Input-Output Analysis of Channel Flows

and Implications for Flow Control

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Transition & Turbulence as Natural Phenomena

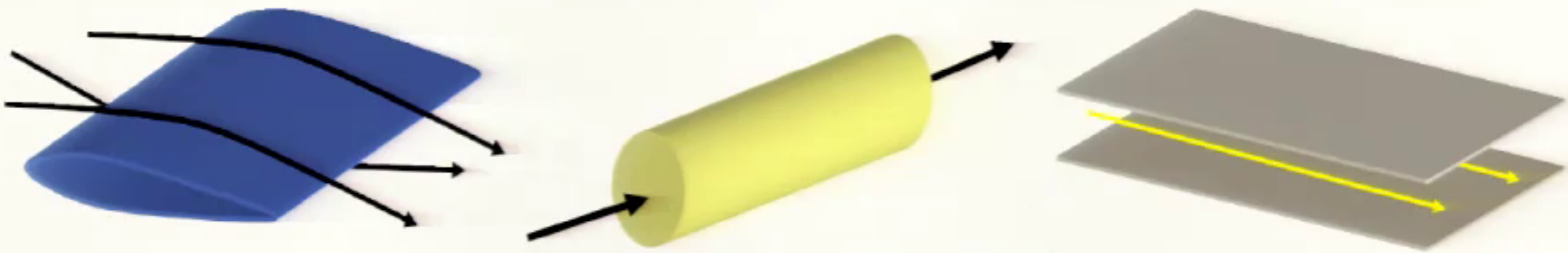
All fluid flows *transition* (as $0 \xrightarrow{R} \infty$) from laminar to turbulent flows

- Bluff bodies

dominant phenomenon: *separation*

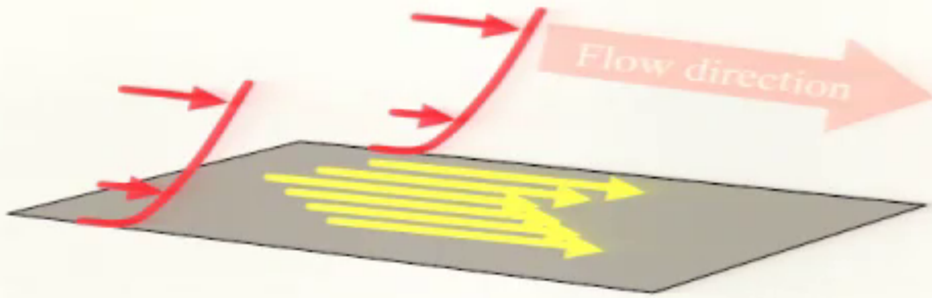
- Streamlined bodies

dominant phenomenon: *friction with walls*

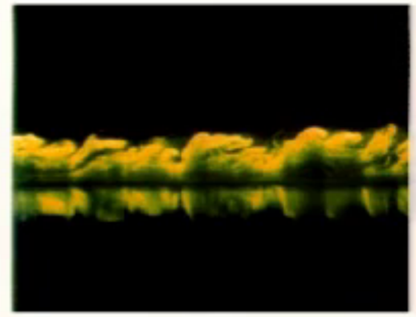


wall-bounded shear flows

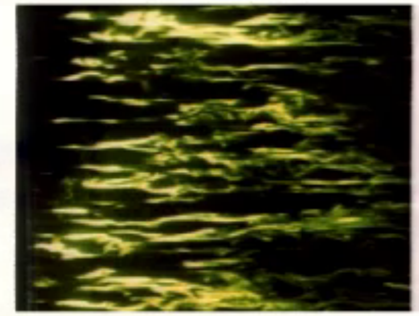
Boundary Layer Turbulence



boundary layer turbulence



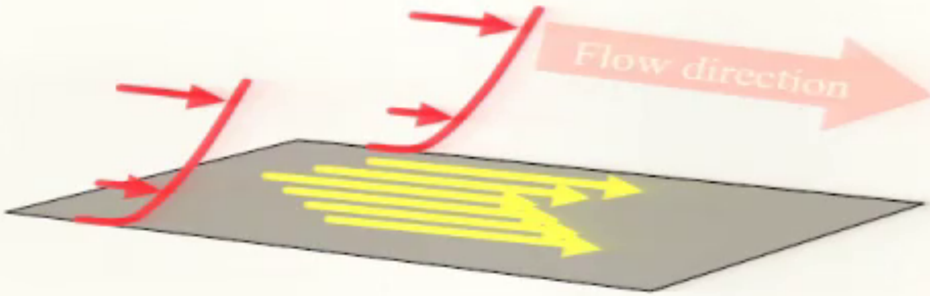
side view



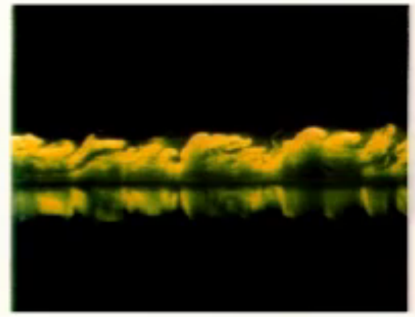
top view

- Transition & Turbulence in Boundary Layer and Channel Flows

Boundary Layer Turbulence



boundary layer turbulence



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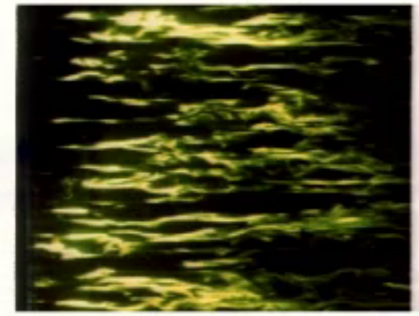

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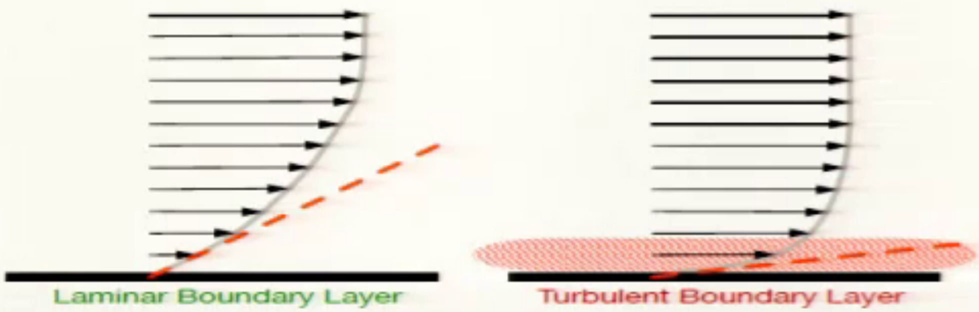
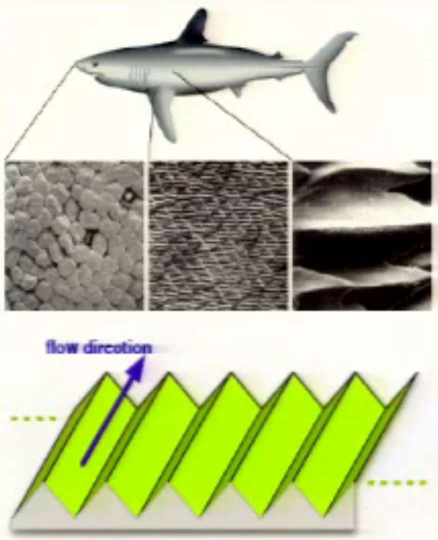


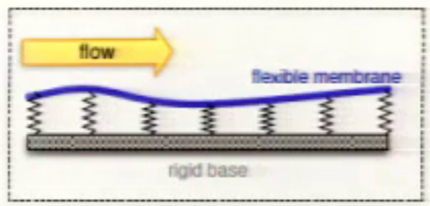
skin-friction drag: laminar vs. turbulent

- Transition & Turbulence in Boundary Layer and Channel Flows
- Technologically Important: *Skin-Friction Drag*

Control of Boundary Layer Turbulence

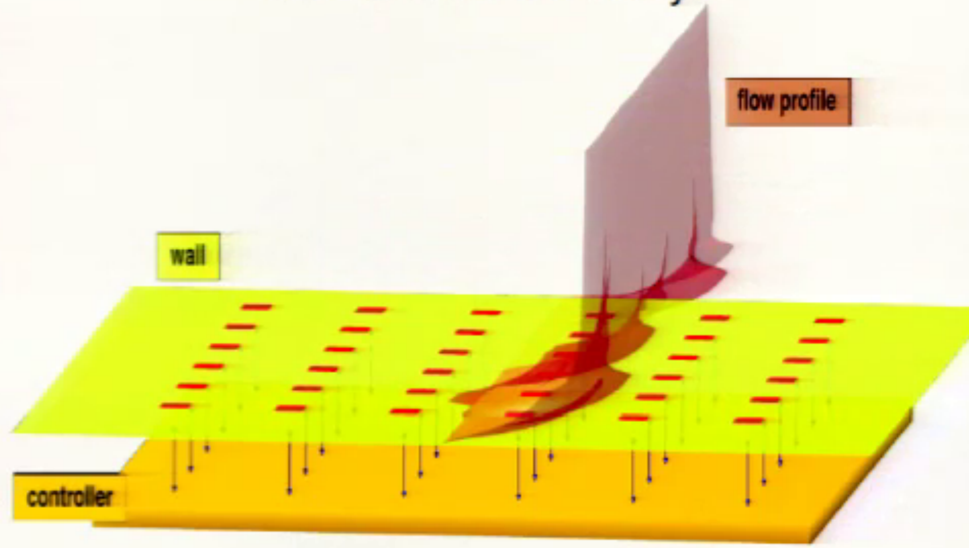
"passive" control





corrugated skin　　　　compliant skin

active control with
sensor/actuator arrays



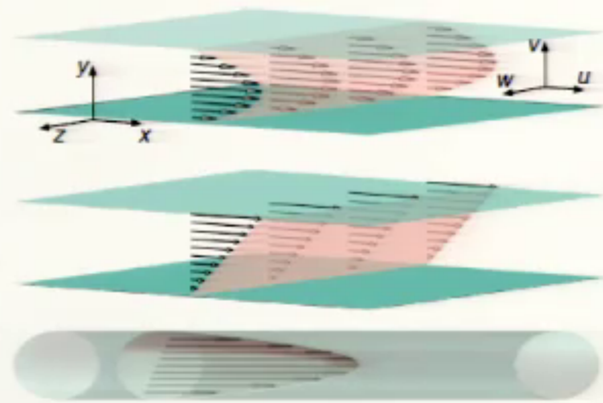
- Other "open loop" schemes:
  - Oscillating walls
  - Body force traveling waves

Caveat: *Plant's dynamics are not well understood*

obstacles { not only device technology
also: *dynamical modeling and control design*

Mathematical Modeling of Transition: Hydrodynamic Stability

The Navier-Stokes (NS) equations:



$$
\begin{aligned}
\partial_t \mathbf{u} &= -\nabla_{\mathbf{u}} \mathbf{u} - \text{grad}\, p + \tfrac{1}{R} \Delta \mathbf{u} \\
0 &= \text{div}\, \mathbf{u}
\end{aligned}
$$

- Hydrodynamic Stability: view NS as a dynamical system
- *laminar flow* $\bar{\mathbf{u}}_R :=$ a stationary solution of the NS equations (an *equilibrium*)

Mathematical Modeling of Transition: Hydrodynamic Stability

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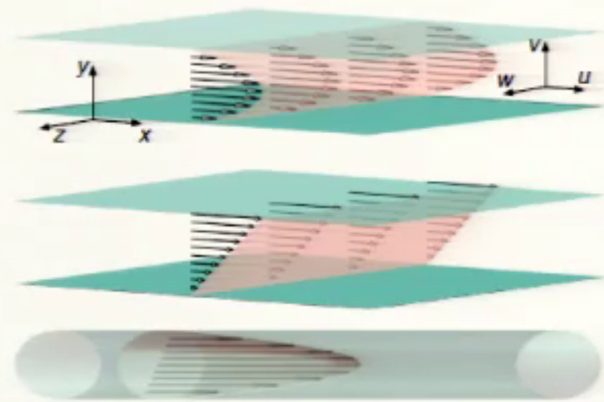

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- Hydrodynamic Stability: view NS as a dynamical system
- *laminar flow* $\bar{\mathbf{u}}_R :=$ a stationary solution of the NS equations (an *equilibrium*)

laminar flow $\bar{\mathbf{u}}_R$ stable $\longleftrightarrow$ i.c. $\mathbf{u}(0) \neq \bar{\mathbf{u}}_R$, $\mathbf{u}(t) \stackrel{t\to\infty}{\longrightarrow} \bar{\mathbf{u}}_R$

- typically done with dynamics linearized about $\bar{\mathbf{u}}_R$
- various methods to track further "non-linear behavior"

Mathematical Modeling of Transition: Hydrodynamic Stability

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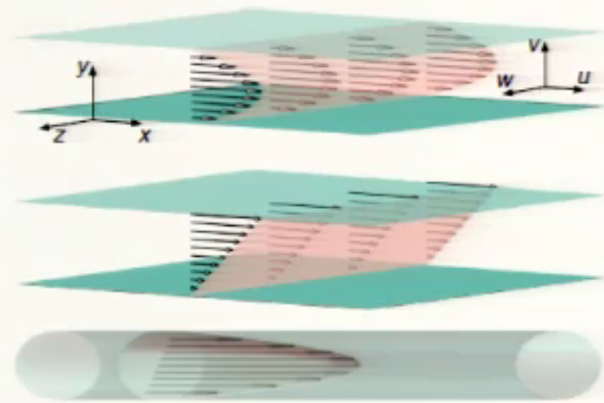

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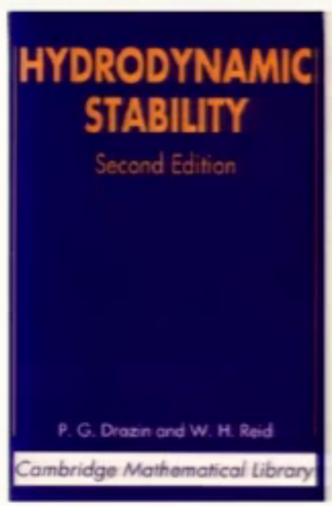
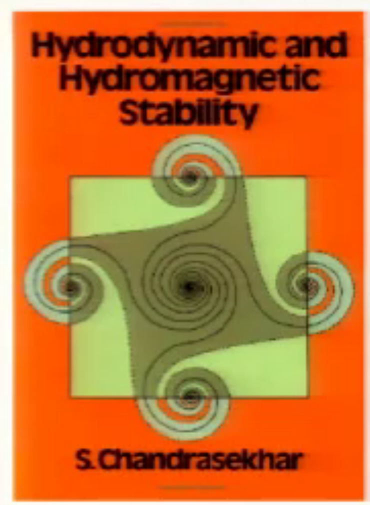
- Hydrodynamic Stability: view NS as a dynamical system
- A very successful (*phenomenologically predictive*) approach for many decades





Mathematical Modeling of Transition: Hydrodynamic Stability

The Navier-Stokes (NS) equations:

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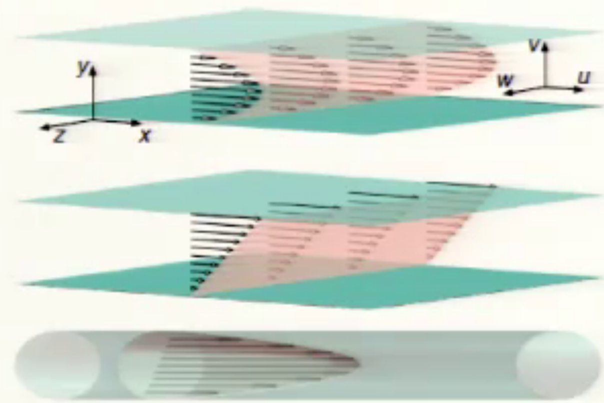

- Hydrodynamic Stability: view NS as a dynamical system
- however: problematic for wall-bounded shear flows

does not predict transition *Reynolds numbers*

Flow type	Classical linear theory R_c	Experimental R_c
Channel Flow	5772	≈ 1,000-2,000
Plane Couette	∞	≈ 350
Pipe Flow	∞	≈ 2,200-100,000

Mathematical Modeling of Transition: Hydrodynamic Stability

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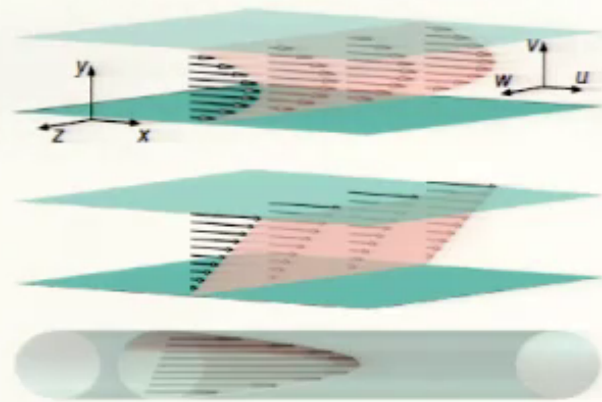

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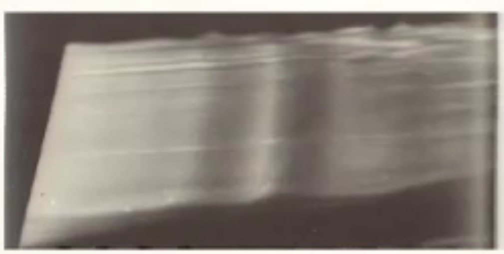
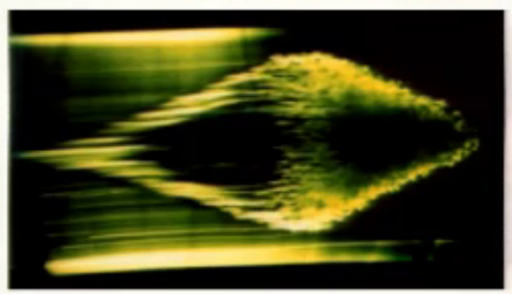
- Hydrodynamic Stability: view NS as a dynamical system
- however: problematic for wall-bounded shear flows

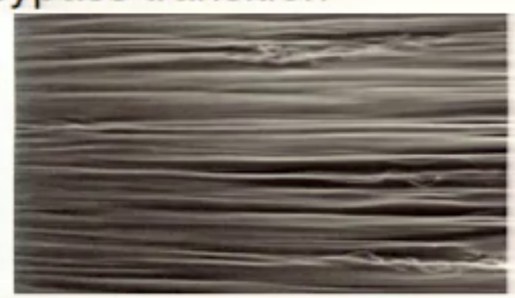
does not predict most transition *flow structures*

prediction: T-S waves



more common: bypass transition





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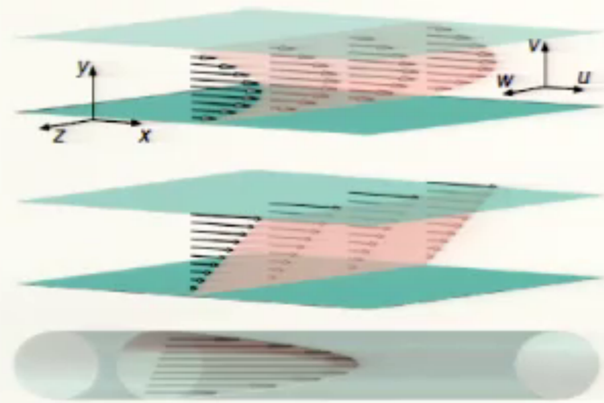


$$\begin{aligned} \partial_t \mathbf{u} &= -\nabla_{\mathbf{u}} \mathbf{u} - \text{grad}\, p + \tfrac{1}{R} \Delta \mathbf{u} \\ 0 &= \text{div}\, \mathbf{u} \end{aligned}$$

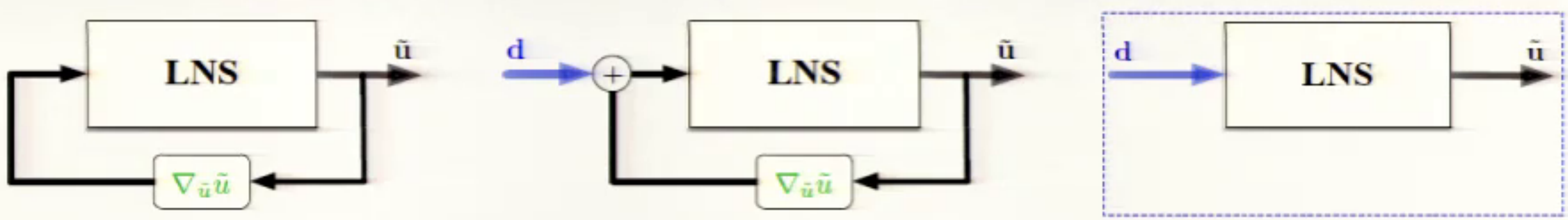
- Hydrodynamic Stability: view NS as a dynamical system
- however: problematic for wall-bounded shear flows
  - was widely believed: *this theory fails because it is* linear *and "nonlinear effects" are important even for infinitesimal i.c.*
  - *however, since 90's: story is actually more interesting than that*

Nonmodal Stability Theory, *Schmid, ARFM '07*

Mathematical Modeling of Transition: Incorporating Uncertainty

- Decompose the fields as $\mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$, where $\bar{\mathbf{u}}$ ↑ laminar and $\tilde{\mathbf{u}}$ ↑ fluctuations
- Add a time-varying *exogenous disturbance* field $\mathbf{d}$ (e.g. random body forces)

$$\begin{aligned} \partial_t \tilde{\mathbf{u}} &= -\nabla_{\bar{\mathbf{u}}}\tilde{\mathbf{u}} - \nabla_{\tilde{\mathbf{u}}}\bar{\mathbf{u}} - \text{grad}\,\tilde{p} + \tfrac{1}{R}\Delta\tilde{\mathbf{u}} - \nabla_{\tilde{u}}\tilde{u} + \mathbf{d} \\ 0 &= \text{div}\,\tilde{\mathbf{u}} \end{aligned}$$



Input-Output view of the Linearized NS Equations

Farrell, Ioannou, '93 PoF

BB, Dahleh, '01 PoF

Jovanovic, BB, '05 JFM

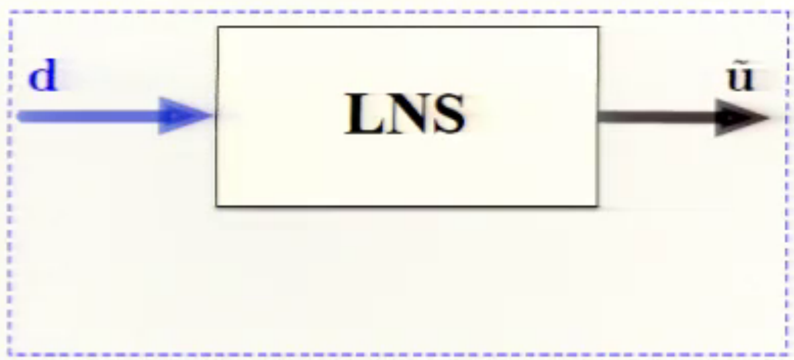
Mathematical Modeling of Transition: Incorporating Uncertainty

- Decompose the fields as $\mathbf{u} \;=\; \underset{\text{laminar}}{\underset{\uparrow}{\bar{\mathbf{u}}}} \;+\; \underset{\text{fluctuations}}{\underset{\uparrow}{\tilde{\mathbf{u}}}}$

- Add a time-varying *exogenous disturbance* field $\mathbf{d}$ (e.g. random body forces)

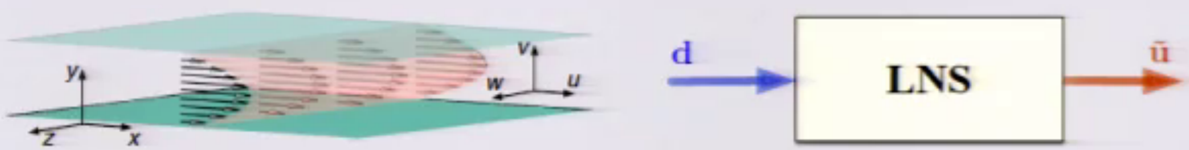
$$\begin{aligned} \partial_t \tilde{\mathbf{u}} &= -\nabla_{\bar{\mathbf{u}}} \tilde{\mathbf{u}} - \nabla_{\tilde{\mathbf{u}}} \bar{\mathbf{u}} - \text{grad}\, \tilde{p} + \tfrac{1}{R} \Delta \tilde{\mathbf{u}} - \nabla_{\tilde{u}} \tilde{u} + \mathbf{d} \\ 0 &= \text{div}\, \tilde{\mathbf{u}} \end{aligned}$$

- Neglect the feedback $\nabla_{\tilde{u}} \tilde{u}$



$$\partial_t \begin{bmatrix} \Delta\tilde{v} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} U''\partial_x - U\Delta\partial_x + \frac{1}{R}\Delta^2 & 0 \\ -U'\partial_z & -U\partial_x + \frac{1}{R}\Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_z & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \left(\partial_x^2 + \partial_z^2\right)^{-1} \begin{bmatrix} \partial_{xy} & -\partial_z \\ \partial_x^2 + \partial_z^2 & 0 \\ \partial_{zy} & \partial_x \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix}$$



$$\begin{aligned} \partial_t \Psi &= \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\,\Psi \end{aligned}$$

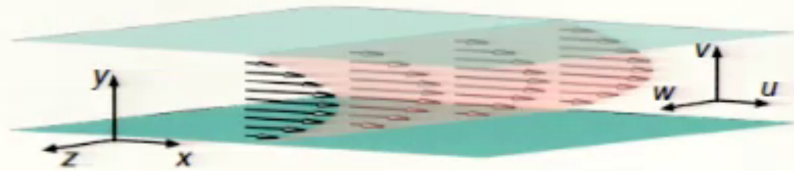
Surprises:

- Even when $\mathcal{A}$ is stable — the gain $\mathbf{d} \longrightarrow \tilde{\mathbf{u}}$ can be very large
  ($(H^2$ norm$)^2$ scales with R^3)
- Input-output resonances — very different from least-damped modes of $\mathcal{A}$

has important implications for control design!

Spatio-temporal Impulse and Frequency Responses

Translation invariance in x & z implies



- *Impulse Response* (*Green's Function*)

$$\tilde{\mathbf{u}}(t,x,y,z) \;=\; \int G(t-\tau, x-\xi, \text{ y,y' }, z-\zeta)\; \mathbf{d}(\tau,\xi,y',\zeta)\; d\tau d\xi dy' d\zeta$$

$$\tilde{\mathbf{u}}(t,x,.,z) \;=\; \int \mathcal{G}(t-\tau, x-\xi, z-\zeta)\; \mathbf{d}(\tau,\xi,.,\zeta)\; d\tau d\xi d\zeta$$

$\mathcal{G}(t,x,z)$: *Operator-valued* impulse response

- *Frequency Response*

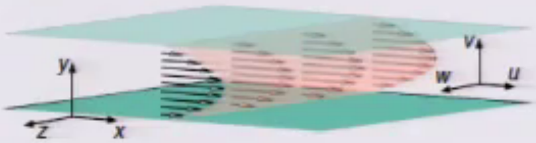
$$\tilde{\mathbf{u}}(\omega,k_x,k_z) \;=\; \mathcal{G}(\omega,k_x,k_z)\; \mathbf{d}(\omega,k_x,k_z)$$

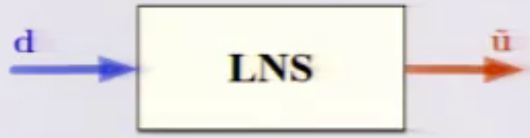
$\mathcal{G}(\omega,k_x,k_z)$: *Operator-valued* frequency response (Packs lots of information!)

- *Spectrum of* $\mathcal{A}$:

$$\sigma(\mathcal{A}) \;=\; \overline{\bigcup_{k_x,k_z} \sigma\left(\hat{A}(k_x,k_z)\right)}$$

Modal vs. Input-Output Analysis





- $$\begin{aligned} \partial_t \Psi &= \mathcal{A}\, \Psi + \mathcal{B}\, \mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\, \Psi \end{aligned}$$
- IR: $\mathcal{G}(t, x, z)$
- FR: $\mathcal{G}(\omega, k_x, k_z)$

Modal Analysis: Look for unstable eigs of $\mathcal{A}$ $\left(\bigcup_{k_x,k_z} \sigma\left(\hat{\mathcal{A}}(k_x, k_z)\right)\right)$

Flow type	Classical linear theory R_c	Experimental R_c
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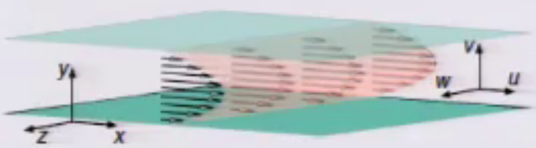

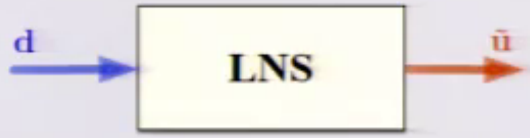

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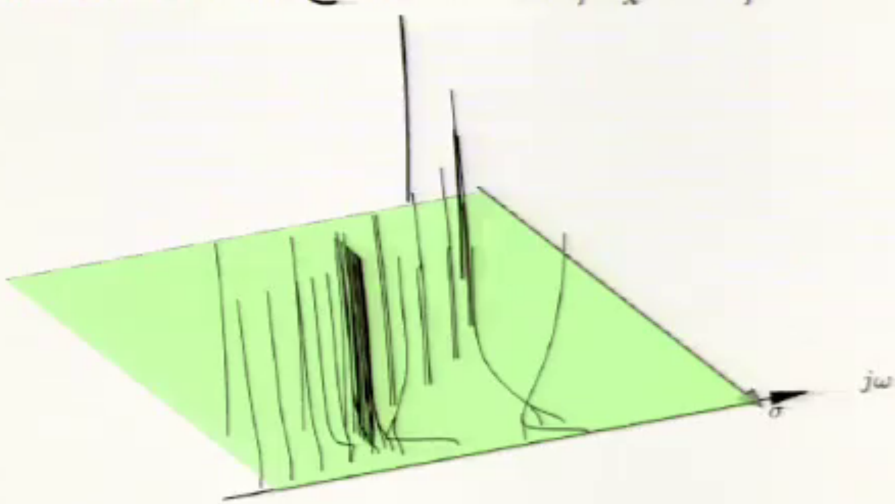
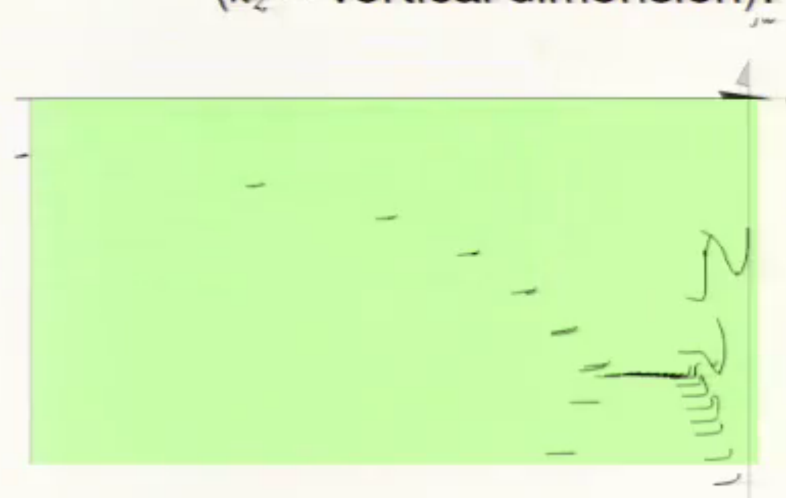
Modal Analysis: Look for unstable eigs of $\mathcal{A}$ $\left(\bigcup_{k_x,k_z} \sigma\left(\hat{\mathcal{A}}(k_x, k_z)\right)\right)$

- Channel Flow @ $R = 2000$, $k_x = 1$, (k_z = vertical dimension):





top view

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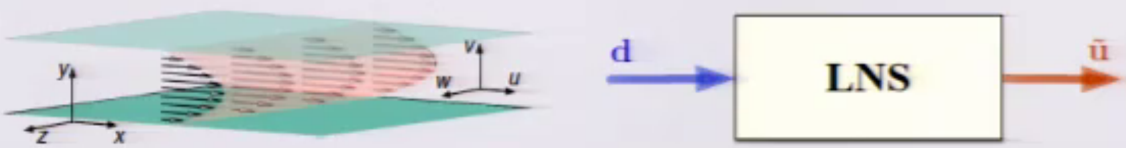

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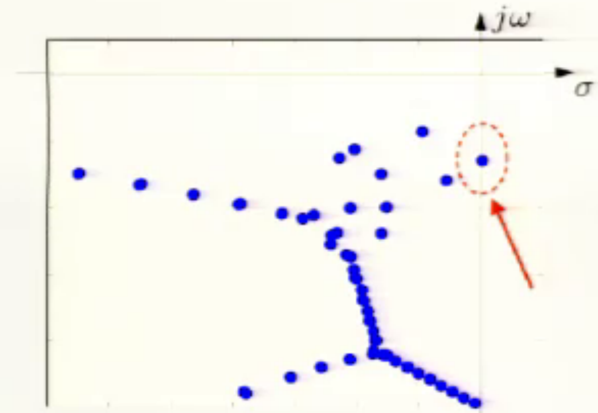
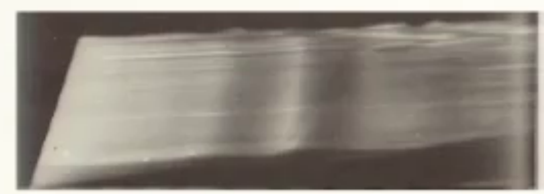
- Channel Flow @ $R = 6000$, $k_x = 1$, $k_z = 0$:



- Flow structure of corresponding eigenfunction: Tollmein-Schlichting (TS) waves



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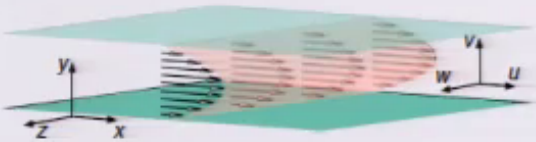

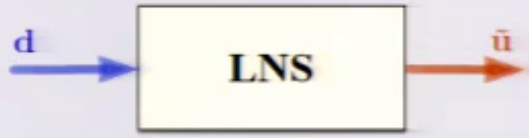

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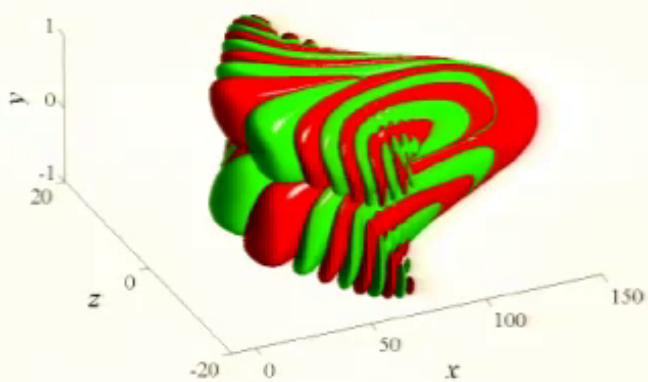
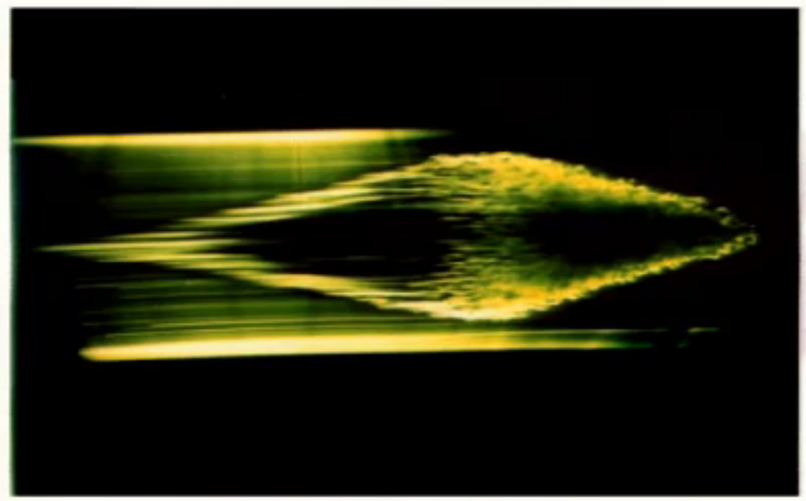
Impulse Response Analysis: Channel Flow @ $R = 2000$



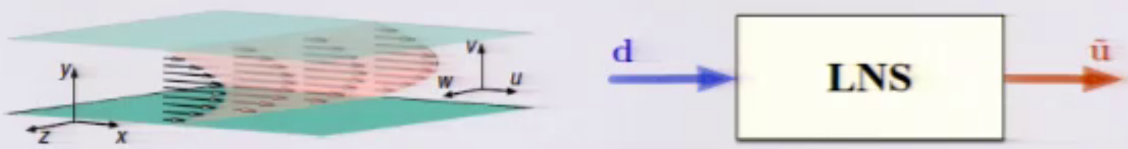


cf. "turbulent spots"

Jovanovic, BB, '01 ACC,

more movies and pics at `http://engineering.ucsb.edu/~bamieh/pics/impulse_page.html`

Modal vs. Input-Output Analysis



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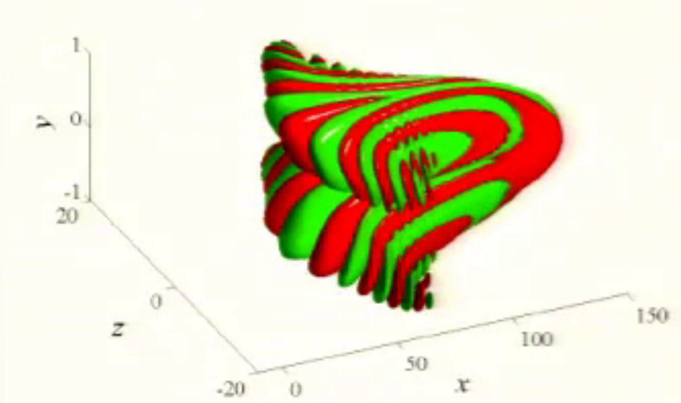
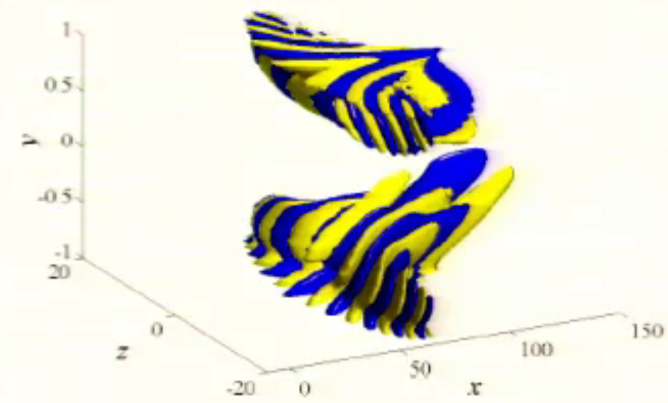


streamwise velocity



streamwise vorticity

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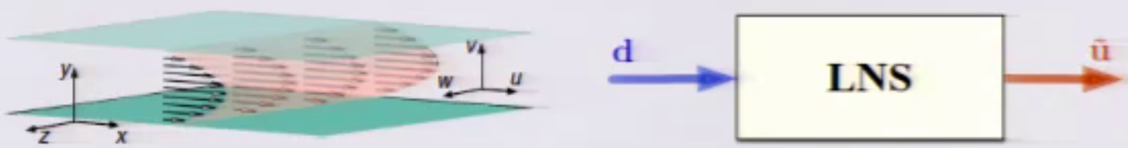

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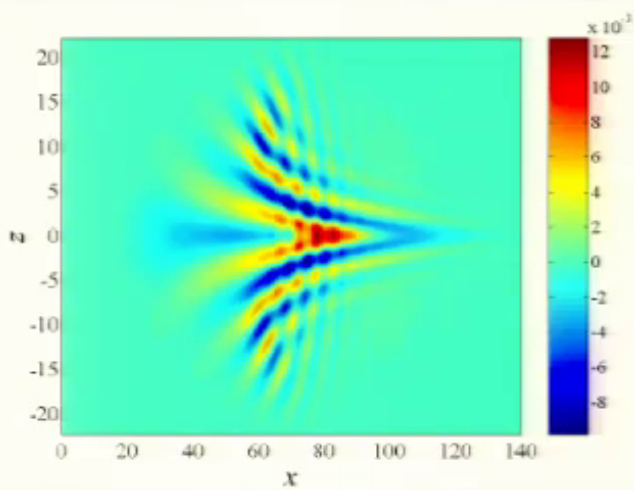
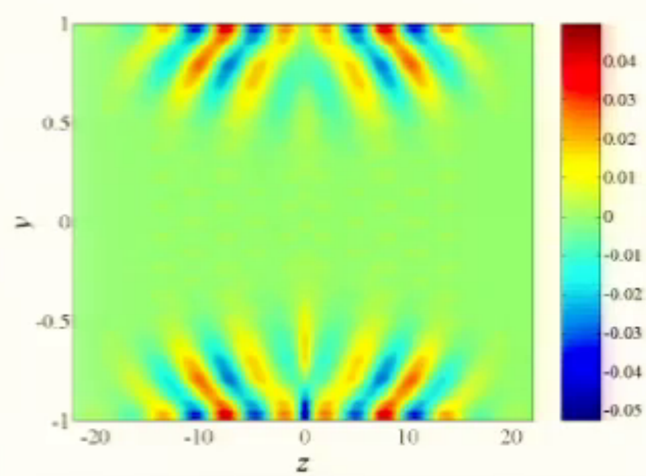
Impulse Response Analysis: Channel Flow @ $R = 2000$

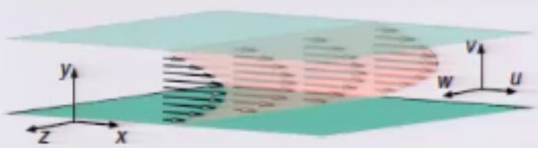


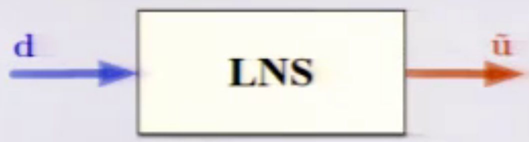
u in a horizontal plane



u in a vertical plane

Modal vs. Input-Output Analysis





- $$\begin{aligned} \partial_t \Psi &= \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\,\Psi \end{aligned}$$
- IR: $\mathcal{G}(t, x, z)$
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- *Frequency Response*

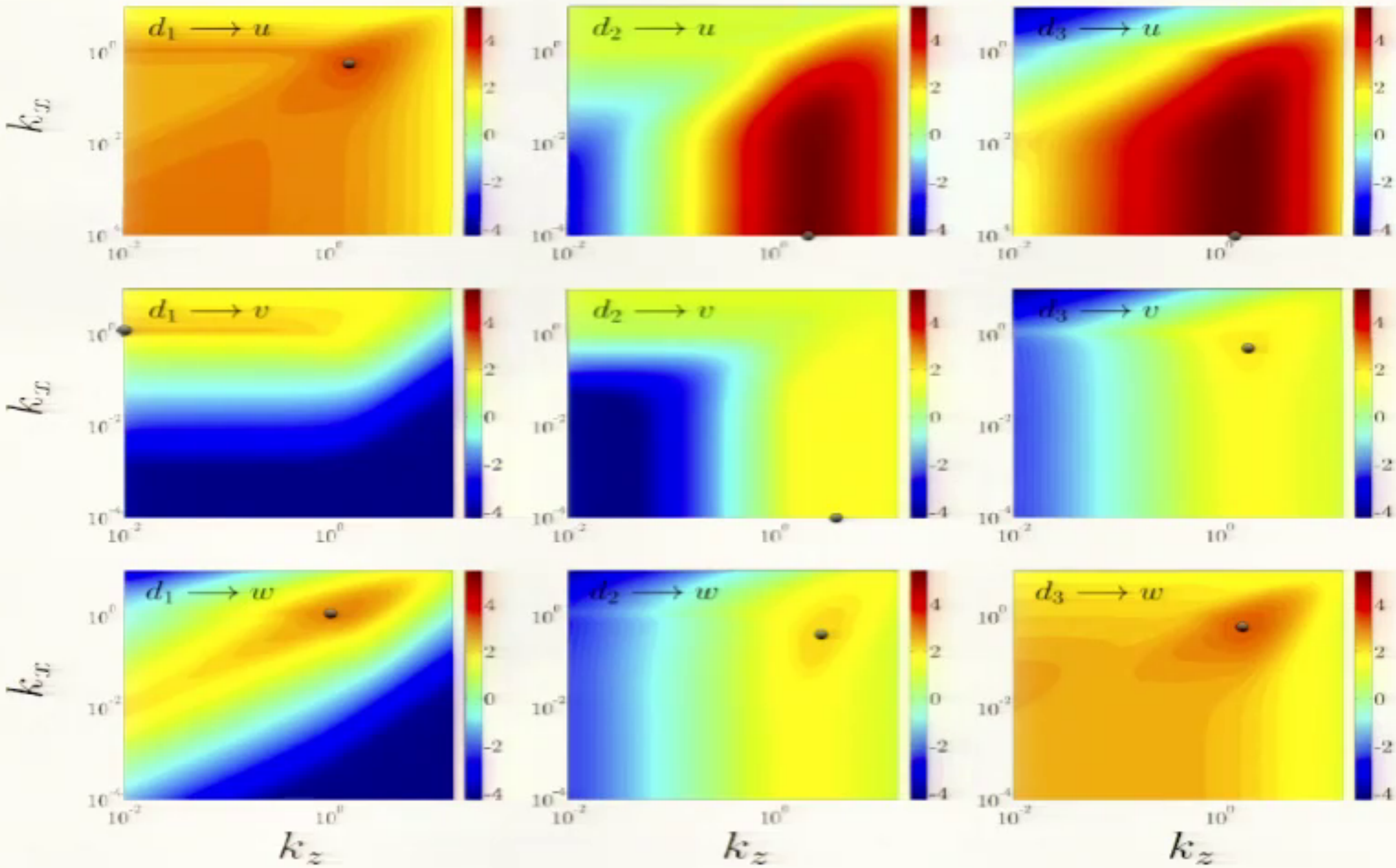
$$\tilde{\mathbf{u}}(\omega, k_x, k_z) = \mathcal{G}(\omega, k_x, k_z)\ \mathbf{d}(\omega, k_x, k_z)$$

$\mathcal{G}(\omega, k_x, k_z)$: *Operator-valued* frequency response (Packs lots of information!)

Spatio-temporal Frequency Response

$\mathcal{G}(\omega, k_x, k_z)$ is a *LARGE* object! (very "data rich"!)

one visualization: $\sup_\omega \sigma_{\max}\left(\mathcal{G}(\omega, k_x, k_z)\right)$ (max over *temporal frequency* *& wall-normal dimension*)



Jovanovic, BB, '05 JFM

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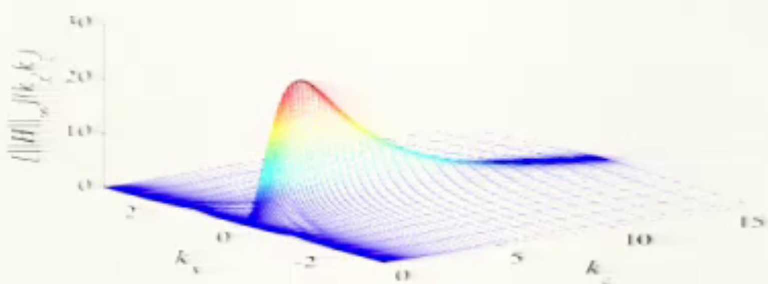
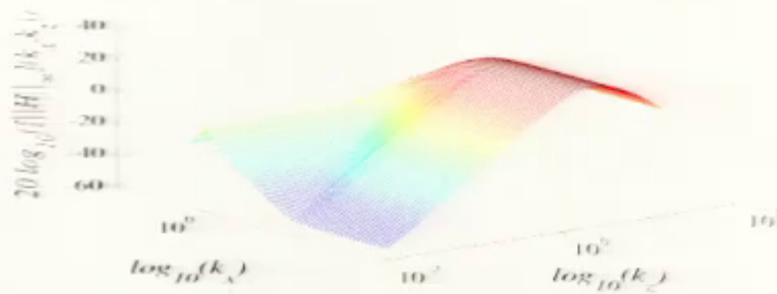
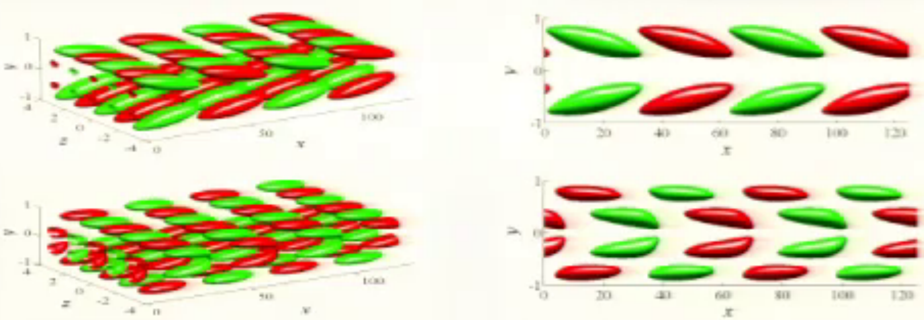
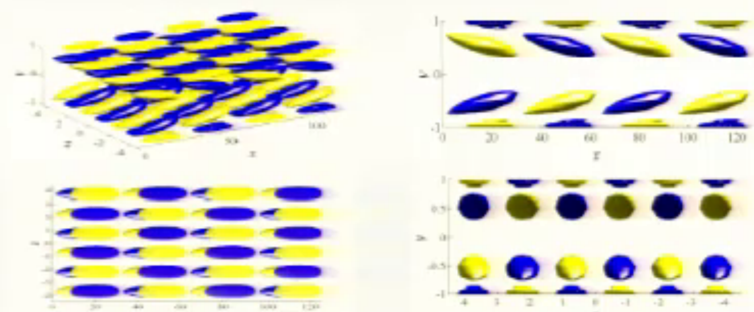




flow structure corresponding to peak:



streamwise velocity isosurfaces



streamwise vorticity isosurfaces

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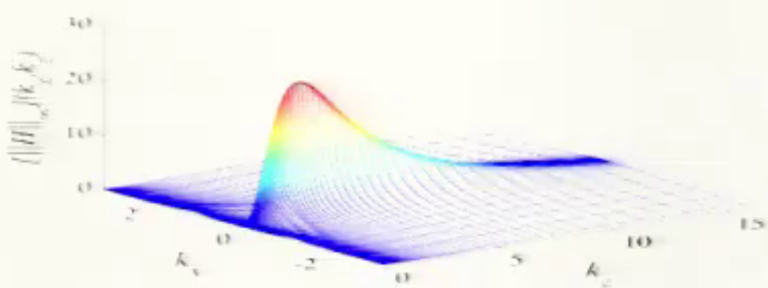
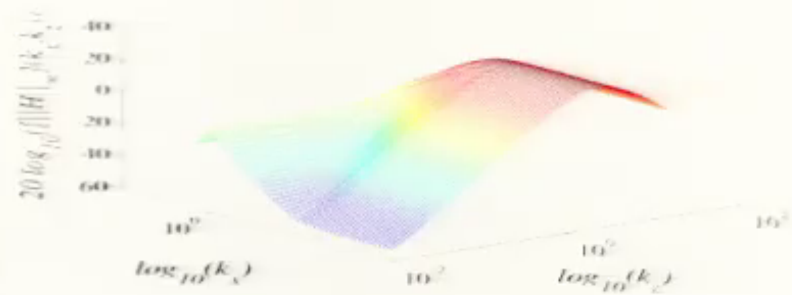
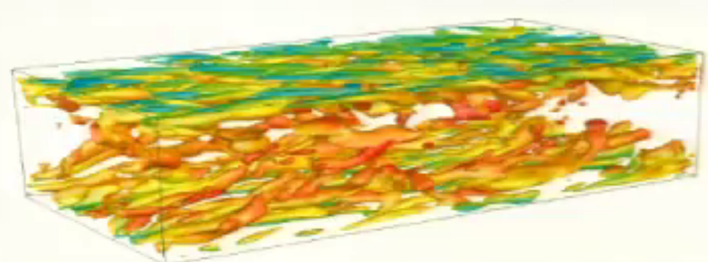
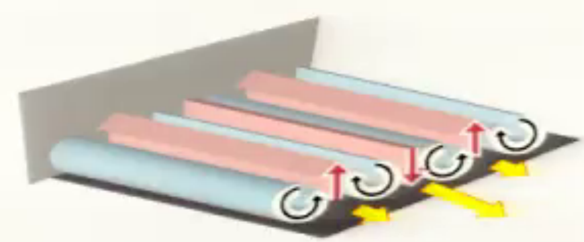




flow structure corresponding to peak:

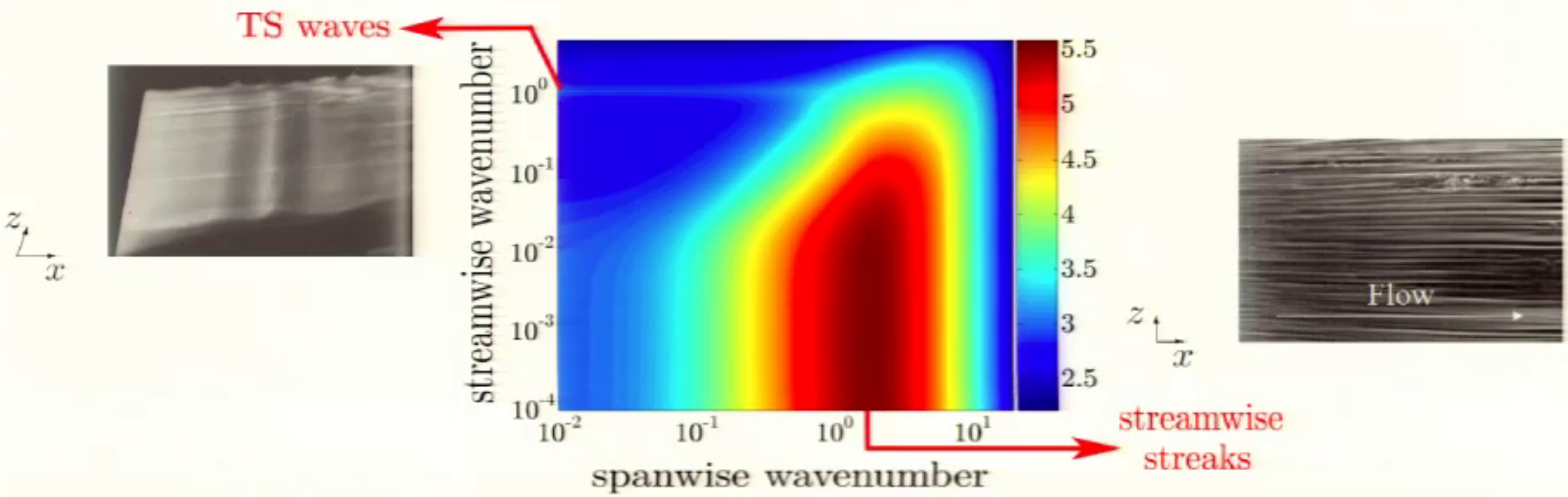
similar to observed near-wall structures





Underdamped Modes vs. Large IO Resonances

channel flow with $Re = 2000$:



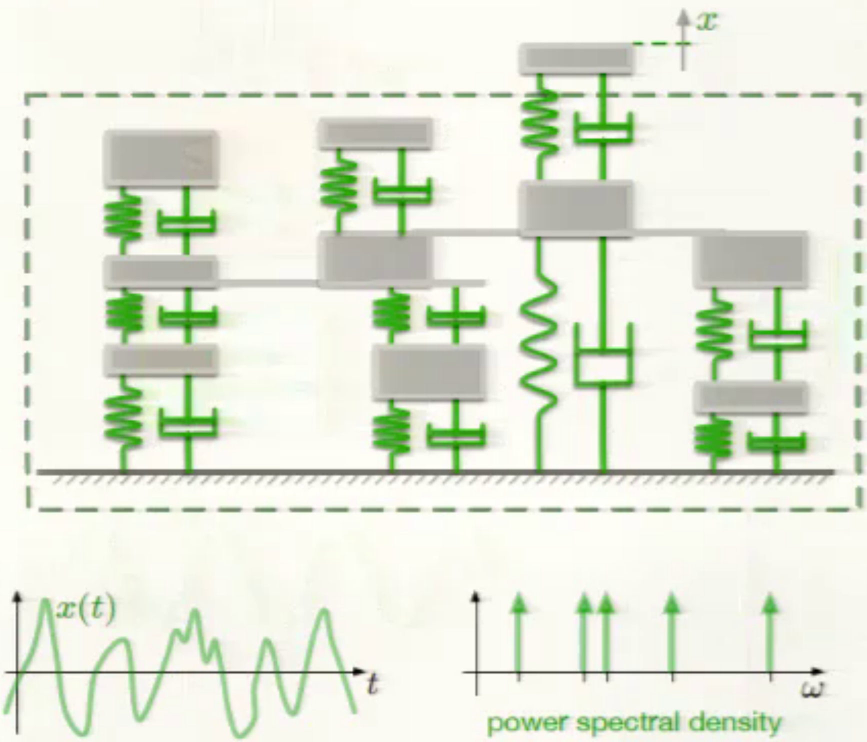
- highly underdamped TS modes barely register in the IO response
- large peaks of IO response *do not* correspond to any particular pure mode

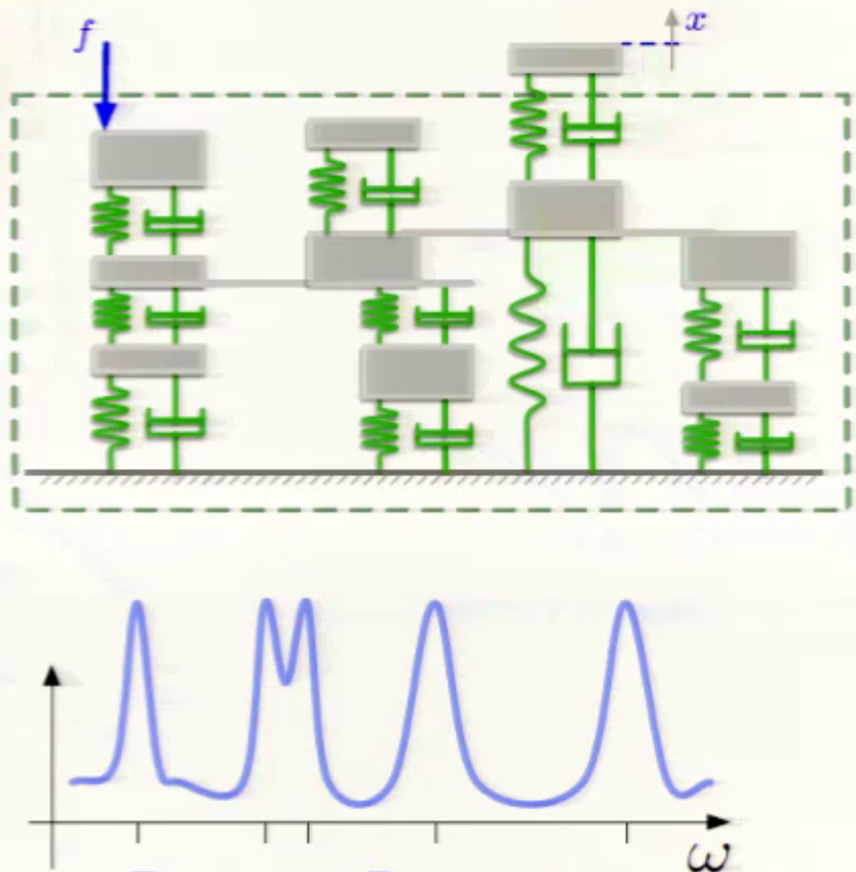
Internal Modes vs. External Resonances

A Detour

Internal Modes vs. External Resonances

Does this correspondence hold for large-scale systems?





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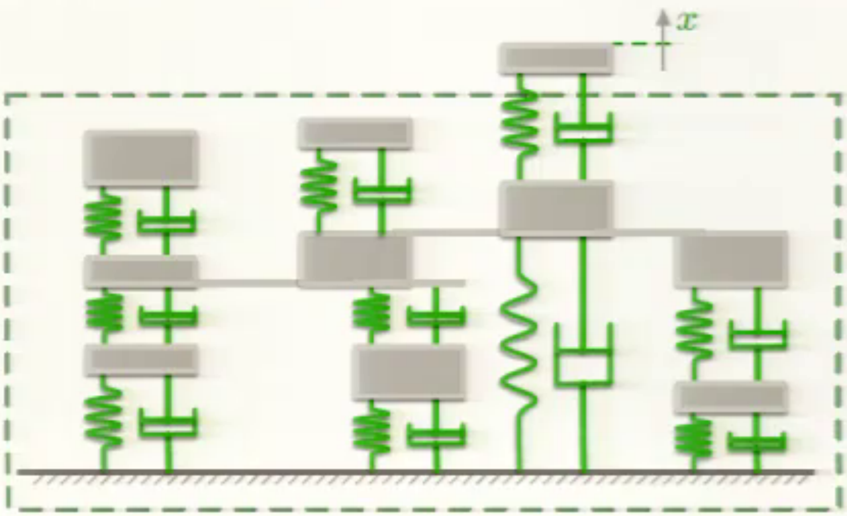
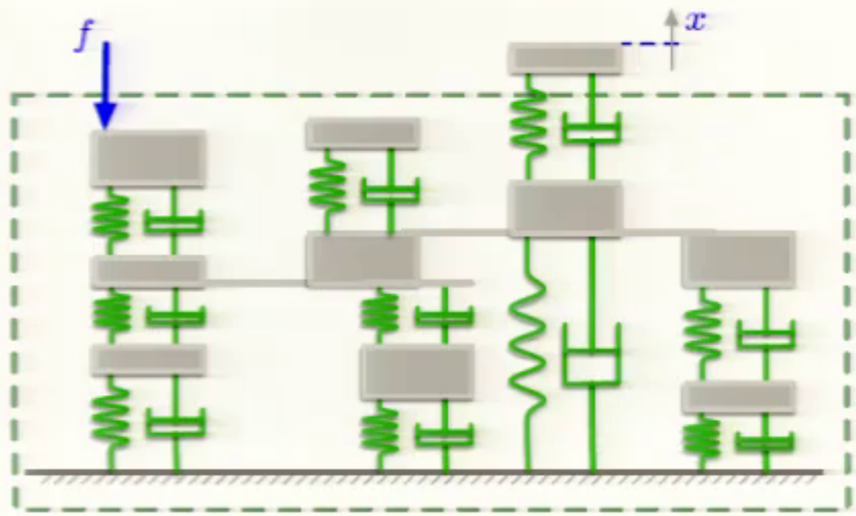
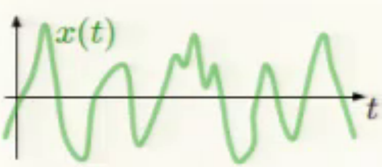
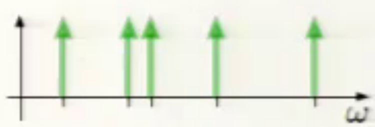
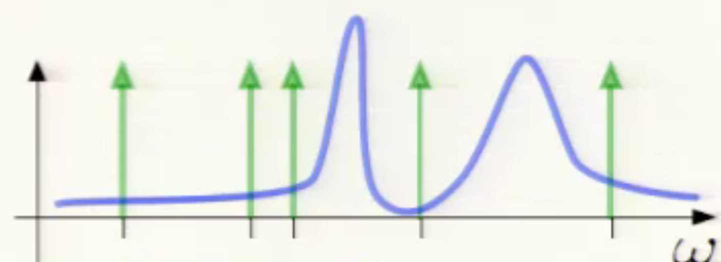










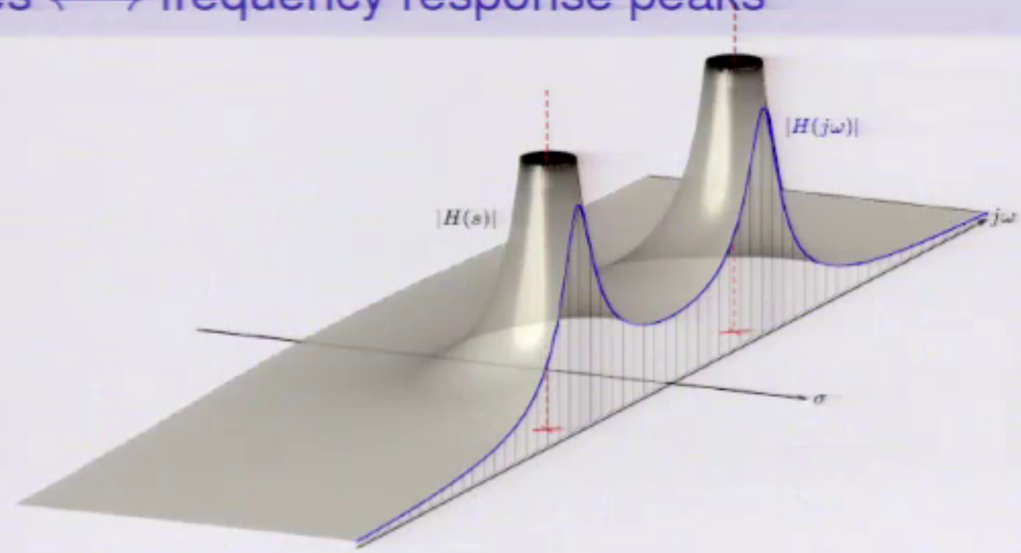
However: this may not hold in general

even in linear systems!

Modal vs. Input-Output Response

Typically: underdamped poles $\longleftrightarrow$ frequency response peaks

cf. The "rubber sheet analogy":



ODE (state space model)

$$\begin{aligned} \dot{\psi}(t) &= A\,\psi(t) + B\,d(t) \\ \tilde{\mathbf{u}}(t) &= C\,\psi(t) \end{aligned}$$

Transfer Function

$$H(s) = C\,(sI - A)^{-1}\,B$$

- eigs(A) = poles$(H(s))$

Modal vs. Input-Output Response

However: Pole Locations $\not\leftrightarrow$ Frequency Response Peaks

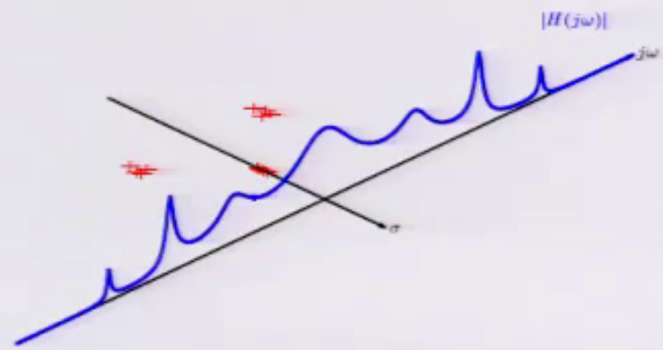
Theorem: *Given any desired pole locations*

$$z_1, \ldots, z_n \in \mathbb{C}_- \text{ (LHP)},$$

and any stable frequency response $H(j\omega)$, arbitrarily close approximation is achievable with

$$\left\| H(s) - \left(\sum_{i=1}^{N_1} \frac{\alpha_{1,i}}{(s-z_1)^i} + \cdots + \sum_{i=1}^{N_n} \frac{\alpha_{n,i}}{(s-z_n)^i} \right) \right\|_{\mathcal{H}^2} \leq \epsilon$$

by choosing any of the N_k's large enough



Remarks:

- No necessary relation between *pole locations* and *system resonances*
- $(\epsilon \to 0 \Rightarrow N_k \to \infty)$, i.e. this is a *large-scale systems* phenomenon
- Large-scale systems: IO behavior not always predictable from modal behavior

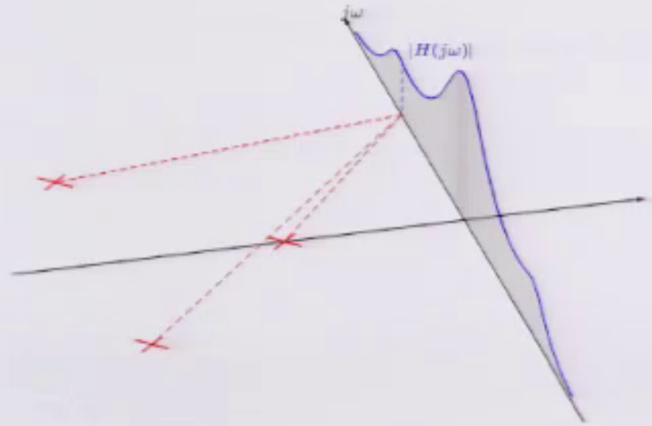
Modal vs. Input-Output Response

However: Pole Locations $\not\leftrightarrow$ Frequency Response Peaks

MIMO case: $H(s) = (sI - A)^{-1}$

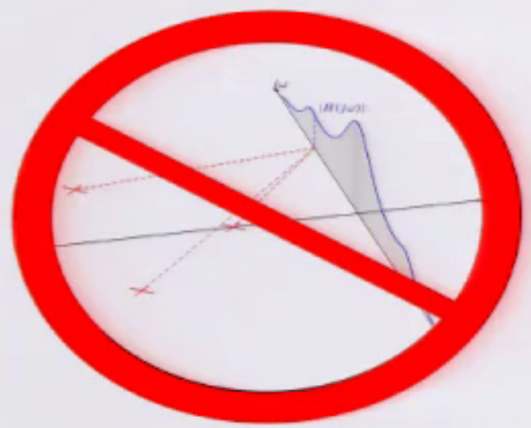
- *If A is normal* (has orthogonal eigenvectors), then

$$\sigma_{\max}\left((j\omega I - A)^{-1}\right) = \frac{1}{\text{distance}\left(j\omega, \text{nearest pole}\right)}$$



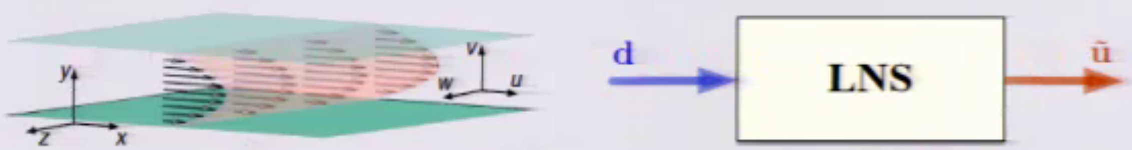
- *If A is non-normal* : no clear relation between

  singular value plot $\not\leftrightarrow$ *eigs(A)*



Implications for Turbulence Modeling

For large-scale systems: IO behavior not predictable from modal behavior



- $$\begin{aligned} \partial_t \Psi &= \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\,\Psi \end{aligned}$$
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Implications for Turbulence Modeling

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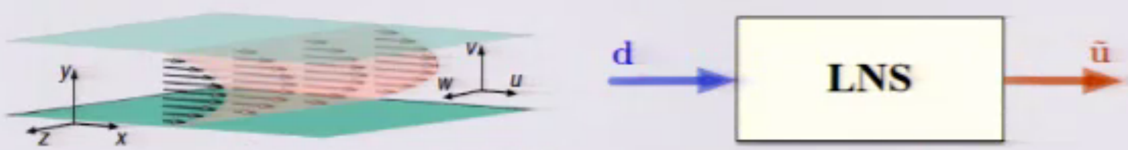

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- "modal behavior": Stability due to *initial condition uncertainty*
- "IO behavior": behavior in the presence of *ambient uncertainty*
  - forcing terms from wall roughness and/or vibrations
  - Free-stream disturbances in boundary layers
  - Thermal (Langevin) forces
  - uncertain dynamics