A new approach for curve matching with second-order Sobolev Riemannian metrics

M. Bauer, M. Bruveris, N. Charon, J. Møller-Andersen

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Sobolev metrics on the shape space of closed curves

Varifold representations and metrics

Matching algorithm

Results

The Manifold of Curves

Let $d \geq 2$. The space of closed, parametrized curves is

$$\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) = \{c \in C^\infty(\mathbb{S}^1, \mathbb{R}^d) : c'(\theta) \neq 0\} \subset C^\infty(\mathbb{S}^1, \mathbb{R}^d)\,.$$

The tangent space of $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$ at a curve c is the set of all vector fields along c,

$$T_c \,\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) = \left\{ h : \begin{tikzcd} & T\mathbb{R}^d \arrow[d, "p"] \\ \mathbb{S}^1 \arrow[ur, "h"] \arrow[r, "c"] & \mathbb{R}^d \end{tikzcd} \right\} \cong \left\{ h \in C^\infty(\mathbb{S}^1, \mathbb{R}^d) \right\}\,.$$

Arclength differentiation and integration

$$D_s = \frac{1}{|c'|}\partial_\theta\,, \qquad \mathrm{d}s = |c'(\theta)|\,\mathrm{d}\theta\,.$$

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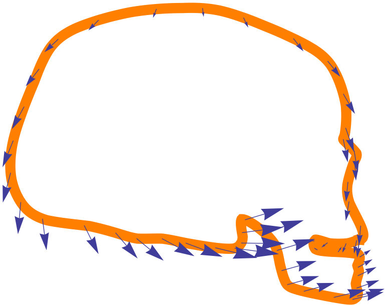
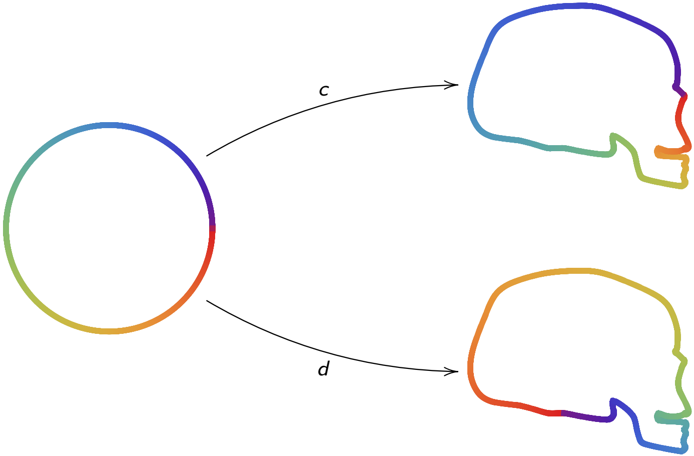


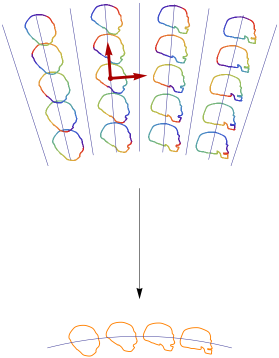
Figures taken from [M. Bauer, P. Harms, P.W. Michor, 2012]

Different Parameterizations



$$c, d : \mathbb{S}^1 \to \mathbb{R}^2, \qquad c = d \circ \varphi, \qquad \varphi \in \mathrm{Diff}(\mathbb{S}^1)$$

Definition of shape space

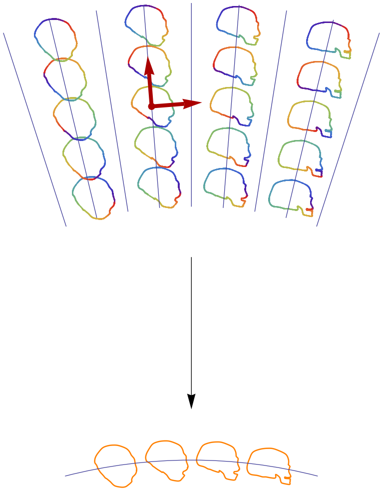


$$\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$$

$$\downarrow$$

$$\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) / \mathrm{Diff}(\mathbb{S}^1) = B_i(\mathbb{S}^1, \mathbb{R}^d)$$

Reparametrization Invariance



$$\begin{array}{c} \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) \\ \Big\downarrow \pi \\ \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)/\,\mathrm{Diff}(\mathbb{S}^1) \end{array}$$

A $\mathrm{Diff}(\mathbb{S}^1)$-equivariant metric "above" induces a metric "below" such that π is a Riemannian submersion.

$$G_c(h, k) = G_{c\circ\varphi}(h \circ \varphi, k \circ \varphi)$$

Sobolev Metrics and Geodesic Distance

- A Sobolev metric on $\operatorname{Imm}(\mathbb{S}^1, \mathbb{R}^d)$ is a metric of the form

$$G_c(h, k) = \int_{\mathbb{S}^1} a_0 \langle h, k \rangle + a_1 \langle D_s h, D_s k \rangle + \cdots + a_n \langle D_s^n h, D_s^n k \rangle \,\mathrm{d}s\,,$$

with $a_i \in \mathbb{R}^+$, $a_0 > 0$.

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- Sobolev metrics satisfy the reparametrization-equivariance property:

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  for all $\varphi \in \operatorname{Diff}(\mathbb{S}^1)$.
- They are in addition equivariant to the action on the left by the group of rigid motions:

$$G_{Rc+b}(Rh, Rk) = G_c(h,k)$$

  for all $(R,b) \in SO(d) \ltimes \mathbb{R}^d$

Sobolev Metrics and Geodesic Distance

The distance between two paramerized curves is then defined as the infimum over all path lengths

$$\mathrm{dist}(c_1, c_2) = \inf_c \int_0^1 \sqrt{G_c(c_t, c_t)}\, dt$$

subject to $c \in C^\infty([0,1], \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d))$ with $c(0) = c_1, c(1) = c_2$. This pathlength metric separates curves in $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$ provided G is stronger than H^1.

Induced quotient metric

On the shape space of unparametrized curves, the induced distance becomes

$$\operatorname{dist}([c_1],[c_2]) = \inf_{\varphi \in \operatorname{Diff}(\mathbb{S}^1)} \operatorname{dist}(c_1, c_2 \circ \varphi)$$

Considering the space of free immersions $\operatorname{Imm}_f(\mathbb{S}^1,\mathbb{R}^d) = \{c \in \operatorname{Imm}(\mathbb{S}^1,\mathbb{R}^d) \mid c \circ \varphi = c \Rightarrow \varphi = \operatorname{Id}\}$ and its quotient $B_{i,f}(\mathbb{S}^1,\mathbb{R}^d) \doteq \operatorname{Imm}_f(\mathbb{S}^1,\mathbb{R}^d)/\operatorname{Diff}(\mathbb{S}^1)$, one obtains

Theorem

For $d \geq 2$, a Sobolev metric with constant coefficients on $\operatorname{Imm}(\mathbb{S}^1,\mathbb{R}^d)$ induces a metric on $B_{i,f}(\mathbb{S}^1,\mathbb{R}^d)$ such that the projection $\pi : \operatorname{Imm}_f(\mathbb{S}^1,\mathbb{R}^d) \to B_{i,f}(\mathbb{S}^1,\mathbb{R}^d)$ is a Riemannian submersion.

Computing the distance and geodesics

Finding the distance between two given unparametrized closed curves $[c_1]$ and $[c_2]$ amounts in solving the following variational problem over all paths $c(t,\cdot) \in \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$ and reparametrization functions $\varphi \in \mathrm{Diff}(\mathbb{S}^1)$:

$$\mathrm{dist}([c_1],[c_2]) = \inf_{c,\varphi} \left\{ \int_0^1 \sqrt{G_c(c_t,c_t)}\, dt \ , \ c(0) = c_1, c(1) = c_2 \circ \varphi \right\}$$

Numerically, the approach of [Møller-Andersen 2017] discretizes both the curves and reparametrization functions using B-splines, which involves an extra projection step on $c_1 \circ \varphi$.

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Idea: reformulate the problem as a minimization over c only with a constraint of the form $\tilde{d}(c(1), c_2) = 0$, where $\tilde{d}$ is a **parametrization-invariant** distance between curves.

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Immersed curves as varifolds

Definition

A 1-dimensional (oriented) varifold of $\mathbb{R}^d$ is a distribution in W^*, where $W \hookrightarrow C^0(\mathbb{R}^d \times \mathbb{S}^{d-1})$ is a Banach space of test functions on $\mathbb{R}^d \times \mathbb{S}^{d-1}$.

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For any $c \in \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$, we define $\mu_c \in W^*$ such that for all $\omega \in W$:

$$\mu_c(\omega) = \int_{\mathbb{S}^1} \omega\left(c(\theta), \frac{c'(\theta)}{|c'(\theta)|}\right) ds$$

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One can check that for any $\varphi \in \mathrm{Diff}^+(\mathbb{S}^1)$, $\mu_{c\circ\varphi} = \mu_c$

Immersed curves as varifolds (oriented)

This leads to the diagram

$$
\begin{tikzcd}
\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) \arrow[rr, "\mu"] \arrow[d, "\pi^+"] & & W^* \\
\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)/\,\mathrm{Diff}^+(\mathbb{S}^1) \arrow[urr, "{[\mu]}"'] & &
\end{tikzcd}
$$

Immersed curves as varifolds (unoriented)

If, in addition, W is restricted to a space of antipodal-symmetric functions, i.e $\forall \omega \in W,\ \omega(x,-u) = \omega(x,u)$ for all $(x,u) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$, then:

$$\begin{tikzcd}
\mathrm{Imm}(\mathbb{S}^1,\mathbb{R}^d) \arrow[r, "\mu"] \arrow[d, "\pi"] & W^* \\
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\end{tikzcd}$$

The varifold distance on unparametrized curves

Principle: obtain an induced distance between curves from a simple metric on the varifold space W^*.

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We construct a particular class of test function space W as follows:

- Let $k_{pos}(x, y) \doteq \rho(|x - y|^2)$ for $x, y \in \mathbb{R}^d$ be a continuous radial positive kernel on $\mathbb{R}^d$.
- Let $k_{tan}(u, v) \doteq \gamma(u \cdot v)$ for $u, v \in \mathbb{S}^{d-1}$ be a continuous zonal positive kernel on $\mathbb{S}^{d-1}$.
- Define $k(x, u, y, v) \doteq \rho(|x - y|^2).\gamma(u \cdot v)$. Then k is a continuous positive kernel on $\mathbb{R}^d \times \mathbb{S}^{d-1}$. We define W to be the **Reproducing Kernel Hilbert Space** (RKHS) associated to k. By construction $W \hookrightarrow C^0(\mathbb{R}^d \times \mathbb{S}^{d-1})$.

The varifold distance on unparametrized curves

We can now define for all $c_1, c_2 \in \mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$:

$$d^{\mathrm{Var}}(c_1, c_2)^2 = \|\mu_{c_1} - \mu_{c_2}\|^2_{W^*} = \|\mu_{c_1}\|^2_{W^*} - 2\langle \mu_{c_1}, \mu_{c_2}\rangle_{W^*} + \|\mu_{c_2}\|^2_{W^*}$$

and thanks to the reproducing kernel property, we have explicitly:

$$\langle \mu_{c_1}, \mu_{c_2}\rangle_{W^*} = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \rho(|c_1(\theta_1) - c_2(\theta_2)|^2)\gamma\left(\frac{c_1'(\theta_1)}{|c_1'(\theta_1)|} \cdot \frac{c_2'(\theta_2)}{|c_2'(\theta_2)|}\right) ds_1 ds_2$$

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- d^{Var} is invariant to positive reparametrization and defines a **pseudo-distance** on $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)/\,\mathrm{Diff}^+(\mathbb{S}^1)$.
- If k_{pos} is a c^0-universal kernel and $\gamma(1) > 0$ then d^{Var} is a **distance** on the space of **embedded** unparametrized curves $\mathrm{Emb}(\mathbb{S}^1, \mathbb{R}^d)/\,\mathrm{Diff}^+(\mathbb{S}^1)$.
- d^{Var} is equivariant to rigid motions: $d^{\mathrm{Var}}(Rc_1 + b, Rc_2 + b) = d^{\mathrm{Var}}(c_1, c_2)$.

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- d^{Var} is invariant to all reparametrization and defines a **pseudo-distance** on $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) / \mathrm{Diff}(\mathbb{S}^1)$ if $\gamma(-t) = t$.
- If k_{pos} is a c^0-universal kernel, $\gamma(1) > 0$ and $\gamma(-t) = t$ then d^{Var} is a **distance** on the space of **embedded** unparametrized curves $\mathrm{Emb}(\mathbb{S}^1, \mathbb{R}^d) / \mathrm{Diff}(\mathbb{S}^1)$.
- d^{Var} is equivariant to rigid motions: $d^{\mathrm{Var}}(Rc_1 + b, Rc_2 + b) = d^{\mathrm{Var}}(c_1, c_2)$.

A relaxed variational problem

Geodesics are the minimizers of the energy functional

$$E(c) = \int_0^1 G_c(c_t, c_t)\,dt, \quad s.t. \quad c(0) = c_1, c(1) = c_2.$$

We can compute the distance on shape space by minimizing

$$\min_c E(c) \quad s.t. \quad c(0) = c_1, d^{\mathrm{Var}}(c(1), c_2) = 0.$$

For simplicity we consider the relaxed functional

$$\min_{c, c(0)=c_1} E(c) + \lambda d^{\mathrm{Var}}(c(1), c_2)^2$$

for fixed λ. This should solve the problem as $\lambda \to \infty$.

Discretization

We use B-splines in time (t) and space (θ) of order n_t and n_θ,

$$c(t,\theta) = \sum_{i=1}^{N_t} \sum_{j=1}^{N_\theta} c_{i,j} B_i(t) C_j(\theta)\,.$$

Advantages:

- Analytic expressions for derivatives are available.
- Can control global smoothness

$$B_i \in C^{n_t - 1}([0,1]),\ C_j \in C^{n_\theta - 1}([0,2\pi])\,.$$

- The basis functions B_i, C_j have local support.

Drawbacks:

- Reparametrization $(c,\varphi) \mapsto c \circ \varphi$ does not preserve order of B-spline.

Discretization - Varifold distance

We write $c(1)(\theta) = \sum_{j=1}^{N_\theta} c_{N_t,j} C_j(\theta)$ and $c_2(\theta) = \sum_{j=1}^{N_\theta} \tilde{c}_j C_j(\theta)$ with the derivatives:

$$c(1)'(\theta) = \sum_{j=1}^{N_\theta} c_{N_t,j} C_j'(\theta), \quad c_2'(\theta) = \sum_{j=1}^{N_\theta} \tilde{c}_j C_j'(\theta)$$

With $u_1(\theta) = c(1)'(\theta)/|c(1)'(\theta)|$, $u_2(\theta) = c_2'(\theta)/|c_2'(\theta)|$:

$$\begin{aligned} d^{\mathrm{Var}}(c(1), c_2)^2 &= \|\mu_{c_1}\|_{W^*}^2 - 2\langle \mu_{c_1}, \mu_{c_2}\rangle_{W^*} + \|\mu_{c_2}\|_{W^*}^2 \\ &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \rho(|c(1)(\theta_1) - c(1)(\theta_2)|^2)\gamma\left(u_1(\theta_1)\cdot u_1(\theta_2)\right) ds_1 ds_2 \\ &- 2\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \rho(|c(1)(\theta_1) - c_2(\theta_2)|^2)\gamma\left(u_1(\theta_1)\cdot u_2(\theta_2)\right) ds_1 ds_2 \\ &+ \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \rho(|c_2(\theta_1) - c_2(\theta_2)|^2)\gamma\left(u_2(\theta_1)\cdot u_2(\theta_2)\right) ds_1 ds_2 \end{aligned}$$

Discretization - Varifold distance

$$d^{\mathrm{Var}}(c(1), c_2)^2 = \|\mu_{c_1}\|^2_{W^*} - 2\langle \mu_{c_1}, \mu_{c_2} \rangle_{W^*} + \|\mu_{c_2}\|^2_{W^*}$$
$$= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \rho(|c(1)(\theta_1) - c(1)(\theta_2)|^2)\gamma\left(u_1(\theta_1) \cdot u_1(\theta_2)\right) ds_1 ds_2 \quad - 2 \dots$$

- No closed form expression for the integrals: these are approximated using quadrature methods.
- Gradient w.r.t the $(c_{N_t,j})_{j=1,\dots,N_\theta}$ is computed by chain rule.
- In the experiments, we use $\rho(s) = e^{-\frac{s^2}{\sigma^2}}$ (Gaussian kernel), $\gamma(s) = s^2$ (Binet kernel).

The inexact matching functional

The discretized optimization problem becomes:

$$\min_{c_{ij}} E(c_{ij}) + \lambda d^{\mathrm{Var}}(c(1), c_2)^2$$

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- Limited memory quasi-Newton method: L-BFGS (HANSO library)
- Initialization by constant path ($c_{i,j} = c_0$)
- (Optional) Multi-grid and multiscale speed-up
- We can also recover a rotation/translation invariant distance by also optimizing over $(R, b) \in SO(d) \ltimes \mathbb{R}^d$:

$$\min_{c_{ij},R,b} E(c_{ij}) + \lambda d^{\mathrm{Var}}(c(1), Rc_2 + b)^2$$

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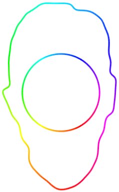
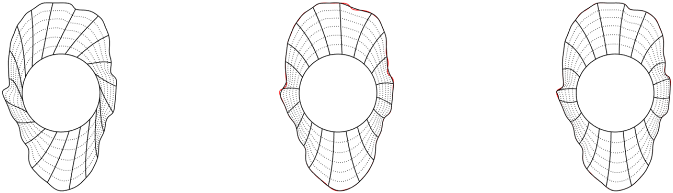
A simple example

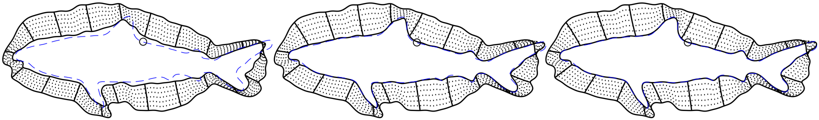




Parametrized H^2 Unparametrized H^2 Varifold H^2

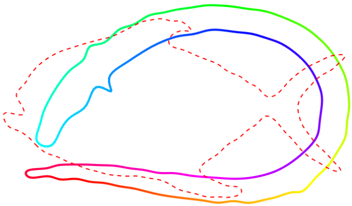
Influence of λ

3 minimizers for $\lambda = 0.3, 1$ and 5. Target curve in blue.



Intrinsic vs extrinsic models

Self-intersections can appear in this model:



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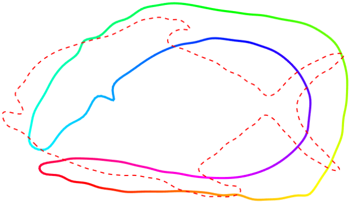

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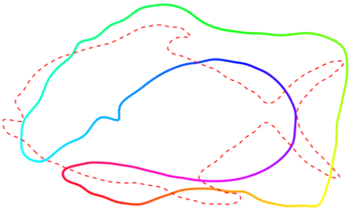

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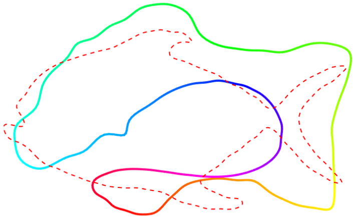

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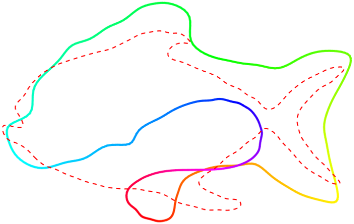

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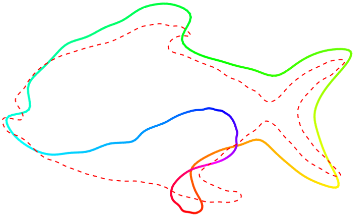

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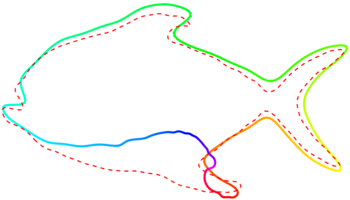

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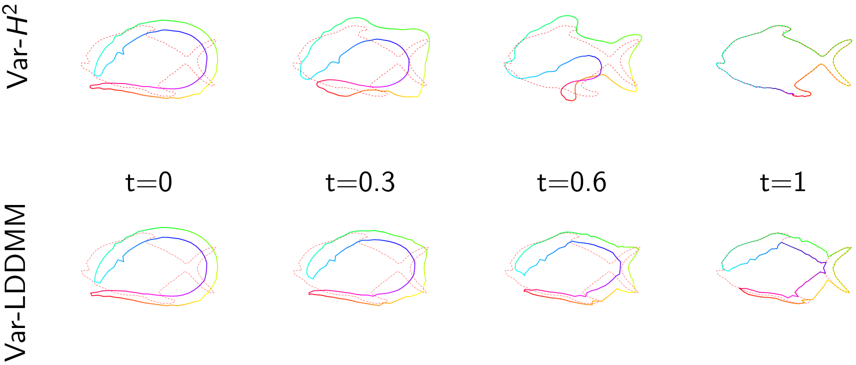
Intrinsic vs extrinsic models

Unlike with extrinsic deformation frameworks like LDDMM



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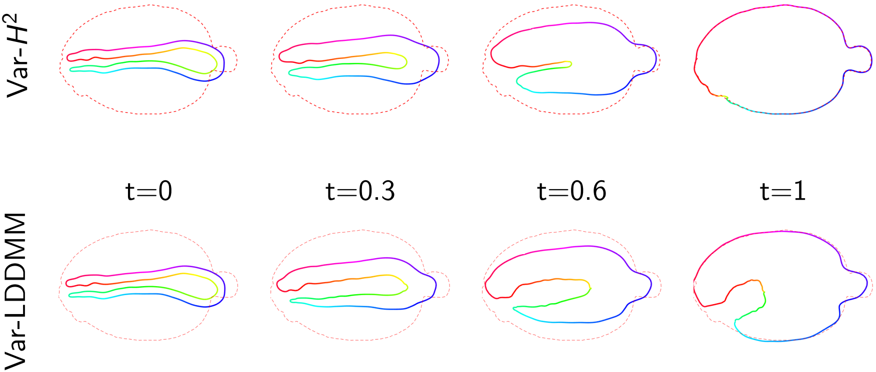
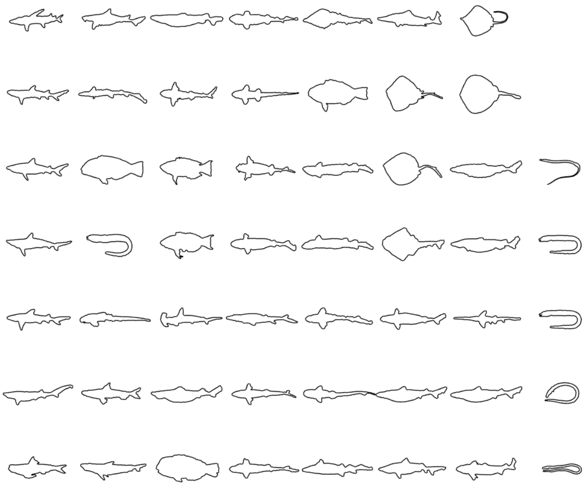


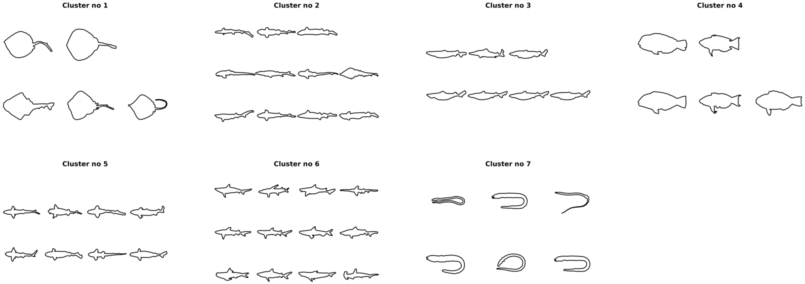
Shape clustering

54 shapes from the Surrey fish database



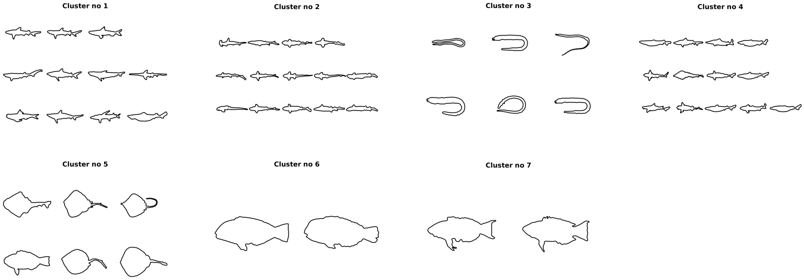
Shape clustering

Spectral clustering based on the estimated pairwise H^2 distances:

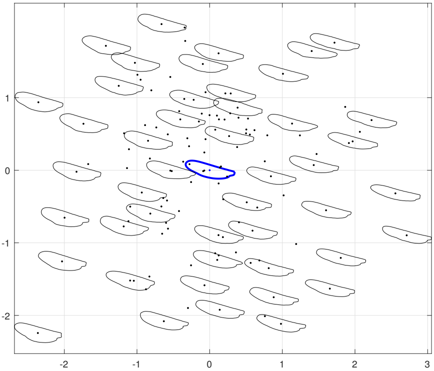


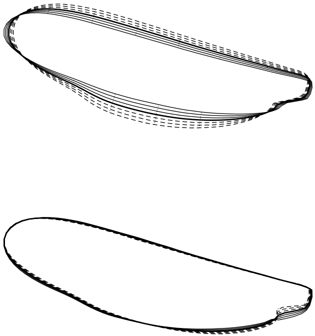
Shape clustering

Spectral clustering based on the pairwise varifold metric (modulo rigid motions):



Mosquito wings: PCA analysis





Conclusions and outlook

- We have proposed a new mathematical and numerical formulation of the distance/geodesic estimation problem for Sobolev metrics on unparametrized curves.
- This allows to do non-linear statistical analysis on shape spaces.
- The method is robust and decently fast.

Ongoing and future work

- Extend the approach to other Riemannian metrics on curves.
- The method is easier to generalize to surfaces.
- Augmented Lagrangian method in the space of varifolds in order to select λ automatically.
- Scale invariance.