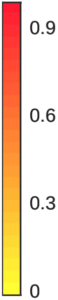
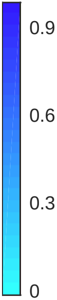
Calibration

This is a colourful sentence. □ × △





Isochrons for Saddle-Type Periodic Orbits in Three-Dimensional Space

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Isochrons

For any system of ordinary differential equations (ODE's), e.g.,

$$\dot{x} = \mu a x - y - b x(x^2 + y^2),$$
$$\dot{y} = x + \mu(a + c)y - (b + d)y(x^2 + y^2),$$

$$a = 0.1, \quad b = -0.05, \quad c = 0.9, \quad d = 0.45, \quad \mu = 2.0,$$

which contains an attracting periodic orbit, we can assign an asymptotic phase to all initial conditions which tend towards that orbit. An 'isochron' is a unique object that connects all initial conditions that have identical asymptotic phase.

- Isochrons were introduced by Winfree in 1974.
  A.T. Winfree, *Patterns of phase compromise in biological cycles,* J. Math. Biol., 1 (1974) pp73-93.
- Guckenheimer formalised the isochron definition as the stable manifold of the time-T_Γ map of the point γ_θ on the periodic orbit Γ.
  J. Guckenheimer, *Isochrons and Phaseless Sets,*J. Math. Biol., 1 (1975) pp259-273.
- The isochrons of a periodic orbit Γ define a set of $(n-1)$-dimensional smooth manifolds that foliate its n-dimensional basin of attraction.

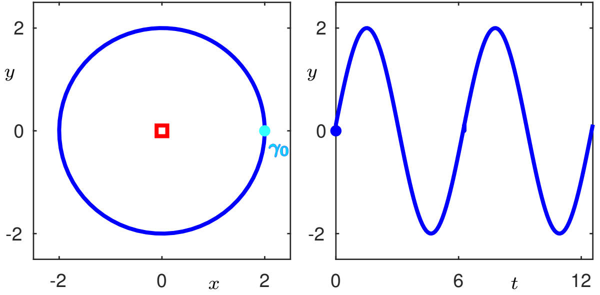
Motivation

Isochrons have been applied in the study a variety of phenomena including,

- Phase resetting in cardiac cells
- Neuronal bursting
- Models of chemical reactions
- Electronics.

They are particularly useful when considering phase resetting experiments often encountered in biology, and phase reductions of models.

A notion of phase



- For an attracting periodic orbit, the convention is to choose the **zero-phase** point γ_0 as the maximum in x.
- Phase is defined on $[0, 1)$, such that a phase $\theta = 0$ corresponds to a time of nT_Γ, $n \in \mathbb{Z}$.
- Γ is defined such that it begins and ends at γ_0; it lies on the **zero-phase** isochron.

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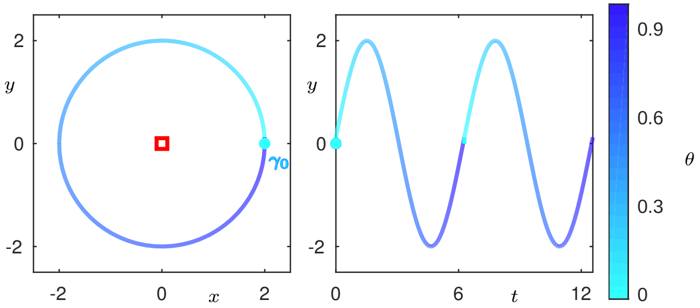
A notion of phase



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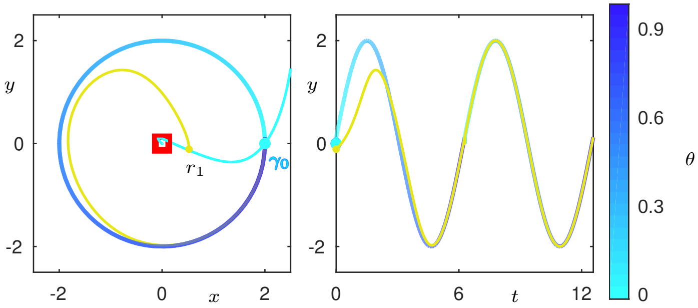
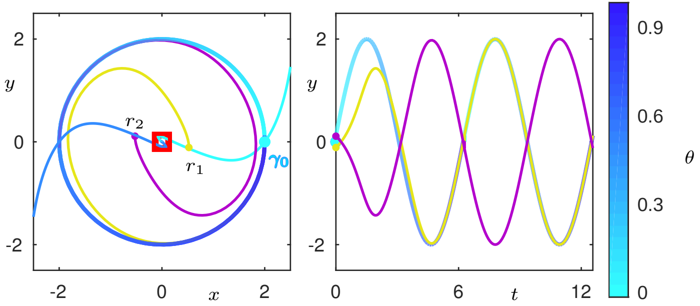


- Γ is defined such that it begins and ends at γ_0; it lies on the **zero-phase** isochron.
- The **zero-phase** isochron intersects the periodic orbit at γ_0.
- r_1 also starts on the **zero-phase** isochron; r_1 must syncrhonise with Γ.
- Any trajectory that starts on the **zero-phase** isochron will synchronise with the periodic orbit with phase $\theta = 0$.

A notion of phase



- Γ is defined such that it begins and ends at γ_0; it lies on the zero-phase isochron.
- Any trajectory that starts on the zero-phase isochron will synchronise with the periodic orbit with phase $\theta = 0$.
- r_2 starts on the half-phase isochron, and so remains identically out of phase with Γ and r_1 in asymptotic time.

Isochron computation by Numerical Continuation

We use the numerical continuation of a two-point boundary value problem as an effective and accurate method for computing isochrons.[1] [2] This boundary value problem is a direct result of the definition of isochrons as the stable manifold of the associated time-T_Γ map for a phase point $\gamma_\theta \in \Gamma$. The eigenvector associated with this stable manifold is the **linear approximation** $\vec{w}$ of the associated isochron. The two point boundary value problem that we will continue requires that the end point $\vec{u}(T_\Gamma)$ lies on the **linear approximation**.

$$(\vec{u}(T_\Gamma) - \vec{\gamma}_\theta) \cdot \vec{w} = \eta$$
$$(\vec{u}(T_\Gamma) - \vec{\gamma}_\theta) \cdot \vec{w}^\perp = 0$$

[1] H.M. Osinga, J. Moehlis, *Continuation-based computation of global isochrons*, SIAM Journal on Applied Dynamical Systems, 9(4) (2010)

[2] P. Langfield, B. Krauskopf, H.M. Osinga, *Solving Winfree's puzzle: the isochrons in the FitzHugh-Nagumo model*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 24 (2014)

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$$(\vec{u}(T_\Gamma) - \vec{\gamma}_\theta) \cdot \vec{s} = \tau$$
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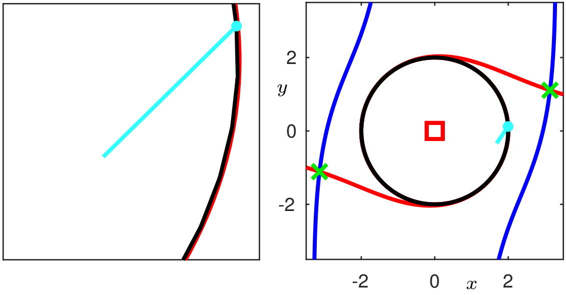

- We start with an attracting periodic orbit Γ, and the **linear approximation** of the isochron at γ_0.
- The Periodic orbit is a trajectory that satisfies the two-point boundary value problem.
  - Γ begins on the **linear approximation** of the isochron.
  - Γ has an integration time T_Γ.

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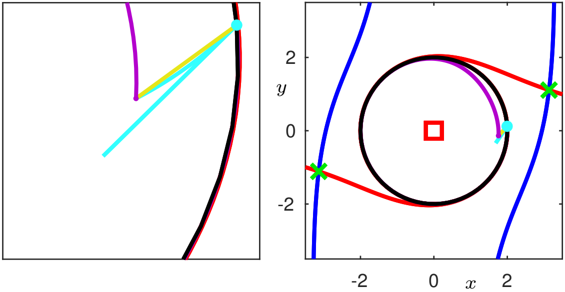
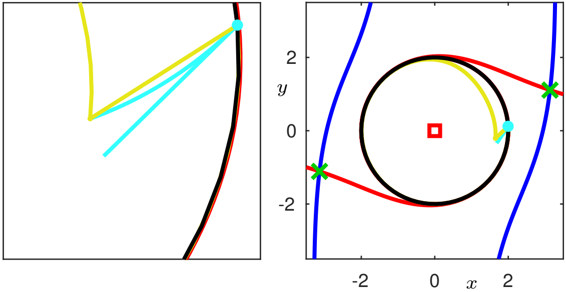
Isochrons by numerical continuation



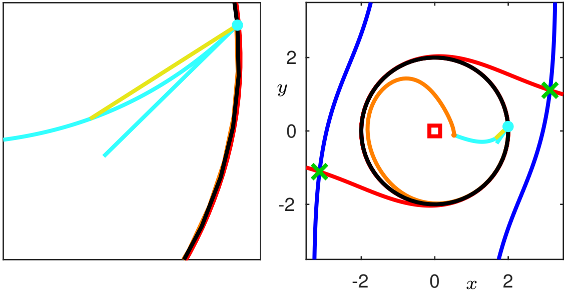
- By moving the end point of Γ along the **linear approximation**, a new trajectory is created.
  - This new trajectory returns to the **linear approximation** at time T_Γ.
  - The trajectory's start point must lie on the isochron.
- We monitor the distance of the start point from the **linear approximation** until it reaches δ_{max}.

Isochrons by numerical continuation



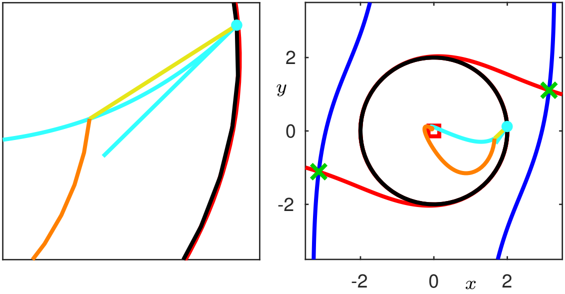
- We monitor the distance of the start point from the linear approximation until it reaches δ_{max}.
- This trajectory defines the fundamental domain, a closer approximation to the isochron than the linear approximation.
- The start point of the trajectory has swept out the zero phase isochron as it was continued.

Isochrons by numerical continuation



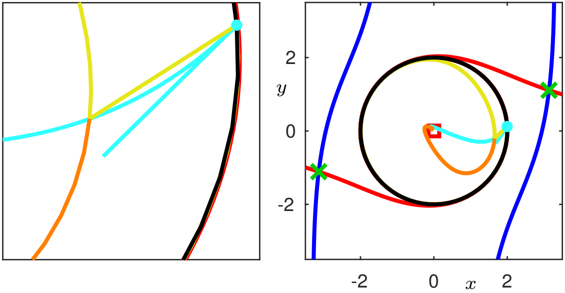
- The start point of the trajectory has swept out the **zero phase** isochron as it was continued.
- We continue the trajectory such that it's end point lies on the **fundamental domain**.

Isochrons by numerical continuation



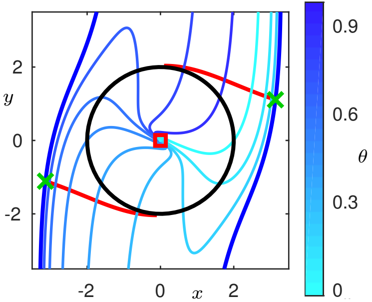
- When the trajectory's end point reaches the fundamental domain's length:
  - Stop continuation.

Isochrons by numerical continuation



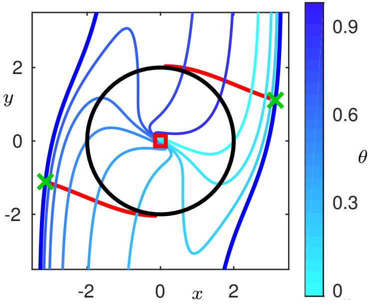
- When the trajectory's end point reaches the fundamental domain's length:
  - Stop continuation.
  - Append the trajectory that defines the fundamental domain.
  - Increase the time interval for the trajectory to $2T_\Gamma$.
  - Continue the new trajectory over the fundamental domain.

Isochrons by numerical continuation



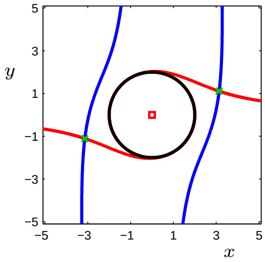
- Repeat for different phases.

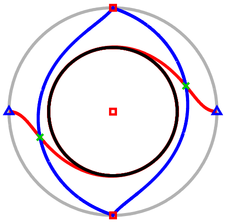
Isochrons by numerical continuation



- Isocrhons must accumulate on the basin boundary.
- The unstable invariant manifolds of the saddle points must intersect each isochron infinitely many times.

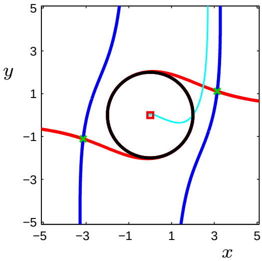
Non-compact basin boundaries

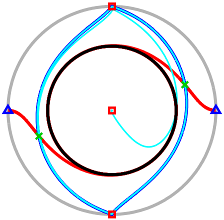




- We can compactify $\mathbb{R}^2$ onto $\mathbb{D}$ in order to apply our method effectively far away from Γ.
- This compactification preserves geometry, invariant dynamics, and introduces equilibria at infinity.

Non-compact basin boundaries





- We can compute global isochrons effectively and accurately, and visualise their geometries near infinity.[a]
- For this example, the isochrons must be computed to very large arclengths in order to confirm phase sensitivity at the basin boundary.

[a] J. Hannam, B. Krauskopf, H.M. Osinga, *Global isochrons of a planar system near a phaseless set with saddle equilibria,* The European Physical Journal Special Topics, 225(13-14) (2016)

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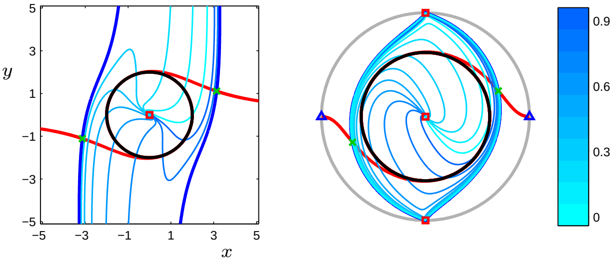

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Adaptation of method for saddle-type periodic orbits

We can compute the isochrons of a saddle-type periodic orbit in a three-dimensional system by modifying the method used in the plane to account for the extra degrees of freedom.

Fundamental Domain

$$(\vec{u}(T_\Gamma) - \vec{\gamma}_\theta) \cdot \vec{w} = \eta$$
$$(\vec{u}(T_\Gamma) - \vec{\gamma}_\theta) \cdot \vec{w}^\perp = 0$$
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Isochron

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$$(\vec{u}(T_\Gamma) - \vec{\gamma}_\theta) \cdot \vec{w}^n = 0$$
$$(\vec{u}(0) - \vec{\gamma}_\theta) \cdot \vec{w}^\times = \delta_n$$
$${\delta_\theta}^2 + {\delta_n}^2 = {\delta_n}^2$$

Isochron

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Backward time isochrons

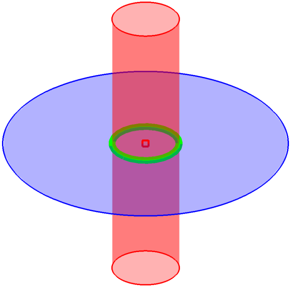
The phase θ of an initial condition $\vec{u}_0$ is given by the asymptotic phase function $\Theta(\vec{u}_0) \in [0, 1)$ assigned by the condition,

$$\lim_{t\to\infty} \| \Phi(t, \vec{u}_0) - \Phi(t + \Theta(\vec{u}_0)T_\Gamma, \gamma_0) \| = 0.$$

For the unstable manifolds of periodic orbits, we can define backward-time isochrons – objects equivalent to the forward-time isochrons of that periodic orbit under the transformation $t = -t$. Thus the asymptotic phase of an 'initial condition' $\vec{u}_0$ on a backwards-time isochron governed by the condition,

$$\lim_{t\to\infty} \| \Phi(-t, \vec{u}_0) - \Phi(\Theta(\vec{u}_0)T_\Gamma - t, \gamma_0) \| = 0.$$

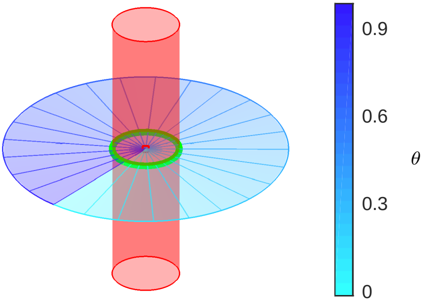
Simple isochrons on orientable manifolds



$$\dot{x} = \beta x - \omega y - x(x^2 + y^2)$$
$$\dot{y} = \omega x + \beta y - y(x^2 + y^2)$$
$$\dot{z} = \alpha z$$

- The stable and unstable invariant manifolds or Γ are known analytically, and serve as a good test case.
- The basin of attraction of Γ is its stable invariant manifold.

Simple isochrons on orientable manifolds



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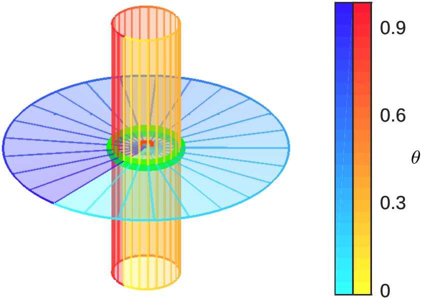

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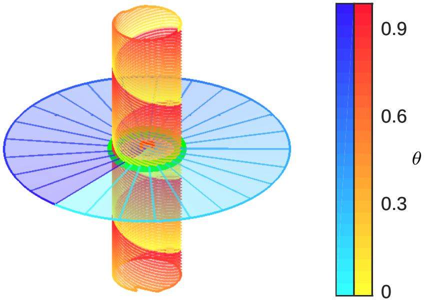
$$\dot{z} = \alpha z$$

- The stable and unstable invariant manifolds or Γ are known analytically, and serve as a good test case.
- In reverse time the unstable invariant manifold forms the basin of attraction of Γ.
- The Backward-time isochrons of Γ foliate its unstable invariant manifold.
- Since ω has no dependence on x, y, z, the isochrons are straight lines.

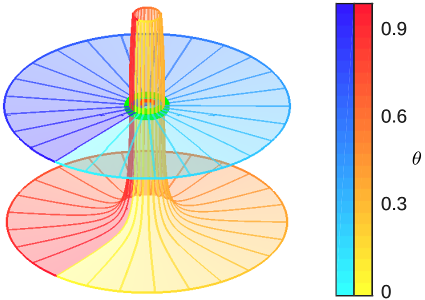
Simple isochrons on orientable manifolds



$$\dot{x} = \beta x - (1 - \kappa z)\omega y - x(x^2 + y^2)$$
$$\dot{y} = (1 - \kappa z)\omega x + \beta y - y(x^2 + y^2)$$
$$\dot{z} = \alpha z$$

- By changing ω to depend on z, the isochrons on the unstable invariant manifold are no longer straight lines.
- The geometry of the unstable invariant manifold is the same, but the geometry of its isochrons change due to the new dynamics.

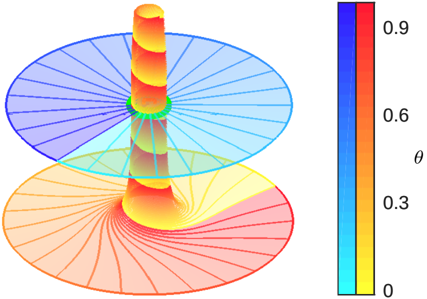
Simple isochrons on orientable manifolds



$$\dot{x} = \beta x - \omega y - x\frac{x^2+y^2}{1-z\zeta}$$
$$\dot{y} = \omega x + \beta y - y\frac{x^2+y^2}{1-z\zeta}$$
$$\dot{z} = \alpha z$$

- We can change the geometry of the unstable invariant manifold so that it is no linger a cylinder.
- The geometry of the isochrons also change to account for the new geometry of the unstable invariant manifold.

Simple isochrons on orientable manifolds



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- The geometry of the isochrons also change to account for the new geometry of the unstable invariant manifold.

Sanstede's System

For parameter values,

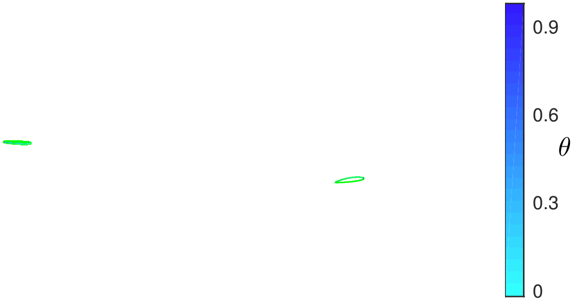
$$a = 0.22, \quad b = 1.0, \quad c = -2.0, \quad \alpha = 0.3, \quad \beta = 1.0, \quad \gamma = 2.0, \quad \mu = 0.004, \widetilde{\mu} = 0.0$$

this system[3] contains a saddle-type periodic orbit.

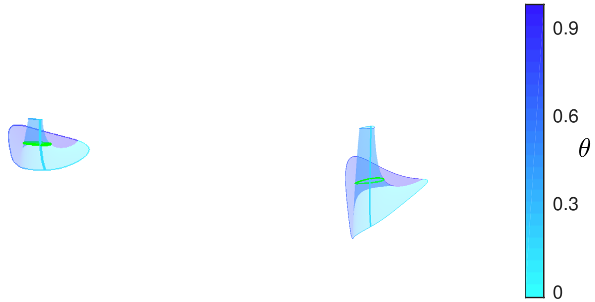
$$\begin{aligned} \dot{x} &= ax + by - ax^2 + x(2 - 3x)(\widetilde{\mu} - \alpha z) \\ \dot{y} &= bx + ay - 1.5bx^2 - 1.5axy - 2y(\widetilde{\mu} - \alpha z) \\ \dot{z} &= cz + \mu x + \gamma xz + \alpha\beta(x^2(1 - x) - y^2) \end{aligned}$$

[3] B. Sandstede, *Constructing dynamical systems having homoclinic bifurcation points of codimension two,* Journal of Dynamics and Differential Equations, 9 (1997)

Visulalising the stable invariant manifold?

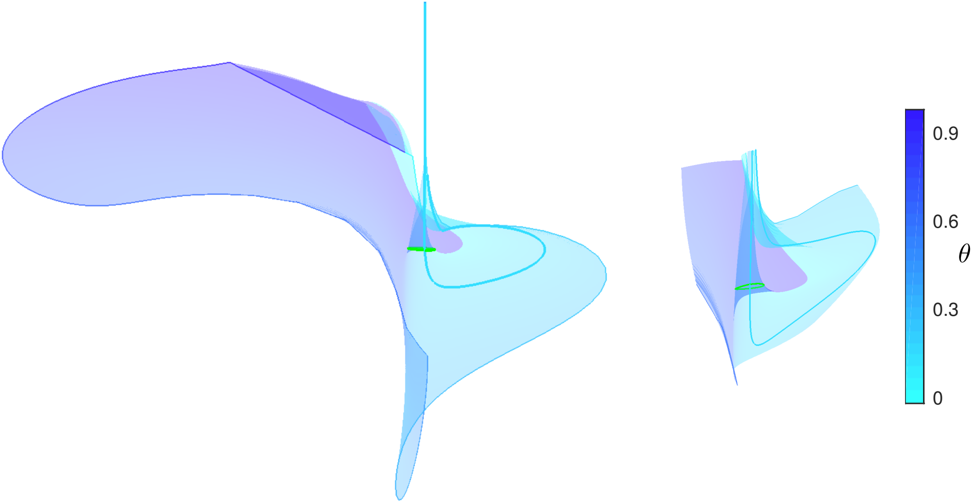


Visulalising the stable invariant manifold?

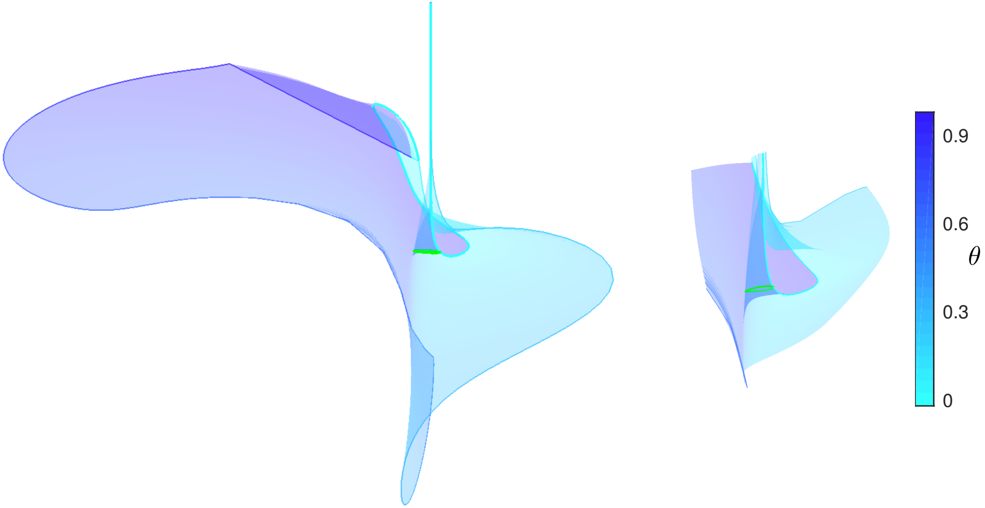


Visulalising the stable invariant manifold?

Visulalising the stable invariant manifold?



Seeing complicated geometry with isochrons



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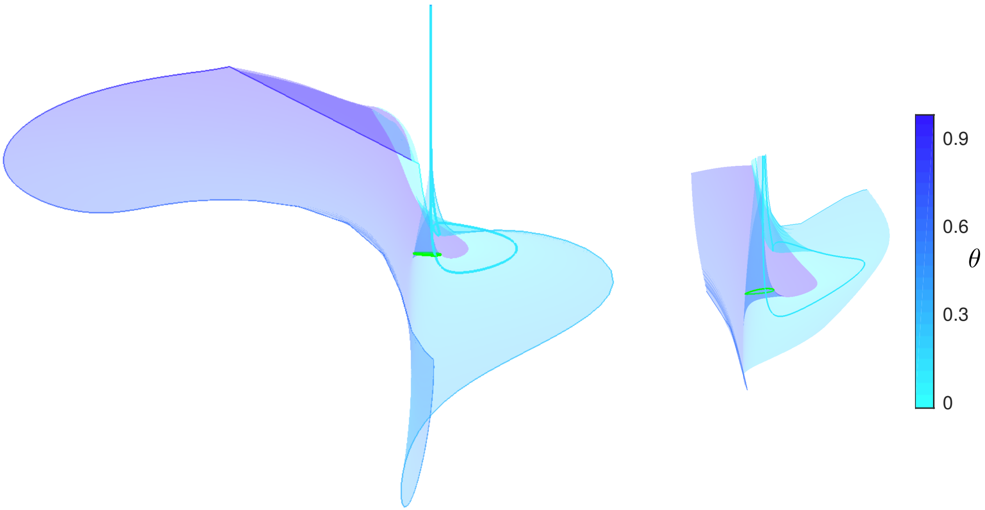

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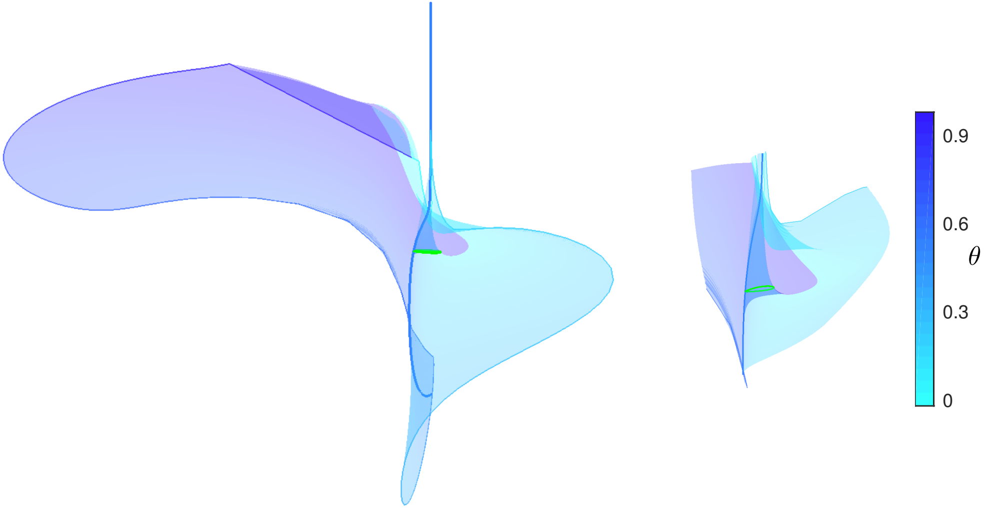

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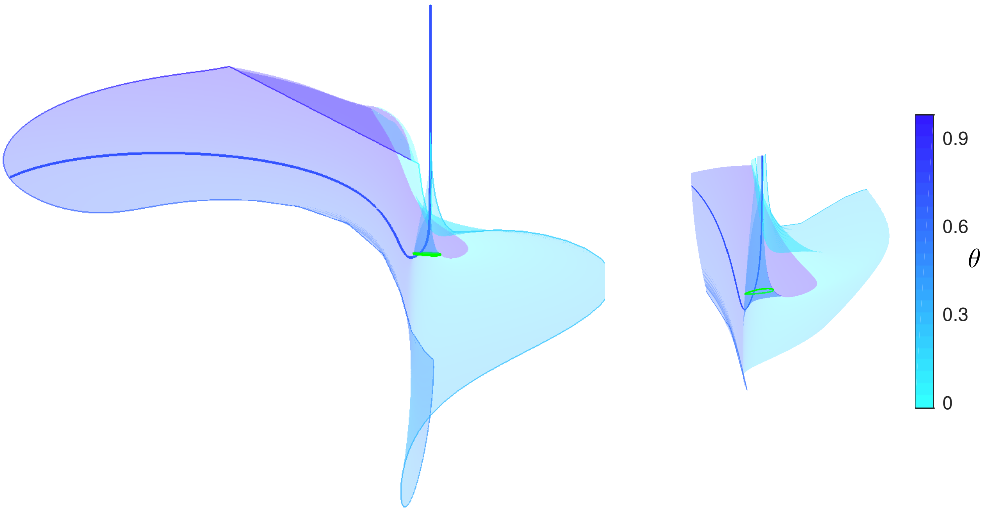

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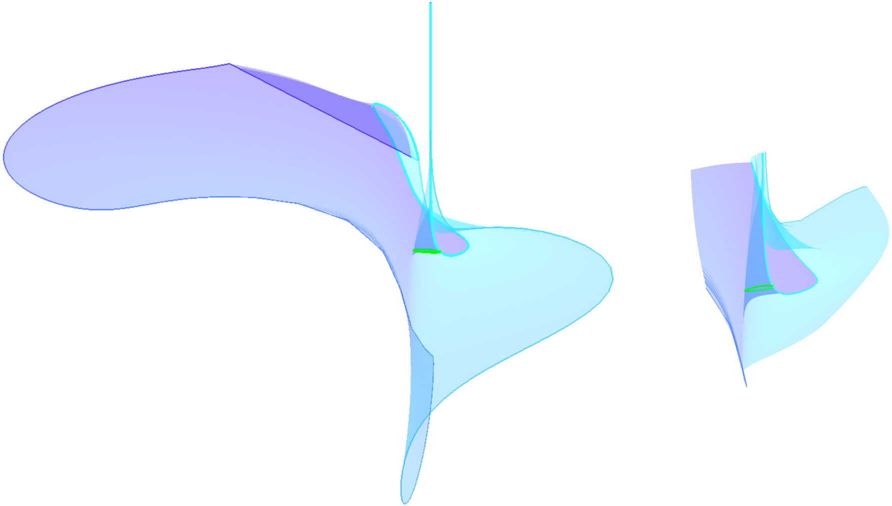
Seeing complicated geometry with isochrons



Arneodo's System

For parameter values,

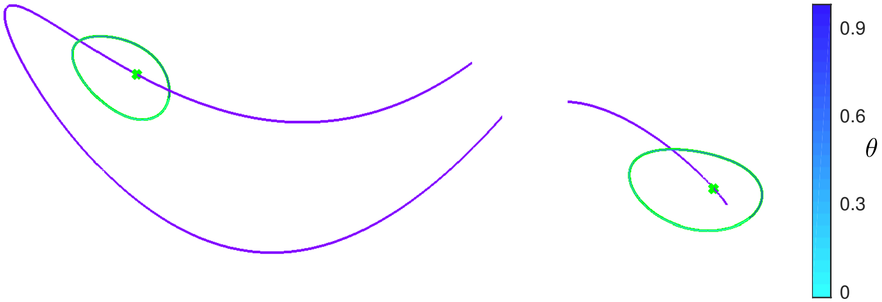
$$\alpha = 3.2, \quad \beta = 2.0,$$

this system[4] contains a non-orientable saddle-type periodic orbit.

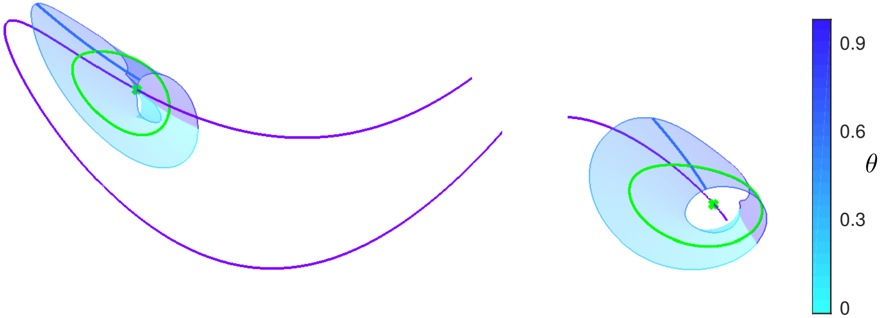
$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= (\alpha - x)x - \beta y - z\end{aligned}$$

[4] A. Arneodo, P.H. Coullet, E.A.Spiegel, C. Tresser, *Asymptotic chaos,* Physica, D14 (1985)

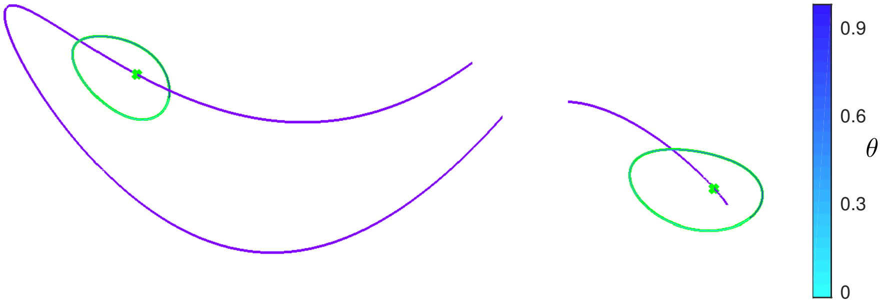
How does a Mobiüs strip grow?



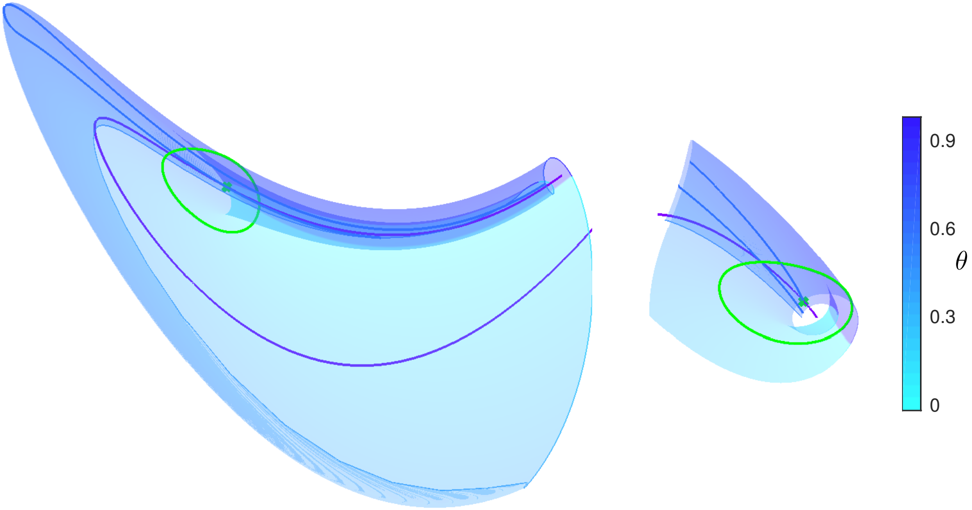
How does a Mobiüs strip grow?



How does a Mobiüs strip grow?



How does a Mobiüs strip grow?



THE UNIVERSITY OF AUCKLAND
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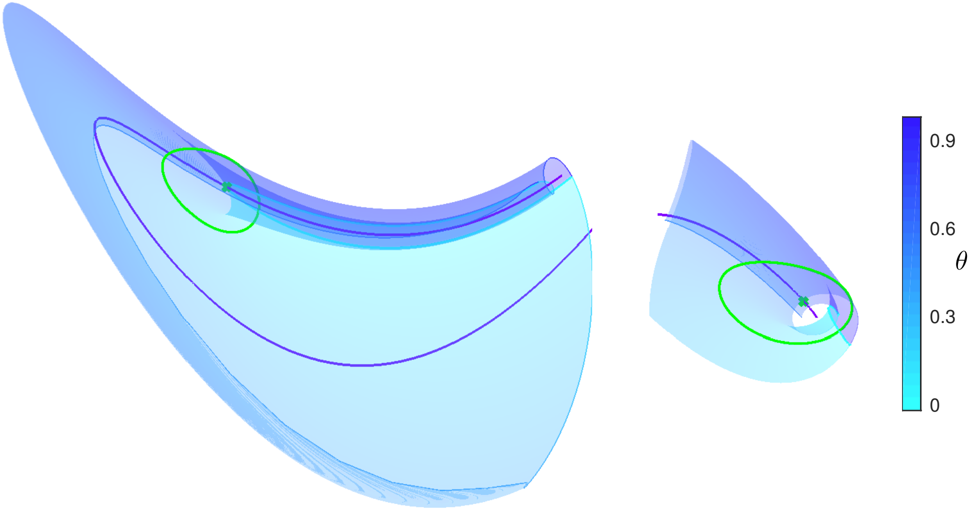

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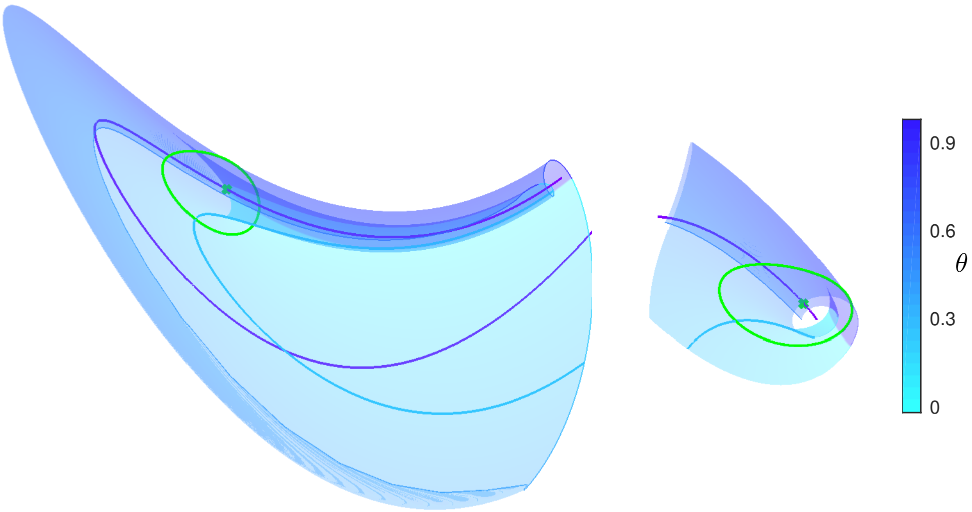

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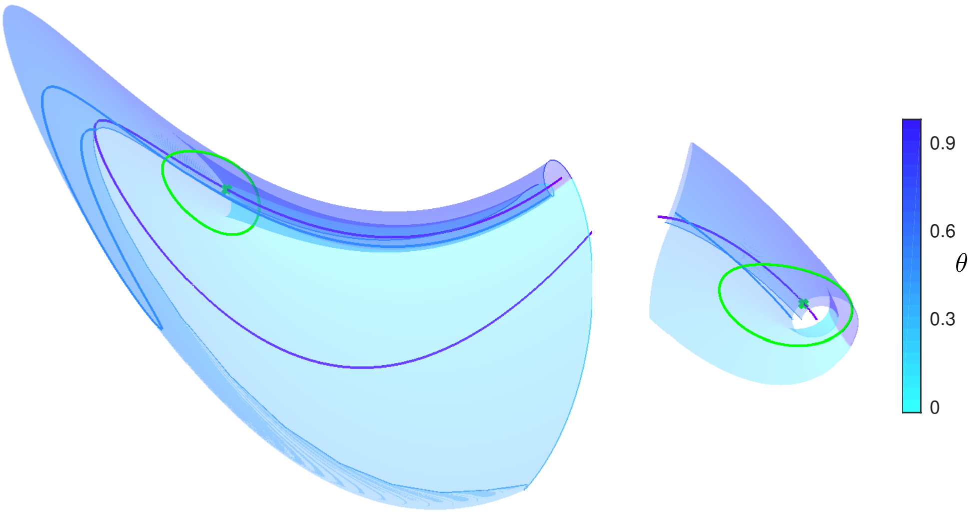

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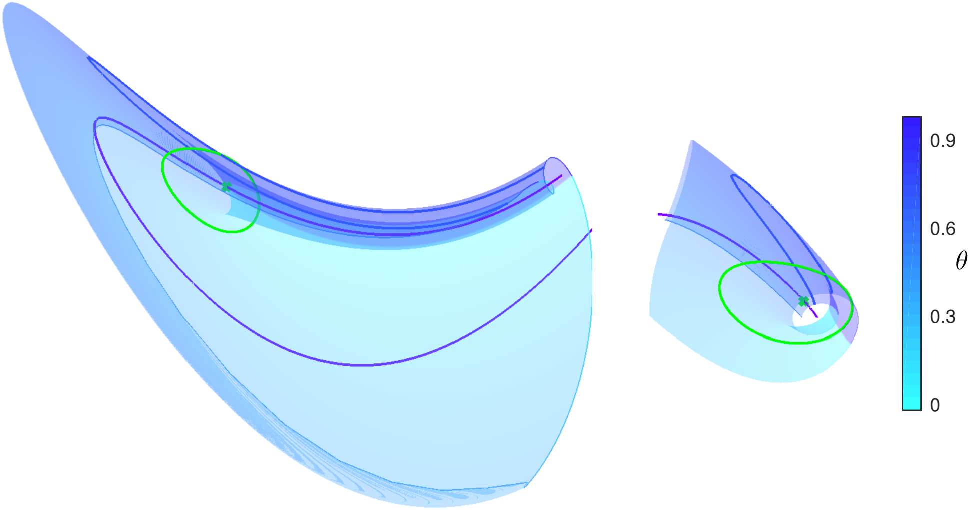

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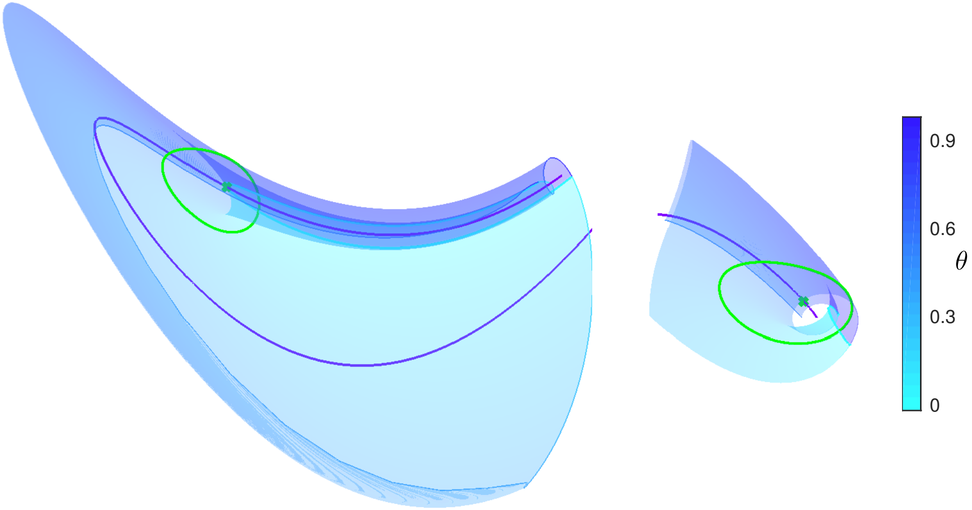
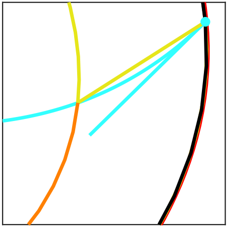
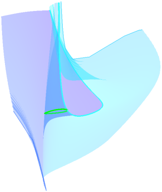
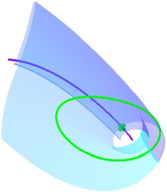


Conclusion

Work so far

- Compactification is a useful tool in the realisation of global isochron geometry.
- We can compute isochrons on the invariant manifolds of saddle type periodic orbits.
- Visualising manifolds in terms of their isochrons is useful in determining their geometry and embedded dynamics.

Future endeavours

- Investigate the interactions of forward and backward time isochrons in 3D.
- Compute the isochrons of purely attracting periodic orbits in 3D.