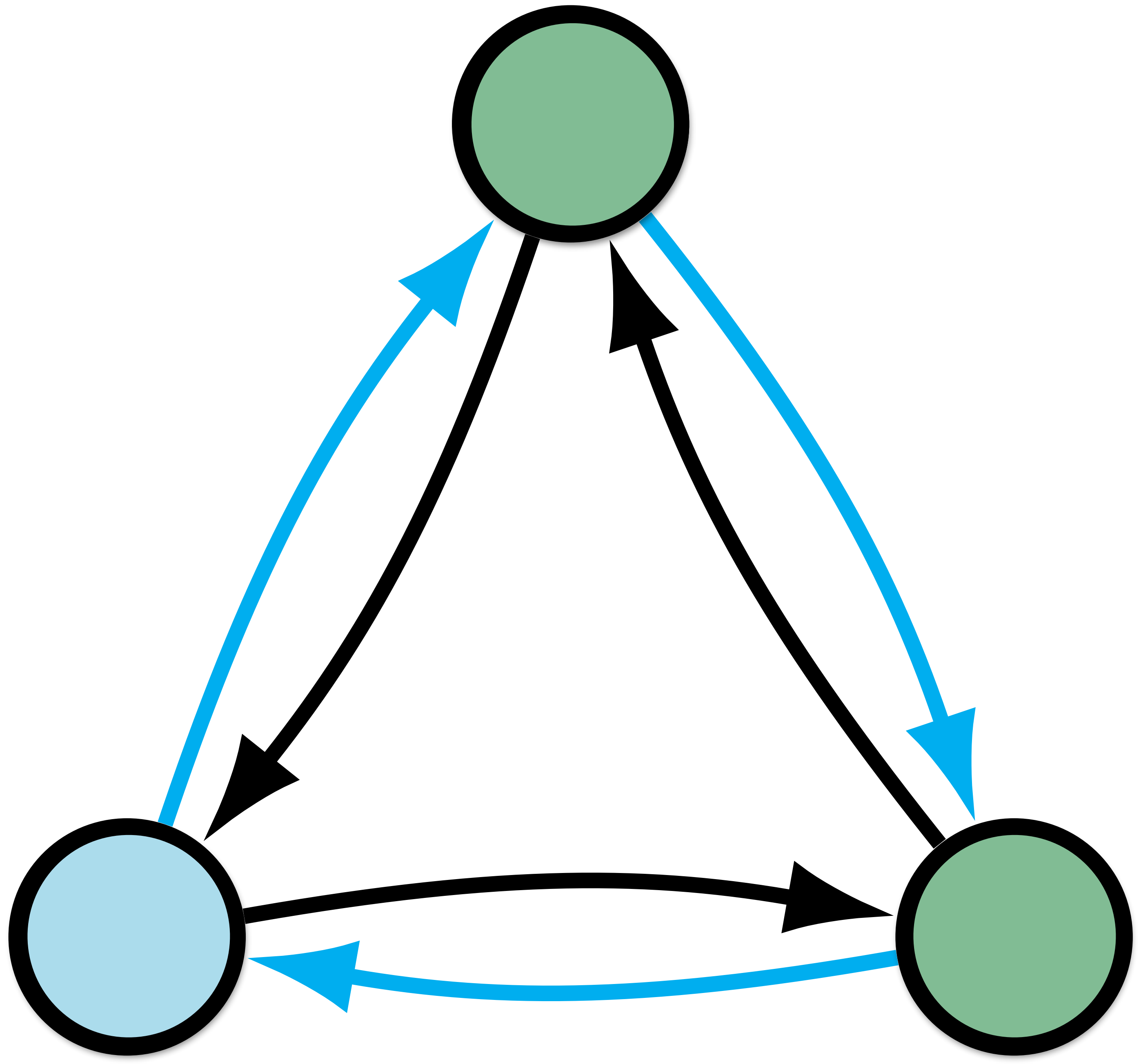
Internal links with strength a

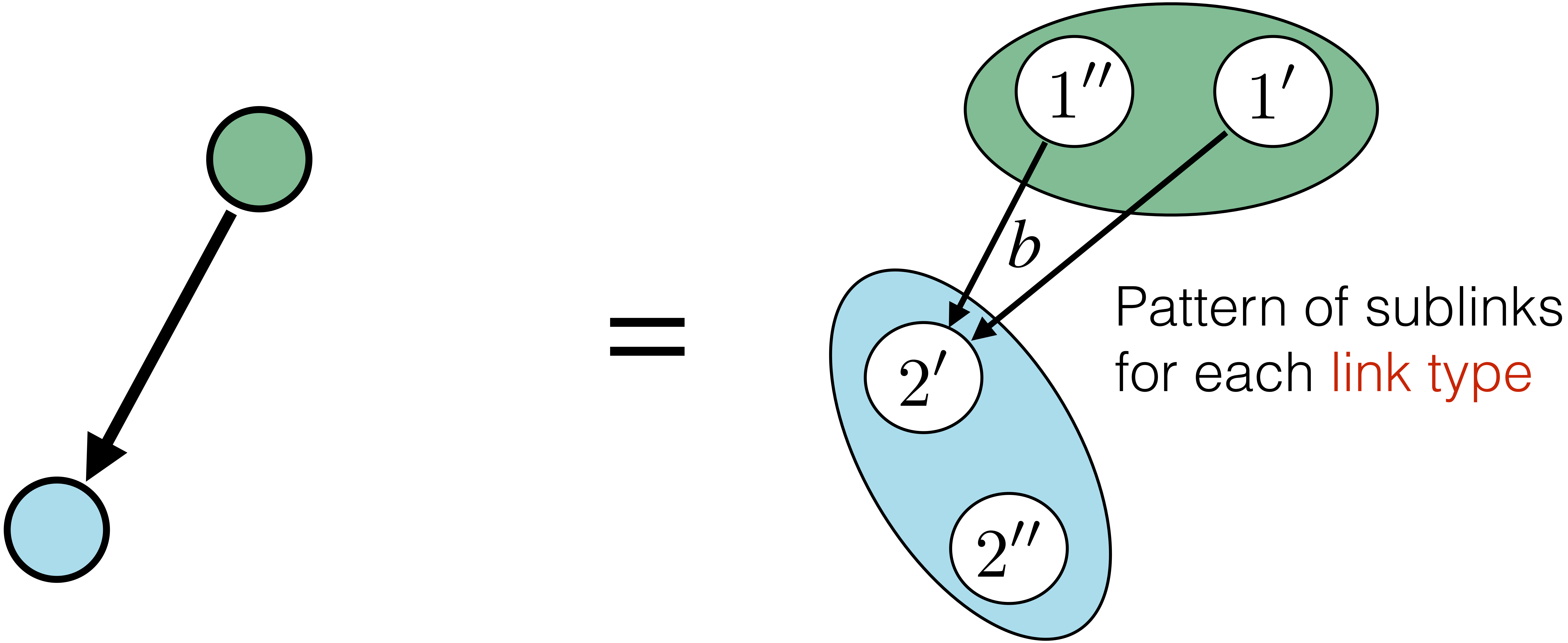


Internal link pattern
defines node type

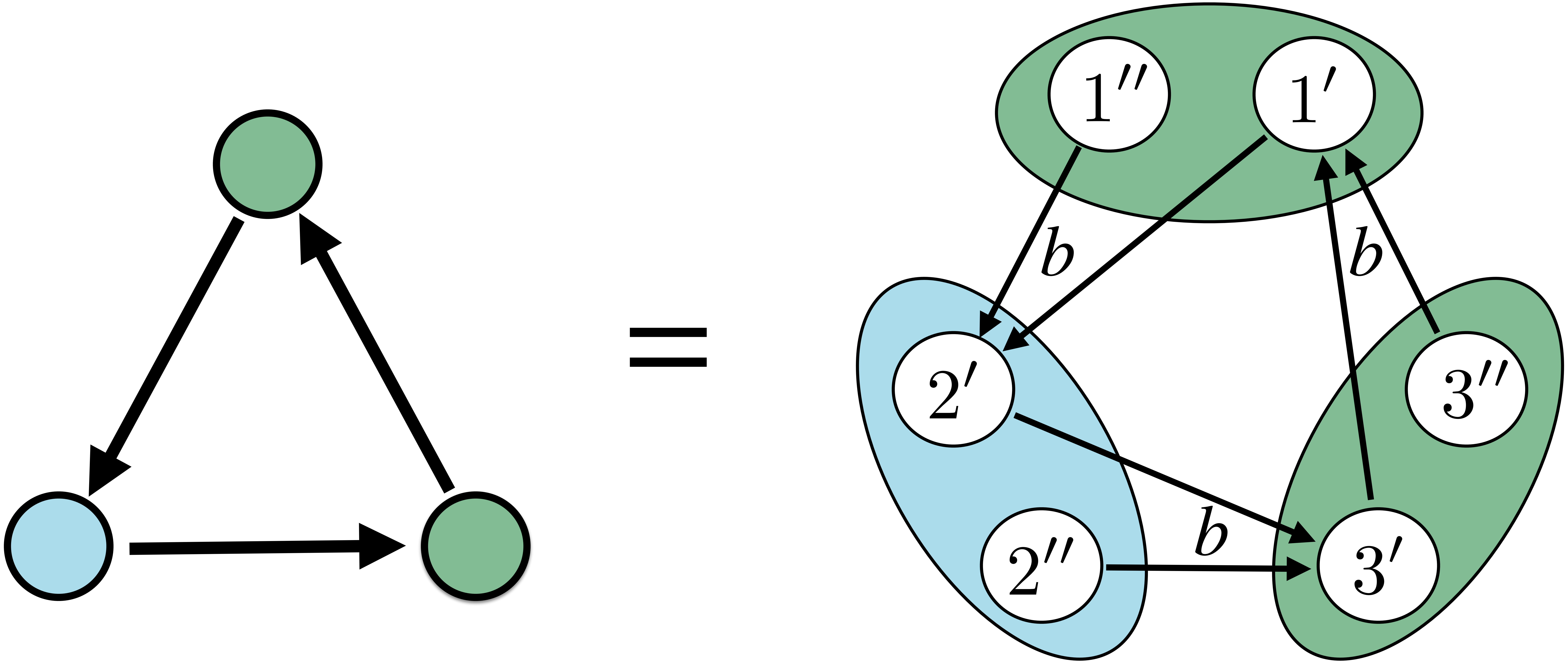


External links?

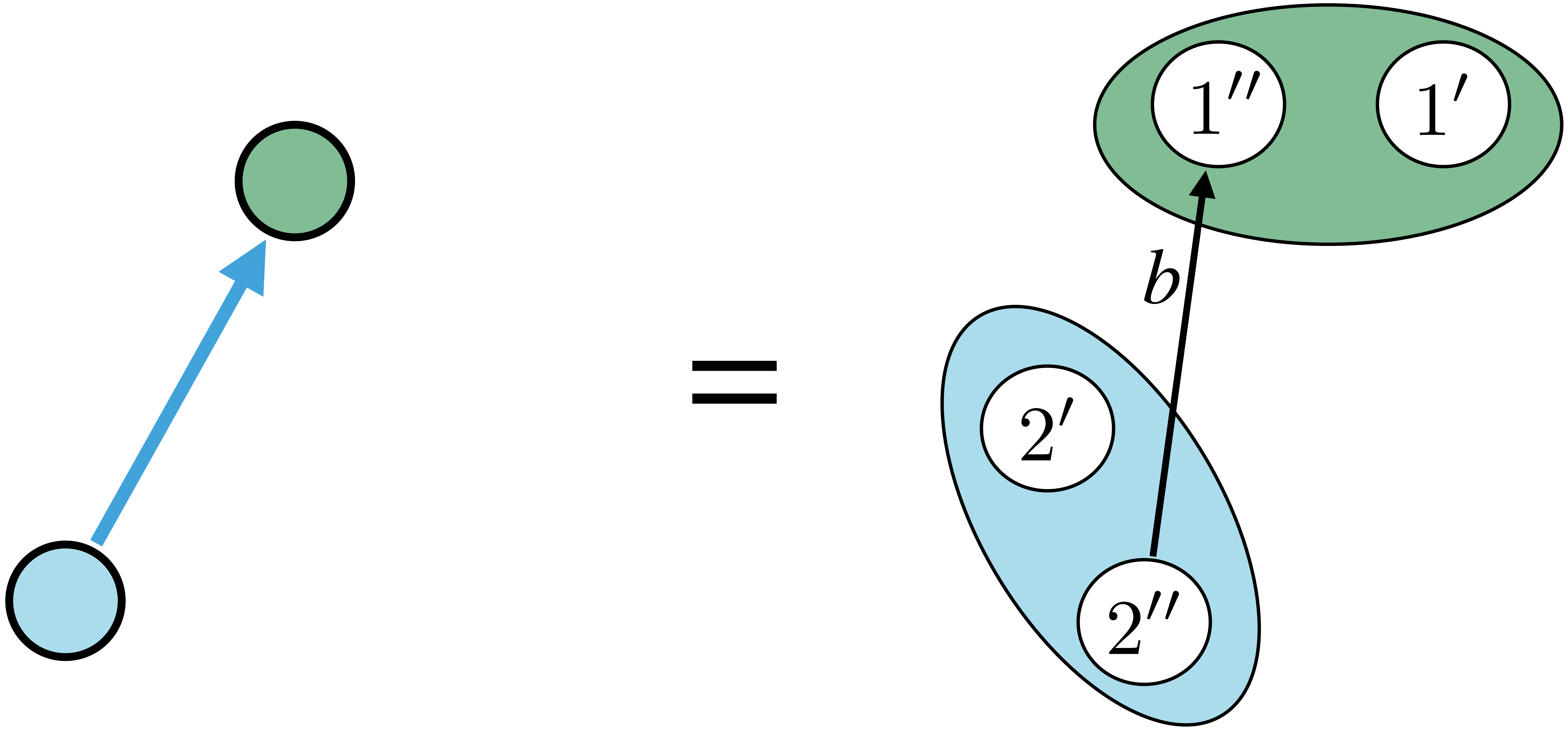




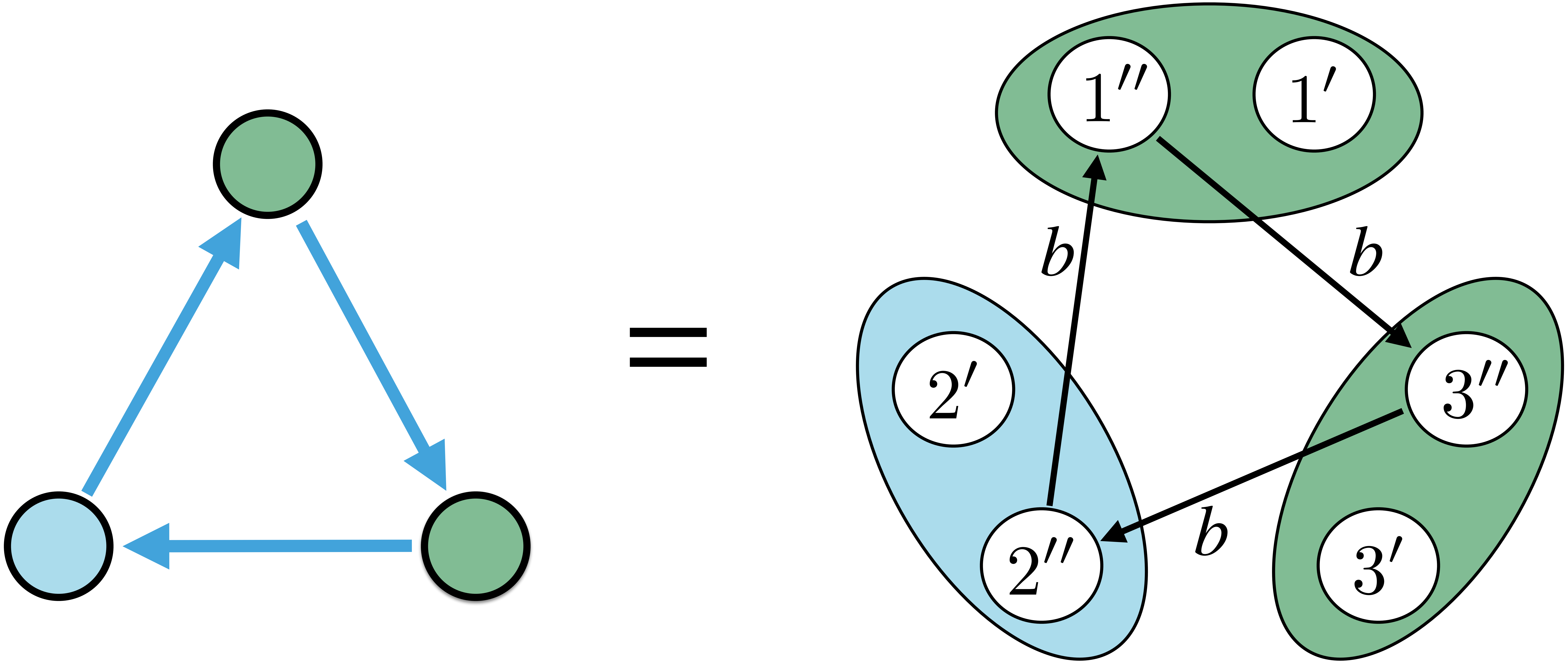
=
1''
1'
b
2'
2''
Pattern of sublinks
for each link type



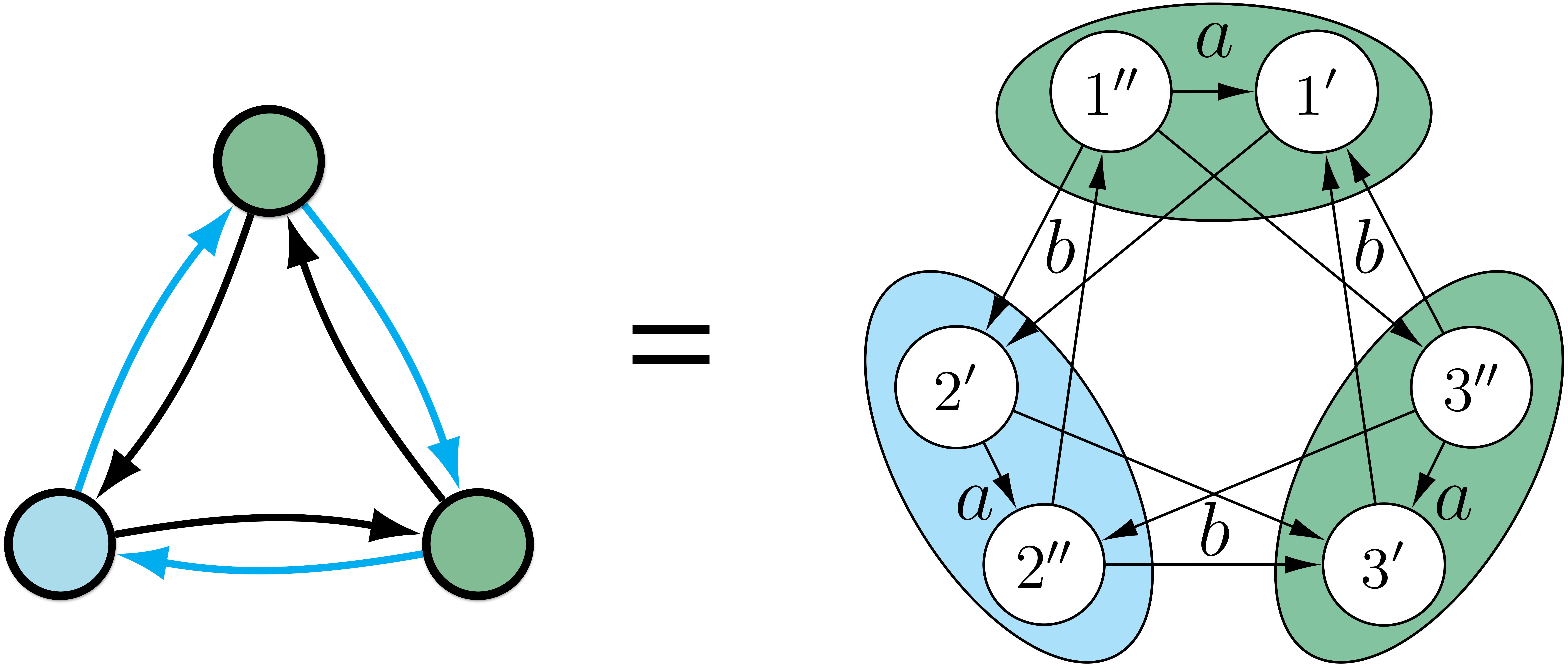
=
1″
1′
b
b
2′
3″
b
2″
3′



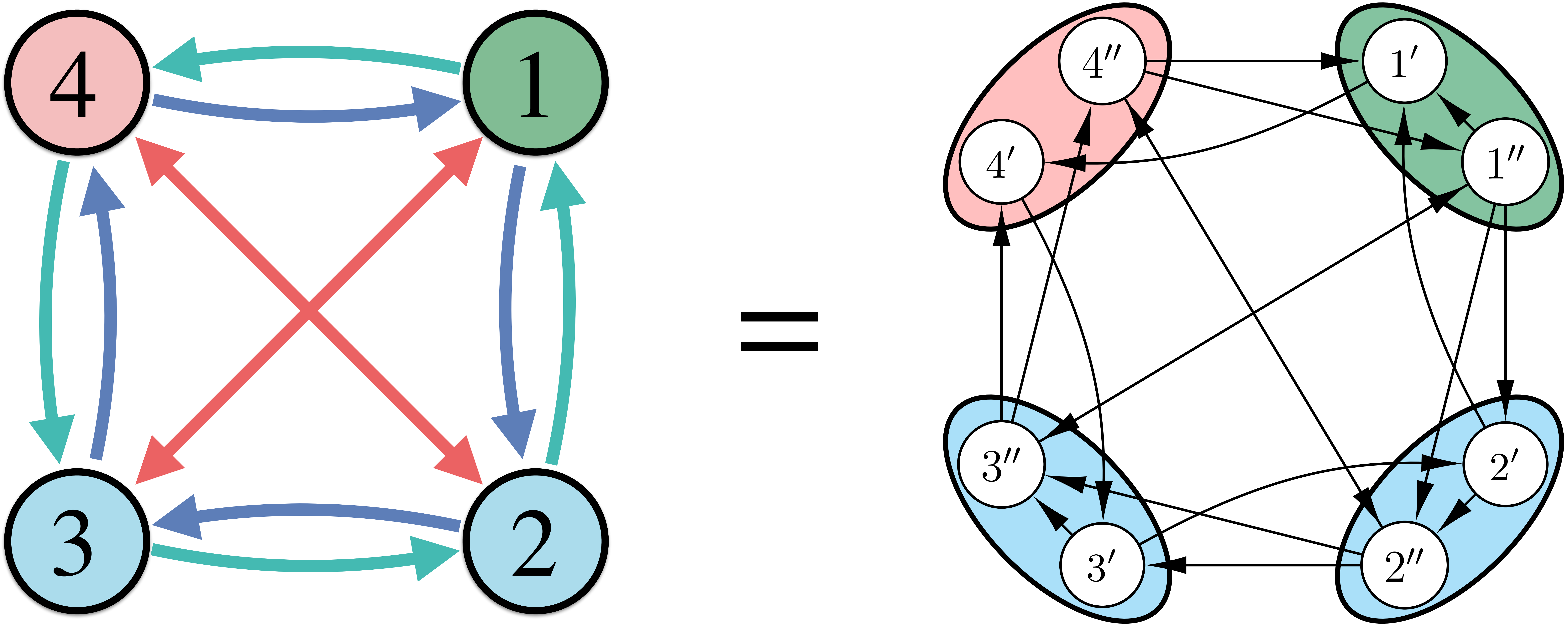
=
$1''$
$1'$
b
$2'$
$2''$

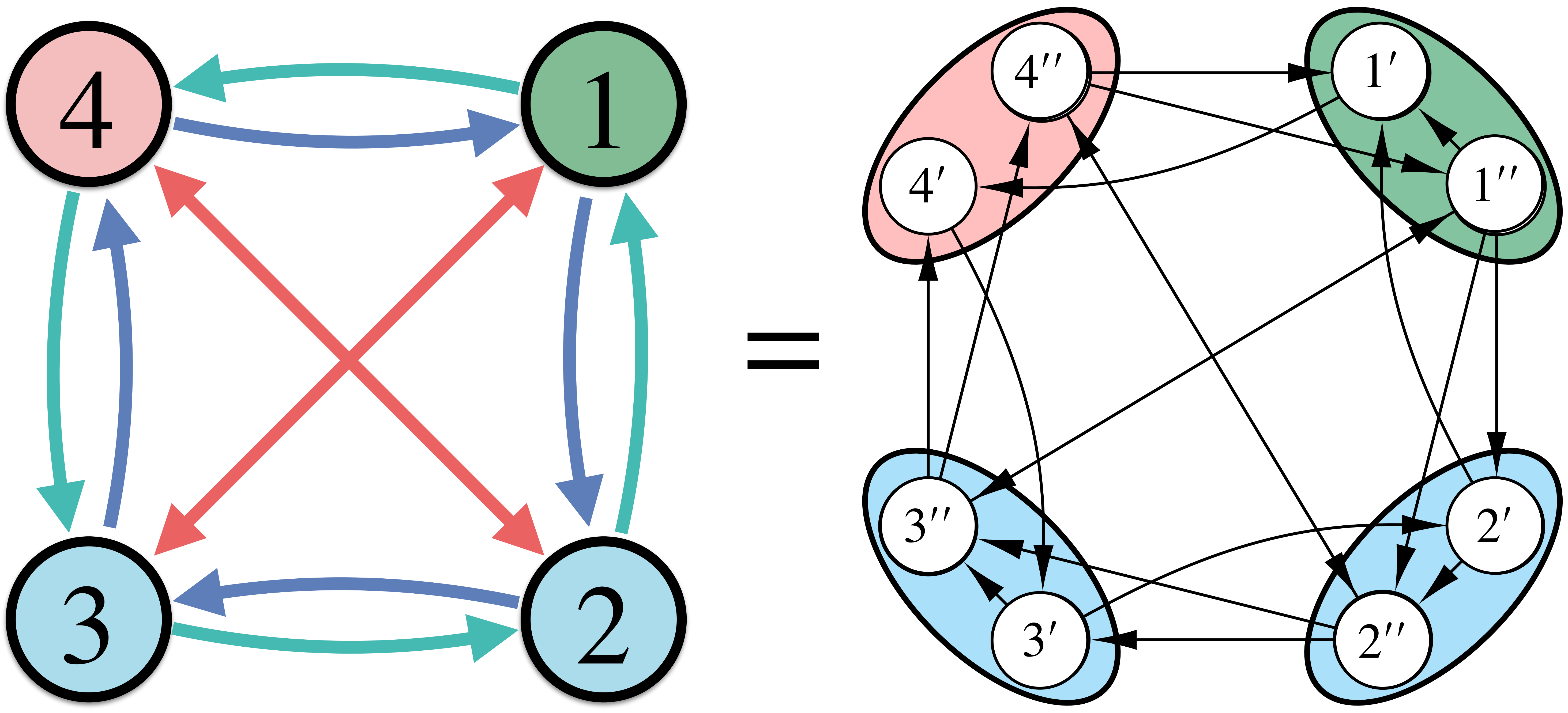


=
$1''$
$1'$
b
b
$2'$
$3''$
b
$2''$
$3'$



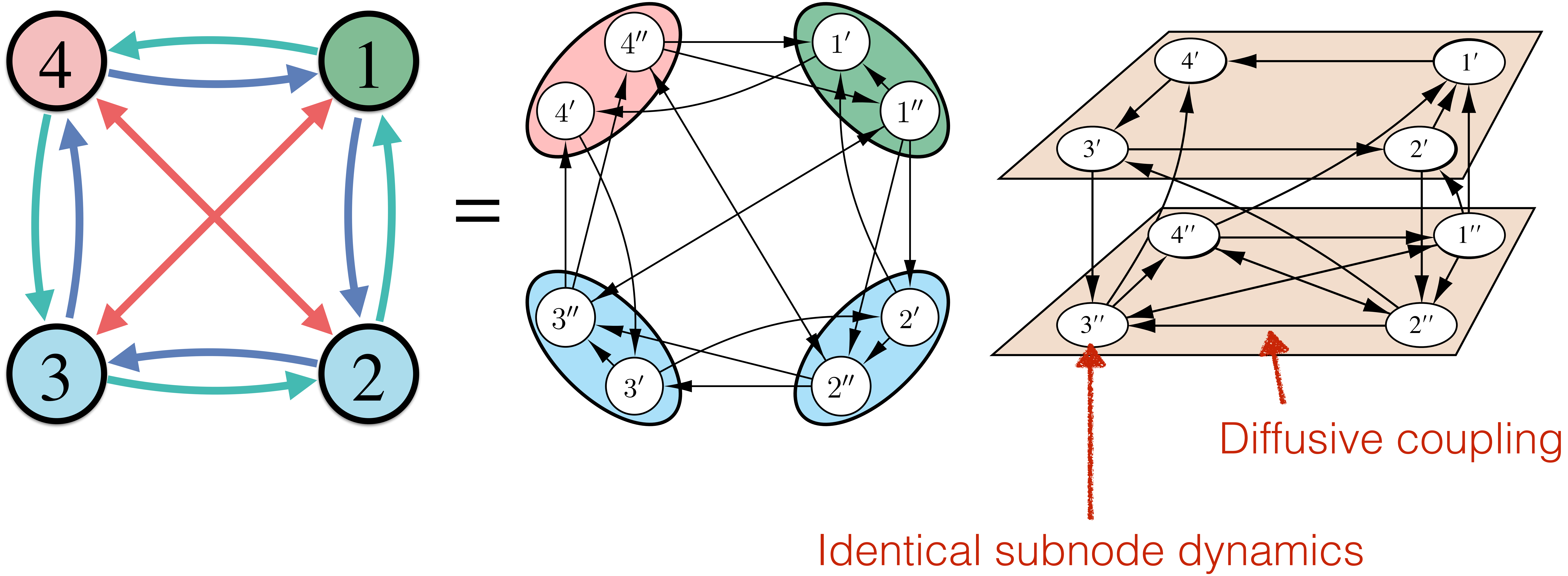
=
a
1″
1′
b
b
2′
3″
a
a
b
2″
3′


4
1
3
2
=
4″
1′
4′
1″
3″
2′
3′
2″

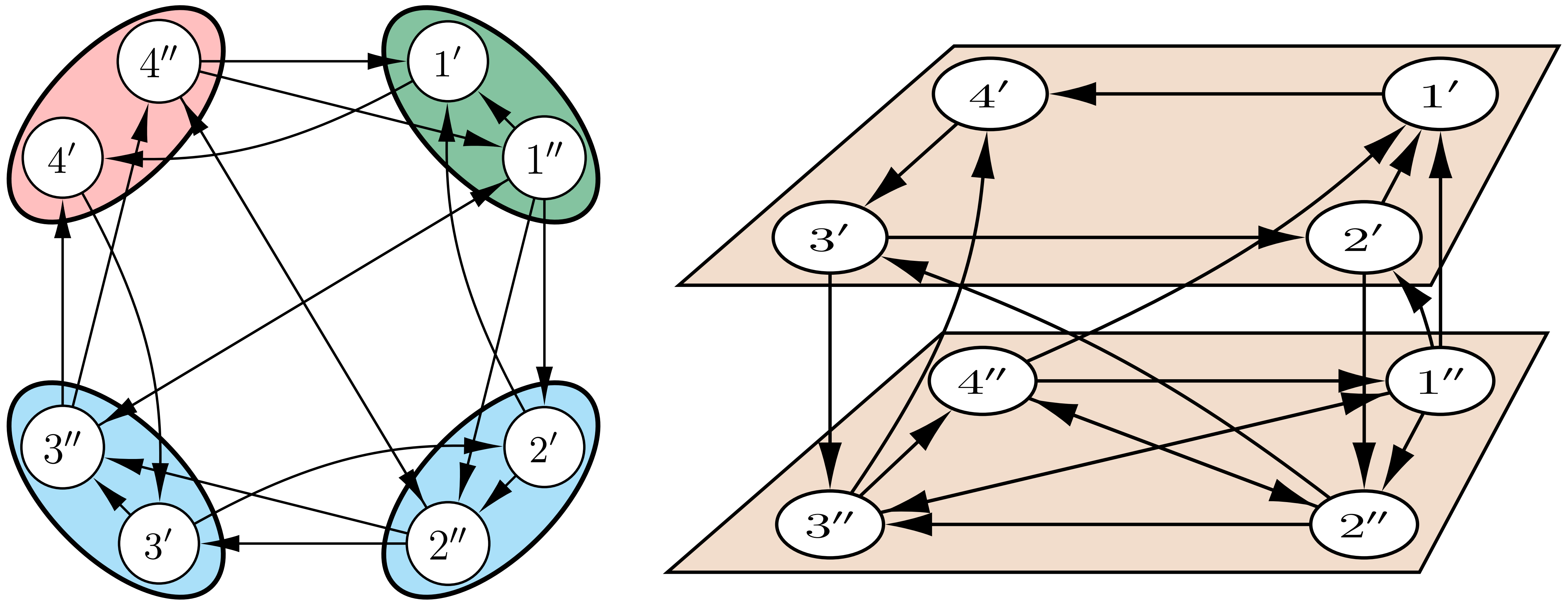


4
1
3
2
=
4′′
1′
4′
1′′
3′′
2′
3′
2′′

Multilayer network



Multilayer network of subnodes and sublinks

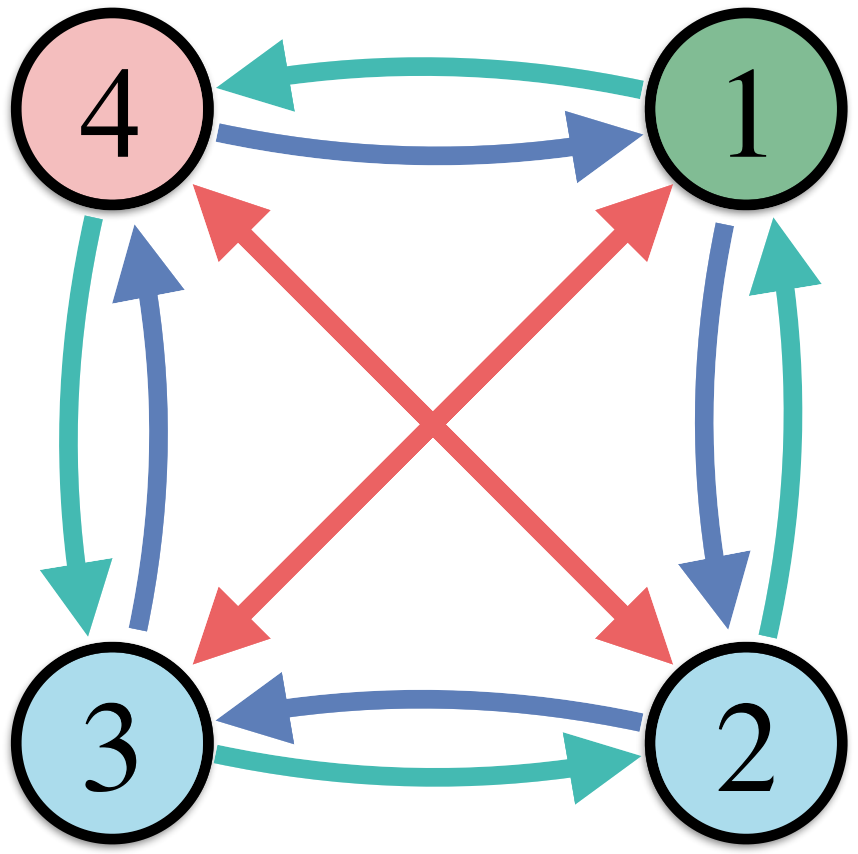


$$\dot{\boldsymbol{x}}_\ell^{(i)} = \boldsymbol{f}(\boldsymbol{x}_\ell^{(i)}) + \sum_{i'=1}^{N} \sum_{\ell'=1}^{L} \widetilde{A}_{\ell\ell'}^{(ii')} \left[\boldsymbol{h}(\boldsymbol{x}_{\ell'}^{(i')}) - \boldsymbol{h}(\boldsymbol{x}_\ell^{(i)})\right]$$

- Completely synchronous state is guaranteed
- Stability readily computed using Master Stability Function

  L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998)
- Valid for arbitrary $\boldsymbol{f}$ and $\boldsymbol{h}$

Network of nodes and links



node dynamics is different
if
internal link pattern is different

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}_i(\boldsymbol{X}_i) + \sum_{\alpha=1}^{K} \sum_{\substack{i'=1 \\ i' \neq i}}^{N} A_{ii'}^{(\alpha)} \boldsymbol{H}^{(\alpha)}(\boldsymbol{X}_i, \boldsymbol{X}_{i'})$$

- Synchronous state is guaranteed
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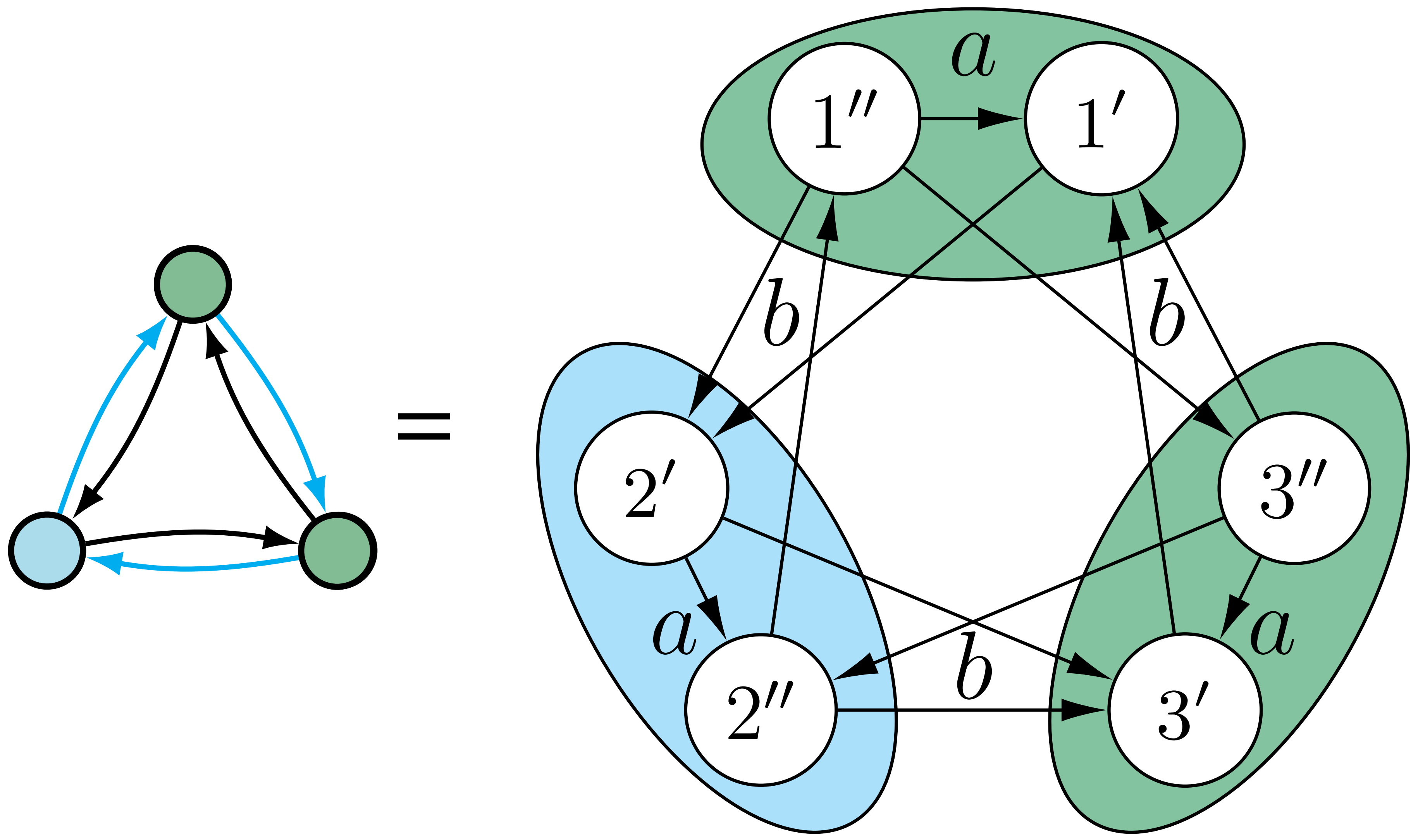
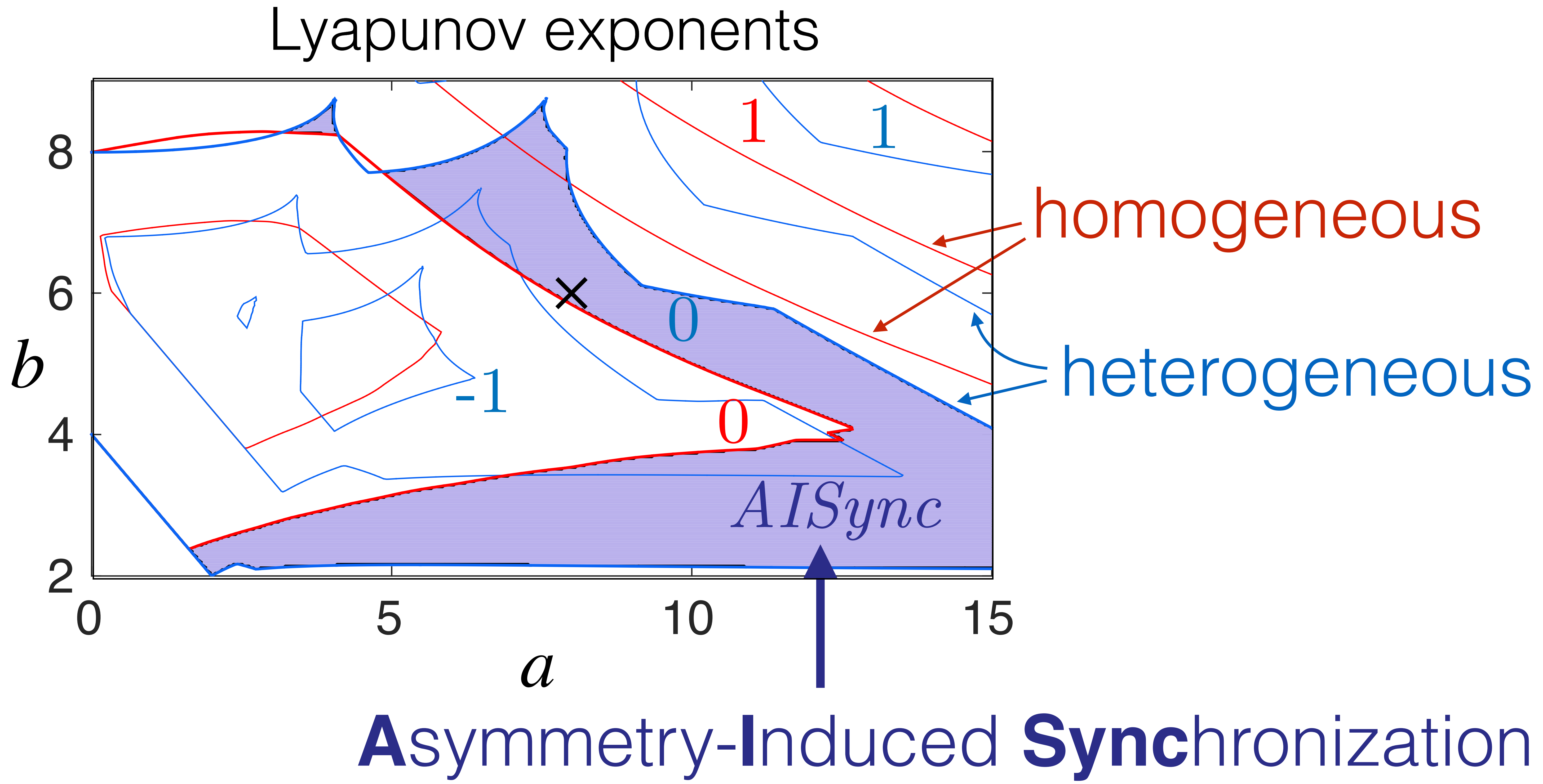
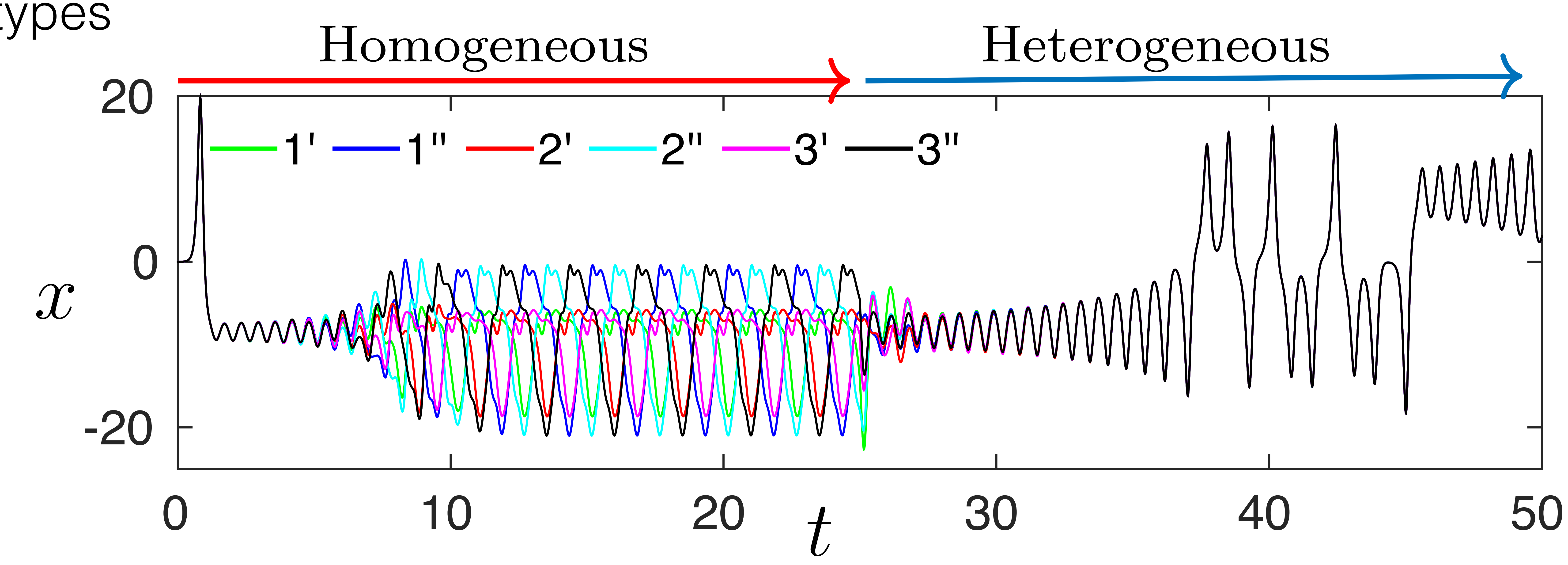




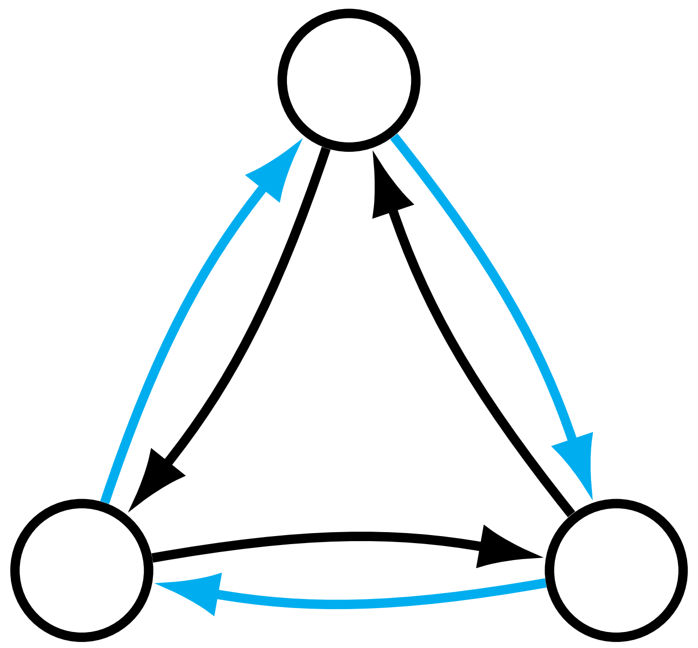
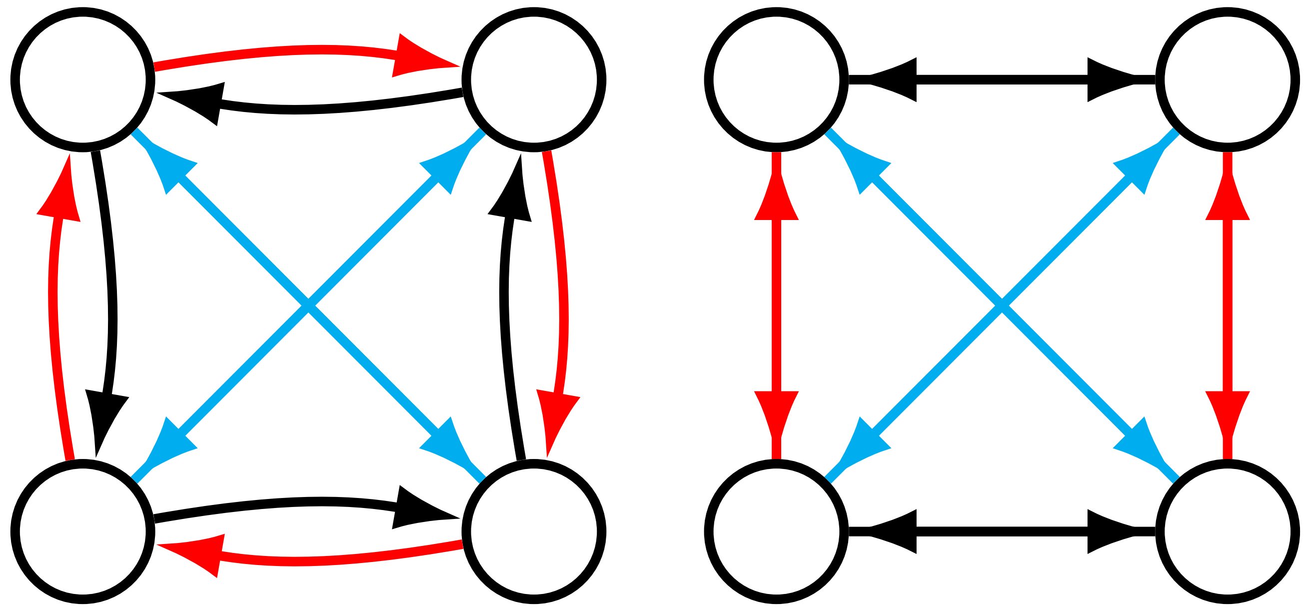
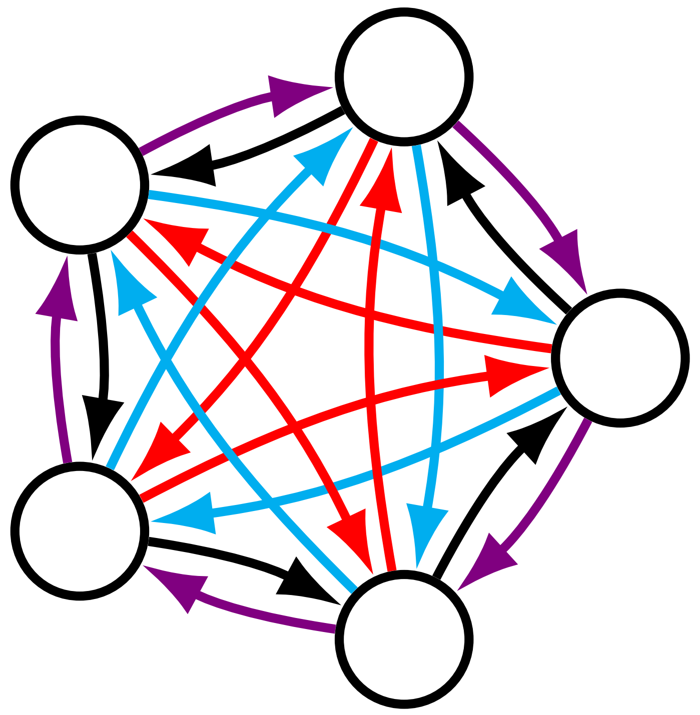
subnode = chaotic Lorenz oscillator
fixed external sublink pattern (strength b)
binary node types



What about other symmetric networks?

AISync strength r quantifies the degree to which a network structure favors AISync.

- $r = 0 \Rightarrow$ No AISync
- Larger $r \Rightarrow$ Favors AISync more strongly
- $r = 1 \Rightarrow$ There is an optimal heterogenous system.

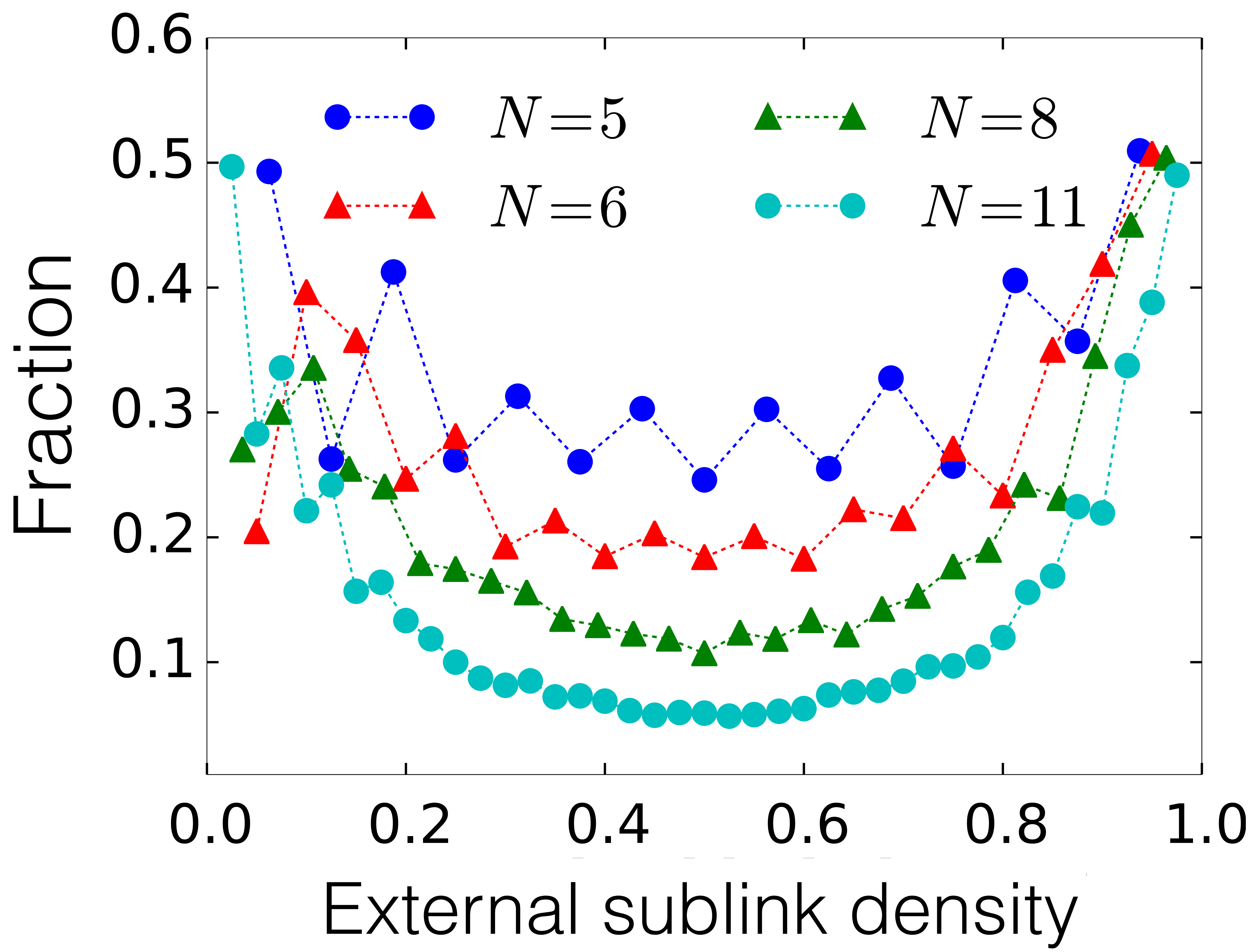
	$N = 3$	$N = 4$	$N = 5$
symmetric networks			
All 4 types (optimal)	9	14	21
All 4 types ($r > 0.2$)	11	81	254
All 4 types ($r > 0.05$)	29	318	2154
Binary ($r > 0.2$)	11	101	204
Binary ($r > 0.05$)	31	400	2406

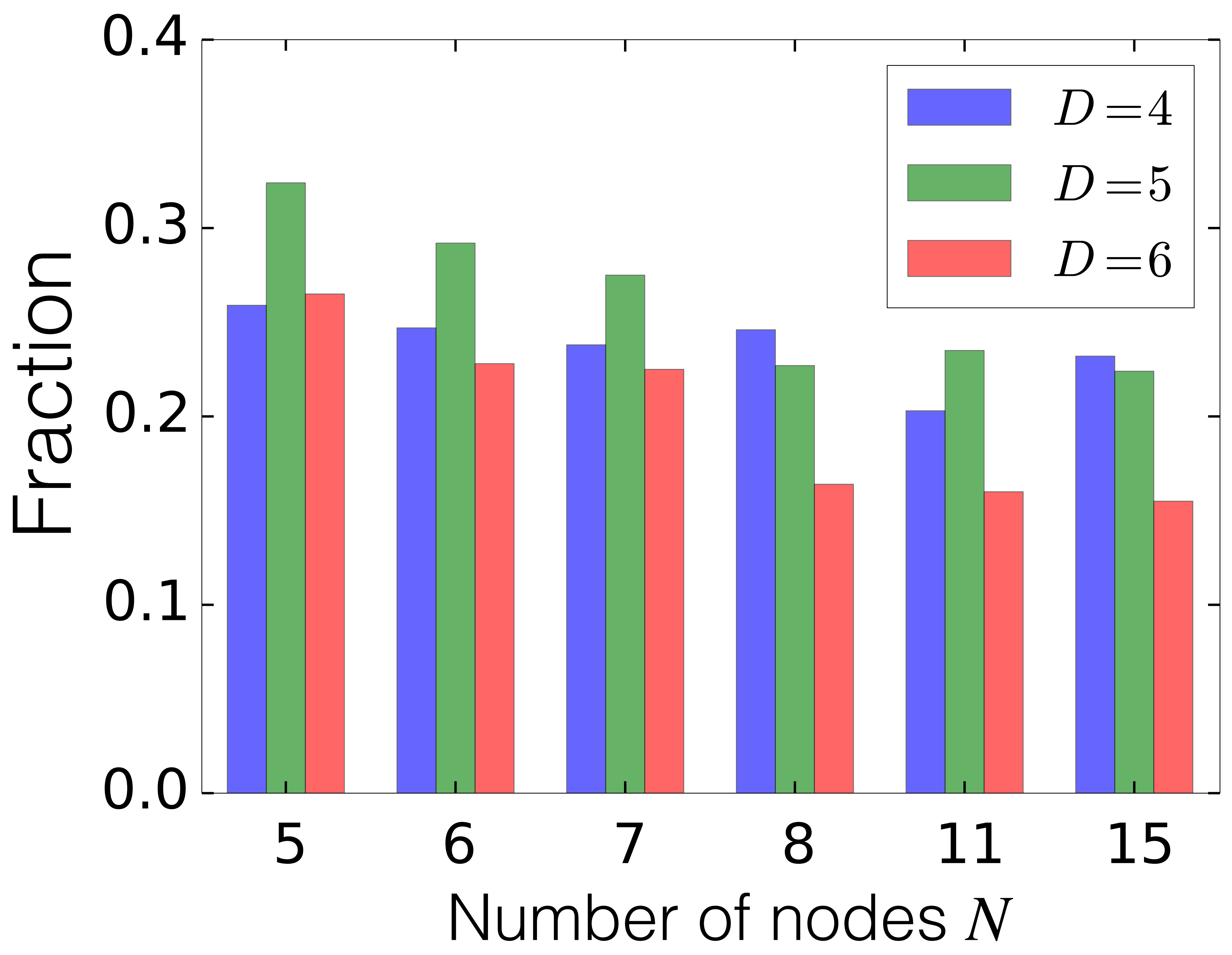
↑

Node types

Fraction of networks with $r > 0.05$

Within class of circulant-graphs (= all symmetric networks, if N is prime)





Significant fraction of systems are AISync-favoring for a range of system parameters

Summary

Symmetric states requiring system asymmetry
(converse of symmetry breaking)

- In network synchronization: fully synchronous state stable only when the oscillators are non-identical
- Observed quite often in the class of multilayer networks we considered

TN & AEM, *Symmetric states requiring system asymmetry*, Phys. Rev. Lett. **117**, 114101 (2016)
YZ, TN, & AEM, *Asymmetry-induced synchronization in oscillator networks*, to appear in Phys. Rev. E, arXiv:1705.07907